On the Globally Concavized Filled Function Method *

WENXING ZHU[†]

Department of Computer Science and Technology, Fuzhou University, Fuzhou 350002, China

Abstract. In this paper we present a new definition on the globally concavized filled function for the continuous global minimization problem, which was modified from that by Ge [3]. A new class of globally concavized filled functions are constructed. These functions contain two easily determinable parameters, which are not dependent on the radius of the basin at the current local minimizer. A randomized algorithm is designed to solve the box constrained continuous global minimization problem basing on the globally concavized filled functions, which can converge asymptotically with probability one to a global minimizer of the problem. Preliminary numerical experiments are presented to show the practicability of the algorithm.

Key words: Box constrained continuous global minimization problem, Globally concavized filled function, Asymptotic convergence, Stopping rule

1 Introduction

Global optimization problems arise in many fields of science and technology. Many methods have been proposed to search for global minimizers of continuous global optimization problems. Usually these approaches can be divided into two classes: stochastic global optimization methods and deterministic global optimization methods. Many deterministic methods, including the filled function method and the tunnelling method, try to move from one local minimizer to another one with lower function value of the objective function by minimizing an auxiliary function [12].

The filled function method was initially proposed by Ge and Qin [2-5] to search for a global minimizer of a multimodal function $f(x)$ on R^n, and has been reconsidered recently in [6-11, 13-17]. The basic idea of the filled function method is to construct a filled function $F(x)$, and by minimizing $F(x)$ to escape from a given local minimizer x_1^* of the original objective function $f(x)$ to one lower than x_1^*. The filled functions constructed up to now can be classified into global concavized and globally convexized filled functions [5].

A filled function $F(x)$ is called a globally concavized filled function if $F(x) \to -\infty$ as $\|x\| \to +\infty$, and is called a globally convexized filled function if $F(x) \to +\infty$ as $\|x\| \to +\infty$.

This paper is concerned with the globally concavized filled function method for continuous global optimization. Suppose that x_1^* is the current minimizer of the objective function $f(x)$. The definition of globally concavized filled function is as follows.

DEFINITION 1([3]). A continuous function $F(x)$ is called a globally concavized filled function of $f(x)$ at x_1^* if $F(x)$ has the following properties:

(1) x_1^* is a maximizer of $F(x)$ and the whole basin B_1^* of $f(x)$ at x_1^* becomes a part

of a hill of $F(x)$;

(2) $F(x)$ has no minimizers or saddle points in any higher basin of $f(x)$ than B_1^*;

(3) if $f(x)$ has a lower basin than B_1^*, then there is a point x' in such a basin that minimizes $F(x)$ on the line through x' and x_1^*.

Up to now, all constructed globally concavized filled functions are based on the above definition. Ge [3] proposed the following globally concavized filled function,

$$P(x, x_1^*, r, \rho) = \frac{1}{r + f(x)} exp(-\frac{\|x - x_1^*\|^2}{\rho^2}), \tag{1}$$

where r and ρ are parameters needed to be chosen appropriately such that the above function satisfies the three properties of Definition 1. The function $P(x, x_1^*, r, \rho)$ has two disadvantages: (1) it contains two parameters which are difficult to adjust in algorithm, and (2) it contains an exponential term such that the rapid increasing value of the exponential term will result in failure of computation even if the size of the feasible region is moderate.

To overcome the first disadvantage, Ge and Qin [4] constructed the following globally concavized filled function,

$$Q(x, A) = -(f(x) - f(x_1^*))exp(A\|x - x_1^*\|^2), \tag{2}$$

which contains only one parameter, but still contains an exponential term. Xu *et al* [13] tried to construct a more general class of globally concavized filled functions which contain functions (1) and (2) as special cases. To overcome the second disadvantage, Liu [7-11] proposed several new globally concavized filled functions, *i.e.*, the H-function, Q-function, L-function and M-function. But his filled functions can not be defined for some part of the feasible region, and how to choose appropriately the parameters in his filled functions are not presented. Recently, Zhang *et al* [14] gave a new globally

concavized filled function,

$$P(x, x_1^*, \rho, \mu) = f(x_1^*) - \min[f(x_1^*), f(x)] - \rho\|x - x_1^*\|^2 + \mu\{\max[0, f(x) - f(x_1^*)]\}^2, \ (3)$$

which contains two parameters ρ and μ and has no exponential terms, but is nonsmooth at $x \in \{x : f(x) = f(x_1^*)\}$.

Since the globally concavized filled functions proposed by [2, 3, 4] are based on the assumption that $f(x)$ has a finite number of local minimizers, and the values of the parameters in these functions must rely on the radius of the basin B_1^* of $f(x)$ at x_1^* to ensure that these functions be globally concavized filled functions, we [15, 16] modified the globally concavized filled functions (1) and (2) such that we do not need the assumption of a finite number of local minimizers of $f(x)$. Moreover, the values of the parameters in the modified filled functions do not rely on the radius of the basin B_1^*, but the modified filled functions still contain exponential terms.

All works on the globally concavized filled function mathod are for global minimization of the objective function $f(x)$ on R^n. However, we have proved that the continuous global minimization problem over unbounded domain can not be solved by any algorithm [17], so we will consider in this paper the boxed constrained continuous global minimization problem.

Up to now, all works on the globally concavized filled function method have two disadvantages. Firstly, the convergence property of the globally concavized filled function method has not been proved, and the implementations of the method use specified initial points. Secondly, all globally concavized filled functions are based on Definition 1, which uses the definition of basin and hill. This makes the construction of a globally concavized filled function need the assumption that $f(x)$ has a finite number of local minimizers, and the values of the parameters in constructed globally concavized filled

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functions rely on the radius of the basin B_1^* of $f(x)$ at x_1^*.

Now we have some comments on the definition of globally concavized filled function. In the second property of Definition 1, for any point x in all basins higher than B_1^* of $f(x)$ at x_1^*, it holds that $f(x) > f(x_1^*)$. Moreover, property 2 of Definition 1 implies that this x is not a minimizer or a saddle point of the concavized filled function $F(x)$. So property 2 could be modified as: for all x such that $f(x) > f(x_1^*)$, x is not a minimizer or a saddle point of $F(x)$. The third property of Definition 1 means that if $f(x)$ has a basin lower than B_1^*, then a globally concavized filled function $F(x)$ does have a minimizer along the direction $x - x_1^*$ for x in such a basin. This implies that $F(x)$ can not guarantee to have a minimizer y such that $f(y) < f(x_1^*)$, and makes the globally concavized filled function method not very efficient in practice [5]. Since we are only interested in a minimizer x of $F(x)$ such that $f(x) < f(x_1^*)$, it would be promising if we can construct a globally concavized filled function such that it has a minimizer in the solution space where $f(x) < f(x_1^*)$.

The above discussions suggest that we need a new definition of globally concavized filled function, and construction of a new globally concavized filled function. In section 2 of this paper, we present a new definition of globally concavized filled function for the continuous global minimization problem, which is based on the above comments. The properties of globally concavized filled functions are also analyzed in this section. In section 3 we construct a new class of globally concavized filled functions, which contains two parameters. The way to determine the values of the two parameters are presented such that the constructed functions are globally concavized filled functions. We present an algorithm in section 4 to solve the box constrained continuous global minimization problem basing on the globally concavized filled functions. We prove the asymptotic

convergence of the algorithm, and present a stopping rule for it. In the last section of this paper the algorithm is tested on several standard testing problems to demonstrate its practicability.

2 Definition of globally concavized filled function

Consider the following box constrained global minimization problem

$$(P) \begin{cases} \min \quad f(x) \\[2ex] s.t. \quad x \in X, \end{cases}$$

where $f(x)$ is continuously differentiable on X, X is a bounded closed box in R^n, *i.e.*, $X = \{x | l_i \leq x_i \leq u_i, i = 1, 2, \cdots, n\}$ with $l_i, u_i \in R$.

Suppose that x_1^* is the current local minimizer of problem (P). Before solving problem (P), we can get x_1^* using any local minimization method to minimize $f(x)$ in X.

By the comments in section 1, a promising definition of the globally concavized filled function of problem (P) is as follows.

DEFINITION 2. The function $F(x)$ is called a globally concavized filled function of problem (P) at its minimizer x_1^* if $F(x)$ is a continuous function and has the following properties:

(1) x_1^* is a maximizer of $F(x)$;

(2) $F(x)$ has no stationary point in the region $S_1 = \{x \in R^n : f(x) > f(x_1^*)\}$;

(3) $F(x)$ does have a minimizer in the region $S_2 = \{x \in R^n : f(x) < f(x_1^*)\}$ if $S_2 \neq \emptyset$.

In Definition 2, $F(x)$ is defined on R^n, and the second property implies that all minimizers of $F(x)$ in R^n are not in the set S_1. However, since we consider the box

constrained continuous global minimization problem, we should restrict the minimization of $F(x)$ in the box X, and in this case $F(x)$ on X does have at least one minimizer. So a definition of the globally concavized filled function must consider this situation, which will be presented in Definition 3.

Construct the following auxiliary global minimization problem

$$(AP) \quad \begin{cases} \min \quad F(x) \\ \\ s.t. \quad x \in X, \end{cases}$$

where $F(x)$ is a globally concavized filled function of problem (P) defined as follows.

DEFINITION 3. The function $F(x)$ is called a globally concavized filled function of problem (P) at its minimizer x_1^* if $F(x)$ is a continuous function and has the following properties:

(1) x_1^* is a maximizer of $F(x)$;

(2) Problem (AP) has no stationary point in the set $S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$ except x_1^* and the minimizer(s) of problem (AP) in S_1;

(3) $F(x)$ does have a minimizer in the set $S_2 = \{x \in X : f(x) < f(x_1^*)\}$ if $S_2 \neq \emptyset$.

In Definition 3, properties 1 and 2 suggest "the concavity structure" of $F(x)$. Moreover, it must be remarked that in property 2 a stationary point of problem (AP) is a point $x \in X$ which satisfies the following necessary conditions:

$$\frac{\partial F(x)}{\partial x_i} \geq 0, \quad \text{if} \quad x_i = l_i;$$

$$\frac{\partial F(x)}{\partial x_i} \leq 0, \quad \text{if} \quad x_i = u_i;$$

$$\frac{\partial F(x)}{\partial x_i} = 0, \quad \text{if} \quad l_i < x_i < u_i.$$

Hence the maximizer x_1^* of $F(x)$ and minimizers of problem (AP) are stationary points of problem (AP).

Furthermore, property 2 of Definition 3 implies that problem (AP) might have minimizers in the set S_1. However, is it possible that we can construct a globally concavized filled function for problem (P) such that problem (AP) has no minimizers in S_1? The following two theorems present a negative answer to this question.

THEOREM 1. *Suppose that $F(x)$ is a globally concavized filled function of problem (P), and x_1^* is already a global minimizer of problem (P). Then all minimizers of problem (AP) are in the set S_1.*

Proof. If x_1^* is already a global minimizer of problem (P), then the set S_2 defined in the third property of Definition 3 is empty. Thus all minimizers of problem (AP) are in the set S_1. $\qquad\square$

THEOREM 2. *For problem (AP), without at least one minimizer in the set S_1, it is impossible to find a globally concavized filled function $F(x)$ for problem (P) which satisfies all properties of Definition 3.*

Proof. We prove this theorem by contradiction. Suppose that such a globally concavized filled function $F(x)$ exists. That is, Problem (AP) has no minimizers in S_1, but it does have a minimizer in S_2. Thus when x_1^* is already a global minimizer of problem (P), the set S_2 is empty, and problem (AP) has no minimizer. But the continuity of $F(x)$ and the boundedness and closedness of the feasible domain X imply that problem (AP) has at least one minimizer. This contradicts the above conclusion about problem (AP). $\qquad\square$

3 New class of globally concavized filled functions

In this section, we construct a new class of globally concavized filled functions for problem (P) which was derived from [5]. As before, suppose that x_1^* is the current

minimizer of problem (P), which can be found by any local minimization method. Moreover, suppose that x^* is a global minimizer of problem (P).

Construct the following function

$$F(x, A, h) = G(\frac{1}{\|x - x_1^*\| + c})H(A[f(x) - f(x_1^*) + h]), \tag{4}$$

where c is a positive constant, $A > 0$ is a large parameter, $h > 0$ is a small parameter, $G(t)$ is a continuously differentiable univariate function which satisfies that ([5])

$$G(0) = 0, \quad G'(t) \geq a > 0 \quad \text{for all} \quad t \geq 0, \tag{5}$$

and $H(t)$ is a continuously differentiable univariate function which satisfies the following three properties ([5]):

(1) $H'(t) > 0$, *i.e.*, $H(t)$ is a monotonically increasing function of t;

(2) $H'(t)$ and $tH'(t)$ are monotonically decreasing to 0;

(3) $H(0) = 0$, $\lim_{t \to +\infty} H(t) = B > 0$.

By [5], two successful examples of this kind of functions could be

$$F(x, A, h) = \frac{1}{\|x - x_1^*\| + c} \arctan(A[f(x) - f(x_1^*) + h]),$$

$$F(x, A, h) = \frac{1}{\|x - x_1^*\| + c} \tanh(A[f(x) - f(x_1^*) + h]).$$

In the following, we show that the function $F(x, A, h)$ is a globally concavized filled function of problem (P) by proving that it satisfies the properties of Definition 3 if parameter A is large enough and h is small enough.

LEMMA 1. *For positive parameters A and h, suppose that A is large enough such that*

$$AH'(Ah) < \frac{a}{L(D + c)^2 G(\frac{1}{c})} H(Ah), \tag{6}$$

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where $L \geq \|\triangledown f(x)\|$ and $D \geq \|x - x_1^*\|$ for all $x \in X$. Then for all $x \in S_1 = \{x \in X :$ $f(x) \geq f(x_1^*)\}$, $x \neq x_1^*$, $x - x_1^*$ is a descent direction of $F(x, A, h)$ at x.

Proof. For all $x \neq x_1^*$, the derivative of $F(x, A, h)$ is

$$
\begin{aligned}
\triangledown F(x, A, h) = \quad & G'(\tfrac{1}{\|x-x_1^*\|+c}) \cdot [-\tfrac{1}{(\|x-x_1^*\|+c)^2}] \cdot \tfrac{x-x_1^*}{\|x-x_1^*\|} \cdot H(A[f(x) - f(x_1^*) + h]) + \\
& G(\tfrac{1}{\|x-x_1^*\|+c}) H'(A[f(x) - f(x_1^*) + h]) \cdot A \cdot \triangledown f(x) \\
= \quad & AG(\tfrac{1}{\|x-x_1^*\|+c}) H'(A[f(x) - f(x_1^*) + h]) \triangledown f(x) - \\
& \tfrac{G'(\tfrac{1}{\|x-x_1^*\|+c})}{(\|x-x_1^*\|+c)^2} H(A[f(x) - f(x_1^*) + h]) \cdot \tfrac{x-x_1^*}{\|x-x_1^*\|}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& (x - x_1^*)^T \triangledown F(x, A, h) \\
& = AG(\tfrac{1}{\|x-x_1^*\|+c}) H'(A[f(x) - f(x_1^*) + h])(x - x_1^*)^T \triangledown f(x) - \qquad (7) \\
& \quad \tfrac{G'(\tfrac{1}{\|x-x_1^*\|+c})}{(\|x-x_1^*\|+c)^2} H(A[f(x) - f(x_1^*) + h]) \|x - x_1^*\|.
\end{aligned}
$$

For all $x \in X$ such that $f(x) \geq f(x_1^*)$ and $x \neq x_1^*$, by the assumptions on $G(t)$ and $H(t)$ we have

$$
\begin{aligned}
& AG(\tfrac{1}{\|x-x_1^*\|+c}) H'(A[f(x) - f(x_1^*) + h])(x - x_1^*)^T \triangledown f(x) \\
& \leq AG(\tfrac{1}{c}) H'(Ah) \cdot \|x - x_1^*\| \cdot \|\triangledown f(x)\| \\
& \leq ALG(\tfrac{1}{c}) H'(Ah) \|x - x_1^*\|,
\end{aligned}
$$

and

$$
\begin{aligned}
& \tfrac{G'(\tfrac{1}{\|x-x_1^*\|+c})}{(\|x-x_1^*\|+c)^2} H(A[f(x) - f(x_1^*) + h]) \|x - x_1^*\| \\
& \geq \tfrac{a}{(D+c)^2} H(Ah) \|x - x_1^*\|.
\end{aligned}
$$

So the equality (7) leads to the following inequalities,

$$(x - x_1^*)^T \triangledown F(x, A, h)$$

$$\leq ALG(\tfrac{1}{c})H'(Ah)\|x - x_1^*\| - \tfrac{a}{(D+c)^2}H(Ah)\|x - x_1^*\|$$

$$= [ALG(\tfrac{1}{c})H'(Ah) - \tfrac{a}{(D+c)^2}H(Ah)]\|x - x_1^*\|.$$

Since parameter A is large enough such that it satisfies inequality (6), it holds that

$$[ALG(\frac{1}{c})H'(Ah) - \frac{a}{(D + c)^2}H(Ah)]\|x - x_1^*\| < 0.$$

Thus $(x - x_1^*)^T \triangledown F(x, A, h) < 0$, and for all $x \in S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$, $x \neq x_1^*$,

$x - x_1^*$ is a descent direction of $F(x, A, h)$ at x. $\qquad\qquad\qquad\qquad\qquad\square$

With Lemma 1, we prove in the following that $F(x, A, h)$ satisfies the first property

of Definition 3.

THEOREM 3. *Suppose that parameter A is large enough such that inequality (6)*

holds. Then x_1^* is a strict local maximizer of $F(x, A, h)$.

Proof. Since x_1^* is a local minimizer of $f(x)$, there exists a neighborhood $N(x_1^*)$ of

x_1^* such that for all $x \in N(x_1^*) \cap X$, $f(x) \geq f(x_1^*)$. Then it must holds that $F(x, A, h) <$

$F(x_1^*, A, h)$ for all $x \in N(x_1^*) \cap X$. In fact, consider two points on the line segment

between x and x_1^*: $x_1^* + t_1(x - x_1^*)$ and $x_1^* + t_2(x - x_1^*)$, where $0 \leq t_1 < t_2 \leq 1$. By

Lemma 1, $x - x_1^*$ is a descent direction of $F(x, A, h)$ at $x_1^* + t_1(x - x_1^*)$ and $x_1^* + t_2(x - x_1^*)$.

Thus it holds that

$$F(x_1^* + t_1(x - x_1^*), A, h) > F(x_1^* + t_2(x - x_1^*), A, h).$$

Moreover, by the continuity of $F(x, A, h)$, we have

$$F(x_1^*, A, h) = \lim_{t_1 \to 0} F(x_1^* + t_1(x - x_1^*), A, h)$$

$$> \lim_{t_2 \to 1} F(x_1^* + t_2(x - x_1^*), A, h) = F(x, A, h).$$

Hence x_1^* is a strict local maximizer of $F(x, A, h)$. □

Furthermore, it is obvious that Lemma 1 implies the following theorem.

THEOREM 4. *For positive parameters A and h, suppose that A is large enough such that inequality (6) holds. Then problem (AP) has no stationary point in the set $S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$ except x_1^* and the minimizer(s) of problem (AP) in S_1.*

Theorem 4 means that $F(x, A, h)$ satisfies the second property of Definition 3 if parameter A is large enough such that inequality (6) holds.

However, one may doubt that whether there exists a value of parameter A such that inequality (6) holds. In fact, by the assumptions on functions $G(t)$ and $H(t)$, the left hand side of inequality (6) is a monotonically decreasing function of A, the right hand side of inequality (6) is a monotonically increasing function of A, and the limits of the both hand sides of inequality (6) are

$$\lim_{A \to +\infty} AH'(Ah) = \frac{1}{h} \lim_{A \to +\infty} AhH'(Ah) = 0,$$

$$\lim_{A \to +\infty} \frac{a}{L(D+c)^2 G(\frac{1}{c})} H(Ah) = \frac{a}{L(D+c)^2 G(\frac{1}{c})} \lim_{A \to +\infty} H(Ah)$$

$$= \frac{a}{L(D+c)^2 G(\frac{1}{c})} B > 0.$$

So if A is large enough, then inequality (6) will hold.

Next, we prove that if parameter h is small enough, then the function $F(x, A, h)$ satisfies the third property of Definition 3.

THEOREM 5. *Suppose that x_1^* is not a global minimizer of problem (P). If positive parameter h satisfies that*

$$h < f(x_1^*) - f(x^*), \tag{8}$$

then problem (AP) takes its global minimal value in the set $S_2 = \{x \in X : f(x) < f(x_1^*)\}$.

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Proof. Since $F(x, A, h)$ is a continuous function in the closed bounded box X, problem (AP) has a global minimizer. For all $x \in S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$, we have $f(x) \geq f(x_1^*)$ and $\|x - x_1^*\| \geq 0$. Therefore,

$$\frac{1}{\|x - x_1^*\| + c} > 0,$$

$$f(x) - f(x_1^*) + h > 0,$$

and by the assumptions on $G(t)$ and $H(t)$, it holds that $F(x, A, h) > 0$. For the set S_2, the inequality (8) implies that S_2 is not empty, and there is a point $y \in S_2$ such that $f(y) < f(x_1^*) - h$, which implies that $A[f(y) - f(x_1^*) + h] < 0$, and $F(y, A, h) < 0$. Hence, problem (AP) takes its global minimal value in the set S_2. $\qquad \square$

So by Theorems 3-5, the function $F(x, A, h)$ is a globally concavized filled function of problem (P) at x_1^*.

However, during construction of a globally concavized filled function, one may ask how to choose the value of h to satisfy inequality (8), since we do not know the global minimal value of problem (P). In fact, for practical consideration, problem (P) might be considered solved if we can find an $x \in X$ such that $f(x) \leq f(x^*) + \epsilon$, where $\epsilon > 0$ is a given desired optimality tolerance. Let $h = \epsilon$. Then for any current local minimizer x_1^* such that $f(x_1^*) > f(x^*) + \epsilon = f(x^*) + h$, i.e., $h < f(x_1^*) - f(x^*)$, by Theorem 5, problem (AP) takes its global minimal value in the set $S_2 = \{x \in X : f(x) < f(x_1^*)\}$. Moreover, if we choose the value of A such that it satisfies inequality (6), then $F(x, A, h)$ is a globally concavized filled function of problem (P).

In the other respect, if the current local minimizer x_1^* of problem (P) satisfies that $f(x_1^*) \leq f(x^*) + \epsilon$, then inequality (8) does not hold, and $F(x, A, h)$ may not be a globally concavized filled function of problem (P). But in this case $f(x_1^*)$ is very close

to the global minimal value of problem (P) and we can output x_1^* as an approximate global minimizer.

According to the above discussions, we have the following theorem.

THEOREM 6. *Given any desired tolerance $\epsilon > 0$, let $h = \epsilon$. If A satisfies inequality (6), then $F(x, A, h)$ is a globally concavized filled function of problem (P) at its current minimizer x_1^* if $f(x_1^*) > f(x^*) + \epsilon$.*

Thus if we use a local minimization method to solve problem (AP) from any initial point on X, then by Lemma 1, it is obvious that the minimization sequence converges either to a minimizer of problem (AP) on the boundary of the bounded closed box X or to a point $x' \in X$ such that $f(x') < f(x_1^*)$. If we find such an x', then using a local minimization method to minimize $f(x)$ in X from initial point x', we can find a point $x'' \in X$ such that $f(x'') \leq f(x')$, which is better than x_1^*. This is the main idea of the algorithm presented in the next section to find an approximate global minimizer of problem (P).

4 The algorithm and its asymptotic convergence

In this section, we present an algorithm to solve problem (AP) to find a local minimizer of problem (P) better than the current one x_1^*, and prove its asymptotic convergence to a global minimizer of problem (P) with probability one. The algorithm is described as follows.

ALGORITHM.

Step 1. Select randomly a point $x \in X$, and start from which to minimize $f(x)$ on X to get a minimizer x_1^* of problem (P). Let h be a small positive number and A be a

large positive number such that it satisfies inequality (6). Let N_L be a sufficiently large

integer.

Step 2. Construct a globally concavized filled function $F(x, A, h)$ of problem (P) at

x_1^*. *Set $N = 0$.*

Step 3. If $N \geq N_L$, then go to Step 6.

Step 4. Set $N = N + 1$. Draw randomly an initial point near x_1^*, and start from

which to minimize $F(x, A, h)$ in X using any local minimization method. Suppose that

x' *is an obtained local minimizer. If $f(x') \geq f(x_1^*)$, then go to Step 3, otherwise go to*

Step 5.

Step 5. Minimize $f(x)$ in X from initial point x', and obtain a local minimizer x_2^*

of $f(x)$. Let $x_1^* = x_2^*$ and go to Step 2.

Step 6. Stop the algorithm, output x_1^* and $f(x_1^*)$ as an approximate global minimal

solution and global minimal value of problem (P) respectively.

In Step 1 of the above algorithm, parameter N_L is the maximal number of mini-

mizing problem (AP) between Steps 3 and 4.

In the following two subsections, we discuss the asymptotic convergence of the

algorithm and present a stopping rule for it.

4.1   Asymptotic convergence

With a little loss of generality, suppose that problem (P) has a finite number of local

minimal values, and f_L^* is the least one which is larger than the global minimal value

$f(x^*)$ of problem (P). Since $f(x)$ is a continuous function, it is obvious that the

Lebesgue measure of the set $S_L^* = \{x \in X : f(x) < f_L^*\}$ is $m(S_L^*) > 0$.

Suppose that the local minimization method used in Step 5 of the algorithm is

strictly descent and can converge to a local minimizer of the problem being solved. Thus with an initial point $x' \in S_L^* = \{x \in X : f(x) < f_L^*\}$, the minimization sequence generated by the minimization of $f(x)$ in X will converge to a global minimizer of $f(x)$ on X.

In Step 4 of the algorithm, by Lemma 1, for any initial point x, $x - x_1^*$ is a descent direction of $F(x, A, h)$ at x. So we suppose that the local minimization method used in Step 4 of the algorithm to minimize $F(x, A, h)$ in X is just a line search along the direction $x - x_1^*$ from x_1^*, in which the initial step length is drawn uniformly in $[0, D]$, where $D = \max_{x_1^* + t(x - x_1^*) \in X} t$. Obviously, by the above discussions the line search will converge to a point on the boundary of X or find a point in $\{x \in X : f(x) < f(x_1^*)\}$, and is this case the algorithm will go to Step 5 and find another local minimizer of $f(x)$ which is lower than x_1^*.

Let x_k be the k-th random point drawn randomly near x_k^*, i.e., $x_k - x_k^*$ is a random search direction at x_k^*, t_k be the k-th random number drawn uniformly in $[0, D]$, where $D = \max_{x_k^* + t(x_k - x_k^*) \in X} t$, and let x_{k+1}^* be a local minimizer of $f(x)$ in X which is such that if the line search from x_k^* along $x_k - x_k^*$ goes out of X, then $x_{k+1}^* = x_k^*$, otherwise the algorithm goes to Step 5 and x_{k+1}^* is the local minimizer found at this step, $k = 1, 2, \cdots$.

Thus we have three sequences x_k, t_k, x_k^*, $k = 1, 2, \cdots$. Obviously, $f(x_1^*) \geq f(x_2^*) \geq \cdots \geq f(x_k^*) \geq \cdots \geq f(x^*)$, and obviously $x_k^* + t_k(x_k - x_k^*)$ is a random point drawn in the solution space X. Moreover, $x_k^* + t_k(x_k - x_k^*)$, $k = 1, 2, \cdots$, are i.i.d., and

$$P\{x_k^* + t_k(x_k - x_k^*) \notin S_L^*\} = P\{x_{k+1}^* + t_{k+1}(x_{k+1} - x_{k+1}^*) \notin S_L^*\}, \quad k = 1, 2, \cdots, \quad (9)$$

and by $m(S_L^*) > 0$,

$$0 < P\{x_k^* + t_k(x_k - x_k^*) \notin S_L^*\} < 1, \quad k = 1, 2, \cdots. \quad (10)$$

THEOREM 7. *Let $N_L = +\infty$. x_k^* converges to a global minimizer of problem* (P) *with probability 1, i.e., $P\{\lim_{k\to\infty} f(x_k^*) = f(x^*)\} = 1$.*

Proof. To prove Theorem 7 is equivalent to prove that

$$P\{\bigcap_{k=1}^{\infty} \bigcup_{l=k}^{\infty} (f(x_l^*) - f(x^*) \geq \epsilon)\} = 0, \forall \epsilon > 0. \tag{11}$$

Let $q = P\{x_k^* + t_k(x_k - x_k^*) \notin S_L^*\}$. By equalities (9) and the monotonicity of $f(x_k^*)$, $k = 1, 2, \cdots$, for all $\epsilon > 0$, it holds that

$$P\{f(x_l^*) - f(x^*) \geq \epsilon\}$$

$$= P\{\bigcap_{i=1}^{l}(f(x_i^*) - f(x^*) \geq \epsilon)\}$$

$$\leq P\{\bigcap_{i=1}^{l-1}(x_i^* + t_i(x_i - x_i^*) \notin S_L^*)\}$$

$$= \prod_{i=1}^{l-1} P\{x_i^* + t_i(x_i - x_i^*) \notin S_L^*\}$$

$$= \prod_{i=1}^{l-1} q$$

$$= q^{l-1}.$$

Thus,

$$P\{\bigcap_{k=1}^{\infty} \bigcup_{l=k}^{\infty}(f(x_l^*) - f(x^*) \geq \epsilon)\}$$

$$\leq \lim_{k\to\infty} P\{\bigcup_{l=k}^{\infty}(f(x_l^* - f(x^*) \geq \epsilon)\}$$

$$\leq \lim_{k\to\infty} \sum_{l=k}^{\infty} P\{f(x_l^*) - f(x^*) \geq \epsilon\}$$

$$\leq \lim_{k\to\infty} \sum_{l=k}^{\infty} q^{l-1}$$

$$= \lim_{k\to\infty} \frac{q^{k-1}}{1-q}.$$

By inequalities (10), we have $0 < q < 1$, and $\frac{q^{k-1}}{1-q} \to 0$ as $k \to \infty$. Hence equality (11) holds. $\square$

4.2 Stopping rule

We consider the globally concavized filled function $F(x, A, h)$ with parameter h small enough, and A large enough. In the algorithm, after getting a current local minimizer x_1^* of problem (P), the algorithm draws an initial point randomly near x_1^*, and start from which to minimize $F(x, A, h)$ in X. If the above process does not find a minimizer of $f(x)$ which is better than x_1^*, then it is repeated.

Roughly speaking, the above process is a way of multistart local search to solve problem (AP). So we use a Bayesian stopping rule developed by Boender and Rinnooy Kan [1] for multistart local search method to estimate the value of N_L, *i.e.*, the maximum number of minimization of $F(x, A, h)$ in X between Steps 3 and 4.

Assume that w is the number of different local minimizers of $F(x, A, h)$ in X having been discovered, which are all in the set S_1, and N is the number of minimizations of $F(x, A, h)$ for finding these w local minimizers. Boender and Rinnooy Kan [1] discovered that the Bayesian estimate of the total number of local minimizers of $F(x, A, h)$ in X is $\frac{w(N-1)}{N-w-2}$. Hence if the value $\frac{w(N-1)}{N-w-2}$ is very close to w, then one can probabilistically say that $F(x, A, h)$ has only w local minimizers in X, which have already been found, and we can terminate the algorithm. Therefore a simple stopping rule is to terminate the algorithm if N satisfies that

$$\frac{w(N-1)}{N-w-2} \leq w + \frac{1}{2},$$

which leads to $N \geq 2(w^2 + w) + (w + 2)$.

The above discussion means that after $2(w^2 + w) + (w + 2)$ minimizations of $F(x, A, h)$ with initial points drawn randomly near x_1^*, if we can only find the minimizers of problem (AP) in the set S_1, then we can conclude approximately that problem (AP) has only w local minimizers, and a global minimizer of problem (P) has been found.

So in the algorithm we set $N_L = 2(w^2 + w) + (w + 2)$, and terminate the algorithm if

$N \geq N_L = 2(w^2 + w) + (w + 2)$.

5   Test results of the algorithm

Although the focus of this paper is more theoretical than computational, we still test our algorithm on several standard global minimization problems to have an initial feeling of the practical interest of the globally concavized filled function method.

We choose

$$F(x, A, h) = \frac{1}{\|x - x_1^*\| + 1} \arctan(A[f(x) - f(x_1^*) + h]),$$

where $h = 10^{-3}$, and $A = 10^3$. We use the BFGS local minimization method with inexact line search to minimize both the objective function and the globally concavized filled functions. The stopping criterion of the local minimization method is that the norm of the derivative is less than 10^{-5}. Each test problem has been solved ten times. The obtained results are reported in Table 1.

PROBLEM 1. *The three hump camel function*

$$f(x) = 2x_1^2 - 1.05x_1^4 + \frac{x_1^6}{6} - x_1 x_2 + x_2^2$$

has three local minimizers in the domain $-3 \leq x_i \leq 3$, $i = 1, 2$, and the global minimizer is $x^* = (0, 0)^T$. The global minimal value is $f(x^*) = 0$.

PROBLEM 2. *The six hump camel function*

$$f(x) = 4x_1^2 - 2.1x_1^4 + \frac{x_1^6}{3} + x_1 x_2 - 4x_2^2 + 4x_2^4$$

has six local minimizers in the domain $-3 \leq x_1 \leq 3$, $-1.5 \leq x_2 \leq 1.5$, and two of them are global minimizers: $x^* = (-0.089842, 0.712656)^T$, $x^* = (0.089842, -0.712656)^T$. The global minimal value is $f(x^*) = -1.031628$.

PROBLEM 3. *The Treccani function*

$$f(x) = x_1^4 + 4x_1^3 + 4x_1^2 + x_2^2$$

has two local minimizers $x^* = (-2, 0)^T$ and $x^* = (0, 0)^T$ in the domain $-3 \leq x_i \leq 3$, $i = 1, 2$. The global minimal value is $f(x^*) = 0$.

PROBLEM 4. *The Goldstein-Price function*

$$f(x) = [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)]$$

$$\times [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)]$$

has 4 local minimizers in the domain $-2 \leq x_i \leq 2$, $i = 1, 2, 3, 4$, but only one global minimizer $x^* = (0, -1)^T$ with the global minimal value $f(x^*) = 3$.

PROBLEM 5. *The two dimensional Shubert function*

$$f(x) = \{\sum_{i=1}^{5} i \cos[(i + 1)x_1 + i]\}\{\sum_{i=1}^{5} i \cos[(i + 1)x_2 + i]\}$$

has 760 local minimizers in the domain $-10 \leq x_i \leq 10$, $i = 1, 2$, and eighteen of them are global minimizers. The global minimal value is $f(x^*) = -186.730909$.

PROBLEM 6. *The two dimensional Shubert function*

$$f(x) = \{\sum_{i=1}^{5} i \cos[(i+1)x_1+i]\}\{\sum_{i=1}^{5} i \cos[(i+1)x_2+i]\}+\frac{1}{2}[(x_1+0.80032)^2+(x_2+1.42513)^2]$$

has roughly the same behavior as the function presented in Problem 5 in the domain $-10 \leq x_i \leq 10$, $i = 1, 2$, but has a unique global minimizer $x^* = (-0.80032, -1.42513)^T$. The global minimal value is $f(x^*) = -186.730909$.

PROBLEM 7. *The two dimensional Shubert function*

$$f(x) = \{\sum_{i=1}^{5} i \cos[(i+1)x_1+i]\}\{\sum_{i=1}^{5} i \cos[(i+1)x_2+i]\}+[(x_1+0.80032)^2+(x_2+1.42513)^2]$$

in the domain $-10 \le x_i \le 10$, $i = 1, 2$ *has roughly the same behavior and the same global minimizer and global minimal value as the function presented in Problem 6, but with steeper slope around global minimizer.*

PROBLEM 8. *The function*

$$f(x) = \frac{\pi}{n}\{10\sin^2(\pi x_1) + \sum_{i=1}^{n-1}(x_i - 1)^2[1 + 10\sin^2(\pi x_{i+1})] + (x_n - 1)^2\}$$

has roughly 30^n *local minimizers in the domain* $-10 \le x_i \le 10$, $i = 1, \cdots, n$, *but only one global minimizer* $x^* = (1, 1, \cdots, 1)^T$ *with the global minimal value* $f^* = 0$.

The test results of the above 8 problems are presented in Table 1.

Table 1. Numerical results of the algorithm									
Problem No.	n	NF	NG	NFF	NFG	LNF	LNG	LNFF	LNFG	Fail
1	2	35	12	737	640	35	12	340	289	0
2	2	31	11	665	525	31	11	545	425	0
3	2	25	13	589	518	22	12	576	505	0
4	2	64	23	789	575	61	22	547	416	0
5	2	92	25	743	592	92	25	120	78	0
6	2	90	25	826	678	90	25	327	238	0
7	2	88	23	816	693	88	23	194	127	0
8	2	163	41	296	267	123	34	98	82	2
	5	560	120	1243	1218	560	120	471	448	3
	10	1309	292	21080	20822	1309	291	3674	3417	3

In the above table,

n= the dimension of the tested problem,

NF= the mean number of objective function evaluations to satisfy the stopping rule,

NG= the mean number of evaluations of the gradient of the objective function to satisfy the stopping rule,

NFF= the mean number of the filled functions evaluations to satisfy the stopping rule,

NFG= the mean number of evaluations of the gradients of the filled functions to satisfy the stopping rule,

LNF= the mean number of objective function evaluations needed to get the global minimal value,

LNG= the mean number of evaluations of the gradient of the objective function needed to get the global minimal value,

LNFF= the mean number of the filled functions evaluations needed to get the global minimal value,

LNFG= the mean number of evaluations of the gradients of the filled functions needed to get the global minimal value,

Fail= the number of times that the stopping rule is satisfied but no global minima are located.

All the mean values have been computed without considering the failures, and have been rounded to integers.

6   Conclusions

The definition of a globally concavized filled function $F(x)$ of problem (P) has been improved as: (1) x_1^* is a maximizer of $F(x)$, where x_1^* is the current local minimizer of problem (P); (2) Problem (AP) has no stationary point in the set $S_1 = \{x \in X : f(x) \geq f(x_1^*)\}$ except x_1^* and the minimizer(s) of problem (AP) in S_1; (3) $F(x)$ does

have a minimizer in the set $S_2 = \{x \in X : f(x) < f(x_1^*)\}$ if $S_2 \neq \emptyset$. In this definition properties 1 and 2 imply "the concavity structure" of the filled function. We have developed a new class of globally concavized filled functions, which was modified from the globally convexized filled functions by Ge and Qin [5]. The values of the parameters in the globally concavized filled functions do not rely on the radius of the basin at x_1^*. Based on the globally concavized filled functions, an algorithm has been designed to solve the box constrained continuous global minimization problem. If we use a line search method with random initial step lengths to minimize the globally concavized filled functions, the algorithm can converge with probability one to a global minimizer of the problem being solved. Preliminary numerical experiments have been presented to show the practicability of the algorithm.

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