A global optimization problem in portfolio selection

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Abstract
This paper deals with the issue of buy-in thresholds in portfolio optimization using the Markowitz approach. Optimal values of invested fractions calculated using, for instance, the classical minimum-risk problem can be unsatisfactory in practice because they imply that very small amounts of certain assets are purchased. Realistically, we want to impose a discrete restriction on each invested fraction y_i such as

$$y_i > l \quad \text{or} \quad y_i = 0.$$

We shall describe an approach which uses a combination of local and global optimization to determine satisfactory solutions. The approach could also be applied to other discrete conditions - for instance in dealing with assets that can only be purchased in units of a certain size (roundlots).

1 Introduction

In this paper we consider the standard Markowitz approach to portfolio optimization [1, 2], using forms of the minimum-risk and maximum-return problem which exclude the possibility of short-selling. We suppose that, for a set of n assets, we have the mean returns $\bar{r}_1, ..., \bar{r}_n$ and the $n \times n$ variance-covariance matrix Q based on a past performance history. If a portfolio is defined by the invested fractions $y_1, ..., y_n$ then its expected return, R, and its risk V are denoted by

$$R = \bar{r}^T y \quad \text{and} \quad V = y^T Q y. \tag{1.1}$$

Initially, we shall require the invested fractions to satisfy

$$\sum_{i=1}^{n} y_i = 1 \quad \text{(or equivalently } \; e^T y = 1 \quad \text{where} \;\; e^T = (1, 1, ..., 1) \;). \tag{1.2}$$

Later in the paper we shall relax this condition to $e^T y \leq 1$ and in this case the definition of risk becomes

$$V = \frac{y^T Q y}{(e^T y)^2}. \tag{1.3}$$

The problem of choosing invested fractions to obtain a minimum-risk portfolio with an expected return $R = R_p$ can be tackled by minimizing

$$y^T Qy + \frac{\rho}{R_p^2}(\bar{r}^T y - R_p)^2 + \rho(e^T y - 1)^2 \tag{1.4}$$

where ρ is a suitably large positive penalty parameter. The corresponding problem of maximizing the return from a portfolio with a specified level of risk V_a is equivalent to minimizing

$$\bar{r}^T y + \frac{\rho}{V_a^2}(y^T Qy - V_a)^2 + \rho(e^T y - 1)^2. \tag{1.5}$$

If we want to solve these problems while also excluding the possibility of short-selling we can introduce new variables $x_1, .., x_n$ such that $y_i = x_i^2$. This transformation ensures that the invested fractions y_i are non-negative. Expressing (1.4) and (1.5) as functions of $x_1, .., x_n$ is straightforward. If σ_{ij} is the (i, j)-th element of Q we obtain the problems

$$\text{Minimize } \frac{\rho}{R_p^2}(\sum_{i=1}^{n} \bar{r}_i x_i^2 - R_p)^2 + \rho(\sum_{i=1}^{n} x_i^2 - 1)^2 + \sum_{i=1}^{n} x_i^2(\sum_{j=1}^{n} \sigma_{ij} x_j^2). \tag{1.6}$$

$$\text{Minimize } \frac{\rho}{V_a^2}(\sum_{i=1}^{n} x_i^2 - 1)^2 + \rho(\sum_{i=1}^{n} x_i^2(\sum_{j=1}^{n} \sigma_{ij} x_j^2) - V_a)^2 - \sum_{i=1}^{n} \bar{r}_i^2 x_i^2. \tag{1.7}$$

The $y = x^2$ transformation is not the only way (or even necessarily the best way) of forcing the invested fractions to be non-negative. However it provides a convenient formulation with which to illustrate an approach to portfolio problems in which, as well as avoiding short-selling, we also want to prevent *very small* amounts being invested in any asset. In other words, we want each y_i to satisfy

$$\text{either } y_i = 0 \quad \text{or} \quad y_i \geq y_{min}. \tag{1.8}$$

This kind of *buy-in threshold* constraint is discussed, for instance, in [3].

In this paper we shall suggest an extension of (1.6) which can be used to solve the minimum-risk problem subject to a restriction of the form (1.8). Taking the optimization variables as $x_1, ..., x_n$ and setting $y_i = x_i^2$, $i = 1, .., n$, we shall minimize

$$F = y^T Qy + \rho(e^T y - 1)^2 + \rho(\frac{\bar{r}^T y}{R_p} - 1)^2 + \mu \sum_{i=1}^{n} \psi(y_i)^2 \tag{1.9}$$

where

$$\psi(y_i) = \begin{cases} 0 & \text{if } y_i \geq y_{min} \\ 4\frac{y_i(y_{min} - y_i)}{y_{min}^2} & \text{if } 0 \leq y_i < y_{min} \end{cases} \tag{1.10}$$

The function $\psi(y_i)$ takes values between zero and one and its presence in (1.9) penalises any y-values which lie in the forbidden zone $0 < y_i < y_{min}$. Since the function (1.9) is likely to have several local minima – each corresponding to some of the y_i being close to zero or close to y_{min} – we shall need to use a global optimization technique. In particular, we shall choose the DIRECT algorithm proposed by Jones et al [4].

We can use a similar approach to solve maximum return problems with buy-in constraints by minimizing the function

$$F = -\bar{r}^T y + \rho(e^T y - 1)^2 + \rho(\frac{y^T Qy}{V_a} - 1)^2 + \mu \sum_{i=1}^{n} \psi(y_i)^2. \qquad (1.11)$$

The next section gives a brief outline of DIRECT and then, in section 3, we present some numerical results obtained by applying DIRECT to (1.9) and (1.11). In section 4 we discuss the extension of the ideas of this paper to deal with the *roundlot* problem.

2 DIRECT

In practice, most methods which seek the global minimum of a function $F(x)$ are applied in some restricted region of variable-space, typically in a "hyperbox" defined by

$$l_i \leq x_i \leq u_i.$$

The algorithm DIRECT (Jones et al [4]) relies on the use of such rectangular bounds on the variables. It works by systematic exploration of sub-boxes in the region of interest. In the limit, as the number of iterations becomes infinite, it will sample the whole region and, in that sense, the algorithm is guaranteed to converge. The good performance of the method in practice depends on the way in which it chooses which sub-boxes to explore first, because this will determine whether the global minimum can be approximated in an acceptable number of iterations.

To describe the method we consider first the one-variable problem of finding the global minimum of $F(x)$ for $0 \leq x \leq 1$. (Any problem can be reduced to this form by a simple transformation.) Initially, the range $[0,1]$ is divided into three equal parts and the function is evaluated at their midpoints. The sub-range which has the least function value is then trisected and F is calculated at the centre point of all the new ranges. We then have a situation like that shown in Figure 1.

There are now trial ranges of two different widths, namely, $\frac{1}{3}$ and $\frac{1}{9}$. For *each* of these widths, the one with the smallest value of F at the centre is chosen and trisected. A typical outcome of this is depicted in Figure 2.

After this second iteration there are *three* candidate range-sizes, $\frac{1}{3}$, $\frac{1}{9}$ and $\frac{1}{27}$. For each of these, the one with smallest central F-value can be selected for further

```
                             F

                                                          F
          F
                                                   F
                                           F

0__________________|__________________|_______|_______|______1
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Figure 1: **One iteration of DIRECT on a one-variable problem**

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                              F

                                                                F
           F
                   F                                    F
    F                                         F   F
                                          F
0_____|_______|_______|_________________|___|___|___|_______|______1
A     B       C       D                 E   F   G   H       I      J
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Figure 2: **Two iterations of DIRECT on a one-variable problem**

subdivision. (In Figure 2, the intervals DE, AB and EF would be trisected.) Continuation of this process amounts to a systematic exploration the whole range in a way that gives priority to the most promising regions. Thus the aim is to find a good estimate of the global optimum before the iteration count gets too high.

DIRECT is based on the ideas outlined above but it uses a more sophisticated and efficient way of identifying "promising" ranges. Suppose that $d_1,..,d_p$ are the p different range-sizes at the start of an iteration. Suppose also that F_j denotes the smallest of the mid-point function values in ranges of width d_j. DIRECT will trisect the range containing F_j *only* if a "potential optimality" test is satisfied. The argument behind this test is based upon *Lipschitz constants* (i.e. bounds on the magnitude of the first derivative of F). If $F(x)$ has Lipschitz constant L then, within the range containing F_j, the objective function can be bounded by

$$F_j + \frac{1}{2}Ld_j \geq F(x) \geq F_j - \frac{1}{2}Ld_j.$$

We can deduce from these inequalities whether the range containing F_j can also include a function value less than the current best estimate of the global minimum, F_{min}. If this is the case then the range containing F_j is "potential optimal" and merits further exploration. On the other hand, if $F(x) < F_{min}$ cannot hold within the range then the cost of trisecting it can be avoided. Unfortunately, the implementation of this idea is usually impractical because a Lipschitz constant for F is not known. DIRECT gets around this problem by treating the range containing F_j as potentially optimal if *there exists* a Lipschitz constant L such that

$$F_j - \frac{1}{2}Ld_j < F_i - \frac{1}{2}Ld_i \quad \text{for} \quad i = 1,..,p; \quad i \neq j. \tag{2.1}$$

These inequalities imply that, within the range containing F_j, there *could be* a function value smaller than what is reachable in all the other sub-ranges. For (2.1) to hold we need

$$L > 2 \times max\{\frac{F_j - F_i}{d_j - d_i}\} \quad \text{for all } i : d_i < d_j \tag{2.2}$$

and also

$$L < 2 \times min\{\frac{F_i - F_j}{d_i - d_j}\} \quad \text{for all } i : d_i > d_j \tag{2.3}$$

If conditions (2.2) and (2.3) are inconsistent then the range containing F_j cannot be considered potentially optimal and hence it need not be subdivided. This consideration enables DIRECT to economize on function evaluations when there are many different candidate ranges.

DIRECT also employs another "filter" to reduce the number of subdivisions on each iteration. If F_{min} is the smallest function value found so far then a range for which there is an L satisfying (2.2) and (2.3) will only be trisected if, in addition,

$$F_j - \frac{1}{2}Ld_j < F_{min} - \varepsilon|F_{min}|. \tag{2.4}$$

In (2.4) ε is a user specified parameter (typically about 0.001). Condition (2.4) suggests that subdivision of the range containing F_j can be expected to produce a non-trivial improvement in the current best function value.

The ideas outlined so far can be extended fairly easily to produce a version of DIRECT for problems in several variables [4]. The original search region is now a hyperbox rather than a line segment and the initial subdivision is into three sub-boxes by trisection along the longest edge. The objective function is evaluated at the centre of each sub-box and the size of each box is taken as the length of its diagonal. The box with the smallest central value of F is subdivided by trisection along its longest side and the process of identification and subdivision of potentially optimal hyperboxes then continues as in the one-variable case. (There are refinements for dealing with the subdivison of boxes which have several longest sides [4].)

DIRECT can get good estimates of global optima quite quickly. Since it only uses function values, it can be applied to non-smooth problems or to those where the computation of derivatives is difficult. One drawback, however, is that there is no hard-and-fast convergence test for stopping the algorithm. One can simply let it run for a fixed number of iterations or else choose to terminate if there is no improvement in the best function value after a prescribed number of function evaluations. Neither of these strategies, however, will *guarantee* that enough iterations have been done to identify the neighbourhood of the global optimum.

It should be emphasised, of course, that many other global optimization algorithms exist. In our experience, DIRECT has proved easy to use and has given encouraging results: but the implementation of the ideas behind (1.9) and (1.11) does not rely on the use of this, or any other particular method.

3 Numerical results

We now give some demonstration examples involving small portfolios. Specifically, for a group of five real-life assets we use historical stock market data to generate mean returns $\bar{r}$ and variance-covariance matrix Q. We then solve problems (1.6) or (1.7) and, on the basis of the computed solutions, we specify a value for y_{min} and determine a modified portfolio to satisfy (1.8) by seeking global minima of the functions (1.9) or (1.11).

3.1 Minimum-risk solutions with buy-in threshold

For the five assets in our sample problem, the vector of mean returns is

$$\bar{r} = (-0.056,\ 0.324,\ 0.343,\ 0.132,\ 0.108)^T.$$

A reasonable value for the target return is therefore $R_p = 0.25\%$ and the corresponding solution to (1.6) (using $\rho = 10^3$) has invested fractions

$$y_1 \approx 0.132,\ y_2 \approx 0.368,\ y_3 \approx 0.345,\ y_4 \approx 0.117,\ y_5 \approx 0.037. \tag{3.1}$$

The portfolio risk is $V \approx 0.6894$.

We note that (3.1) includes a relatively small investment in asset five. Hence we now consider the extended function (1.9) with $y_{min} = 0.05$ and penalty parameter $\mu = 1$. At the minimum of (1.9) we expect *either* that y_5 will be near zero *or* that $y_5 \approx 0.05$. That is, we expect a change of about ± 0.03 in y_5. In order to maintain the total investment $\sum y_i = 1$, this means there could be a compensating change of up to ± 0.03 in any of the other invested fractions. Therefore we shall seek the *global* optimum of (1.9) in the hyperbox defined by

$$0.1 \le y_1 \le 0.16;\ 0.34 \le y_2 \le 0.4;\ 0.31 \le y_3 \le 0.37; \tag{3.2}$$

$$0.09 \le y_4 \le 0.15;\ 0 \le y_5 \le 0.06. \tag{3.3}$$

As a preliminary step we apply a standard quasi-Newton *local* minimization method to (1.9). Specifically we use a BFGS algorithm [5, 6]. If we start from the midpoint of the region (3.2), (3.3), where (1.9) has a value of about 2.49, the quasi-Newton method converges to a minimum with

$$y_1 \approx 0.152,\ y_2 \approx 0.381,\ y_3 \approx 0.348,\ y_4 \approx 0.118,\ y_5 \approx 0 \tag{3.4}$$

where (1.9) has a value approximately 0.7005. The portfolio risk is 0.7001 and the return is 0.25%, as required.

We now apply the global algorithm DIRECT to (1.9) within the same hyperbox (3.2), (3.3). After 10 iterations it gives a point where (1.9) has a value of about 0.6949. This is already appreciably better than the function value at (3.4) which implies that the optimum found by the quasi-Newton method is only a local solution. The invested fractions given by DIRECT after 20 iterations are

$$y_1 \approx 0.122,\ y_2 \approx 0.369,\ y_3 \approx 0.339,\ y_4 \approx 0.111,\ y_5 \approx 0.058 \tag{3.5}$$

with a portfolio risk of 0.6935. This approximate solution has $y_5 \approx 0.05$ in contrast to (3.4), which has $y_5 \approx 0$.

Although DIRECT has fairly easily obtained a better solution than the quasi-Newton method, it may not be very efficient at finding optima to high accuracy. This is partly because it does not use derivatives and partly because it only samples function values at the centres of hyperboxes. A common strategy, therefore, is to run a quasi-Newton method again from the best point located by DIRECT, in order to refine the approximate solution. If we apply this strategy to the estimate (3.5), we find that an accurate global minimum (1.9) has a value of about 0.691 and the invested fractions are

$$y_1 \approx 0.125,\ y_2 \approx 0.364,\ y_3 \approx 0.344,\ y_4 \approx 0.116,\ y_5 \approx 0.05. \tag{3.6}$$

Here the target return 0.25% is still achieved and the portfolio risk is 0.6906. This is only slightly worse than the risk for the portfolio (3.1) which was obtained without considering the buy-in threshold constraint ($y_i = 0$ or $y_i \geq 0.05$).

3.2 Maximum return with buy-in threshold

For this example we use the same dataset as in the previous section and take the acceptable risk as $V_a = 0.75$ which is slightly higher than the minimum risk associated with an expected return of 0.25%. With $\rho = 10^3$ the solution of (1.7) is

$$y_1 = 0.0462,\ y_2 = 0.4022,\ y_3 = 0.4162,\ y_4 = 0.1053,\ y_5 = 0.0299 \tag{3.7}$$

giving a portfolio return of 0.2877%.

To obtain a portfolio in which all non-zero invested fractions are greater than or equal to 0.05 we minimize (1.11) with $y_{min} = 0.05$ and $\mu = 1$. A reasonable box to search in is

$$0 \leq y_1 \leq 0.06;\ 0.36 \leq y_2 \leq 0.44;\ 0.38 \leq y_3 \leq 0.46; \tag{3.8}$$

$$0.06 \leq y_4 \leq 0.12;\ 0 \leq y_5 \leq 0.06. \tag{3.9}$$

When started from the midpoint of the region (3.8), (3.9), the quasi-Newton method finds a minimum at

$$y_1 \approx 0.0034,\ y_2 \approx 0.4037,\ y_3 \approx 0.4112,\ y_4 \approx 0.1529,\ y_5 \approx 0.0030 \tag{3.10}$$

where (1.11) has a value of about 0.6472. Twenty iterations of DIRECT in the same box, however, produce the much lower value, $F = -0.276$, at

$$y_1 \approx 0.0578,\ y_2 \approx 0.399,\ y_3 \approx 0.419,\ y_4 \approx 0.0746,\ y_5 \approx 0.0494. \tag{3.11}$$

This indicates that it is better for y_1 and y_5 to be near 0.05 rather than near zero, as in the *local* solution (3.10) produced by the quasi-Newton method.

If we use the quasi-Newton method to refine the approximate global minimum (3.11) as given by DIRECT we get the more accurate result

$$y_1 = 0.05,\ y_2 \approx 0.396,\ y_3 \approx 0.417,\ y_4 \approx 0.0869,\ y_5 = 0.05. \tag{3.12}$$

Here the portfolio return is about 0.2855%, which is only slightly worse than was possible when (1.8) is not enforced.

3.3 Discussion and extensions

The two previous examples provide *prima facie* evidence that global minimization of the penalty functions (1.9) and (1.11) can be used to solve portfolio optimization problems involving disjoint constraints like (1.8). To strengthen that evidence we first point out that the success of DIRECT as a global minimizer of (1.9) need not depend on us restricting the search to the rather small hyperbox (3.2), (3.3) which is obtained by first solving the simpler problem (1.6). For instance, if we apply DIRECT to (1.9) using the same data as in section 3.1 but with the much larger search region

$$0 \leq y_i \leq 0.5, \quad \text{for}\ \ i = 1,..,5 \tag{3.13}$$

then, after about 100 iterations, it reaches a point similar to (3.5). This can be refined to the accurate solution (3.6) by a few quasi-Newton iterations. In a similar way the maximum-return problem in section 3.2 can also be solved by applying DIRECT to (1.11) in the hyperbox (3.13) rather than (3.8), (3.9). This takes about 150 iterations of DIRECT followed by quasi-Newton refinement.

We now describe a more general way of using (1.9) for problems with larger numbers of assets when constraints (1.8) are involved. In spite of the comments in the preceding paragraph, the approach we propose does make use of a preliminary solution of (1.6), as outlined below.

Find values $\hat{x}_i$ to solve (1.6). Hence obtain invested fractions $\hat{y}_i = x_i^2$.
Obtain new trial values $\tilde{y}_i$ by

$$\tilde{y}_i = \begin{cases} 0 & \text{if } \hat{y}_i = 0 \\ y_{min} & \text{if } 0 < \hat{y}_i \leq y_{min} \\ \hat{y}_i & \text{if } \hat{y}_i > y_{min} \end{cases}$$

Solve (1.9) *using* $y_1,..,y_n$ *as variables* by applying DIRECT in the hyperbox

$$\tilde{y}_i - \delta y_i \leq y_i \leq \tilde{y}_i + \delta y_i \quad \text{where } \delta y_i = \begin{cases} 0 & \text{if } \tilde{y}_i = 0 \\ y_{min} & \text{if } \tilde{y}_i \geq y_{min} \end{cases}$$

We can avoid the $y_i = x_i^2$ transformation in the solution of (1.9) because the hyperbox limits ensure that short-selling will not occur.

We now apply the strategy to a ten-asset problem, using $y_{min} = 0.05$ and the results are summarised in table 1.

From (1.6) with $\rho = 1000$ (using q-N)
y=(0.054, 0.227, 0.185, 0.055, 0.035, 0.098, 0.007, 0.099, 0.2, 0.041)
Risk$\approx$ 0.384
From (1.9) with $\rho = 1000, \mu = 1$ (using DIRECT)
y=(0.071, 0.227, 0.180, 0.055, 0.05, 0.066, 0.0, 0.104, 0.197, 0.05)
Risk$\approx$ 0.392
From (1.9) with $\rho = 1000,\ \mu = 1$ (using DIRECT + q-N)
y=(0.058, 0.215, 0.173, 0.06, 0.05, 0.096, 0.0, 0.098, 0.2, 0.05)
Risk$\approx$ 0.387

Table 1: **Ten asset problem with buy-in threshold constraint**

The final section of the table shows that the constraint (1.8) can be satisfied for a relatively small increase in risk. We emphasise again that the use of the global optimizer DIRECT is important. If we had attempted to solve the problem by applying a quasi-Newton method to (1.9) using $x_i = \sqrt{(\tilde{y}_i)}$ as a starting point then we would only have obtained a local solution with $y_1,\ y_4,\ y_5$ and y_7 all at the limiting value 0.05 and a substantially higher risk value of 0.411.

As our final example in this section we consider a 50 asset problem with $y_{min} = 0.03$. Calculated portfolios to give minimum risk for an expected return of 0.1% are summarised in Table 2. The entries in the table show how the distribution of non-zero invested fractions changes as the threshold constraint (1.8) is taken into account.

	Assets with $0 < y_i < 0.03$	Assets with $y_i = 0.03$	Assets with $y_i > 0.03$
From (1.6) (Q-N) with $\rho = 5000$ Risk ≈ 0.744 & 50	1,8,15, 34,44,47,	-	12,17,18, 19,20,31
From (1.9) (Q-N only) with $\rho = 5000, \mu = 1$ Risk ≈ 0.778	-	8,12,15, 34,44,50	1,17,19,20, 31,47
From (1.9) (DIRECT) with $\rho = 5000, \mu = 1$ Risk ≈ 0.764	-	47, 50	1,8,12,15, 17,18,19,20, 31
From (1.9) (DIRECT + Q-N) with $\rho = 5000, \mu = 1$ Risk ≈ 0.757	-	15,47	1,8,12,17, 18,19,20,31, 50

Table 2: **Fifty asset problem with buy-in threshold constraint**

In the results in Table 2 we observe how the smaller investments in the solution of (1.6) are re-allocated when we consider the extended function (1.9). When (1.9) is minimized by the quasi-Newton method we only get a local solution with six of the y_i fixed at the threshold value 0.03. Fifty iterations of DIRECT yield a better portfolio with only two of the $y_i = 0.03$. From here, a further local refinement gives the still better result in the last section of the table.

The numerical experience reported in this section is quite promising. However we must acknowledge that the approach we have described is a simple and rather pragmatic one. It relies on two plausible but unchecked assumptions, namely:
(a) Any assets which are excluded from the portfolio obtained from (1.6) will not figure in the solution which takes account of buy-in thresholds.
(b) In order to accomodate the constraints (1.8), the changes to non-zero invested fractions $\hat{y}_i$ will be confined to the range $[-y_{min}, y_{min}]$.
In our (limited) numerical tests so far, these assumptions seem to be justified when y_{min} is small. Further investigation may show a need for more development of the ideas we have outlined.

4 Roundlot constraints

The invested fractions y_i obtained at the solution of a minimum-risk problem must, in practice, be converted to actual numbers of purchased shares or bonds. If the total investment is M and if the price of asset i is p_i then the number of assets to be acquired is

$$a_i = \frac{My_i}{p_i}. \tag{4.1}$$

Obviously a_i must be an integer: and, more likely, it must be a multiple of some *lot size* such as 10 or 100. This will mean that we will have to round the values of y_i that are given by (1.6). In effect we want to apply a constraint which amounts to

$$a_i \text{ is a multiple of some integer lot size } L_i. \tag{4.2}$$

If we define

$$\theta(y_i) = \frac{My_i}{p_i} - \lfloor \frac{My_i}{p_i} \rfloor, \tag{4.3}$$

where $\lfloor v \rfloor$ denotes the integer part of a real value v, then we want the invested fractions to satisfy the constraints

$$\phi(y_i) = \theta(y_i)(1 - \theta(y_i)) = 0 \quad \text{for} \quad i = 1,..,n. \tag{4.4}$$

Therefore, following the ideas proposed in the first part of this paper, we could consider solving the minimum risk problem with roundlot constraints by minimizing

$$y^T Qy + \rho(e^T y - 1)^2 + \rho(\frac{\bar{r}^T y}{R_p} - 1)^2 + \mu \sum_{i=1}^{n} \phi(y_i)^2. \tag{4.5}$$

However, if the optimal y_i are to be adjusted to satisfy the roundlot constraints then the condition $e^T y = 1$ may not hold precisely – i.e., we may not be able to convert all of our investment into assets. We can only require that $e^T y \leq 1$; and this in turn means that we use (1.3) as the definition of risk (see, for instance, [7]). Therefore a penalty function that can be used for the roundlot constrained problem is

$$\frac{y^T Qy}{(e^T y)^2} + \rho[Min(0, 1 - e^T y)]^2 + \rho(\frac{\bar{r}^T y}{R_p} - 1)^2 + \mu \sum_{i=1}^{n} \phi(y_i)^2. \tag{4.6}$$

We can now approach the minimum risk problem involving constraint (4.4) as follows.

Find values $\hat{x}_i$ to solve (1.6). Hence obtain invested fractions $\hat{y}_i = x_i^2$.
Obtain new trial values $\tilde{y}_i$ from

$$\tilde{y}_i = \begin{cases} 0 & \text{if } \hat{y}_i = 0 \\ L_i & \text{if } 0 < \hat{y}_i \leq L_i \\ \hat{y}_i & \text{if } \hat{y}_i > L_i \end{cases}$$

Solve (4.6) *using* $y_1, .., y_n$ *as variables* by applying DIRECT in the hyperbox

$$\tilde{y}_i - \delta y_i \leq y_i \leq \tilde{y}_i + \delta y_i \quad \text{where} \quad \delta y_i = \begin{cases} 0 & \text{if } \tilde{y}_i = 0 \\ L_i & \text{if } \tilde{y}_i \geq L_i \end{cases}$$

As an illustrative example we consider the five- and ten-asset problem introduced in sections 3.1 and 3.3 respectively. For simplicity we shall take the values $M = 1000$ and $p_i = 1,\ L_i = 10$ in (4.1), (4.2) for $i = 1,..,n$. This means that, ideally, we want the y_i which minimize (4.6) to have zeros in the third and subsequent decimal places. Some solutions are given in Tables 3 and 4, both of which compare portfolios calculated with and without roundlot constraints.

From (1.6) with $\rho = 100$
y=(0.1319, 0.3686, 0.3452, 0.1168, 0.0374)
$\sum y_i = 1$, Risk = 0.69
From (4.6) with $\rho = 1000,\ \mu = 1$
y=(0.140, 0.370, 0.350, 0.110, 0.030)
$\sum y_i = 1$, Risk = 0.6908

Table 3: **Five asset problem with roundlot constraint**

From (1.6) with $\rho = 1000$
y=(0.054, 0.227, 0.185, 0.055, 0.035, 0.098, 0.007, 0.099, 0.2, 0.041)
$\sum y_i = 1$, Risk = 0.3843
From (4.6) with $\rho = 1000,\ \mu = 1$
y=(0.050, 0.230, 0.190, 0.050, 0.030, 0.10, 0.010, 0.090, 0.20, 0.040)
$\sum y_i = 0.99$, Risk = 0.3847

Table 4: **Ten asset problem with roundlot constraint**

In each case, the inclusion of constraint (4.4) only produces a small increase in risk compared with the solution of (1.6). It is interesting to note, however, that the portfolios obtained using (4.6) are *not* what would be obtained simply by rounding the invested fractions from (1.6) to the nearest multiple of 10. For these two examples, the y_i obtained in this way are not acceptable because they do not give $e^T y \leq 1$ and because the expected portfolio return is not the target value 0.25%. Therefore the minimization of (4.6) offers a reasonable basis for determining the best way to obtain a practical solution from the invested fractions which solve (1.6).

5 Discussion and conclusions

In this paper we have suggested a way of handling disjoint constraints (buy-in thresholds and roundlot constraints) occurring in portfolio selection problems. The approach, which is justified by some preliminary numerical experience, involves constructing penalty functions (1.9), (1.11) and (4.6). Since these may have several local minima, we need to use global optimization techniques. We have chosen to use DIRECT [4], but other global methods could be used instead. DIRECT has the advantage of being a non-gradient method, which is useful for the problem in section 4 where the formulation of the roundlot constraint involves the non-differentiable function $\theta(y_i)$ (4.3).

The solution techniques proposed in this paper are essentially prototypes which demonstrate that the basic idea we are proposing does have some merit. Further work is planned with a view to improving both the formulation and the computational algorithms. For instance, the problems (1.6) and (1.7) are by no means the best way of solving minimum-risk or maximum-return problems since they are sensitive to the choice of weighting parameters ρ and μ. Moreoever, all the functions we have used feature the classical squared penalty term for violated constraints and therefore their minima are only approximations to the true constrained solutions. This objection could be overcome by replacing the squared-penalty terms by absolute values, as in

$$F = y^T Qy + \rho|e^T y - 1| + \rho|\frac{\bar{r}^T y}{R_p} - 1| + \mu \sum_{i=1}^{n} |\psi(y_i)|. \tag{5.1}$$

For ρ and μ sufficiently large the minimum of (5.1) coincides with the constrained solution of the minimum risk problem. It is worth noting that DIRECT would still be a suitable algorithm for seeking the global minimum of this non-differentiable function. At the expense of a little more work, a differentiable exact penalty function could be used, which would facilitate the use of quasi-Newton refinement as mentioned in this paper.

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