ON THE RELATIONSHIP BETWEEN CONVERGENCE RATES OF DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS

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Abstract. Considering iterative sequences that arise when the approximate solution x_k to a numerical problem is updated by $x_{k+1} = x_k + v(x_k)$, where v is a vector field, we derive necessary and sufficient conditions for such discrete methods to converge to a stationary point of v at different Q-rates in terms of the differential properties of v and in terms of the asymptotic dynamical behaviour of the associated continuous dynamical system.

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1. Introduction. In this paper we study sequences $(x_k)_{\mathbb{N}_0}$ which, given a starting point x_0, consist of points that are iteratively related to one another via the rule

$$x_{k+1} = x_k + v(x_k), \tag{1.1}$$

where $v : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field with Jacobian $J = \mathrm{D}v$. We are interested in the situation where x_k converges to a stationary point x^* of v.

Sequences of the form (1.1) appear in many areas of numerical analysis where an approximate solution x_k is iteratively improved, notably in unconstrained optimisation and in zero-finding problems. One of the major objectives in designing iterative schemes of this kind is to assure that they converge at a provably fast rate to a point x^* which represents a solution of interest.

A widely used notion of fast convergence is the following: we say that the method (1.1) converges to x^* at the Q-convergence rate $q > 1$ uniformly in the ball $\mathcal{B}_\rho(x^*)$ of radius $\rho < 1$ if there exist $\beta > 0$ such that

$$\|x + v(x) - x^*\| \leq \beta \|x - x^*\|^q \tag{1.2}$$

for all $x \in \mathcal{B}_\rho(x^*)$. That is, if the sequence $(x_k)_{\mathbb{N}}$ ever enters the ball $\mathcal{B}_\rho(x^*)$, then it converges to x^*, and each iteration starting from within the ball improves the accuracy of x_k as an approximation of x^* to about q times as many correct digits as beforehand. The constant β is called the convergence factor. For example, the well-known Kantorovich theorem [5] shows that Newton's method for zero-solving converges Q-quadratically under standard regularity assumptions.

A weaker notion of fast convergence is the following: we say that the method (1.1) converges to x^* uniformly Q-superlinearly if

$$\lim_{x \to x^*} \frac{\|x + v(x) - x^*\|}{\|x - x^*\|} = 0. \tag{1.3}$$

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That is, asymptotically, each iteration adds more than any fixed number of additional correct digits to the current approximation of x^*.

The goal of this paper is to derive exact characterisations of the vector fields that make the method (1.1) converge fast under either notion of Q-convergence.

A classical theorem of Ortega-Rheinboldt [6] shows that if $v(x^*) = 0$ and $J(x^*) = -\mathrm{I}$, and if J is sufficiently smooth, then the method (1.1) converges at a quantifiable Q-convergence rate. In Theorem 2.2 we show that, conversely, if (1.1) converges Q-superlinearly then $v(x^*) = 0$ and $J(x^*) = -\mathrm{I}$, and we use this result in Corollary 2.3 to show that the criteria derived by Ortega-Rheinboldt are both necessary and sufficient.

The discrete dynamical system (1.1) has a continuous analogue obtained when damping with an infinitesimal step size is applied. The associated flow is defined by

$$\dot{\varphi}(x_0, t) = v(\varphi(x_0, t)), \qquad \varphi(x_0, 0) = x_0. \tag{1.4}$$

In other words, in the continuous method one chooses a starting point x_0 and follows the flux line $t \mapsto \varphi(x_0, t)$, obtained by integrating the ODE (1.4) to a limit point $x^* = \lim_{t\to\infty} \varphi(x_0, t)$. Continuous methods for solving zero-finding and optimisation problems have been proposed by many authors, see e.g. [1, 2, 3, 7, 8]. Some of these methods have theoretical advantages over discrete methods in overcoming ill-posedness, getting around singularities, and resolving the dilemma of local versus global optimisation.

Although in Section 3 we consider the continuous problem (1.4), we do not propose it as a replacement for the discrete method (1.1) as was done in the cited publications. Our interest in (1.4) is merely to investigate the link between the convergence rates of the discrete system and those of the continuous system, and thereby to gain a deeper geometric understanding of what distinguishes different Q-rates. For this purpose we develop notions of exponential convergence and p-exponential convergence for the continuous system (1.4), based on its dynamics, and we show that these notions are equivalent to Q-superlinear convergence and Q-convergence of rate $p+1$ for the associated discrete system (1.1).

In Section 4 we show how our characterisation of Q-convergence rates can be used to prove that the convergence rate of the method (1.1) can usually not be improved simply by rescaling the vector field v.

Theoretical tools for the analysis of discrete methods in terms of associated continuous methods are interesting because continuous dynamical systems are generally better understood than discrete ones. Results of this kind are relatively rare in the literature. A notable exception is a paper by Janovsky-Seige [4] who analysed the basins of attraction of Newton's method in terms of those of the continuous Newton method.

To conclude this section, let us introduce two short technical lemmas which we will use in our analysis. The following property of differential inequalities is well-known, see e.g. [9]:

Lemma 1.1. *Let* $\frac{d}{dt}\, y(t) = g(t, y(t))$, $y(0) = y_0$, $t \geq 0$, *where* g *is continuous, and let* $\frac{d}{dt}\, z(t) \lesseqqgtr g(t, z(t))$, $z(0) = y_0$, $t \geq 0$. *If* $g(t, x)$ *is monotone increasing in* x *or* *of the form* $g(t,x) = h(t)$ *or* $g(t,x) = h(t)x$*, then* $z(t) \lesseqqgtr y(t)$ *for all* $t \geq 0$*.*

We will be interested in the normalised vector

$$\mathfrak{n}(\varphi(x_0,t) - x^*) := \frac{\varphi(x_0,t) - x^*}{\|\varphi(x_0,t) - x^*\|} \in S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$$

and the speed $\|\frac{d}{dt}\,\mathfrak{n}(\varphi(x_0,t) - x^*)\|$ which we will call the *angular speed* and which can be interpreted as the absolute angle traversed by the flux line $\varphi(x_0,t)$ with respect to x^* per unit time. Using (1.4), the following result is obtained via straightforward calculation:

LEMMA 1.2. *For* $\varphi(x_0,t)$ *as defined by* (1.4) *it is true that*

$$\frac{d}{dt}\,\mathfrak{n}(\varphi(x_0,t) - x^*) = \frac{(\mathrm{I} - P(t))\, v(\varphi(x_0,t))}{\|\varphi(x_0,t) - x^*\|}, \tag{1.5}$$

where $P(t)$ *denotes the orthogonal projection of* $\mathbb{R}^n$ *onto* $\mathrm{span}\{\varphi(x_0,t) - x^*\}$*.*

2. Fast Convergence of Discrete Systems. In this section we consider discrete dynamical systems. We will see that uniform Q-superlinear convergence of (1.1) is characterised by the differential properties of v in a neighbourhood of x^*.

Let $p > 0$. Recall that the vector field v is called p-Hölder continuous at x^* if there exist constants $\alpha > 0$ and $\varrho > 0$ such that

$$\|J(x) - J(x^*)\| \leq \alpha \|x - x^*\|^p \tag{2.1}$$

for all $x \in \mathcal{B}_\varrho(x^*)$. Ortega and Rheinboldt showed the following result (set $G(x) = x + v(x)$ in Theorem 10.1.6 [6]):

THEOREM 2.1 (Ortega-Rheinboldt). *Let* J *be continuous at* x^**,* $v(x^*) = 0$ *and* $J(x^*) = -\mathrm{I}$*. Then* (1.1) *converges uniformly Q-superlinearly to* x^**. If moreover* J *is* p*-Hölder continuous at* x^* *then* (1.1) *converges uniformly at the Q-convergence rate* $p+1$ *in a neighbourhood of* x^**.*

We are now going to prove a partial inverse of Theorem 2.1:

THEOREM 2.2. *If* (1.1) *converges uniformly Q-superlinearly to* x^* *then* $v(x^*) = 0$ *and* $J(x^*) = -\mathrm{I}$*.*

Proof. Equation (1.3) implies that for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $x \in B_{\delta_\varepsilon}(x^*)$ implies

$$\|x + v(x) - x^*\| \leq \varepsilon \|x - x^*\|. \tag{2.2}$$

Taking limits $x \to x^*$ on both sides and using the continuity of v, we obtain $\|v(x^*)\| \leq 0$, which shows that $v(x^*) = 0$. Next, let $z \in S^{n-1}$ and consider the sequence $(x_n)_{\mathbb{N}}$ defined by $x_n = x^* + z/n$. Then (1.3) and $v(x^*) = 0$ show that

$$\|\big(\mathrm{I} + J(x^*)\big)z\| = \lim_{n\to\infty} \frac{\|x_n - x^* + v(x_n) - v(x^*)\|}{\|x_n - x^*\|} = 0.$$

But this shows that $J(x^*)z = -z$, and since z was an arbitrary unit vector, it follows that $J(x^*) = -\mathrm{I}$. □

If continuity or Hölder continuity of J at x^* is given then the two preceding theorems yield the following exact characterisation of fast convergence:

COROLLARY 2.3. *If J is continuous at x^* then* (1.1) *converges to x^* uniformly Q-superlinearly if and only if $v(x^*) = 0$ and $J(x^*) = -\mathrm{I}$. If J is p-Hölder continuous at x^* then* (1.1) *converges uniformly at the Q-convergence rate $p+1$ in a neighbourhood of x^* if and only if $v(x^*) = 0$ and $J(x^*) = -\mathrm{I}$.*

Proof. Observe that Q-convergence at a rate $p + 1 > 1$ implies Q-superlinear convergence. Everything else follows directly from Theorems 2.1 and 2.2. □

Corollary 2.3 shows that the only difference between Q-superlinear convergence and convergence at Q-rates $p+1 > 1$ consists in the smoothness of the Jacobian of v in a neighbourhood of x^*.

3. Fast Convergence of Continuous Systems. In this section we introduce notions of fast convergence for the continuous dynamical system (1.4) which we will show to be equivalent to uniform Q-superlinear convergence and uniform Q-convergence of rate $p+1$ respectively of the discrete system (1.1).

DEFINITION 3.1. *We say that the continuous dynamical system* (1.4) *associated with the vector field v converges exponentially to the stable attractor x^* if for all $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that $x \in \mathcal{B}_{\rho_\varepsilon}$ and $t \geq 0$ imply*

$$\mathrm{e}^{-(1+\varepsilon)t}\|x - x^*\| \leq \|\varphi(x,t) - x^*\| \leq \mathrm{e}^{-(1-\varepsilon)t}\|x - x^*\|, \tag{3.1}$$

$$\left\| \frac{d}{dt}\, \mathfrak{n}(\varphi(x,t) - x^*) \right\| \leq \varepsilon. \tag{3.2}$$

We say that (1.4) *converges p-exponentially to x^* if* (3.1) *holds and there exist constants $\xi > 0$ and $\gamma > 0$ such that $\rho_\varepsilon > \xi\varepsilon^{1/p}$ for all ε small enough and*

$$\left\| \frac{d}{dt}\, \mathfrak{n}(\varphi(x,t) - x^*) \right\| \leq \gamma \|x - x^*\|^p \mathrm{e}^{-(1-\varepsilon)pt}. \tag{3.3}$$

for all $x \in \mathcal{B}_{\rho_\varepsilon}$ and $t \geq 0$.

LEMMA 3.2. *If* (1.1) *converges uniformly Q-superlinearly to x^* then* (1.4) *converges exponentially to x^*. Moreover, if* (1.1) *converges uniformly at Q-convergence rate $p+1 > 1$ then* (1.4) *converges p-exponentially with $\gamma = \beta$ and $\rho_\varepsilon \geq \min\{(\varepsilon/\beta)^{1/p}, \rho\}$ for all ε, where β and ρ take the same values as in* (1.2).

Proof. Let δ_ε be chosen as in the proof of Theorem 2.2 and let $\varepsilon \in (0,1)$. If $\varphi(x,t) \in B_{\delta_\varepsilon}$ then

$$\begin{aligned} \frac{d}{dt}\left\|\varphi(x,t) - x^*\right\| &= \big\langle v\big(\varphi(x,t)\big), \mathfrak{n}\big(\varphi(x,t) - x^*\big)\big\rangle \\ &\leq -(1-\varepsilon)\|\varphi(x,t) - x^*\| < 0. \end{aligned} \tag{3.4}$$

We now claim that $x \in B_{\delta_\varepsilon}$ implies $\varphi(x,t) \in B_{\delta_\varepsilon}$ for all $t \geq 0$. Indeed, if the contrary holds then there exists $\tau > 0$ such that $\varphi(x,\tau) \in \partial B_{\delta_\varepsilon}$ and $\varphi(x,t) \in B_{\delta_\varepsilon}$ for all $t \in [0,\tau)$. But this leads to the following contradiction:

$$\delta_\varepsilon = \|\varphi(x,\tau) - x^*\| = \|x - x^*\| + \int_0^\tau \frac{d}{dt}\Big|_{t=\theta} \|\varphi(x,t) - x^*\|\, d\theta < \delta_\varepsilon + \int_0^\tau 0\, d\theta.$$

Thus, we find that $x \in B_{\delta_\varepsilon}$ implies that (3.4) holds for all $t \geq 0$.

On the other hand, $x \in B_{\delta_\varepsilon}$ implies

$$\dot{\varphi}(x,t) = v\big(\varphi(x,t)\big) = -(\varphi(x,t) - x^*) + R(t)\|\varphi(x,t) - x^*\|, \tag{3.5}$$

where (1.3) and $\varphi(x,t) \in B_{\delta_\varepsilon}$ imply

$$\|R(t)\| = \frac{\big\|\varphi(x,t) + v\big(\varphi(x,t)\big) - x^*\big\|}{\big\|\varphi(x,t) - x^*\big\|} < \varepsilon. \tag{3.6}$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\|\varphi(x,t) - x^*\| &= \big\langle v(\varphi(x,t)), \mathfrak{n}(\varphi(x,t) - x^*)\big\rangle \\ &= \big(-1 + \big\langle R(t), \mathfrak{n}(\varphi(x,t) - x^*)\big\rangle\big) \times \|\varphi(x,t) - x^*\| \\ &\geq -(1+\varepsilon)\|\varphi(x,t) - x^*\| \end{aligned}$$

for all $t \geq 0$. We thus find

$$-(1+\varepsilon)\|\varphi(x,t) - x^*\| \leq \frac{d}{dt}\left\|\varphi(x,t) - x^*\right\| \leq -(1-\varepsilon)\|\varphi(x,t) - x^*\| \tag{3.7}$$

for all $t \geq 0$, and Lemma 1.1 implies that (3.1) holds with $\rho_\varepsilon \geq \delta_\varepsilon$ for $\varepsilon \in (0,1)$.

Moreover, if (1.2) holds, then instead of (3.7) we obtain

$$\begin{aligned} &-(1+\beta\|\varphi(x,t) - x^*\|^p)\|\varphi(x,t) - x^*\| \\ &\qquad\leq \frac{d}{dt}\left\|\varphi(x,t) - x^*\right\| \\ &\qquad\qquad\leq -(1-\beta\|\varphi(x,t) - x^*\|^p)\|\varphi(x,t) - x^*\| \end{aligned} \tag{3.8}$$

for $x \in B_r(x^*)$ and $t \geq 0$, where $r = \min\{(\beta)^{-1/p}, \rho\}$ now assures that if $x \in B_r(x^*)$ then $\varphi(x,t) \in B_r(x^*)$ for all $t \geq 0$. It follows from (3.8) that for $r_\varepsilon = \min\{(\varepsilon/\beta)^{1/p}, r\} = \min\{(\varepsilon/\beta)^{1/p}, \rho\}$, $x \in \mathcal{B}_{r_\varepsilon}(x^*)$ and $t \geq 0$ it is the case that

$$-(1+\varepsilon)\|\varphi(x,t) - x^*\| \leq \frac{d}{dt}\,\|\varphi(x,t) - x^*\| \leq -(1-\varepsilon)\|\varphi(x,t) - x^*\|.$$

Lemma 1.1 therefore shows that (3.1) holds with $\rho_\varepsilon \geq r_\varepsilon$.

Equations (1.5) and (3.5) imply that for $x \in B_{\delta_\varepsilon}(x^*)$,

$$\left\|\frac{d}{dt}\,\mathfrak{n}\big(\varphi(x,t) - x^*\big)\right\| = \|R(t)\|, \tag{3.9}$$

and together with (3.6) this implies (3.2).

Finally, if (1.2) holds, then $x \in B_{r_\varepsilon}(x^*)$ implies $\varphi(x,t) \in B_{r_\varepsilon}(x^*) \subset B_\rho(x^*)$ for all $t \geq 0$ and then

$$\|R(t)\| \leq \beta\|\varphi(x,t) - x^*\|^p.$$

Equations (3.1) and (3.9) therefore show that (3.3) holds. □

LEMMA 3.3. *If* (1.4) *converges exponentially to* x^*, *then* (1.1) *converges uniformly Q-superlinearly to* x^*. *Moreover, if* (1.4) *converges p-exponentially then* (1.1) *converges uniformly at Q-convergence rate* $p+1$.

Proof. Let ρ_ε be as in Definition 3.1. Suppose that $\frac{d}{dt}\ \|\varphi(x_0,t_0) - x^*\| > -\|\varphi(x_0,t_0) - x^*\|(1-\varepsilon)$ for some $x_0 \in \mathcal{B}_{\rho_\varepsilon}(x^*)$ and $t_0 > 0$. By continuity there exists $\delta > 0$ such that

$$\frac{d}{dt}\ \|\varphi(x_0,t) - x^*\| > -\|\varphi(x_0,t) - x^*\|(1-\varepsilon)$$

for all $t \in [t_0, t_0+\delta]$. By Lemma 1.1 we then have

$$\|\varphi(x_0,t) - x^*\| > \|\varphi(x_0,t_0) - x^*\|\mathrm{e}^{-(1-\varepsilon)t}$$

for $t \in [t_0, t_0+\delta]$, contradicting the upper bound in (3.1) when $\varphi(x_0,t_0)$ is used in place of x. This and a similar argument using the lower bound in (3.1) show that

$$-\|\varphi(x,t) - x^*\|(1+\varepsilon) \leq \frac{d}{dt}\ \|\varphi(x,t) - x^*\| \leq -\|\varphi(x,t) - x^*\|(1-\varepsilon)$$

for all $t \geq 0$ and $x \in \mathcal{B}_{\rho_\varepsilon}(x^*)$, and hence,

$$\begin{aligned} &\big\|P(t)\big(v(\varphi(x,t)) + \varphi(x,t) - x^*\big)\big\| \\ &\qquad = \left|\frac{d}{dt}\ \|\varphi(x,t) - x^*\| + \|\varphi(x,t) - x^*\|\right| \\ &\qquad \leq \varepsilon\|\varphi(x,t) - x^*\|. \end{aligned} \tag{3.10}$$

On the other hand, (1.5) and (3.2) show that for all $t \geq 0$ and $x \in \mathcal{B}_{\rho_\varepsilon}(x^*)$,

$$\begin{aligned} &\big\|(\mathrm{I} - P(t))\big(v(\varphi(x,t)) + \varphi(x,t) - x^*\big)\big\| \\ &\qquad = \left\|\frac{d}{dt}\ \mathfrak{n}(\varphi(x,t) - x^*)\right\| \times \|\varphi(x,t) - x^*\| \\ &\qquad \leq \varepsilon\|\varphi(x,t) - x^*\|. \end{aligned} \tag{3.11}$$

Inequalities (3.10) and (3.11) finally show that

$$\begin{aligned} &\frac{\big\|x + v(x) - x^*\big\|}{\|x - x^*\|} = \frac{\big\|v\big(\varphi(x,0)\big) + \varphi(x,0) - x^*\big\|}{\|\varphi(x,0) - x^*\|} \\ &\leq \frac{\big\|P(0)\big(v(\varphi(x,0)) + \varphi(x,0) - x^*\big)\big\|}{\|\varphi(x,0) - x^*\|} + \frac{\big\|\big(\mathrm{I} - P(0)\big)\big(v(\varphi(x,0)) + \varphi(x,0) - x^*\big)\big\|}{\|\varphi(x,0) - x^*\|} \end{aligned} \tag{3.12}$$

$$\leq 2\varepsilon$$

for all $x \in B_{\rho_\varepsilon}(x^*)$. Since this holds true for any $\varepsilon > 0$, we find that (1.3) holds, showing the first claim.

If (1.4) converges p-exponentially then the estimate (3.11) improves to

$$\left\| \frac{d}{dt}\, \mathfrak{n}(\varphi(x,t)-x^*) \right\| \times \|\varphi(x,t)-x^*\| \le \gamma \|x-x^*\|^{p+1} \mathrm{e}^{-(1-\varepsilon)(p+1)t},$$

which leads to the inequality

$$\left\| (\mathrm{I}-P(0)) \big(v(\varphi(x,0)) + \varphi(x,0) - x^* \big) \right\| \le \gamma \|x-x^*\|^{p+1}. \tag{3.13}$$

Likewise, the estimate (3.10) improves, because there exists $\delta > 0$ such that for all $x \in B_\delta(x^*)$ we have $x \in B_{\rho_\varepsilon}(x^*)$ for some $\varepsilon \le \xi^{-p}\|x-x^*\|^p$. But then (3.1) shows that for $t \ge 0$,

$$\mathrm{e}^{-(1+\xi^{-p}\|x-x^*\|^p)t}\|x-x^*\| \le \|\varphi(x,t)-x^*\| \le \mathrm{e}^{-(1-\xi^{-p}\|x-x^*\|^p)t}\|x-x^*\|,$$

and by a similar argument as used above,

$$\begin{aligned} &-\big(1+\xi^{-p}\|x-x^*\|^p\big)\|\varphi(x,t)-x^*\| \\ &\qquad \le \frac{d}{dt}\,\|\varphi(x,t)-x^*\| \le -\big(1-\xi^{-p}\|x-x^*\|^p\big)\|\varphi(x,t)-x^*\| \end{aligned}$$

for all $t \ge 0$ and $x \in B_\delta(x^*)$, so that

$$\left| \frac{d}{dt}\,\|\varphi(x,0)-x^*\| + \|\varphi(x,0)-x^*\| \right| \le \xi^{-p}\|x-x^*\|^{p+1}.$$

Using this in (3.10), we obtain

$$\left\| P(0) \big(v(\varphi(x,0)) + \varphi(x,0) - x^* \big) \right\| \le \xi^{-p} \|x-x^*\|^{p+1}. \tag{3.14}$$

Finally, substituting (3.13) and (3.14) in (3.12), we obtain (1.2) with $\beta = \gamma + \xi^{-p}$ and $q = p$, and hence the second claim is true. ☐

Combining Lemmas 3.2 and 3.3, we obtain the following result:

THEOREM 3.4. *The notion of exponential convergence of the continuous dynamical system* (1.4) *is equivalent to the notion of uniform Q-superlinear convergence of the discrete system* (1.1)*. Likewise, the notion of p-exponential convergence of* (1.4) *is equivalent to the notion of uniform Q-convergence of rate* $p+1$.

Note that in contrast to Corollary 2.3, Theorem 3.4 does not make any assumptions on the Hölder continuity of J at x^*.

4. Discussion. At first sight, the notion of exponential convergence seems to suggest that Q-convergence of the discrete dynamical system (1.1) is essentially due to two factors:

(i) the associated continuous dynamical system $\varphi(x,t)$ converges exponentially fast to x^*,

(ii) the angular speed of $\varphi(x,t)$ relative to x^* decays to zero at a fast enough rate.

There remains however the more subtle point that these rates have to be observable in neighbourhoods of x^* that are not too small. This implies that a pointwise rescaling $w(x) = \lambda(x)v(x)$ of the vector field $v(x)$ generally cannot increase the Q-convergence of the discrete method (1.1), see the section below. As a consequence, the combination of a descent method for unconstrained minimisation with a line-search procedure cannot in general improve the convergence speed of the method in small neighbourhoods of a local minimiser.

We illustrate the above made claim via a simple class of examples. Let $g \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R})$ be such that $\lim_{\zeta\to 0} g(\zeta) = 0$, $\lim_{\zeta\to\infty} g'(\zeta) = 0$ and $g''(\zeta)$ remains bounded for $\zeta \to \infty$. Then the vector field

$$v(x) = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g'\left(-\ln\|x\|\right)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix},$$

extended by continuity at the origin and on S^1, is C^1 on $\overline{B_1(0)}$ with $v(0) = 0$ and $J(0) = -\mathrm{I}$. For $y \in S^1$ the flux-line through y defined by v is then given by

$$\varphi(y,t) = \mathrm{e}^{-t}\begin{bmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{bmatrix}\begin{bmatrix} \cos g(t) \\ \sin g(t) \end{bmatrix},$$

and we observe that $\varphi(U_\theta y, t) = U_\theta \varphi(y,t)$ holds for all rotations

$$U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Since $\lim_{t\to\infty}\varphi(y,t) = 0$, for all $y \in S^1$, this rotational invariance implies that for all $x \in B_1(0)$ there exists $y^{[x]} \in S^1$ such that

$$\varphi(x,t) = \mathrm{e}^{-t}\|x\|\begin{bmatrix} y_1^{[x]} & -y_2^{[x]} \\ y_2^{[x]} & y_1^{[x]} \end{bmatrix}\begin{bmatrix} \cos g(t-\ln\|x\|) \\ \sin g(t-\ln\|x\|) \end{bmatrix}.$$

Furthermore, it is easy to check that

$$\begin{aligned} \left\| \frac{d}{dt}\,\mathfrak{n}(\varphi(x,t)) \right\| &= \left|g'\left(t-\ln\|x\|\right)\right|, \quad \text{and} \\ \|\varphi(x,t)\| &= \mathrm{e}^{-t}\|x\| \end{aligned} \tag{4.1}$$

for all $t \geq 0$ and $x \in B_1(0)$. Now, if

$$\bar{p} := \sup\{p \geq 1 : (1.1) \text{ is Q-convergent with rate } p+1\},$$

then for general g we have $1 \leq \bar{p} < \infty$. If $p > \bar{p}$ this implies that the method (1.1) defined by $v(x)$ is not Q-convergent with rate $p+1$. Let $\lambda \in C^1(\mathbb{R}^2, \mathbb{R}_+)$ be a positive scalar field and consider the vector field

$$w(x) = \lambda(x)v(x).$$

We claim that the method (1.1) defined by w is also not Q-convergent with rate $p+1$, that is, a simple rescaling of v cannot improve the convergence rate.

In fact, if the contrary holds for some choice of λ then the flux $\psi(x,t)$ associated with $w(x)$ converges p-exponentially to the origin, that is, there exist constants $\xi > 0$, $\gamma_p > 0$ such that for all $\varepsilon > 0$ small enough there exists $\rho_\varepsilon > \xi\varepsilon^{1/p}$ with the property that for all $x \in B_{\rho_\varepsilon}(0)$ and $t \geq 0$,

$$\mathrm{e}^{-(1+\varepsilon)t}\|x\| \leq \|\psi(x,t)\| \leq \mathrm{e}^{-(1-\varepsilon)t}\|x\|, \tag{4.2}$$

$$\left\| \frac{d}{dt}\ \mathfrak{n}(\psi(x,t)) \right\| \leq \gamma_p \|x\|^p \mathrm{e}^{-(1-\varepsilon)pt}. \tag{4.3}$$

On the other hand, $\varphi(x,t)$ does not converge p-exponentially, and since (4.1) holds for all $x \in B_1(0)$, this must be due to the fact that for $0 < \varepsilon \ll 1$ there exists $z_\varepsilon \in B_{\rho_\varepsilon}(0)$ and $T \geq 0$ such that

$$\frac{\gamma_p}{1-\varepsilon}\|z_\varepsilon\|^p \mathrm{e}^{-(1-\varepsilon)pT} < \left| g'\left(T - \ln\|z_\varepsilon\|\right)\right|.$$

Replacing z_ε by $\varphi(z_\varepsilon, T)$ it is easy to see that we may assume without loss of generality that $T = 0$, and then by continuity there exist constants $\delta, \tau > 0$ such that

$$\frac{\gamma_p}{1-\varepsilon}\|z_\varepsilon\|^p \mathrm{e}^{-(1-\varepsilon)pt} < \left| g'\left(t - \ln\|z_\varepsilon\|\right)\right|(1-\delta) \tag{4.4}$$

for all $t \in [0,\tau)$. Now let $s(t)$ be defined by the ODE

$$\frac{d}{dt}s(t) = \lambda\bigl(\varphi(z_\varepsilon, t)\bigr), \qquad s(0) = 0.$$

Since $\lambda > 0$, $s(t)$ is a monotone increasing reparameterisation of t constructed so that $\psi(z_\varepsilon, t) = \varphi\bigl(z_\varepsilon, s(t)\bigr)$. Therefore,

$$\left\| \frac{d}{dt}\ \mathfrak{n}(\psi(z_\varepsilon,t)) \right\| = \frac{d}{dt}s(t) \times \left| g'\left(s(t) - \ln\|z_\varepsilon\|\right)\right|,$$

and using (4.3)) and (4.4) we obtain that

$$\begin{aligned} \frac{d}{dt}s(t) \times \frac{\gamma_p}{1-\varepsilon}\|z_\varepsilon\|^p \mathrm{e}^{-(1-\varepsilon)p\times s(t)} &< \frac{d}{dt}s(t) \times \left| g'\left(s(t) - \ln\|z_\varepsilon\|\right)\right|(1-\delta) \\ &\leq \gamma_p\|z_\varepsilon\|^p \mathrm{e}^{-(1-\varepsilon)pt}(1-\delta) \end{aligned}$$

holds for every $t \in [0, s^{-1}(\tau))$, that is,

$$\frac{d}{dt}\left(s(t) - t\right) < \frac{\gamma_p\|z_\varepsilon\|^p \mathrm{e}^{-(1-\varepsilon)pt}(1-\delta)}{\frac{\gamma_p}{1-\varepsilon}\|z_\varepsilon\|^p \mathrm{e}^{-(1-\varepsilon)p\times s(t)}} - 1 = (1-\varepsilon)(1-\delta)\mathrm{e}^{(1-\varepsilon)p\times(s(t)-t)}.$$

Note that the right-hand side is monotone increasing in $s(t) - t$. Therefore, we can apply Lemma 1.1 to find that

$$s(t) - t < -\frac{\ln\left[(1-\varepsilon)(1-\delta) + (\varepsilon+\delta-\varepsilon\delta)\mathrm{e}^{(1-\varepsilon)pt}\right]}{(1-\varepsilon)p}$$

holds for small positive t. Since

$$\ln\left[(1-\varepsilon)(1-\delta) + (\varepsilon+\delta-\varepsilon\delta)\mathrm{e}^{(1-\varepsilon)pt}\right] = (\varepsilon+\delta-\varepsilon\delta)(1-\varepsilon)pt + \mathcal{O}(t^2),$$

it follows that

$$s(t) - t < -\varepsilon t \tag{4.5}$$

for small positive t. On the other hand, (4.1) and (4.2) imply

$$\mathrm{e}^{-s(t)}\|z_\varepsilon\| = \|\varphi(z_\varepsilon, s(t))\| = \|\psi(z_\varepsilon, t)\| \le \mathrm{e}^{-(1-\varepsilon)t}\|z_\varepsilon\|,$$

and hence, $-\varepsilon t \le s(t) - t$ for small positive t. Since this contradicts (4.5), it follows that our claim is true.

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