Necessary and Sufficient Optimality Conditions for Mathematical Programs with Equilibrium Constraints*

Jane J. Ye†

October 15, 2004

Abstract

In this paper we consider a mathematical program with equilibrium constraints (MPEC) formulated as a mathematical program with complementarity constraints. Various stationary conditions for MPECs exist in literature due to different reformulations. We give a simple proof to the M-stationary condition and show that it is sufficient for global or local optimality under some MPEC generalized convexity assumptions. Moreover we propose new constraint qualifications for M-stationary conditions to hold. These new constraint qualifications include piecewise MFCQ, piecewise Slater condition, MPEC weak reverse convex constraint qualification, MPEC Arrow-Hurwicz-Uzawa constraint qualification, MPEC Zangwill constraint qualification, MPEC Kuhn-Tucker constraint qualification and MPEC Abadie constraint qualification.

Key words: mathematical program with equilibrium constraints, necessary optimality conditions, sufficient optimality conditions, constraint qualifications

AMS subject classification:49K10, 90C30, 91A65

1 Introduction

In this paper we study necessary and sufficient optimality conditions for the *mathematical program with equilibrium constraints* (MPEC):

$$
\begin{aligned}
\text{(MPEC)} \quad \min \quad & f(z) \\
\text{s.t.} \quad & g(z) \leq 0, \quad h(z) = 0, \\
& G(z) \geq 0, \quad H(z) \geq 0, \quad G(z)^\top H(z) = 0,
\end{aligned}
$$

where $f : R^n \to R$, $G : R^n \to R^m$, $H : R^n \to R^m$, $g : R^n \to R^p$, $h : R^n \to R^q$ and $^\top$ indicates the transpose. This formulation is equivalent to but more convenient than the nonsymmetric formulation of the *optimization problem with complementarity constraints* (OPCC):

$$
\begin{aligned}
\text{(OPCC)} \quad \min \quad & f(x, y) \\
\text{s.t.} \quad & g(x, y) \leq 0, \quad h(x, y) = 0, \\
& G(x, y) \geq 0, \quad y \geq 0, \quad G(x, y)^\top y = 0,
\end{aligned}
$$

which is the most important special case (where $\Omega = R_+^m$) of the *optimization problem with variational inequality constraints* (OPVIC) :

$$
\begin{aligned}
\text{(OPVIC)} \quad \min \quad & f(x, y) \\
\text{s.t.} \quad & g(x, y) \leq 0, \quad h(x, y) = 0, \\
& y \in \Omega, \quad \langle G(x, y), y - y' \rangle \leq 0 \quad \forall y' \in \Omega,
\end{aligned}
$$

where $f : R^{n+m} \to R$, $G : R^{n+m} \to R^m$, $g : R^{n+m} \to R^p$, $h : R^{n+m} \to R^q$, and Ω is a closed convex subset of R^m. Since x, y can be used to model the upper and lower level variables respectively and y is considered to be a solution of a complementarity or a variational inequality constraint parameterized in x and hence a solution of an equilibrium or an optimization problem, (OPCC) and (OPVIC) are also called a *generalized bilevel programming problem* (see e.g. [25]) or a mathematical program with equilibrium constraints (see e.g., [8]). The reader is referred to [8] and [15] for applications and recent developments.

For MPEC, it is well-known that the usual nonlinear programming constraint qualifications such as Mangasarian-Fromovitz constriant qualification (MFCQ) does

not hold (see [25, Proposition 1.1]). Since there are several different approaches to reformulate MPEC, various stationarity concepts arise (see e.g. [19]). In this paper, we show that the M-stationary condition is the most appropriate stationary condition for MPEC in the sense that it is the second strongest stationary condition (with the strongest one being the S-stationary condition) and it holds under almost all analogues of the constraint qualifications for nonlinear programming problems such as MPEC linear constraint qualification, MPEC weak reverse convex constraint qualification, MPEC Arrow-Hurwicz-Uzawa constraint qualification, MPEC MFCQ, MPEC Zangwill constraint qualification, MPEC Kuhn-Tucker and MPEC Abadie constraint qualification. Also analogues to nonlinear programming, the M-stationary condition becomes a sufficient condition for global or local optimality under some MPEC generalized convexity condition.

In this paper, for simplicity, unless specified we assume that the objective function and all binding constraints are differentiable, all nonbinding constriants are continuous. The results may be extended to include the possibility of nonsmooth or nondifferentiable functions as in [22].

The following notations are used throughout the paper. For a vector $d \in R^n$ and an index sets $I \subseteq \{1, 2, \ldots, n\}$, d_i is the ith component of d and d_I is the subvector composed from the components $d_i, i \in I$. $\langle a, b \rangle$ or $a^\top b$ is the inner product of vectors a and b.

2   Stationary points and constraint qualifications

Given a feasible vector z^* of MPEC, we define the following index sets:

$$I_g := \{i : g_i(z^*) = 0\},$$
$$\alpha := \alpha(z^*) := \{i : G_i(z^*) = 0, H_i(z^*) > 0\},$$
$$\beta := \beta(z^*) := \{i : G_i(z^*) = 0, H_i(z^*) = 0\},$$
$$\gamma := \gamma(z^*) := \{i : G_i(z^*) > 0, H_i(z^*) = 0\}.$$

The set β is known as the *degenerate* set. If β is empty, the vector z^* is said to satisfy the *strict complementarity* condition. This paper focuses on the important case where β is nonempty. We define the set of all partitions of β by:

$$\mathcal{P}(\beta) := \{(\beta_1, \beta_2) : \beta_1 \cup \beta_2 = \beta, \beta_1 \cap \beta_2 = \emptyset\}.$$

Each partition $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$ is associated with a branch of MPEC:

$$\text{MPEC}(\beta_1, \beta_2) \quad \min \quad f(z)$$

$$\text{s.t.} \quad g(z) \leq 0, \quad h(z) = 0,$$

$$G_i(z) = 0 \quad i \in \alpha \cup \beta_2, \quad H_i(z) = 0 \quad i \in \gamma \cup \beta_1$$

$$G_i(z) \geq 0 \quad i \in \beta_1, \quad H_i(z) \geq 0 \ i \in \beta_2.$$

It is obvious that z^* is a local optimal solution of MPEC if and only if it is a local optimal solution to $\text{MPEC}(\beta_1, \beta_2)$ for all partition $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$.

2.1 Primal stationary conditions

In order to define a primal stationary condition for MPEC, we recall the notion of a tangent cone.

Definition 2.1 (Tangent Cone) *Let $\mathcal{Z}$ denote the feasible region of MPEC and $z^* \in \mathcal{Z}$. The Tangent cone of $\mathcal{Z}$ at z^* is the closed cone defined by*

$$\mathcal{T}(z^*) := \{d \in R^n : \exists t_n \downarrow 0, d_n \to d \ s.t. \ z^* + t_n d_n \in \mathcal{Z} \ \forall n\}.$$

The following notion of primal stationary condition for MPEC was first introduced in [9], and studied in depth in the monograph [8]. It is different from the B-stationary condition in [19] which is defined by

$$\nabla f(z^*)^\top d \geq 0, \quad \forall d \in \mathcal{T}_{MPEC}^{lin}(z^*),$$

where $\mathcal{T}_{MPEC}^{lin}(z^*)$ is the MPEC linearization cone defined in Definition 3.1.

Definition 2.2 (B-stationary point) *A feasible point z^* of the MPEC is said to be Bouligand stationary (B-stationary) if*

$$\nabla f(z^*)^\top d \geq 0, \quad \forall d \in \mathcal{T}(z^*).$$

Using the definition of the tangent cone it is easy to see that a local optimal solution of MPEC must be a B-stationary point. Although a B-stationary condition holds at any local optimal solution, the difficulty lies in the characterization of the tangent cone and hence it is more useful to consider dual stationary conditions.

2.2   Dual stationary conditions

Unlike the standard nonlinear programming which has only one dual stationary condition, i.e. the Karush-Kuhn-Tucker condition, there are various stationarity concepts for MPEC. We now summarize them and indicate their connections.

Definition 2.3 (W-stationary point) *A feasible point z^* of MPEC is called weakly stationary if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ such that the following condition hold:*

$$0 = \nabla f(z^*) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^q \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^m [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$

(1)

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0.$$

(2)

It is easy to see that W-stationary condition is the KKT condition for the tightened MPEC:

$$\begin{aligned}
(\text{TMPEC}) \quad \min \quad & f(z) \\
\text{s.t.} \quad & g(z) \leq 0, \quad h(z) = 0, \\
& G_i(z) = 0 \quad i \in \alpha, \quad H_i(z) = 0 \quad i \in \gamma, \\
& G_i(z) = 0 \quad H_i(z) = 0 \quad i \in \beta.
\end{aligned}$$

Definition 2.4 (C-stationary point) *A feasible point z^* of MPEC is called Clarke stationary if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ such that (1)-(2) and the following condition hold:*

$$\forall i \in \beta, \quad \lambda_i^G \lambda_i^H \geq 0.$$

By [19, lemma 1] the C-stationary condition is the nonsmooth KKT condition using the Clarke generalized gradient [4] by reformulating MPEC as a nonsmooth nonlinear programming problem:

$$\begin{aligned}
\min \quad & f(z) \\
\text{s.t.} \quad & g(z) \leq 0, \quad h(z) = 0, \\
& G_i(z) = 0 \quad i \in \alpha, \qquad H_i(z) = 0 \quad i \in \gamma, \\
& \min\{G_i(z), H_i(z)\} = 0 \quad i \in \beta.
\end{aligned}$$

5

Definition 2.5 (A-Stationary point) *A feasible point z^* of MPEC is called alternatively stationary if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ such that (1)-(2) and the following condition hold:*

$$\forall i \in \beta, \quad \lambda_i^G \geq 0 \quad or \quad \lambda_i^H \geq 0.$$

The notion of the A-stationary condition was introduced by Flegel and Kanzow [5]. Actually the A-stationary condition is the KKT conditions for MPEC(β_1, β_2) for a partition $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$.

Definition 2.6 (M-stationary point) *A feasible point z^* of MPEC is called Mordukhovich stationary if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ such that (1)-(2) and the following condition hold:*

$$\forall i \in \beta, \quad either \ \lambda_i^G > 0, \ \lambda_i^H > 0 \quad or \quad \lambda_i^G \lambda_i^H = 0.$$

It will become clear in §2.3 that the M-stationary condition is the nonsmooth KKT condition involving the limiting subgradient for EMPEC, an equivalent formulation of MPEC.
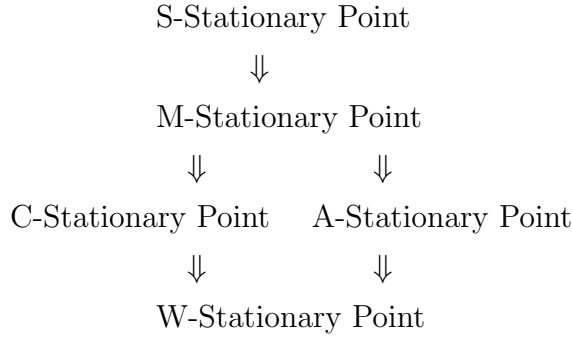
Definition 2.7 (S-stationary point) *A feasible point z^* of MPEC is called strong stationary if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ such that (1)-(2) and the following condition hold:*

$$\forall i \in \beta, \ \lambda_i^G \geq 0 \ \ \lambda_i^H \geq 0.$$

The S-stationary condition is the KKT condition for the relaxed MPEC:

$$
\begin{aligned}
\text{(RMPEC)} \quad \min \quad & f(z) \\
\text{s.t.} \quad & g(z) \leq 0, \quad h(z) = 0, \\
& G_i(z) = 0 \quad i \in \alpha, \quad H_i(z) = 0 \quad i \in \gamma, \\
& G_i(z) \geq 0, \quad H_i(z) \geq 0 \ \ i \in \beta.
\end{aligned}
$$

The following diagram summarizes the relations between the dual stationary concepts that we have discussed.

$$\text{S-Stationary Point}$$
$$\Downarrow$$
$$\text{M-Stationary Point}$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\text{C-Stationary Point} \qquad \text{A-Stationary Point}$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\text{W-Stationary Point}$$

Definition 2.8 (MPEC LICQ) *Let z^* be a feasible point of MPEC where all functions are continuously differentiable at z^*. We say that MPEC linear independence constraint qualification is satisfied at z^* if the gradient vectors of the binding constraints for RMPEC is satisfied, i.e.,*

$$\nabla g_i(z^*) \quad \forall i \in I_g,$$

$$\nabla h_i(z^*) \quad \forall i = 1, 2, \ldots, q,$$

$$\nabla G_i(z^*) \quad \forall i \in \alpha \cup \beta,$$

$$\nabla H_i(z^*) \quad \forall i \in \gamma \cup \beta$$

are linearly independent.

MPEC LICQ is a very strong condition. It is the linear independence constraint qualification for the relaxed MPEC and hence it is a constraint qualification for the S-stationary condition to hold at a local optimal solution. A condition weaker than MPEC LICQ under which the B-stationary condition is equivalent to the S-stationary condition was given in [16, Theorem 3]. For a local optimal solution, under the MPEC LICQ, all concepts of stationary points discussed above including the B-stationary condition coincide. In fact it is easy to see from the proof of [20, Theorem 3.2] that all dual stationary conditions for a local optimal solution coincide under the following weaker condition. Actually under this condition the β components of the multiplier λ^G, λ^H of TMPEC is unique.

Definition 2.9 (Partial MPEC LICQ) *Let z^* be a feasible point of MPEC. The partial MPEC linear independence constraint qualification holds at z^* if for any vectors $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$,*

$$0 = \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^q \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^m [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$

$$\lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0.$$

implies that

$$\lambda_\beta^G = 0, \quad \lambda_\beta^H = 0.$$

2.3 M-stationary conditions

The M-stationary condition was first introduced in Ye and Ye [23, Theorem 3.2] for OPVIC by using Mordukhovich coderivative of set-valued maps (see e.g. [12]), further studied by Ye in [21] and Outrata in [14]. The term "M-stationary condition" was first used in [19]. In [21, Theorem 3.2] a Fritz John type necessary optimality condition involving Mordukhovich coderivative for OPVIC is given. One may derive the Fritz John type M-stationary condition for MPEC by reformulating MPEC as the following OPVIC:

$$(P) \quad \min \quad f(z)$$

$$\text{s.t.} \quad g(z) \le 0, \quad (x, y, w) \in \Omega$$

$$\langle (H(z), G(z) - x, h(z) - w), (x, y, w) - (x', y', w') \rangle \le 0 \quad \forall y' \in \Omega,$$

where $\Omega = R_+^m \times R^m \times R^q$ and applying [21, Theorem 3.2] to the above problem. Note that although the proof of [21, Theorem 3.2] has a gap since the nontriviality of the multipliers was not proved it is known that the theorem itself is correct since it can be proved in various other ways. For example in [24, Theorem 1.3] a more general theorem is given for multiobjective MPEC. We now provide an easy and independent proof. The proof also shows that the M-stationary condition is in fact the generalized multiplier rule in terms of limiting subgradients for the equivalent problem EMPEC.

Theorem 2.1 (Fritz John type M-stationary condition) *Let z^* be a local solution of MPEC where all functions are continuously differentiable at z^*. Then there exists $r \ge 0$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ not all zero such that*

$$0 = r \nabla f(z^*) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^q \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^m [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0,$$

$$either \ \lambda_i^G > 0, \ \lambda_i^H > 0 \quad or \quad \lambda_i^G \lambda_i^H = 0 \quad \forall i \in \beta.$$

Proof. By introducing slack variables, we reformulate MPEC in the following equivalent form:

$$\begin{aligned} (\text{EMPEC}) \quad & \min \quad f(z) \\ & s.t. \quad g(z) \leq 0, \ h(z) = 0, \\ & \qquad G(z) - x = 0, \ H(z) - y = 0, \\ & \qquad (x, y) \in \Omega, \end{aligned}$$

where $\Omega := \{(x, y) \in R^{2m} : x \geq 0, y \geq 0, x^\top y = 0\}$. This is an optimization problem with equalities, inequalities and a nonconvex abstract constraint $(x, y) \in \Omega$ with $(x^*, y^*, z^*) = (G(z^*), H(z^*), z^*)$ as a local solution. Applying the limiting subgradient version of the generalized Lagrange multiplier rule first obtained by Mordukhovich in [11, Theorem 1(b)] (see also [18, Corollary 6.15]), we conclude that there exists $r \geq 0$, λ not all zero and $(\xi, \gamma) \in N_\Omega(x^*, y^*)$, the limiting normal cone of Ω at the point (x^*, y^*) such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = r \begin{pmatrix} 0 \\ 0 \\ \nabla f(z^*) \end{pmatrix} + \sum_{i \in I_g} \lambda_i^g \begin{pmatrix} 0 \\ 0 \\ \nabla g_i(z^*) \end{pmatrix} + \sum_{i=1}^q \lambda_i^h \begin{pmatrix} 0 \\ 0 \\ \nabla h_i(z^*) \end{pmatrix}$$

$$- \sum_{i=1}^m \lambda_i^G \begin{pmatrix} -e_i \\ 0 \\ \nabla G_i(z^*) \end{pmatrix} - \sum_{i=1}^m \lambda_i^H \begin{pmatrix} 0 \\ -e_i \\ \nabla H_i(z^*) \end{pmatrix} + \begin{pmatrix} \xi \\ \gamma \\ 0 \end{pmatrix},$$

$$\lambda_{I_g}^g \geq 0,$$

where e_i denotes the unit vector whose i-th component is equal to 1. It follows that

$$0 = \lambda^G + \xi,$$

$$0 = \lambda^H + \gamma,$$

$$0 = r\nabla f(z^*) + \sum_{i=1}^p \lambda_i^g \nabla g_i(z^*) + \sum_{i \in I_g} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^m [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$

$$\lambda_{I_g}^g \geq 0.$$

Since $(\xi, \gamma) \in N_\Omega(x^*, y^*)$ and

$$N_\Omega(x^*, y^*) = \left\{ (\xi, \gamma) : \begin{array}{ll} \xi_i = 0 & \text{if } x_i^* > 0 \\ \gamma_i = 0 & \text{if } y_i^* > 0 \\ \text{either } \xi_i < 0, \gamma_i < 0 \text{ or } \xi_i\gamma_i = 0 & \text{if } x^* = y^* = 0 \end{array} \right\}$$

(see e.g. [21, Proposition 3.7]), the assertion of the theorem follows.

∎

By the Fritz John type M-stationary condition, if r in the condition is never zero then it can be taken as 1. Hence the following KKT type M-stationary condition follows immediately.

Definition 2.10 (NNAMCQ) *Let z^* be a feasible point of MPEC where all functions are continuously differentiable at z^*. We say that the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) is satisfied at z^* if there is no nonzero vector $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ such that*

$$0 = \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^q \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^m [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0,$$

$$\text{either } \lambda_i^G > 0, \lambda_i^H > 0 \quad \text{or} \quad \lambda_i^G \lambda_i^H = 0 \quad \forall i \in \beta.$$

Corollary 2.1 *Let z^* be a local solution of MPEC where all functions are continuously differentiable at z^*. Suppose that NNAMCQ is satisfied at z^*, Then r in Theorem 2.1 can be taken as 1, i.e., z^* is M-stationary.*

Remark: It is known that for the case of nonlinear programming (i.e. when $m = 0$), NNAMCQ is equivalent to the Mangasarian-Fromovitz constraint qualification. In [14], Outrata introduced a generalized Mangasarian-Fromovitz constraint qualification for OPCC and showed that NNAMCQ is equivalent to the generalized Mangasarian-Fromovitz constraint qualification for OPCC under condition (A) in [14, Proposition 3.3]. In [21, Proposition 4.5], Outrata's result was extended to OPVIC with the condition (A) removed. We now state the generalized Mangasarian-Fromovitz constraint qualification for MPEC as the MPEC GMFCQ. For completeness we include the sketch of the proof for the equivalence of the NNAMCQ and the MPEC GMFCQ in Proposition 2.1. Note that the MPEC GMFCQ defined in Definition 2.11 is weaker than MPEC MFCQ in [19, 5] which is defined to be MFCQ for TMPEC, the tightened MPEC.

Definition 2.11 (MPEC GMFCQ) *Let z^* be a feasible point of MPEC where all functions are continuously differentiable at z^*. We say that MPEC generalized Mangasarian-Fromovitz constraint qualification is satisfied at z^* if*

(i) for every partition of β into sets P, Q, R with $R \neq \emptyset$, there exist d such that

$$\nabla g_i(z^*)^\top d \leq 0 \quad \forall i \in I_g,$$

$$\nabla h_i(z^*)^\top d = 0 \quad \forall i = 1, 2, \ldots, q,$$

$$\nabla G_i(z^*)^\top d = 0 \quad \forall i \in \alpha \cup Q,$$

$$\nabla H_i(z^*)^\top d = 0 \quad \forall i \in \gamma \cup P,$$

$$\nabla G_i(z^*)^\top d \geq 0, \nabla H_i(z^*)^\top d \geq 0 \quad i \in R$$

and for some $i \in R$ either $\nabla G_i(z^*)^\top d > 0$ or $\nabla H_i(z^*)^\top d > 0$;

(ii) for every partition of β into sets P, Q, the gradient vectors

$$\nabla h_i(z^*) \quad \forall i = 1, 2, \ldots, q,$$

$$\nabla G_i(z^*) \quad \forall i \in \alpha \cup Q,$$

$$\nabla H_i(z^*) \quad \forall i \in \gamma \cup P$$

are linearly independent and there exists $d \in R^n$ such that

$$\nabla g_i(z^*)^\top d < 0 \quad \forall i \in I_g,$$

$$\nabla h_i(z^*)^\top d = 0 \quad \forall i = 1, 2, \ldots, q,$$

$$\nabla G_i(z^*)^\top d = 0 \quad \forall i \in \alpha \cup Q$$

$$\nabla H_i(z^*)^\top d = 0 \quad \forall i \in \gamma \cup P.$$

Proposition 2.1 *NNAMCQ is equivalent to MPEC GMFCQ.*

Proof. Note that β can be split into the sets

$$P = \{i \in \beta : \lambda_i^G = 0\},$$

$$Q = \{i \in \beta : \lambda_i^H = 0\},$$

$$R = \{i \in \beta : \lambda_i^G > 0, \lambda_i^H > 0\}$$

and so condition NNAMCQ is equivalent to the following two conditions:

(i) For every partition of β into sets P, Q, R with $R \neq \emptyset$ there are no vectors $\lambda_{I_g}^g$, λ^h, $\lambda_{\alpha \cup Q \cup R}^G$ and $\lambda_{\gamma \cup P \cup R}^H$ satisfying the system

$$0 = \nabla g(z^*)^\top \lambda_{I_g}^g + \nabla h(z^*)^\top \lambda^h - \nabla G(z^*)^\top \lambda_{\alpha \cup Q \cup R}^G - \nabla H(z^*)^\top \lambda_{\gamma \cup P \cup R}^H,$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda_R^G > 0, \quad \lambda_R^H > 0;$$

(ii) For every partition of β into sets P, Q there are no vectors $\lambda_{I_g}^g$, λ^h, $\lambda_{\alpha \cup Q}^G$ and $\lambda_{\gamma \cup P}^H$ satisfying the system

$$0 = \nabla g(z^*)^\top \lambda_{I_g}^g + \nabla h(z^*)^\top \lambda^h - \nabla G(z^*)^\top \lambda_{\alpha \cup Q}^G - \nabla H(z^*)^\top \lambda_{\gamma \cup P}^H,$$

$$\lambda_{I_g}^g \geq 0.$$

The results follow from applying Tucker's and Motzkin's theorems of alternatives (see e.g. [10]) to (i) and (ii) respectively.

∎

In mathematical programming, it is well-known that if all constraint functions are affine then the KKT necessary optimality condition holds without any additional constraint qualification. Since MPEC is a special case of OPVIC, [21, Corollary 4.8] which follows from [21, Theorems 3.6, 4.3 and Proposition 4.2] indicates that the M-stationary condition holds for MPEC under the following MPEC linear CQ.

Definition 2.12 (MPEC linear CQ) *We say the MPEC linear constraint qualification is satisfied if all mappings g, h, G, H are affine.*

The following result may be obtained by reformulating MPEC into (P) and applying the correspoinding results for OPVIC in [21, Corollary 4.8]. Alternatively we may prove the results using the equivalent formulation (EMPEC) and the same proof techniques as in [21]. We sketch the proof here.

Theorem 2.2 (Kuhn-Tucker type necessary M-stationary condition) *Let z^* be a local optimal solution for MPEC where all functions are continuously differentiable at z^*. If either MPEC GMFCQ or MPEC linear CQ is satisfied at z^*, then z^* is M-stationary.*

Proof. The conclusion that z^* is M-stationary under MPEC GMFCQ follows from Corollary 2.1 and Proposition 2.1.

To prove that z^* is M-stationary under MPEC linear CQ, we consider the set of solutions to the perturbed constraint system for EMPEC:

$$\Sigma(p, q, r, s) := \{((x, y, z) \in \Omega \times R^n : \quad g(z) + p \leq 0, \quad h(z) + q = 0$$

$$G(z) - x + r = 0, H(z) - y + s = 0\}.$$

It is easy to see that the graph of the set-valued map Σ is a union of polyhedral convex sets and hence Σ is a polyhedral multifunction. By [17, Proposition 1], Σ is upper Lipschitz at each $(0, 0, 0, 0) \in R^{p+q+2m}$, i.e., there exists a neighborhood U of $(0, 0, 0, 0)$ and $\alpha \geq 0$ such that

$$\Sigma(p, q, r, s) \subseteq \Sigma(0, 0, 0, 0) + \alpha\|(p, q, r, s)\|clB, \quad \forall (p, q, r, s) \in U,$$

where clB denotes the closed unit ball. Equivalently the constraint system of EMPEC has a local error bound, i.e.,

$$d((x, y, z), \Sigma(0, 0, 0, 0)) \leq \alpha\|(p, q, r, s)\| \qquad \forall (p, q, r, s) \in U,$$

$$(x, y, z) \in \Sigma(p, q, r, s),$$

where $d(a, C)$ is the distant from point a to set C. By Clarke's principle of exact panelization [4, Proposition 2.4.3], (x^*, y^*, z^*) is also a local optimal solution to the the unconstrained problem:

$$\min f(z) + \mu_f d((x, y, z), \Sigma(0, 0, 0, 0))$$

where μ_f is the Lipschitz constant of f. Hence by the local error bound property, it is easy to see that $(z, p, q, r, s) = (z^*, 0, 0, 0, 0)$ is a local optimal solution to the following problem:

$$\min \quad f(z) + \mu_f \alpha\|(p, q, r, s)\|$$

$$s.t. \quad g(z) + p \leq 0, \; h(z) + q = 0,$$

$$G(z) + r \geq 0, \; H(z) + s \geq 0, \; (G(z) + r)^\top (H(z) + s) = 0.$$

It can be easily verified that NNAMCQ is satisfied at $(z^*, 0, 0, 0)$ for the above problem. Note that although the objective function has a nonsmooth term $\|(p, q, r, s)\|$, using exactly the same technique one can prove that Theorem 2.1 holds with the usual gradients replaced by the limiting subgradients and the equality in (3) replaced by inclusions. Applying Corollary 2.1 to the above MPEC, it is easy to obtain the

M-stationary condition since z component of equation (3) for the above problem is exactly the same as equation (3) for MPEC. ∎

For nonlinear programming problems, it is known that the KKT necessary condition becomes sufficient if the problem is (generalized) convex (see e.g. [3, Theorem 4.3.8]]). Although in some special cases an MPEC may become a convex programming problem such as in Outrata [13, Proposition 2.8], in general MPEC is a nonconvex problem even when all constraint functions are affine and hence the necessary condition is in general not sufficient for optimality. Moreover some necessary conditions for MPECs are derived through an approximation of the generalized gradient of certain nonsmooth function such as in the case of the C-stationary conditions and in the case of using the implicit programming approach (see the remark before [13, Proposition 2.8]) and hence may be too loose to be sufficient. In Ye [20, Proposition 3.1], it was shown that the S-stationary conditions become sufficient or locally sufficient for optimality when the objective function is pseuduconvex and all constraint functions are affine. In the following theorem, we show that M-stationary condition also turns into a sufficient optimality condition or local sufficient optimality condition under certain MPEC generalized convexity condition.

Theorem 2.3 (Sufficient M-stationary condition) *Let z^* be a feasible point of MPEC and the M-stationary condition holds at z^*, i.e., there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$ such that*

$$0 = \nabla f(z^*) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$

(4)

$$\lambda_{I_g}^g \geq 0, \quad \lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0,$$

$$\forall i \in \beta, \quad either \ \lambda_i^G > 0 \ \lambda_i^H > 0 \quad or \ \lambda_i^G \lambda_i^H = 0.$$

Let

$$J^+ := \{i : \lambda_i^h > 0\}, \quad J^- = \{i : \lambda_i^h < 0\},$$

$$\beta^+ := \{i \in \beta : \lambda_i^G > 0, \lambda_i^H > 0\},$$

$$\beta_G^+ := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H > 0\}, \quad \beta_G^- := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H < 0\},$$

$$\beta_H^+ := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G > 0\}, \quad \beta_H^- := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G < 0\},$$

$$\alpha^+ := \{i \in \alpha : \lambda_i^G > 0\}, \quad \alpha^- := \{i \in \alpha : \lambda_i^G < 0\},$$

$$\gamma^+ := \{i \in \gamma : \lambda_i^H > 0\}, \quad \gamma^- := \{i \in \gamma : \lambda_i^H < 0\}.$$

14

Further suppose that f is pseudoconvex at z^*, $g_i(i \in I_g), h_i(i \in J^+), -h_i(i \in J^-), G_i(i \in \alpha^- \cup \beta_H^-), -G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), H_i(i \in \gamma^- \cup \beta_G^-), -H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are quasiconvex. Then in the case when $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \emptyset$, z^* is a global optimal solution of MPEC; in the case when $\beta_G^- \cup \beta_H^- = \emptyset$ or when z^* is an interior point relative to the set $\mathcal{Z} \cap \{z : G_i(z) = 0, H_i(z) = 0, i \in \beta_G^- \cup \beta_H^-\}$, i.e., for all feasible point z which is close to z^*, it holds that

$$G_i(z) = 0, H_i(z) = 0 \quad \forall i \in \beta_G^- \cup \beta_H^-,$$

z^* is a local optimal solution of MPEC where $\mathcal{Z}$ denotes the set of feasible solutions of MPEC.

Proof. Let z be any feasible point of MPEC. Then for any $i \in I_g$,

$$g_i(z) \leq 0 = g_i(z^*).$$

By quasiconvexity of g_i at z^* it follows that

$$g_i(z^* + t(z - z^*)) = g_i(tz + (1 - t)z^*) \leq g_i(z^*)$$

for all $t \in (0, 1)$. This implies that

$$\langle \nabla g_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in I_g. \tag{5}$$

Similarly, we have

$$\langle \nabla h_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in J^+, \tag{6}$$

$$-\langle \nabla h_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in J^-. \tag{7}$$

Since for any feasible point z, $-G(z) \leq 0, -H(z) \leq 0$, one also have

$$-\langle \nabla G_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in \alpha^+ \cup \beta_H^+ \cup \beta^+, \tag{8}$$

$$-\langle \nabla H_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in \gamma^+ \cup \beta_G^+ \cup \beta^+. \tag{9}$$

In the case when $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \emptyset$, multiplying (5), (6), (7), (8), (9) by $\lambda_i^g \geq 0(i \in I_g), \lambda_i^h > 0(i \in J^+), -\lambda_i^h > 0(i \in J^-), \lambda_i^G > 0(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), \lambda_i^H > 0(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ respectively and adding we get

$$\langle \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)], z - z^* \rangle \leq 0.$$

15

By virtue of (4), the above inequality implies that

$$\langle \nabla f(z^*), z - z^* \rangle \geq 0.$$

By the pseudoconvexity of f at z^*, we must have $f(z) \geq f(z^*)$ for all feasible point z and hence z^* is a global optimal solution of MPEC if $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \emptyset$.

Now we discuss the case when $\alpha^- \cup \gamma^- \neq \emptyset$ and $\beta_G^- \cup \beta_H^- = \emptyset$. For any $i \in \alpha$, since $H_i(z^*) > 0$, $H_i(z) > 0$ for z sufficiently close to z^* and hence by the complementarity condition, $G_i(z) = 0$ for such z. That is, for z sufficiently close to z^*, one has

$$G_i(z) = G_i(z^*) \quad \forall i \in \alpha.$$

By quasiconvexity of G_i $(i \in \alpha^-)$ at z^* it follows that for z sufficiently close to z^*,

$$\langle \nabla G_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in \alpha^-, \tag{10}$$

Similarly one has for z sufficiently close to z^*,

$$\langle \nabla H_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in \gamma^-. \tag{11}$$

Multiplying (5), (6), (7), (8), (9), (10), (11) by $\lambda_i^g \geq 0 (i \in I_g), \lambda_i^h > 0 (i \in J^+), -\lambda_i^h > 0 (i \in J^-), \lambda_i^G > 0 (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), \lambda_i^H > 0 (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+), -\lambda_i^G > 0 (i \in \alpha^-), -\lambda_i^H > 0 (i \in \gamma^-)$ respectively and adding we have that for z sufficiently close to z^*,

$$\left\langle \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)], z - z^* \right\rangle \leq 0.$$

By virtue of (4), the above inequality implies that for z sufficiently close to z^*,

$$\langle \nabla f(z^*), z - z^* \rangle \geq 0.$$

By the pseudoconvexity of f at z^*, we must have $f(z) \geq f(z^*)$ for z sufficiently close to z^*. That is, z^* is a local optimal solution of MPEC if $\alpha^- \cup \gamma^- \neq \emptyset$ and $\beta_G^- \cup \beta_H^- = \emptyset$.

Now suppose z^* is an interior point relative to the set $\mathcal{Z} \cap \{z : G_i(z) = 0, H_i(z) = 0, i \in \beta_G^- \cup \beta_H^-\}$. Then for any feasible point z sufficiently close to z^*, it holds that

$$G_i(z) = 0, H_i(z) = 0 \quad \forall i \in \beta_G^- \cup \beta_H^-$$

and hence by the quasiconvexity of $G_i (i \in \beta_H^-)$ and $H_i (i \in \beta_G^-)$,

$$\langle \nabla G_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in \beta_H^-, \tag{12}$$

$$\langle \nabla H_i(z^*), z - z^* \rangle \leq 0 \quad \forall i \in \beta_G^-. \tag{13}$$

16

Multiplying (5), (6), (7), (8), (9), (10), (12), (11), (13) by $\lambda_i^g \geq 0 (i \in I_g), \lambda_i^h > 0 (i \in J^+), -\lambda_i^h > 0 (i \in J^-), \lambda_i^G > 0 (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+), \lambda_i^H > 0 (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+), -\lambda_i^G > 0 (i \in \alpha^- \cup \beta_H^-), -\lambda_i^H > 0 (i \in \gamma^- \cup \beta_G^-)$ respectively and adding we have that for z sufficiently close to z^*,

$$\langle \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)], z - z^* \rangle \leq 0.$$

By virtue of (4), the above inequality implies that for z sufficiently close to z^*,

$$\langle \nabla f(z^*), z - z^* \rangle \geq 0.$$

By the pseudoconvexity of f at z^*, we must have $f(z) \geq f(z^*)$ for z sufficiently close to z^*. That is, z^* is a local optimal solution of MPEC if z^* is an interior point relative to the set $\mathcal{Z} \cap \{z : G_i(z) = 0, H_i(z) = 0, i \in \beta_G^- \cup \beta_H^-\}$ and the proof is complete.

∎

3 More constraint qualifications for M-stationary condition

In this section we provide more constraint qualifications for M-stationary condition to hold. We first discuss the Abadie constraint qualification introduced by Abadie [1]. For a nonlinear programming problem, the Abadie constraint qualification says that the tangent cone is equal to its linearized cone. For example, consider the nonlinear programming problem $\text{MPEC}(\beta_1, \beta_2)$ associated with any partition $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$. Let $\mathcal{T}_{(\beta_1,\beta_2)}(z^*)$ be the tangent cone of $\text{MPEC}(\beta_1, \beta_2)$ at z^* and $\mathcal{T}_{(\beta_1,\beta_2)}^{lin}(z^*)$ the standard linearized cone of $\text{MPEC}(\beta_1, \beta_2)$ at z^*, i.e.,

$$\mathcal{T}_{(\beta_1,\beta_2)}^{lin}(z^*) := \{d \in R^n | \quad \nabla g_i(z^*)^\top d \leq 0 \quad \forall i \in I_g,$$
$$\nabla h_i(z^*)^\top d = 0 \quad \forall i = 1, 2, \ldots, q,$$
$$\nabla G_i(z^*)^\top d = 0 \quad \forall i \in \alpha \cup \beta_2,$$
$$\nabla H_i(z^*)^\top d = 0 \quad \forall i \in \gamma \cup \beta_1,$$
$$\nabla G_i(z^*)^\top d \geq 0 \quad \forall i \in \beta_1,$$
$$\nabla H_i(z^*)^\top d \geq 0 \quad \forall i \in \beta_2\}.$$

Then it is well known that the inclusion

$$\mathcal{T}_{(\beta_1,\beta_2)}(z^*) \subseteq \mathcal{T}_{(\beta_1,\beta_2)}^{lin}(z^*) \tag{14}$$

always holds and the Abadie constraint qualification for MPEC(β_1, β_2) demands that the equality actually holds in (14). It is obvious that the linearized cone is a polyhedral convex set and so the Abadie CQ demands that the tangent cone is also a polyhedral convex set.

The Abadie constraint qualification for nonlinear programming problem MPEC(β_1, β_2) is a very weak condition since the tangent cone for MPEC(β_1, β_2) is likely to be polyhedral convex, and it is known that it is weaker than the Mangasarian-Fromovitz constraint qualification and the Slater condition.

It is easy to see that the linearized tangent cone of MPEC at z^* is given by:

$$
\begin{aligned}
\mathcal{T}^{lin}(z^*) := \{d \in R^n| \quad & \nabla g_i(z^*)^\top d \leq 0 && \forall i \in I_g, \\
& \nabla h_i(z^*)^\top d = 0 && \forall i = 1, 2, \ldots, q, \\
& \nabla G_i(z^*)^\top d = 0 && \forall i \in \alpha, \\
& \nabla H_i(z^*)^\top d = 0 && \forall i \in \gamma, \\
& \nabla G_i(z^*)^\top d \geq 0 && \forall i \in \beta, \\
& \nabla H_i(z^*)^\top d \geq 0 && \forall i \in \beta\}
\end{aligned}
$$

and hence it is obvious that

$$
\mathcal{T}^{lin}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}^{lin}_{(\beta_1, \beta_2)}(z^*).
$$

Since the feasible set of MPEC is the union of the feasible sets of all branches, one has

$$
\mathcal{T}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{(\beta_1, \beta_2)}(z^*).
$$

It follows from (14) that

$$
\mathcal{T}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{(\beta_1, \beta_2)}(z^*) \subseteq \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}^{lin}_{(\beta_1, \beta_2)}(z^*) = \mathcal{T}^{lin}(z^*) \qquad (15)
$$

always holds and the standard Abadie constraint qualification for nonlinear programming requires equality in (15):

$$
\mathcal{T}(z^*) = \mathcal{T}^{lin}(z^*).
$$

Since the tangent cone $\mathcal{T}(z^*)$ is in general nonconvex if the strict complementarity condition fails to hold and the linearized cone $\mathcal{T}^{lin}(z^*)$ is polyhedral and hence convex,

it is easy to see that the standard Abadie constraint qualification is unlikely to be satisfied by MPEC. Flegel and Kanzow [5] introduced the following modified Abadie constraint qualification for MPEC.

Definition 3.1 (MPEC Abadie CQ) *Let z^* be a feasible point of MPEC. We say that MPEC Abadie constraint qualification holds at z^* if*

$$\mathcal{T}^{lin}_{MPEC}(z^*) = \mathcal{T}(z^*)$$

where

$$
\begin{aligned}
\mathcal{T}^{lin}_{MPEC}(z^*) := \{d \in R^n| \quad & \nabla g_i(z^*)^\top d \leq 0 && \forall i \in I_g, \\
& \nabla h_i(z^*)^\top d = 0 && \forall i = 1, 2, \ldots, q, \\
& \nabla G_i(z^*)^\top d = 0 && \forall i \in \alpha, \\
& \nabla H_i(z^*)^\top d = 0 && \forall i \in \gamma, \\
& \nabla G_i(z^*)^\top d \geq 0 && \forall i \in \beta, \\
& \nabla H_i(z^*)^\top d \geq 0 && \forall i \in \beta, \\
& (\nabla G_i(z^*)^\top d) \cdot (\nabla H_i(z^*)^\top d) = 0 && \forall i \in \beta\}
\end{aligned}
$$

is the MPEC linearized tangent cone of MPEC.

Note that

$$\mathcal{T}(z^*) \subseteq \mathcal{T}^{lin}_{MPEC}(z^*)$$

and so MPEC Abadie CQ is equivalent to

$$\mathcal{T}(z^*) \supseteq \mathcal{T}^{lin}_{MPEC}(z^*).$$

Flegel and Kanzow [5] showed that under the MPEC Abadie CQ, a local minimum point of MPEC must be A-stationary. We now prove that under the MPEC Abadie CQ, a local minimum point of MPEC is not just A-stationary. It must be M-stationary.

Theorem 3.1 *Let z^* be a local optimal solution of MPEC. Suppose that MPEC Abadie CQ is satisfied at z^*. Then z^* is M-stationary.*

Proof. By definition, the local solution z^* is B-stationary, i.e.,

$$\nabla f(z^*)^\top d \geq 0 \quad \forall d \in \mathcal{T}(z^*)$$

19

which is equivalent to

$$\nabla f(z^*)^\top d \geq 0 \quad \forall d \in \mathcal{T}_{MPEC}^{lin}(z^*)$$

under MPEC Abadie CQ. Hence z^* is B-stationary if and only if $d = 0$ is a solution to the following problem which is also a MPEC:

$$\min \quad \nabla f(z^*)^\top d$$
$$s.t. \quad d \in \mathcal{T}_{MPEC}^{lin}(z^*).$$

Since the objective function and all constraint functions are linear, by Theorem 2.2, $d = 0$ is a solution to the above problem implies that it is M-stationary, i.e., there exists $\lambda = (\lambda_{I_g}^g, \lambda^h, \lambda_{\alpha \cup \beta}^G, \lambda_{\gamma \cup \beta}^H)$ such that

$$0 = \nabla f(z^*) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^q \lambda_i^h \nabla h_i(z^*)$$
$$- \sum_{i \in \alpha} \lambda_i^G \nabla G_i(z^*) - \sum_{i \in \gamma} \lambda_i^H \nabla H_i(z^*) - \sum_{i \in \beta}[\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$
$$\lambda_{I_g}^g \geq 0,$$
$$\text{either } \lambda_i^G > 0, \lambda_i^H > 0 \quad \text{or} \quad \lambda_i^G \lambda_i^H = 0 \quad \forall i \in \beta.$$

Define $\lambda_\gamma^G = 0, \lambda_\alpha^H = 0$. It is easy to see that the M-stationary condition for the above problem is the M-stationary condition for MPEC. ∎

In the rest of this section, we will try to find sufficient conditions under which the MPEC Abadie CQ holds. First we extend the Kuhn-Tucker constraint qualification introduced by Kuhn and Tucker in [6] and the Zangwill CQ [26] to MPEC (see also [10, 22]). We first recall the notion of the cone of feasible directions and the cone of attainable directions.

Definition 3.2 *Let $\mathcal{Z}$ denote the feasible region of MPEC and $z^* \in \mathcal{Z}$. The cone of feasible directions of $\mathcal{Z}$ at z^* is the cone defined by*

$$\mathcal{D}(z^*) := \{d \in R^n : \exists \delta > 0 \ s.t. \ z^* + td \in \mathcal{Z} \ \forall t \in (0, \delta)\}.$$

The cone of attainable directions of $\mathcal{Z}$ at z^* is the cone defined by

$$\mathcal{A}(z^*) := \{d \in R^n : \quad \exists \delta > 0 \ and \ \alpha : R \to R^n \ s.t. \ \alpha(\tau) \in \mathcal{Z} \quad \forall \tau \in (0, \delta),$$
$$\alpha(0) = z^*, \quad \lim_{\tau \downarrow 0} \frac{\alpha(\tau) - \alpha(0)}{\tau} = d\}.$$

Definition 3.3 (MPEC Kuhn-Tucker CQ and MPEC Zangwill CQ) *Let z^* be a feasible point of MPEC. We say that MPEC Kuhn-Tucker constraint qualification or MPEC Zangwill constraint qualification is satisfied at z^* if*

$$\mathcal{T}^{lin}_{MPEC}(z^*) \subseteq cl\mathcal{A}(z^*) \qquad or \qquad \mathcal{T}^{lin}_{MPEC}(z^*) \subseteq cl\mathcal{D}(z^*)$$

respectively.

Since $\mathcal{D}(z^*) \subseteq \mathcal{A}(z^*) \subseteq \mathcal{T}(z^*)$ and the tangent cone $\mathcal{T}(z^*)$ is closed, it is obvious that:

MPEC Zangwill CQ $\Rightarrow$ MPEC Kuhn-Tucker CQ $\Rightarrow$ MPEC Abadie CQ.

We now extend the Arrow-Hurwicz-Uzawa constraint qualification introduced by Arrow et al. in [2] to MPEC.

Definition 3.4 (MPEC Arrow-Hurwicz-Uzawa CQ) *We say that MPEC Arrow-Hurwicz-Uzawa CQ is satisfied at z^* if $h_i(i = 1, 2, \ldots, q), G_i(i \in \alpha \cup \beta), H_i(i \in \gamma \cup \beta)$ are pseudoaffine at z^* and there exists $d \in R^n$ such that*

$$\nabla g_i(z^*)^\top d < 0 \quad \forall i \in W, \tag{16}$$

$$\nabla g_i(z^*)^\top d \leq 0 \quad \forall i \in V, \tag{17}$$

$$\nabla h_i(z^*)^\top d = 0 \quad \forall i = 1, 2, \ldots, q, \tag{18}$$

$$\nabla G_i(z^*)^\top d = 0 \quad \forall i \in \alpha \tag{19}$$

$$\nabla H_i(z^*)^\top d = 0 \quad \forall i \in \gamma \tag{20}$$

$$\nabla G_i(z^*)^\top d \geq 0 \quad \forall i \in \beta \tag{21}$$

$$\nabla H_i(z^*)^\top d \geq 0 \quad \forall i \in \beta \tag{22}$$

$$(\nabla G_i(z^*)^\top d) \cdot (\nabla H_i(z^*)^\top d) = 0 \quad \forall i \in \beta \tag{23}$$

where

$$V \quad := \quad \{i \in I_g : g_i \text{ is pseudoconcave at } z^*\},$$

$$W \quad := \quad \{i \in I_g : g_i \text{ is not pseudoconcave at } z^*\}.$$

Proposition 3.1 *MPEC Arrow-Hurwicz-Uzawa CQ implies MPEC Zangwill CQ.*

Proof. Suppose d satisfying (16)-(23). For any $i \in W$ by virtue of (16), for all $\tau \in (0, 1]$ small enough,

$$g_i(z^* + \tau d) < g_i(z^*) = 0 \qquad \forall i \in W.$$

For $i \in V$ by virtue of (17), and the definition of pseudoconcavity that $g_i(z^* + \tau d) \leq g_i(z^*) \; \forall \tau \geq 0$ small enough. By the continuity assumptions at z^* for $g_i(i \notin I_g)$, for all $\tau \in (0, 1]$ small enough,

$$g_i(z^* + \tau d) < 0 \quad \forall i \notin I_g.$$

Hence for all $\tau > 0$ small enough,

$$g_i(z^* + \tau d) \leq 0 \qquad i = 1, 2, \ldots, p.$$

Similarly one can prove that

$$
\begin{aligned}
&h_j(z^* + \tau d) = 0 \qquad \forall j = 1, 2, \ldots, q, \\
&G_i(z^* + \tau d) = 0, \quad H_i(z^* + \tau d) > 0 \qquad \forall i \in \alpha, \\
&H_i(z^* + \tau d) = 0, \quad G_i(z^* + \tau d) > 0 \qquad \forall i \in \gamma, \\
&H_i(z^* + \tau d) \geq 0, \quad G_i(z^* + \tau d) \geq 0, \quad H_i(z^* + \tau d)G_i(z^* + \tau d) = 0 \qquad \forall i \in \beta,
\end{aligned}
$$

which implies that $d \in \mathcal{D}(z^*)$ and the proof of the proposition is complete due to the continuity of all functions in d.

∎

Definition 3.5 (MPEC weak reverse convex CQ) *We say that MPEC weak reverse convex constraint qualification holds at z^* if $g_i(i \in I_g)$ are pseudoconcave at z^* and $h_j(j = 1, 2, \ldots, J), G_i(i \in \alpha \cup \beta), H_i(\gamma \cup \beta)$ are pseudoaffine at z^*.*

Since (17)-(23) always has a solution $d = 0$, the following relationship between the MPEC weak reverse convex constraint CQ and MPEC Arrow-Hurwicz-Uzawa CQ is immediate.

Proposition 3.2 *MPEC weak reverse convex constraint CQ implies MPEC Arrow-Hurwicz-Uzawa CQ.*

Now we consider the piecewise constraint qualifications. By virtue of (15), if the Abadie CQ for all MPEC(β_1, β_2) $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$ holds then the MPEC Abadie CQ holds. It is well known that the Slater condition implies the MFCQ which in turn implies the Abadie CQ for problem MPEC(β_1, β_2) and hence the following two piecewise constraint qualifications implies the MPEC Abadie CQ.

Definition 3.6 (Piecewise Slater condition and piecewise MFCQ) *We say that piecewise MPEC Slater condition or piecewise MFCQ is satisfied at a feasible point of MPEC z^* if the Slater condition or MFCQ holds for each $MPEC(\beta_1, \beta_2) \in \mathcal{P}(\beta)$ respectively.*

It was shown in [5] that MPEC MFCQ (i.e., MFCQ for the TMPEC) implies piecewise MFCQ and hence MPEC Abadie CQ. Hence we conclude that MPEC MFCQ is also a constraint qualification for the M-stationarity. This improves the result of [19] in which it was shown that MPEC-MFCQ is a constraint qualification for C-stationarity.

The result of this section is summarized in the following theorem.

Theorem 3.2 *Let z^* be a local solution of MPEC. If one of the MPEC constraint qualifications such as MPEC liner CQ, MPEC weak reverse convex CQ, MPEC Arrow-Hurwicz-Uzawa CQ, piecewise Slater condition, MPEC MFCQ, NNAMCQ, piecewise MFCQ, MPEC GMFCQ, MPEC Zangwill CQ, MPEC Kuhn-Tucker CQ, and MPEC Abadie CQ is satisfied at z^*, then z^* is M-stationary.*

The relationships between the various MPEC constraint qualifications are given in the following diagram.

$$
\begin{array}{ccccc}
& & & MPEC\ Linear\ CQ & \\
& & & \Downarrow & \\
& & & MPEC\ Weak\ Reverse\ Convex\ CQ & \\
& & & \Downarrow & \\
MPEC\ LICQ & & & MPEC\ Arrow\text{-}Hurwicz\text{-}Uzawa\ CQ & \\
\Downarrow & & & \Downarrow & \\
MPEC\ MFCQ & & Piecewise\ Slater & MPEC\ Zangwill\ CQ & \\
\Downarrow & & \Downarrow & \Downarrow & \\
MPEC\ GMFCQ \Leftarrow & Piecewise\ MFCQ & & \text{MPEC Kuhn-Tucker CQ} & \\
\Updownarrow & & \Downarrow & \Downarrow & \\
NNAMCQ & & MPEC\ Abadie\ CQ & &
\end{array}
$$
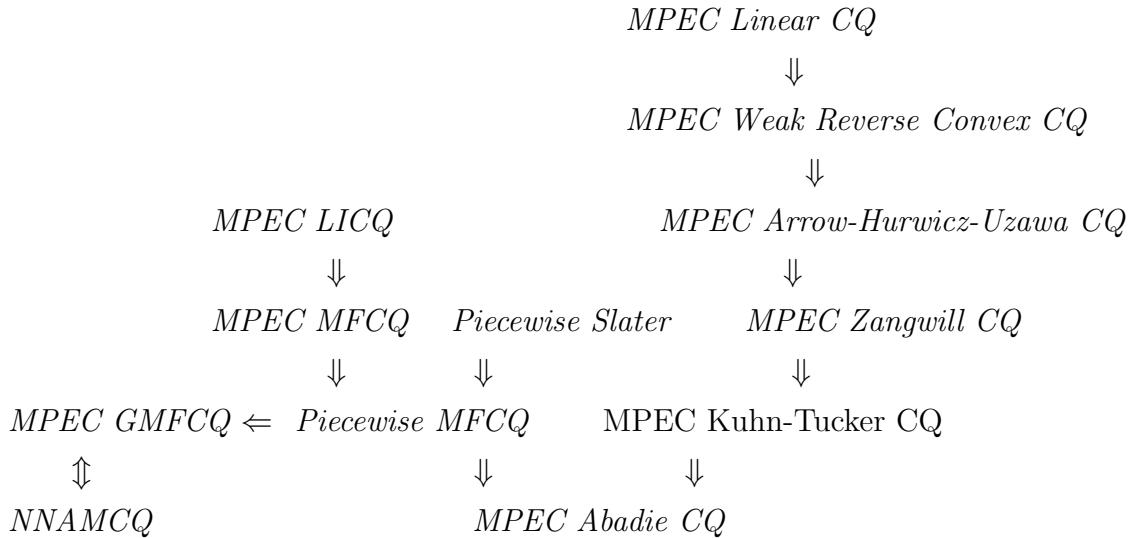
From the above diagram, it is interesting to see that all constraint qualifications except MPEC GMFCQ are stronger than Abadie CQ but there is no connection between MPEC GMFCQ and Abadie CQ.

Ackhowledgements

References

[1] J.M. Abadie, *On the Kuhn-Tucker theorem*, Nonlinear Programming, J. Abadie ed., John Wiley, New York, 1967, pp. 21-36.

[2] K.J. Arrow, L. Hurwicz and H. Uzawa (eds), *Studies in Linear and Nonlinear Programming* Stanford University Press, Stanford, California, 1958.

[3] M.S. Bazaraa, H.D. Sherali and C.M. Shetty, *Nonlinear Programming Theory and Algorithms*, John Wiley & Sons, New York, Second Edition, 1993.

[4] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[5] M.L. Flegel and C. Kanzow, *Optimality conditions for mathematical programs with equilibrium constraints: Fritz John and Abadie-type approaches*, Preprint May 6, 2002.

[6] H.W. Kuhn and A.W. Tucker, *Nonlinear programming*, in Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman (ed.), University of California Press, Berkeley, California, 1951, pp. 481-492.

[7] Y. Lucet and J.J. Ye, *Sensitivity analysis of the value function for optimality problems with variational inequality constraints*, SIAM J. Control Optim. **40**(2001), pp. 699-723.

[8] Z.Q. Luo, J.S. Pang and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.

[9] Z.Q. Luo, J.S. Pang, D. Ralph and S.Q. Wu, *Exact penalization and stationary conditions of mathematical programs with equilibrium constraints*, Mathematical Programming **76**, pp. 19-76, 1996.

[10] O.L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969; SIAM, Philadelphia, 1994.

[11] B.S. Mordukhovich, *Metric approximation and necessary optimality conditions for general classes of nonsmooth extremal problems*, Soviet Math. Dokl. **22**(1980), pp. 526-530.

[12] B.S. Mordukhovich, *Generalized differential calculus for nonsmooth and set-valued mappings*, J. Math. Anal. Appl. **183**(1994), pp. 250-288.

[13] J.V. Outrata, *On optimality problems with variational inequality constraints*, Math. SIAM J. Optim. **4**(1994), pp. 340-357.

[14] J.V. Outrata, *Optimality conditions for a class of mathematical programs with equilibrium constraints*, Math. Oper. Res. **24**(1999), pp. 627-644.

[15] J.V. Outrata, M. Kočvara and J. Zowe, *Nonsmooth Approach to Optimization Problem with Equilibrium Constraints: Theory, Application and Numerical Results,* Kluwer, Dordrecht, The Netherlands, 1998.

[16] J.S. Pang and M. Fukushima, *Complementarity constraint qualifications and simplified B-stationary conditions*, Computational Optimization and Applications, **13**(1999), pp.111-136.

[17] S.M. Robinson, Some continuity properties of polyhedral multifunctions, Math. Programming Stud., **14**(1981), pp. 206-214.

[18] R.T. Rockafellar and R. J-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

[19] H. Scheel and S. Scholtes, *Mathematical programs with complementarity constraints: stationarity, optimality and sensitivity,* Math. Oper. Res. **25**(2000), pp. 1-22.

[20] J.J. Ye, *Optimality conditions for optimization problems with complementarity constraints*, SIAM J. Optim. **9**(1999), pp. 374-387.

[21] J.J. Ye, *Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints*, SIAM J. Optim. **10**(2000), pp. 943-962.

[22] J.J. Ye, *Nondifferentiable multiplier rules for optimization and bilevel optimization problems*, To apprear in SIAM J. Optim.

[23] J.J. Ye and X.Y. Ye, *Necessary optimality conditions for optimization problems with variational inequality constraints*, Math. Oper. Res. **22**(1997), pp. 977-977.

[24] J.J. Ye and Qiji Zhu, *Multiobjective optimization problem with variational inequality constraints*, Math. Programming, Ser. A, **96** (2003), pp. 139-160.

[25] J.J. Ye, D.L. Zhu and Q.J. Zhu, *Exact penalization and necessary optimality conditions for generalized bilevel programming problems*, SIAM J. Optim. **2**(1997), pp. 481-507.

[26] W.I. Zangwill, *Nonlinear Programming: A Unified Approach*, Prentice-Hall, Englewood Cliffs, N.J., 1969.