STATIONARITY AND REGULARITY OF SET SYSTEMS

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Dedicated to R. T. Rockafellar on his 70th birthday

ABSTRACT. Extremality, stationarity and regularity notions for a system of closed sets in a normed linear space are investigated. The equivalence of different abstract "extremal" settings in terms of set systems and multifunctions is proved. The dual necessary and sufficient conditions of weak stationarity (the *Extended extremal principle*) are presented for the case of an Asplund space.

1. INTRODUCTION

Starting with the pioneering work by Dubovitskii and Milyutin [8] it is quite natural when dealing with optimality conditions to reformulate optimality in the original optimization problem as a (some kind of) extremal behaviour of a certain system of sets. An easy example is a problem of unconditional minimization of a real-valued function $\varphi : X \to \mathbb{R}$. If $x^\circ \in X$ one can consider the sets $\Omega_1 = \operatorname{epi} \varphi = \{(x, \mu) \in X \times \mathbb{R} : \varphi(x) \leq \mu\}$ (the epigraph of φ) and $\Omega_2 = X \times \{\mu : \mu \leq \varphi(x^\circ)\}$ (the lower halfspace). The local optimality of x° is then equivalent to the condition $\Omega_1 \cap \operatorname{int} \Omega_2 \cap B_\rho(x^\circ) = \emptyset$ for some $\rho > 0$.

Considering set systems is a rather general scheme of investigating optimization problems. Any set of "extremality" conditions leads to some optimality conditions for the original problem.

When the sets are convex (or admit some convex approximations) extremality conditions are given by the *separation theorem.* In the general case a nonconvex separation theorem (the *generalized Euler equation*) was proved in [22]. By now it is generally referred to as the *Extremal principle* (see [24, 33]) and has numerous applications to optimization, calculus and economics. A different (but in a sense equivalent) scheme of investigating nonconvex set systems was developed in [4].

Any necessary optimality conditions characterize in the nonconvex case not only optimal solutions but some broader set of *stationary* points which can also be of interest. The stationarity notion corresponding to the extremal principle conditions, namely *weak stationarity*, was investigated in [21]. Introducing weak stationarity made possible to reformulate the *(Extended) extremal principle* as a necessary and sufficient condition.

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Thus, Extended extremal principle states equivalence between primal (weak stationarity) and dual (generalized Euler equation) conditions. Another equivalence is true: Extended extremal principle itself is valid if and only if the space under consideration is Asplund. The last statement adds one more line to the list of equivalent extremal characterizations of Asplund spaces in [26].

When a stationarity condition is not true one can speak about (some kind of) regularity of the set system. We are especially interested in investigating a strong regularity property which corresponds to the absence of weak stationarity. It appears to be equivalent to the metric regularity of some multifunction. Another regularity property for set systems is the *metric inequality* developed in [12, 13, 29]. It is proved that regularity as defined in Section 2 is equivalent to the *strong metric inequality* and implies the property from [12, 13, 29].

The paper is organized as follows. The definitions of extremality, stationarity and regularity for the set system are introduced in Section 2. Following [21] some constants characterizing the mutual arrangement of sets in space are used in the definitions. The properties of these constants and corresponding properties of set systems are investigated in this section. More constants are introduced and investigated in Section 3.

Some special cases (convex sets, cones, the case of two sets) are considered in Section 4. It is proved, in particular, that in convex case all extremality and stationarity concepts coincide, and when all but one sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ have nonempty interior, they reduce to the traditional condition $\cap_{i=1}^{n-1} \mathrm{int}\, \Omega_i \cap \Omega_n = \emptyset$. In the case of two sets the notion of weak stationarity (see Section 2) is equivalent to "*extended extremality*" defined in [18].

Sections 5 and 6 contain comparison of extremality and stationarity concepts adopted in the current paper with other "extremal" settings for set systems: the *metric inequality* [12, 13, 29] and the *boundary condition* from [4].

The case of a single set is considered in Section 7. Formally it does not follow from the general setting for a set system.

Section 8 is devoted to the comparison of the extremality, stationarity and regularity properties of set systems with those of multifunctions. Considering multifunctions is another general framework of investigating optimization, complementarity and equilibrium problems. Both approaches (in terms of set systems and in terms of multifunctions) are in a sense equivalent, weak stationarity being a natural counterpart of *metric regularity.*

The final Section 9 presents the *Extended extremal principle*: the dual criterion for weak stationarity in terms of Fréchet normal and strict normal cones.

Mainly standard notations are used throughout the paper. The ball of radios ρ centered at x is denoted $B_\rho(x)$. We write B_ρ if $x = 0$, and simply B if $x = 0$, $\rho = 1$. If Ω is a set then $\mathrm{int}\,\Omega$, $\mathrm{bd}\,\Omega$ and $\mathrm{cl}\,\Omega$ are respectively its interior, the boundary and the closure. When considering product spaces we will always assume that they are equipped with the maximum-type norm: $\|(x_1, x_2)\| = \max(\|x_1\|, \|x_2\|)$.

2. Extremality, stationarity and regularity

Let us consider a system of closed sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ $(n > 1)$ in a normed space X with $x^\circ \in \cap_{i=1}^n \Omega_i$.

The following constants can be used for characterizing the mutual arrangement of sets Ω_1, Ω_2, ..., Ω_n near x° ([21]):

$$\theta_\rho[\Omega_1,\dots,\Omega_n](x^\circ) = \sup\left\{ r\ge 0 : \Big(\bigcap_{i=1}^{n}(\Omega_i - a_i)\Big)\bigcap B_\rho(x^\circ)\ne\emptyset, \forall a_i\in B_r \right\}, \tag{1}$$

$$\theta[\Omega_1,\dots,\Omega_n](x^\circ) = \liminf_{\rho\to+0}\ \theta_\rho[\Omega_1,\dots,\Omega_n](x^\circ)/\rho, \tag{2}$$

$$\hat\theta[\Omega_1,\dots,\Omega_n](x^\circ) = \liminf_{\omega_i\overset{\Omega_i}{\to}x^\circ}\ \theta[\Omega_1-\omega_1,\dots,\Omega_n-\omega_n](0). \tag{3}$$

(1) is defined for $\rho\ge 0$. The denotation $\omega_i \overset{\Omega_i}{\to} x^\circ$ in (3) means that $\omega_i\to x^\circ$ with $\omega_i\in\Omega_i$.

Evidently all the constants (1)–(3) are nonnegative (and can take the value $+\infty$). When investigating extremality-stationarity-regularity properties of the set system one needs to check whether the corresponding constant is zero or strictly positive.

Definition 1. The system of sets Ω_1, Ω_2, ..., Ω_n is

(i) *extremal* at x° if $\theta_\rho[\Omega_1,\dots,\Omega_n](x^\circ)=0$ for all $\rho>0$.
(ii) *locally extremal* at x° if $\theta_\rho[\Omega_1,\dots,\Omega_n](x^\circ)=0$ for some $\rho>0$.
(iii) *stationary* at x° if $\theta[\Omega_1,\dots,\Omega_n](x^\circ)=0$.
(iv) *weakly stationary* at x° if $\hat\theta[\Omega_1,\dots,\Omega_n](x^\circ)=0$.
(v) *regular* at x° if $\hat\theta[\Omega_1,\dots,\Omega_n](x^\circ)>0$.

The next proposition follows immediately from the definitions.

Proposition 1. *(i) ⇒ (ii) ⇒ (iii) ⇒ (iv) in Definition 1.*

Remark 1. *Opposite implications are not true in general.* $\theta[\Omega_1,\dots,\Omega_n](x^\circ)$*, in particular, may be zero even if* $\theta_\rho[\Omega_1,\dots,\Omega_n](x^\circ)>0$ *for all* $\rho>0$*. Take* $\varphi(x)=-x^2$ *in the two-set system mentioned in the Introduction. If to take* $\varphi(x)=x\sin(1/x)$ *for* $x\ne 0$ *and* $\varphi(0)=0$ *then the system will be weakly stationary at* 0 *but not stationary.*

Remark 2. *Condition (iii) corresponds to the traditional notion of stationarity, while (iv) means that arbitrarily close to* x° *there exist points whose properties are arbitrarily close to the stationarity property.*

Remark 3. *All the constants (1)–(3) are local. It is actually sufficient to assume in the rest of the paper that the sets are only locally closed near* x°*. Formally the definitions above can be applied also to nonclosed sets. However, to obtain meaningful results one needs to assume closedness. Another possibility is to replace* Ω_i *by* $\operatorname{cl}\Omega_i$ *in the right-hand side of (1).*

Remark 4. *The condition* $\theta[\Omega_1,\dots,\Omega_n](x^\circ)>0$ *also defines a kind of regularity which is weaker than the one defined in part (iv) of Definition 1. It can be referred to as* weak regularity*. We will not use this concept in the current paper.*

The above definition of local extremality can be equivalently reformulated in terms of sequences.

Proposition 2. *The system of sets Ω_1, Ω_2, ..., Ω_n is locally extremal at x° if and only if there exists a number $\rho > 0$ and sequences $\{a_{ik}\} \subset X$ tending to zero, such that $\cap_{i=1}^n (\Omega_i - a_{ik}) \cap B_\rho(x^\circ) = \emptyset$, $k = 1, 2, \ldots$.*

As it can be seen from Proposition 2, the notion of local extremality defined above is equivalent to the initial one introduced in [22]: an arbitrarily small shift makes the sets unintersecting in a neighborhood of x° (see examples in [21]). It defines a general notion of extremality embedding different solution notions in optimization problems.

The simplest sufficient condition of regularity at x° is $x^\circ \in \text{int} \cap_{i=1}^n \Omega_i$.

Proposition 3. *The following assertions are equivalent:*

(i) $\lim_{\rho \to +0} \theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) > 0$.
(ii) $x^\circ \in \text{int} \cap_{i=1}^n \Omega_i$.

Under these conditions $\theta[\Omega_1, \ldots, \Omega_n](x^\circ) = \hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) = +\infty$.

Proof. (ii) $\Rightarrow$ (i). Let $x^\circ \in \text{int} \cap_{i=1}^n \Omega_i$. Then $B_r(x^\circ) \subset \Omega_i$ for some $r > 0$ and all $i = 1, 2, \ldots, n$. Consequently $x^\circ \in \Omega_i - a_i$ for any $a_i \in B_r$, and $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) \geq r$ for all $\rho > 0$. This implies (i), because the function $\rho \to \theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$ is monotone on the set of positive numbers.

(i) $\Rightarrow$ (ii). Let $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) > r > 0$ for all $\rho > 0$. Then for any $\rho > 0$ and any $a_i \in B_r$, $i = 1, 2, \ldots, n$, there exists

$$x \in (\bigcap_{i=1}^n (\Omega_i - a_i)) \bigcap B_\rho(x^\circ).$$

Consequently $a_i \in \Omega_i - x$, $x^\circ + a_i \in \Omega_i + x^\circ - x$ and $B_r(x^\circ) \subset \Omega_i + B_\rho$. Since this holds true for all $\rho > 0$ and Ω_i is closed one has $B_r(x^\circ) \subset \Omega_i$, $i = 1, 2, \ldots, n$, and $B_r(x^\circ) \subset \cap_{i=1}^n \Omega_i$.

If $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) > r > 0$ for all $\rho > 0$ then it follows from (2) that $\theta[\Omega_1, \ldots, \Omega_n](x^\circ) = +\infty$. If $x^\circ \in \text{int} \cap_{i=1}^n \Omega_i$ then

$$0 \in \text{int} \bigcap_{i=1}^n (\Omega_i - \omega_i)$$

for all $\omega_i \in \Omega_i$ sufficiently close to x°. Thus $\theta[\Omega_1 - \omega_1, \ldots, \Omega_n - \omega_n](0) = +\infty$ and consequently $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) = +\infty$. □

The next proposition gives a weaker sufficient condition for $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$ to be positive, though it does not guarantee regularity.

Proposition 4. *Let the following condition be true:*

$$\bigcap_{i=1}^{n-1} \text{int}\, \Omega_i \cap \Omega_n \neq \emptyset. \tag{4}$$

Then $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) > 0$ for some $\rho > 0$.

Proof. It follows from (4) that there exists an $x \in \Omega_n$ and an $r > 0$, such that $B_{2r}(x) \subset \Omega_i$, $i = 1, 2, \ldots, n-1$. If $a_i \in B_r$ then $y + a_i \in \Omega_i$, $i = 1, 2, \ldots, n-1$, where $y = x - a_n$. Thus $y \in \bigcap_{i=1}^n (\Omega_i - a_i)$ and $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) \geq r > 0$ for $\rho = \|x - x^\circ\| + r$. □

The next two propositions give some relations between extremality-stationarity-regularity properties of a set system and some its subsystem.

Proposition 5 (Reduction). *Let $n > 2$. The following assertions hold:*

(i) $\theta_\rho[\Omega_1, \ldots, \Omega_{n-1}](x^\circ) \geq \theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$.
(ii) $\theta_\rho[\cap_{i=1}^{n-1}\Omega_i, \Omega_n](x^\circ) \geq \theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$.

Proof. The assertions are evident if to notice the next two simple facts:
(i) If $(\bigcap_{i=1}^n(\Omega_i - a_i)) \cap B_\rho(x^\circ) \neq \emptyset$ then $(\bigcap_{i=1}^{n-1}(\Omega_i - a_i)) \cap B_\rho(x^\circ) \neq \emptyset$.
(ii) If $(\bigcap_{i=1}^n(\Omega_i - a_i)) \cap B_\rho(x^\circ) \neq \emptyset$ for all $a_i \in B_r$, $i = 1, 2, \ldots, n$, then

$$(\bigcap_{i=1}^{n-1}(\Omega_i - a_1)) \cap (\Omega_n - a_2) \cap B_\rho(x^\circ) \neq \emptyset$$

for all $a_1, a_2 \in B_r$. □

Proposition 6. *Let $n > m \geq 2$. The following assertions hold:*

(i) *If one of the systems $\{\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_m\}$ or $\{\bigcap_{i=1}^m \Omega_i,\ \Omega_{m+1},\ \ldots,\ \Omega_n\}$ is (locally) extremal at x° then the system $\{\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n\}$ is (locally) extremal at x°.*
(ii) *If one of the systems $\{\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_m\}$ or $\{\bigcap_{i=1}^m \Omega_i,\ \Omega_{m+1},\ \ldots,\ \Omega_n\}$ is (weakly) stationary at x° then the system $\{\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n\}$ is (weakly) stationary at x°.*
(iii) *If the system $\{\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n\}$ is regular at x° then each of the systems $\{\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_m\}$ or $\{\bigcap_{i=1}^m \Omega_i,\ \Omega_{m+1},\ \ldots,\ \Omega_n\}$ is regular at x°.*

Combining (1)–(3), one can get the following representation:

$$\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) = \liminf_{\substack{\omega_i \xrightarrow{\Omega_i} x^\circ \\ \rho \to +0}} \sup\Big\{r \geq 0 : \Big(\bigcap_{i=1}^n(\Omega_i - \omega_i - a_i)\Big) \bigcap B_\rho \neq \emptyset,\ \forall a_i \in B_r\Big\}/\rho. \quad (5)$$

The next assertion is an immediate consequence of (5).

Proposition 7. $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) > \alpha > 0$ *if and only if there exists a $\delta > 0$, such that*

$$\Big(\bigcap_{i=1}^n(\Omega_i - \omega_i - a_i)\Big) \bigcap B_\rho \neq \emptyset \quad (6)$$

for any $\rho \in (0, \delta]$, $\omega_i \in \Omega_i \cap B_\delta(x^\circ)$, $a_i \in B_{\alpha\rho}$, $i = 1, 2, \ldots, n$.

Due to Proposition 3 $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ)$ can be finite only if $x^\circ \notin \operatorname{int} \cap_{i=1}^n \Omega_i$. The next proposition gives a more precise estimate.

Proposition 8. *If $x^\circ \in \operatorname{bd} \cap_{i=1}^n \Omega_i$ then $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) \leq 1$.*

Proof. If $x^\circ \in \text{bd} \cap_{i=1}^n \Omega_i$ then $x^\circ \in \text{bd}\,\Omega_i$ for some $i \in \{1, 2, \ldots, n\}$ and there exists a sequence $\{x_k\} \not\subset \Omega_i$ approaching x°. Denote by r_k the distance from x_k to Ω_i. One has $r_k > 0$ since Ω_i is closed. Without loss of generality we will assume that $r_k < 1$. Select $\omega_k \in \Omega_i$ such that

$$\|x_k - \omega_k\| \le r_k + r_k^2$$

and denote $a_k = x_k - \omega_k$, $\rho_k = r_k - r_k^2$. Then $\Omega_i \cap B_{\rho_k}(x_k) = \emptyset$ or

$$(\Omega_i - \omega_k - a_k) \cap B_{\rho_k} = \emptyset$$

and it follows from Proposition 7 that

$$\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) \le \lim_{k\to\infty} \|a_k\| / \rho_k = 1.$$

□

Thus, one always has either $0 \le \hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) \le 1$ or $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) = +\infty$ (when $x^\circ \in \text{int} \cap_{i=1}^n \Omega_i$). At the same time $\theta[\Omega_1, \ldots, \Omega_n](x^\circ)$ can obviously take any nonnegative values.

3. More Constants

Let us consider two more constants related to the behavior of the set system $\{\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n\}$ near x°:

$$\theta'_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = \sup\{r \ge 0 :\ B_r \subset (\Omega_i \cap B_\rho(x^\circ)) - (\Omega_j \cap B_\rho(x^\circ)),\ \forall i, j \in \{1, 2, \ldots, n\},\ i \ne j\}, \quad (7)$$

$$\theta''_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = \sup\{r \ge 0 : (\Omega_i - a_1) \cap (\Omega_j - a_2) \cap B_\rho(x^\circ) \ne \emptyset,\ \forall a_1, a_2 \in B_r,\ i, j \in \{1, 2, \ldots, n\},\ i \ne j\}. \quad (8)$$

The difference of sets in (7) is understood in the algebraic sense:

$$\Omega_1 - \Omega_2 = \{\omega_1 - \omega_2 :\ \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\}.$$

Obviously $0 \le \theta'_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) \le 2\rho$ for any $\rho > 0$. This means that, in contrast to $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$ (see Proposition 3), $\theta'_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$ always tends to zero when $\rho \to 0$.

One always has the inequality

$$\theta''_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) \ge \theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$$

and it can be strict. Take e.g. three halfspaces in $\mathbb{R}^2$:

$$\Omega_1 = \{(x, y) : y \ge 0\},\ \Omega_2 = \{(x, y) : y \le x\},\ \Omega_3 = \{(x, y) : y \le -x\}.$$

Then $\Omega_1 \cap \Omega_2 \cap \Omega_3 = \{(0, 0)\}$ and $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$ while $\theta''_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$ is positive for all $\rho > 0$.

Some relations between (1), (7) and (8) are given by the following proposition.

Proposition 9. *The following assertions hold:*

(i) *If* $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) > r \ge 0$ *then* $\theta'_{\rho+r}[\Omega_1, \ldots, \Omega_n](x^\circ) \ge 2r$.

(ii) *If* $\theta'_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) > r \ge 0$ *then* $\theta''_{\rho+r/2}[\Omega_1,\ldots,\Omega_n](x^\circ) \ge r/2$.

Proof. (i) If $\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) > r \ge 0$ then by the definition (1)

$$\Big(\bigcap_{i=1}^{n}(\Omega_i - a_i)\Big) \cap B_\rho(x^\circ) \neq \emptyset$$

for any $a_i \in B_r$, $i = 1,2,\ldots,n$. In particular, for any $i,j \in \{1,2,\ldots,n\}$, $i \neq j$ there exists

$$x \in [(\Omega_i - a_i) \cap B_\rho(x^\circ)] \cap [(\Omega_j - a_j) \cap B_\rho(x^\circ)]$$

and consequently

$$0 \in [(\Omega_i - a_i) \cap B_\rho(x^\circ)] - [(\Omega_j - a_j) \cap B_\rho(x^\circ)]$$

and

$$a_i - a_j \in [\Omega_i \cap B_\rho(x^\circ + a_i)] - [\Omega_j \cap B_\rho(x^\circ + a_j)] \subset (\Omega_i \cap B_{\rho+r}(x^\circ)) - (\Omega_j \cap B_{\rho+r}(x^\circ)).$$

Since a_i and a_j are arbitrary, one has

$$B_{2r} \subset (\Omega_i \cap B_{\rho+r}(x^\circ)) - (\Omega_j \cap B_{\rho+r}(x^\circ))$$

and $\theta'_{\rho+r}[\Omega_1,\ldots,\Omega_n](x^\circ) \ge 2r$.

(ii) If $\theta'_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) > r \ge 0$, $i,j \in \{1,2,\ldots,n\}$, $i \neq j$, and $a_1, a_2 \in B_{r/2}$ then $a_1 - a_2 \in B_r$ and by definition (7)

$$a_1 - a_2 \in (\Omega_i \cap B_\rho(x^\circ)) - (\Omega_j \cap B_\rho(x^\circ))$$

and consequently

$$0 \in ((\Omega_i - a_1) \cap B_\rho(x^\circ - a_1)) - ((\Omega_j - a_2) \cap B_\rho(x^\circ - a_2)) \subset$$
$$((\Omega_i - a_1) \cap B_{\rho+r/2}(x^\circ)) - ((\Omega_j - a_2) \cap B_{\rho+r/2}(x^\circ)).$$

This means that

$$((\Omega_i - a_1) \cap B_{\rho+r/2}(x^\circ)) \cap ((\Omega_j - a_2) \cap B_{\rho+r/2}(x^\circ)) \neq \emptyset.$$

The set in the left-hand side of the last inequality is exactly

$$(\Omega_i - a_1) \cap (\Omega_j - a_2) \cap B_{\rho+r/2}(x^\circ).$$

Thus $\theta''_{\rho+r/2}[\Omega_1,\ldots,\Omega_n](x^\circ) \ge r/2$. □

The next proposition summarizes some easy corollaries of Proposition 9.

Proposition 10. *The following assertions hold:*

(i) *If* $\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) < \infty$ *then* $\theta'_r[\Omega_1,\ldots,\Omega_n](x^\circ) \ge 2\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)$ *for any* $r \ge \rho + \theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)$.

(ii) *If* $\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) = \infty$ *for some* $\rho > 0$ *then* $\sup_{r>0}\theta'_r[\Omega_1,\ldots,\Omega_n](x^\circ) = \infty$.

(iii) $\theta'_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \min(2\theta_{\rho/2}[\Omega_1,\ldots,\Omega_n](x^\circ), \rho)$.

(iv) $\theta''_r[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \theta'_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)/2$ *for any* $r \ge \theta'_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)/2 + \rho$.

(v) $\theta''_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \theta'_{\rho/2}[\Omega_1,\ldots,\Omega_n](x^\circ)/2$.

It is possible to define some more constants based on (7) and (8) in the same way as (2) and (3) were defined on the base of (1).

$$\theta'[\Omega_1,\ldots,\Omega_n](x^\circ) = \liminf_{\rho\to+0}\ \theta'_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)/\rho, \tag{9}$$

$$\theta''[\Omega_1,\ldots,\Omega_n](x^\circ) = \liminf_{\rho\to+0}\ \theta''_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)/\rho, \tag{10}$$

$$\hat\theta'[\Omega_1,\ldots,\Omega_n](x^\circ) = \liminf_{\omega_i \xrightarrow{\Omega_i} x^\circ}\ \theta'[\Omega_1-\omega_1,\ldots,\Omega_n-\omega_n](0), \tag{11}$$

$$\hat\theta''[\Omega_1,\ldots,\Omega_n](x^\circ) = \liminf_{\omega_i \xrightarrow{\Omega_i} x^\circ}\ \theta''[\Omega_1-\omega_1,\ldots,\Omega_n-\omega_n](0). \tag{12}$$

The next proposition follows from estimates (iii) and (v) of Proposition 10.

Proposition 11. *The following assertions hold:*

(i) $4\theta''[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \theta'[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \min(\theta[\Omega_1,\ldots,\Omega_n](x^\circ),1)$.
(ii) $4\hat\theta''[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \hat\theta'[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \min(\hat\theta[\Omega_1,\ldots,\Omega_n](x^\circ),1)$.

The “zero” case is of special interest.

Proposition 12. *The following assertions hold:*

(i) $(\theta''_{2\rho}[\Omega_1,\ldots,\Omega_n](x^\circ)=0) \Rightarrow (\theta'_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)=0) \Rightarrow (\theta_{\rho/2}[\Omega_1,\ldots,\Omega_n](x^\circ)=0)$.
(ii) $(\theta''[\Omega_1,\ldots,\Omega_n](x^\circ)=0) \Rightarrow (\theta'[\Omega_1,\ldots,\Omega_n](x^\circ)=0) \Rightarrow (\theta[\Omega_1,\ldots,\Omega_n](x^\circ)=0)$.
(iii) $(\hat\theta''[\Omega_1,\ldots,\Omega_n](x^\circ)=0) \Rightarrow (\hat\theta'[\Omega_1,\ldots,\Omega_n](x^\circ)=0) \Rightarrow (\hat\theta[\Omega_1,\ldots,\Omega_n](x^\circ)=0)$.

4. Special Cases

4.1. **Convex sets.** In the convex case, as one could expect, the concepts of extremality and local extremality coincide and appear to be equivalent to both stationarity and weak stationarity.

Proposition 13. *Let Ω_1, Ω_2, …, Ω_n be convex.*

(i) *If $\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) > 0$ for some $\rho > 0$ then $\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) > 0$ for all $\rho > 0$.*
(ii) *The function $\rho \to \theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)/\rho$, considered on the set of positive numbers, is nonincreasing.*
(iii) $\theta[\Omega_1,\ldots,\Omega_n](x^\circ) = \sup_{\rho>0}\ \theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)/\rho$.
(iv) $\hat\theta[\Omega_1,\ldots,\Omega_n](x^\circ) = \theta[\Omega_1,\ldots,\Omega_n](x^\circ)$.

Proof. (i) Let $\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ) > 0$ for some $\rho > 0$. Since the function $\rho \to \theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)$ is nondecreasing it is sufficient to show that $\theta_{\rho'}[\Omega_1,\ldots,\Omega_n](x^\circ)$ is positive for any positive $\rho' < \rho$. Denote $t = \rho'/\rho$. It follows from (1) that for any positive $r < \theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)$ and any $a_i \in B_r$, $i = 1,2,\ldots,n$, one has

$$\Big(\bigcap_{i=1}^n(\Omega_i - a_i)\Big)\bigcap B_\rho(x^\circ) \ne \emptyset.$$

Take arbitrary $a_i \in B_{tr}$, $i = 1, 2, \ldots, n$, and select some

$$x \in \Big(\bigcap_{i=1}^{n} (\Omega_i - a_i/t)\Big) \bigcap B_\rho(x^\circ).$$

Then $x + a_i/t \in \Omega_i$ and, since Ω_i is convex, $x_t + a_i \in \Omega_i$, where $x_t = x^\circ + t(x - x^\circ)$. Thus

$$x_t \in \Big(\bigcap_{i=1}^{n} (\Omega_i - a_i)\Big) \bigcap B_{\rho'}(x^\circ)$$

and consequently $\theta_{\rho'}[\Omega_1, \ldots, \Omega_n](x^\circ) \geq t\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$.

(ii) The last inequality can be rewritten as

$$\theta_{\rho'}[\Omega_1, \ldots, \Omega_n](x^\circ)/\rho' \geq \theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)/\rho.$$

(iii) The assertion follows from (ii) and the definition (2).

(iv) The inequality $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) \leq \theta[\Omega_1, \ldots, \Omega_n](x^\circ)$ follows from the definition (3). Let $0 < \alpha < \beta < \theta[\Omega_1, \ldots, \Omega_n](x^\circ)$. Then it follows from (iii) that there exists $\rho > 0$, such that $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) > \beta\rho$ and consequently

$$\Big(\bigcap_{i=1}^{n} (\Omega_i - a_i)\Big) \bigcap B_\rho(x^\circ) \neq \emptyset$$

for any $a_i \in B_{\beta\rho}$, $i = 1, 2, \ldots, n$. The last condition can be rewritten as

$$\Big(\bigcap_{i=1}^{n} (\Omega_i - x^\circ - a_i)\Big) \bigcap B_\rho \neq \emptyset.$$

Take arbitrary $\omega_i \in \Omega_i \cap B_{(\beta-\alpha)\rho}(x^\circ)$. Then

$$\Big(\bigcap_{i=1}^{n} (\Omega_i - \omega_i - a_i)\Big) \bigcap B_\rho \neq \emptyset$$

for any $a_i \in B_{\alpha\rho}$ and consequently

$$\theta_\rho[\Omega_1 - \omega_1, \ldots, \Omega_n - \omega_n](0) \geq \alpha\rho$$

and

$$\theta[\Omega_1 - \omega_1, \ldots, \Omega_n - \omega_n](0) \geq \alpha.$$

The conclusion follows from the definition (3). □

Proposition 14. *Let Ω_1, Ω_2, $\ldots$, Ω_n be convex. Then (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) in Definition 1.*

If $\operatorname{int}\Omega_i \neq \emptyset$, $i = 1, 2, \ldots, n-1$, then the first four conditions in Definition 1 are equivalent to the following one:

$$\bigcap_{i=1}^{n-1} \operatorname{int}\Omega_i \cap \Omega_n = \emptyset, \tag{13}$$

while condition (v) is equivalent to

$$\bigcap_{i=1}^{n-1} \operatorname{int}\Omega_i \cap \Omega_n \neq \emptyset, \tag{14}$$

Proof. The first assertion follows from Proposition 13. Condition (i) in Definition 1 implies (13) (Proposition 4).

Let $\operatorname{int}\Omega_i \neq \emptyset$, $i = 1, 2, \ldots, n-1$, and (13) hold true. Choose $\omega_i^\circ \in \Omega_i$, $i = 1, 2, \ldots, n-1$, and $\rho > 0$, such that $B_{2\rho}(\omega_i^\circ) \subset \operatorname{int}\Omega_i$, $i = 1, 2, \ldots, n-1$. Denote

$$a_{ik} = k^{-1}(x^\circ - \omega_i^\circ),\ i = 1, 2, \ldots, n-1,\ a_{nk} = 0.$$

We shall show that

$$\Big(\bigcap_{i=1}^{n}(\Omega_i - a_{ik})\Big) \cap B_\rho(x^\circ) = \emptyset$$

for all sufficiently large k. If $\|a_{ik}\| \leq \rho$ then for $i = 1, 2, \ldots, n-1$ one has

$$(\Omega_i - a_{ik}) \cap B_\rho(x^\circ) \subset \Omega_i \cap B_{2\rho}(x^\circ) - a_{ik} = \{\omega - a_{ik} : \omega \in \Omega_i \cap B_{2\rho}(x^\circ)\} =$$
$$\{\omega + k^{-1}((\omega_i^\circ + \omega - x^\circ) - \omega) : \omega \in \Omega_i \cap B_{2\rho}(x^\circ)\}.$$

In the last expression $\omega_i^\circ + \omega - x^\circ \in B_{2\rho}(\omega_i^\circ) \subset \operatorname{int}\Omega_i$ and

$$\omega + k^{-1}((\omega_i^\circ + \omega - x^\circ) - \omega) \in \operatorname{int}\Omega_i$$

since Ω_i is convex. Thus

$$(\Omega_i - a_{ik}) \cap B_\rho(x^\circ) \subset \operatorname{int}\Omega_i$$

and it follows from (v) that

$$\Big(\bigcap_{i=1}^{n}(\Omega_i - a_{ik})\Big) \cap B_\rho(x^\circ) = \emptyset.$$

This implies (ii) in Definition 1. □

Remark 5. *As it follows from Proposition 14, for convex sets all of which except maybe one are solid, the concept of extremality/stationarity takes the traditional form (13). From the other hand, the concepts investigated in the current paper make sense and are applicable for convex sets which are not necessarily solid.*

4.2. **Cones.** Another special case that can be important for applications is the case of a system of cones. It follows from the next proposition that all the properties under consideration are determined by the constant θ_ρ calculated for $\rho = 1$ (possibly for shifted cones).

Proposition 15. *Let Ω_1, Ω_2, ..., Ω_n be cones. Then the following assertions hold.*

(i) *If $\omega_i \in \Omega_i$, $i = 1, 2, \ldots, n$ and $\rho > 0$ then*

$$\theta_\rho[\Omega_1 - \omega_1, \ldots, \Omega_n - \omega_n](0) = \rho\theta_1[\Omega_1 - \omega_1/\rho, \ldots, \Omega_n - \omega_n/\rho](0).$$

In particular, $\rho \to \theta_\rho[\Omega_1, \ldots, \Omega_n](0)$ is positively homogeneous:

$$\theta_\rho[\Omega_1, \ldots, \Omega_n](0) = \rho\theta_1[\Omega_1, \ldots, \Omega_n](0).$$

(ii) *If* $\omega_i \in \Omega_i$, $i = 1, 2, \ldots, n$ *then*

$$\theta[\Omega_1 - \omega_1, \ldots, \Omega_n - \omega_n](0) = \liminf_{t\to\infty} \theta_1[\Omega_1 - t\omega_1, \ldots, \Omega_n - t\omega_n](0).$$

In particular, $\theta[\Omega_1, \ldots, \Omega_n](0) = \theta_1[\Omega_1, \ldots, \Omega_n](0)$.

(iii) $\hat{\theta}[\Omega_1, \ldots, \Omega_n](0) = \inf_{\omega_i\in\Omega_i} \theta_1[\Omega_1 - \omega_1, \ldots, \Omega_n - \omega_n](0)$.

Proof. (i) Let $\omega_i \in \Omega_i$, $i = 1, 2, \ldots, n$ and $\rho > 0$. Since Ω_i, $i = 1, 2, \ldots, n$ are cones inequality (6) is equivalent to the condition

$$\Big(\bigcap_{i=1}^{n}(\Omega_i - \omega_i/\rho - a_i/\rho)\Big) \cap B \neq \emptyset.$$

The conclusion follows from Proposition 7.

(ii) The assertion follows from (i) and the definition (2) by substituting $\rho = 1/t$.

(iii) The assertion follows from (ii) and the definition (3) since $t\omega_i$ when $t \to \infty$ and $\omega_i \to 0$ can determine any point in Ω_i. □

Proposition 16. *Let* Ω_1, Ω_2, $\ldots$, Ω_n *be cones,* $x^\circ = 0$.

(i) *(i)* ⇔ *(ii)* ⇔ *(iii) in Definition 1 and these conditions are equivalent to* $\theta_1[\Omega_1, \ldots, \Omega_n](0) = 0$.

(ii) *The system of sets* Ω_1, Ω_2, $\ldots$, Ω_n *is weakly stationary at* x° *if and only if*

$$\inf_{\omega_i\in\Omega_i} \theta_1[\Omega_1 - \omega_1, \ldots, \Omega_n - \omega_n](0) = 0.$$

Proposition 17. *Let* Ω_1, Ω_2, $\ldots$, Ω_n *be cones. If* $\theta'_\rho[\Omega_1, \ldots, \Omega_n](0) > 0$ *for some* $\rho > 0$ *then* $\Omega_i - \Omega_j = X$ *for any* $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$.

Proof. Let $\theta'_\rho[\Omega_1, \ldots, \Omega_n](0) > r > 0$ for some $\rho > 0$ and $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. It follows from (7) that $B_r \subset (\Omega_i \cap B_\rho) - (\Omega_j \cap B_\rho)$. Then $B_r \subset \Omega_i - \Omega_j$ and consequently $\Omega_i - \Omega_j = X$ since $\Omega_i - \Omega_j$ is a cone. □

4.3. **Case** $n = 2$**.** Considering (7) (and (9) and (11)) can be especially useful in the case of two sets ($n = 2$). In this case one has $\theta''_\rho[\Omega_1, \Omega_2](x^\circ) = \theta_\rho[\Omega_1, \Omega_2](x^\circ)$ and the "derivate" constants (10) and (12) reduce respectively to (2) and (3). It follows from Propositions 10 – 12 that the properties of $\theta'_\rho[\Omega_1, \Omega_2](x^\circ)$ are very similar to those of $\theta_\rho[\Omega_1, \Omega_2](x^\circ)$.

Proposition 18. *The following assertions hold:*

(i) $(\theta_\rho[\Omega_1, \Omega_2](x^\circ) = 0) \Rightarrow (\theta'_{\rho/2}[\Omega_1, \Omega_2](x^\circ) = 0)$.

(ii) $(\theta'_\rho[\Omega_1, \Omega_2](x^\circ) = 0) \Rightarrow (\theta_{\rho/2}[\Omega_1, \Omega_2](x^\circ) = 0)$.

(iii) $(\theta'[\Omega_1, \Omega_2](x^\circ) = 0) \Leftrightarrow (\theta[\Omega_1, \Omega_2](x^\circ) = 0)$.

(iv) $(\hat{\theta}'[\Omega_1, \Omega_2](x^\circ) = 0) \Leftrightarrow (\hat{\theta}[\Omega_1, \Omega_2](x^\circ) = 0)$.

Thus in the case of two sets all the concepts in Definition 1 can be equivalently described in terms of (7), (9) and (11) instead of (1), (2) and (3). In particular, weak stationarity is equivalent to "*extended extremality*" as defined in [18] (see also [19], [20]).

On the other hand, the general case $n \geq 2$ can be easily reduced to the case of two sets: one can consider the system of two sets $\tilde{\Omega}_1 = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$ and $\tilde{\Omega}_2 = \{(x, x, \ldots, x) : \ x \in X\}$ in X^n and the point $\tilde{x}^\circ = (x^\circ, x^\circ, \ldots, x^\circ) \in \tilde{\Omega}_1 \cap \tilde{\Omega}_2$.

Proposition 19. *The following inequalities hold:*

$$\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) \geq \theta_\rho[\tilde{\Omega}_1, \tilde{\Omega}_2](\tilde{x}^\circ) \geq \min(\theta_{2\rho}[\Omega_1, \ldots, \Omega_n](x^\circ)/2, \rho).$$

Proof. If $0 \leq r < \theta_\rho[\tilde{\Omega}_1, \tilde{\Omega}_2](\tilde{x}^\circ)$ then

$$(\tilde{\Omega}_1 - \tilde{a}) \cap \tilde{\Omega}_2 \cap B_\rho(\tilde{x}^\circ) \neq \emptyset$$

for any $\tilde{a} = (a_1, a_2, \ldots, a_n) \in B_r$. Due to the definitions of $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ this means that for any $a_i \in B_r$, $i = 1, 2, \ldots, n$, there exists $x \in B_\rho(x^\circ)$, such that $x \in \Omega_i - a_i$, $i = 1, 2, \ldots, n$. Thus,

$$\Big(\bigcap_{i=1}^{n} (\Omega_i - a_i)\Big) \cap B_\rho(x^\circ) \neq \emptyset$$

and consequently $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) \geq r$. This proves the first inequality.

If $0 \leq r < \theta_{2\rho}[\Omega_1, \ldots \Omega_n](x^\circ)/2$ then

$$\Big(\bigcap_{i=1}^{n} (\Omega_i - a_i)\Big) \cap B_{2\rho}(x^\circ) \neq \emptyset$$

for any $a_i \in B_{2r}$, $i = 1, 2, \ldots, n$. In other words,

$$(\tilde{\Omega}_1 - \tilde{a}) \cap \tilde{\Omega}_2 \cap B_{2\rho}(\tilde{x}^\circ) \neq \emptyset$$

for any $\tilde{a} \in B_{2r}$. Assume additionally that $r \leq \rho$. Then

$$(\tilde{\Omega}_1 - \tilde{a}_1) \cap \tilde{(}\Omega_2 - \tilde{a}_2) \cap B_\rho(\tilde{x}^\circ) \neq \emptyset$$

for any $\tilde{a}_1, \tilde{a}_2 \in B_r$. This proves the second inequality. □

Proposition 20. *The system of sets Ω_1, Ω_2, $\ldots$, Ω_n is extremal (locally extremal, stationary, weakly stationary, regular) at x° if and only if the system of sets $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ in X^n is extremal (locally extremal, stationary, weakly stationary, regular) at $\tilde{x}^\circ$.*

Remark 6. *The analogs of Propositions 19 and 20 are true for the following system of two sets in X^{n-1}: $\tilde{\Omega}_1 = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_{n-1}$ and $\tilde{\Omega}_2 = \{(x, x, \ldots, x) : \ x \in \Omega_n\}$.*

5. Metric Inequality

Some other approaches based on comparing distances can be used for characterizing stationarity/regularity properties of set systems. Let $d(\cdot, \cdot)$ be the distance function in X associated with the norm. We will keep the same notation for point-to-set distances. Thus,

$d(x,\Omega) = \inf_{\omega\in\Omega} \|x-\omega\|$ is the distance from a point x to a set Ω and $d(x,\emptyset)=\infty$. Let us introduce two more constants:

$$\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = \limsup_{x\to x^\circ} \left(d(x, \bigcap_{i=1}^n \Omega_i) \;/\; \max_{1\le i\le n} d(x,\Omega_i) \right)_\circ , \tag{15}$$

$$\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = \limsup_{\substack{x\to x^\circ \\ x_i\to 0}} \left(d(x, \bigcap_{i=1}^n (\Omega_i - x_i)) \;/\; \max_{1\le i\le n} d(x+x_i,\Omega_i) \right)_\circ . \tag{16}$$

The "extended" division operation $(\cdot/\cdot)_\circ$ is used in (15), (16) to simplify the definitions. It makes division by zero legal. The formal rules are as follows: 1) $(\alpha/\beta)_\circ = \alpha/\beta$, if $\beta\ne 0$; 2) $(\alpha/0)_\circ = +\infty$, if $\alpha>0$; 3) $(\alpha/0)_\circ = -\infty$, if $\alpha<0$; 4) $(0/0)_\circ = 0$. The first three rules are quite usual (the second and the third cases will never occur in (15), (16)). The fourth rule is the most important one here. In the case $x^\circ \in \operatorname{int} \cap_{i=1}^n \Omega_i$ it automatically leads to $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = \vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = 0$. Otherwise the points where the numerator and the denominator are both zero in the right-hand side of (15) or (16) can be ignored when calculating the value of the upper limit.

Some easy corollaries of the definitions (15), (16) are summarized in the next two propositions.

Proposition 21. *The following assertions hold:*

(i) $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = \limsup_{x_i\to 0} \vartheta[\Omega_1-x_1,\ldots,\Omega_n-x_n](x^\circ)$.

(ii) *If* $x^\circ \notin \operatorname{int} \cap_{i=1}^n \Omega_i$ *then* $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) \ge \vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) \ge 1$.
Otherwise $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = \vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = 0$.

Proposition 22. *The following assertions hold:*

(i) $\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) < \infty$ *if and only if there exists a* $\beta>0$ *and a* $\delta>0$ *such that*

$$d(x, \bigcap_{i=1}^n \Omega_i) \le \beta \max_{1\le i\le n} d(x,\Omega_i) \tag{17}$$

for all $x\in B_\delta(x^\circ)$. $\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ)$ *coincides with the exact lower bound of all such* β.

(ii) $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) < \infty$ *if and only if there exists a* $\beta>0$ *and a* $\delta>0$ *such that*

$$d(x, \bigcap_{i=1}^n (\Omega_i - x_i)) \le \beta \max_{1\le i\le n} d(x+x_i,\Omega_i) \tag{18}$$

for all $x\in B_\delta(x^\circ)$, $x_i\in B_\delta$, $i=1,2,\ldots,n$. $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ)$ *coincides with the exact lower bound of all such* β.

The main "regularity" question is whether the corresponding constant is finite.

Remark 7. *The condition formulated in part (i) of Proposition 22 is equivalent to the regularity condition known as the* metric inequality [12, 13, 29] *(it is formulated in* [12, 13, 29] *with the sum of the distances in the right-hand side instead of the maximum). The condition* *in part (ii) can be considered as the* strong metric inequality. *If (17) is valid for all $x \in X$ then the system of sets is said to be* linear regular [28]. *Collections of* normal *or* downward [34] *sets are examples of linearly regular set systems.*

Remark 8. *The inequality $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) \geq \vartheta[\Omega_1, \ldots, \Omega_n](x^\circ)$ can be strong. Take e.g. $\Omega_1 = \Omega_2 = \{x^\circ\}$. Then evidently, $\vartheta[\Omega_1, \Omega_2](x^\circ) = 1$, $\hat{\vartheta}[\Omega_1, \Omega_2](x^\circ) = \infty$. Thus, the strong metric inequality is really stronger than its traditional counterpart.*

The constant (16) appears to be closely related to (3).

Theorem 1. $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) = 1/\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ)$.

Proof. Let us show that $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) \geq \alpha$ for any $\alpha < 1/\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ)$. Chose an arbitrary

$$\beta \in (\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ), 1/\alpha).$$

Due to Proposition 22 there exists a $\delta > 0$ such that (18) holds for all $x \in B_\delta(x^\circ)$, $x_i \in B_\delta$, $i = 1, \ldots, n$. Denote $\delta' = \delta/(\alpha + 1)$. Let $0 < \rho \leq \delta'$, $\omega_i \in \Omega_i \cap B_{\delta'}(x^\circ)$, $\|a_i\| \leq \alpha\rho$, $i = 1, \ldots, n$. Then

$$\|\omega_i - x^\circ + a_i\| \leq \delta, \; i = 1, \ldots, n,$$

and it follows from (18) for $x = x^\circ$, $x_i = \omega_i - x^\circ + a_i$ that the inequalities

$$d(0, \bigcap_{i=1}^n (\Omega_i - \omega_i - a_i)) \leq \beta \max_{1 \leq i \leq n} d(\omega_i + a_i, \Omega_i) \leq \beta \max_{1 \leq i \leq n} \|a_i\| < \rho$$

hold true. Thus, (6) is valid, and it follows from Proposition 7 that the inequality $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) \geq \alpha$ holds. Since α is arbitrary one has $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) \geq 1/\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ)$.

Let us show that $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) \leq 1/\alpha$ for any positive $\alpha < \hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ)$. Due to Proposition 7 there exists a $\delta > 0$ such that (6) holds for all $0 < \rho \leq \delta$, $\omega_i \in \Omega_i \cap B_\delta(x^\circ)$, $a_i \in B_{\alpha\rho}$, $i = 1, \ldots, n$. Denote

$$\delta' = \delta \min[\alpha/(\alpha + 1), 1/(\alpha + 3)]$$

and chose arbitrary points $x \in B_{\delta'}(x^\circ)$, $x_i \in B_{\delta'}$, $i = 1, \ldots, n$. Consider two cases.

1) $d(x + x_j, \Omega_j) \geq \delta'(\alpha + 1)$ for some $j \in \{1, \ldots, n\}$. Take $\rho = \delta'/\alpha$, $\omega_i = x^\circ$, $a_i = x_i$. Obviously $\rho < \delta$, $\|a_i\| \leq \alpha\rho$, $i = 1, \ldots, n$, and it follows from (6) that there exists an $x' \in B_{\delta'/\alpha}(x^\circ)$ such that $x' + x_i \in \Omega_i$, $i = 1, \ldots, n$. Thus,

$$d(x, \bigcap_{i=1}^n (\Omega_i - x_i)) \leq \|x - x'\| \leq \delta'(\alpha + 1)/\alpha \leq (1/\alpha) \max_{1 \leq i \leq n} d(x + x_i, \Omega_i),$$

which implies $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) \leq 1/\alpha$ due to Proposition 22.

2) $d(x + x_i, \Omega_i) < \delta'(\alpha + 1)$, $i = 1, \ldots, n$. For any $i = 1, \ldots, n$ chose a point $\omega_i \in \Omega_i$ such that $\|x + x_i - \omega_i\| < \delta'(\alpha + 1)$. Evidently

$$\|\omega_i - x^\circ\| < \delta'(\alpha + 3) \leq \delta.$$

Take $a_i = x + x_i - \omega_i$, $\rho = \max_{1\le i\le n} \|a_i\|/\alpha$. Then $\rho \le \delta$, $\|a_i\| \le \alpha\rho$, $i = 1,\ldots,n$, and one can use condition (6) again: there exists an $x' \in B_\rho(x)$ such that $x'+x_i = x'-x+\omega_i+a_i \in \Omega_i$, $i = 1,\ldots,n$. Consequently

$$d(x, \bigcap_{i=1}^n (\Omega_i - x_i)) \le \|x - x'\| \le (1/\alpha) \max_{1\le i\le n} \|x + x_i - \omega_i\|.$$

Since $\|x + x_i - \omega_i\|$ can be made arbitrary close to $d(x + x_i, \Omega_i)$ by the appropriate choice of $\omega_i \in \Omega_i$, the last inequality implies that

$$d(x, \bigcap_{i=1}^n (\Omega_i - x_i)) \le (1/\alpha) \max_{1\le i\le n} d(x + x_i, \Omega_i)$$

and consequently $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) \le 1/\alpha$.

Since α is arbitrary one has $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) \le 1/\hat\theta[\Omega_1,\ldots,\Omega_n](x^\circ)$. □

Corollary 1.1. *The system of sets Ω_1, Ω_2, $\ldots$, Ω_n is regular at x° if and only if $\hat\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) < \infty$. Under these conditions $\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) < \infty$.*

6. Boundary Condition

Another "extremal" setting based on considering set systems was developed in [4]. Instead of considering the intersection of closed sets Ω_1, Ω_2, $\ldots$, Ω_n the authors consider their sum

$$\sum_{i=1}^n \Omega_i = \left\{\omega = \sum_{i=1}^n \omega_i : \ \omega_i \in \Omega_i, \ i = 1,2,\ldots,n\right\}.$$

The sets are not assumed to have a common point. Let $x_i^\circ \in \Omega_i$, $i = 1,2,\ldots,n$. The main "extremal" property investigated in [4] is whether the *boundary condition*

$$\sum_{i=1}^n x_i^\circ \in \mathrm{bd}\left(\sum_{i=1}^n \Omega_i\right) \tag{19}$$

holds true.

Not surprisingly this condition appears closely related to the extremality concept considered above. In the framework of the current paper we are interested not only in extremality, but also in regularity properties. Therefore a little more general setting will be discussed here. We are going to introduce one more constant:

$$\zeta_\rho[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ) = \sup\left\{r \ge 0 : \ B_r(\sum_{i=1}^n x_i^\circ) \subset \sum_{i=1}^n (\Omega_i \cap B_\rho(x_i^\circ))\right\}. \tag{20}$$

Note that the sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ are considered locally when defining (20). The condition (19) is equivalent to $\zeta_\rho[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ) = 0$ for any $\rho > 0$.

Proposition 23. *Define $\tilde\Omega_1 = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$, $\tilde\Omega_2 = \{(x_1, x_2, \ldots, x_n) \in X^n : \ \sum_{i=1}^n x_i = \sum_{i=1}^n x_i^\circ\}$, $\tilde x^\circ = (x_1^\circ, x_2^\circ, \ldots, x_n^\circ) \in X^n$. The following assertions hold for any $\rho > 0$:*

(i) $\theta_{2\rho}[\tilde\Omega_1, \tilde\Omega_2](\tilde x^\circ) \ge \min(\zeta_\rho[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ)/(2n), \rho)$.

(ii) $\zeta_{2\rho}[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ) \ge \min(\theta_\rho[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ),\rho)$.

Proof. Evidently the sets $\tilde\Omega_1$, $\tilde\Omega_2$ are closed and $\tilde x^\circ \in \tilde\Omega_1 \cap \tilde\Omega_2$.

(i) Let $0 \le r < \zeta_\rho[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ)$. Select arbitrary points $\tilde a_1, \tilde a_2 \in B_{r/(2n)} \subset X^n$. Then $\tilde a = \tilde a_1 - \tilde a_2 \in B_{r/n}$. If $\tilde a = (a_1,a_2,\ldots,a_n)$ then $a_i \in B_{r/n} \subset X$, $i=1,2,\ldots,n$, and $a = \sum_{i=1}^n a_i \in B_r$. It follows from (20) that

$$\sum_{i=1}^n x_i^\circ + a \in \sum_{i=1}^n (\Omega_i \cap B_\rho(x_i^\circ)).$$

This means that there exist $\omega_i \in \Omega_i \cap B_\rho(x_i^\circ)$, $i=1,2,\ldots,n$, such that

$$\sum_{i=1}^n (\omega_i - a_i) = \sum_{i=1}^n x_i^\circ.$$

In other words, $\tilde\omega - \tilde a \in \tilde\Omega_2$, where $\tilde\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in \tilde\Omega_1 \cap B_\rho(\tilde x^\circ)$, and consequently

$$\tilde\omega - \tilde a_1 \in (\tilde\Omega_1 - \tilde a_1) \cap (\tilde\Omega_2 - \tilde a_2) \cap B_{\rho + r/(2n)}(\tilde x^\circ).$$

Assume additionally that $r \le 2n\rho$. Then

$$(\tilde\Omega_1 - \tilde a_1) \cap (\tilde\Omega_2 - \tilde a_2) \cap B_{2\rho}(\tilde x^\circ) \ne \emptyset$$

and it follows from the definition (1) that $\theta_{2\rho}[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ) \ge r/(2n)$. This proves the first assertion.

(ii) Let $0 \le r < \theta_\rho[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ)$. Select arbitrary $a \in B_r \subset X$ and take $\tilde a = (a,0,\ldots,0) \in X^n$. Then $\tilde a \in B_r \subset X^n$ and it follows from (1) that

$$(\tilde\Omega_1 - \tilde a) \cap \tilde\Omega_2 \cap B_\rho(\tilde x^\circ) \ne \emptyset.$$

This means that there exist $\omega_i \in \Omega_i \cap B_{\rho+r}(x_i^\circ)$, $i=1,2,\ldots,n$, such that

$$\sum_{i=1}^n \omega_i - a = \sum_{i=1}^n x_i^\circ.$$

In other words,

$$\sum_{i=1}^n x_i^\circ + a \in \sum_{i=1}^n (\Omega_i \cap B_{\rho+r}(x_i^\circ)).$$

Assume additionally that $r \le \rho$. Then it follows from (20) that

$$\zeta_{2\rho}[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ) \ge r.$$

This proves the second assertion. □

Proposition 23 implies the following "extremality" statement.

Proposition 24. *Let $\tilde\Omega_1$, $\tilde\Omega_2$, $\tilde x^\circ$ be as in Proposition 23. The following assertions hold:*

(i) $\zeta_\rho[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ) = 0$ *for some $\rho > 0$ if and only if the system of sets $\tilde\Omega_1$, $\tilde\Omega_2$ is locally extremal at $\tilde x^\circ$.*

(ii) $\zeta_\rho[\Omega_1,\ldots,\Omega_n](x_1^\circ,\ldots,x_n^\circ)=0$ *for all* $\rho>0$ *(i.e. the boundary condition (19) holds true) if and only if the system of sets* $\tilde\Omega_1$, $\tilde\Omega_2$ *is extremal at* $\tilde x^\circ$.

Thus, the boundary condition for a set system (both in local and global forms) is equivalent to the (corresponding version of) extremality for another set system. The opposite statement is also true.

Proposition 25. *Let* $\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n$ *be closed sets in* X *and* $x^\circ\in\cap_{i=1}^n\Omega_i$. *Define* $\tilde\Omega_1=\Omega_1\times\Omega_2\times\ldots\times\Omega_n$, $\tilde\Omega_2=\{(x,x,\ldots,x)\in X^n:\ x\in X\}$, $\tilde x^\circ=(x^\circ,x^\circ,\ldots,x^\circ)\in\tilde X^n$. *The following assertions hold for any* $\rho>0$*:*

(i) $\zeta_{2\rho}[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ,-\tilde x^\circ)\ge\min(\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ),\rho)$.

(ii) $\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)\ge\zeta_\rho[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ,-\tilde x^\circ)$.

Proof. Evidently the sets $\tilde\Omega_1$, $\tilde\Omega_2$ are closed, $\tilde x^\circ\in\tilde\Omega_1$, $-\tilde x^\circ\in\tilde\Omega_2$ and $0\in\tilde\Omega_1+\tilde\Omega_2$.

(i) Let $0\le r<\theta_\rho[\Omega_1,\ldots,\Omega_n](x^\circ)$. Select arbitrary $\tilde a=(a_1,a_2,\ldots,a_n)\in B_r\subset X^n$. Then $a_i\in B_r\subset X$, $i=1,2,\ldots,n$, and it follows from (1) that

$$\Big(\bigcap_{i=1}^n(\Omega_i-a_i)\Big)\bigcap B_\rho(x^\circ)\ne\emptyset. \tag{21}$$

This means that there exists $\tilde\omega\in\tilde\Omega_1\cap B_{\rho+r}(\tilde x^\circ)$, such that $\tilde\omega-\tilde a\in\tilde\Omega_2\cap B_\rho(\tilde x^\circ)$, or $\tilde a-\tilde\omega\in\tilde\Omega_2\cap B_\rho(-\tilde x^\circ)$. Consequently

$$\tilde a\in\big(\tilde\Omega_1\cap B_{\rho+r}(\tilde x^\circ)\big)+\big(\tilde\Omega_2\cap B_\rho(-\tilde x^\circ)\big).$$

Since $\tilde a$ is arbitrary one has

$$B_r\subset\big(\tilde\Omega_1\cap B_{\rho+r}(\tilde x^\circ)\big)+\big(\tilde\Omega_2\cap B_\rho(-\tilde x^\circ)\big).$$

Assume additionally that $r\le\rho$. Then it follows from (20) that the inequality $\zeta_{2\rho}[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ,-\tilde x^\circ)\ge r$ holds. This proves the first assertion.

(ii) Let $0\le r<\zeta_\rho[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ,-\tilde x^\circ)$. Select arbitrary $a_i\in B_r\subset X$, $i=1,2,\ldots,n$, and set $\tilde a=(a_1,a_2,\ldots,a_n)$. Then $\tilde a\in B_r\subset X^n$ and it follows from (20) that

$$\tilde a\in\big(\tilde\Omega_1\cap B_\rho(\tilde x^\circ)\big)+\big(\tilde\Omega_2\cap B_\rho(-\tilde x^\circ)\big).$$

This means that there exist $\omega_i\in\Omega_i\cap B_\rho(x^\circ)$, $i=1,2,\ldots,n$, and $x\in B_\rho(x^\circ)$, such that $a_i=\omega_i-x$, $i=1,2,\ldots,n$. Thus, $x=\omega_i-a_i\in(\Omega_i-a_i)\cap B_\rho(x^\circ)$, $i=1,2,\ldots,n$. In other words, (21) holds true. Due to (1) this proves the second assertion. □

Proposition 26. *Let* $\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n,\ \tilde\Omega_1,\ \tilde\Omega_2,\ x^\circ,\ \tilde x^\circ$ *be as in Proposition 25. The following assertions hold:*

(i) *The system of sets* $\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n$ *is locally extremal at* x° *if and only if* $\zeta_\rho[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ,-\tilde x^\circ)=0$ *for some* $\rho>0$.

(ii) *The system of sets* $\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n$ *is extremal at* x° *if and only if* $\zeta_\rho[\tilde\Omega_1,\tilde\Omega_2](\tilde x^\circ,-\tilde x^\circ)=0$ *for all* $\rho>0$.

Remark 9. *The four propositions presented in this section extend and improve Proposition 2 from* [4]*, which actually contained the following two implications (for the case* $n=2$*):*

(i) *(19) $\Rightarrow$ the system of sets $\tilde\Omega_1$, $\tilde\Omega_2$ is locally extremal at $\tilde x^\circ$, where $\tilde\Omega_1$, $\tilde\Omega_2$, $\tilde x^\circ$ are as in Proposition 23.*

(ii) *The system of sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ is locally extremal at $x^\circ \Rightarrow \tilde x^\circ \in \mathrm{bd}\,(\tilde\Omega_1 + \tilde\Omega_2)$, where $\Omega_1, \Omega_2, \ldots, \Omega_n$, $\tilde\Omega_1$, $\tilde\Omega_2$, x°, $\tilde x^\circ$ are as in Proposition 25.*

The first assertion is covered by Proposition 24 above. The second one is not true. Firstly, there should be 0 *instead of $\tilde x^\circ$ in the right-hand side of the implication. Secondly, the local extremality leads to the local version of (19):*

$$0 \in \mathrm{bd}\left(\left(\tilde\Omega_1 \cap B_\rho(\tilde x^\circ)\right) + \left(\tilde\Omega_2 \cap B_\rho(-\tilde x^\circ)\right)\right)$$

for some $\rho > 0$ (see Proposition 26).

Remark 10. *Two more constants can be defined based on (20) in the same way as (2) and (3) were defined on the base of (1). One can use them for defining the correspondent stationarity and regularity notions. Due to Propositions 23 and 25 they will be equivalent to the corresponding notions investigated in the preceding sections.*

7. Regularity of a Single Set

The case of a single set is not covered by the definitions and results presented above. Let Ω be a closed set in X with $x^\circ \in \Omega$.

Definition 2. Ω is extremal (locally extremal, stationary, weakly stationary, regular) at x° if the system of two sets $\{\Omega, \Omega\}$ is extremal (locally extremal, stationary, weakly stationary, regular) at x°.

As it follows from Proposition 18 in the case of two sets the constants (7), (9), (11) are convenient for describing extremality-stationarity-regularity properties. In the current setting they take the following form:

$$\theta_\rho[\Omega](x^\circ) = \sup\{r \ge 0 :\ B_r \subset (\Omega \cap B_\rho(x^\circ)) - (\Omega \cap B_\rho(x^\circ))\}, \tag{22}$$

$$\theta[\Omega](x^\circ) = \liminf_{\rho\to+0}\ \theta_\rho[\Omega](x^\circ)/\rho, \tag{23}$$

$$\hat\theta[\Omega](x^\circ) = \liminf_{\omega_1,\omega_2 \xrightarrow{\Omega} x^\circ}\ \theta'[\Omega-\omega_1, \Omega-\omega_2](0). \tag{24}$$

Remark 11. *Note that (24) is defined on the basis of (9), not of (23). This is because two different shifts of a single set make two different sets.*

Remark 12. *The concept of set regularity adopted in Definition 2 is related to the property of $\Omega-\Omega$ to have the nonempty interior. It differs from other existing definitions of tangential or normal regularity (see* [3, 5, 6, 23]*). For example, it follows from Proposition 29 below that any convex set with the empty interior is extremal at each point.*

Application of the results of the previous sections leads to the following statements.

Proposition 27. Ω *is locally extremal at x° if and only if there exists a number $\rho > 0$ and a sequence $\{a_k\} \subset X$ tending to zero, such that*

$$\Omega \cap (\Omega - a_k) \cap B_\rho(x^\circ) = \emptyset,\ k = 1, 2, \ldots.$$

Proposition 28. *The following assertions are equivalent:*

(i) $x^\circ \in \operatorname{bd}\Omega$.
(ii) $\lim_{\rho\to+0}\theta_\rho[\Omega](x^\circ) = 0$.
(iii) $\hat{\theta}[\Omega](x^\circ) \le 1$.

Proposition 29. *Let* Ω *be convex. The following assertions are equivalent:*

(i) Ω *is extremal at* x°.
(ii) Ω *is locally extremal at* x°.
(iii) Ω *is stationary at* x°.
(iv) Ω *is weakly stationary at* x°.
(v) $\operatorname{int}\Omega = \emptyset$.

Proposition 30. *Let* Ω *be a cone. The following assertions are equivalent:*

(i) Ω *is extremal at* 0.
(ii) Ω *is locally extremal at* 0.
(iii) Ω *is stationary at* 0.
(iv) $\theta_1[\Omega](0) = 0$.

Proposition 31. *Let* Ω *be a cone.* Ω *is weakly stationary at* 0 *if and only if*

$$\inf_{\omega_1,\omega_2\in\Omega}\theta'[\Omega-\omega_1,\Omega-\omega_2](0) = 0.$$

Remark 13. *If* Ω *is a cone, the condition* $\theta_\rho[\Omega](0) > 0$ *for some* $\rho > 0$ *means that* Ω *is* nonflattened *(*nonoblate*)* [35] *which in its turn implies that* Ω *is* generating*:* $\Omega - \Omega = X$*. If* X *is Banach these conditions are actually equivalent (see* [1]*).*

8. Extremality, Stationarity and Regularity of Multifunctions

It is not surprising that the extremality-stationarity-regularity concepts defined above for set systems are closely related to the similar notions for set-valued mappings (multifunctions).

Let $F:\ X \rightrightarrows Y$ be a multifunction between normed spaces X and Y with a graph $\operatorname{gph} F = \{(x,y)\in X\times Y:\ y\in F(x)\}$ and $(x^\circ,y^\circ)\in\operatorname{gph} F$.

Similarly to (1)–(3) the following three constants can be defined for characterizing the local behavior of F near (x°,y°):

$$\theta_\rho[F](x^\circ,y^\circ) = \sup\{r\ge 0: B_r(y^\circ)\subset F(B_\rho(x^\circ))\}, \tag{25}$$

$$\theta[F](x^\circ,y^\circ) = \liminf_{\rho\to+0}\frac{\theta_\rho[F](x^\circ,y^\circ)}{\rho}, \tag{26}$$

$$\hat{\theta}[F](x^\circ,y^\circ) = \liminf_{(x,y)\overset{\operatorname{gph} F}{\to}(x^\circ,y^\circ)}\theta[F](x,y). \tag{27}$$

Definition 3. The multifunction $F:\ X\rightrightarrows Y$ is

(i) *extremal* at (x°,y°) if $\theta_\rho[F](x^\circ,y^\circ) = 0$ for all $\rho > 0$.

(ii) *locally extremal* at (x°, y°) if $\theta_\rho[F](x^\circ, y^\circ) = 0$ for some $\rho > 0$.
(iii) *stationary* at (x°, y°) if $\theta[F](x^\circ, y^\circ) = 0$.
(iv) *weakly stationary* at (x°, y°) if $\hat\theta[F](x^\circ, y^\circ) = 0$.
(v) *regular* at (x°, y°) if $\hat\theta[F](x^\circ, y^\circ) > 0$.

Remark 14. *The constants (25)–(27) are nonnegative. They characterize the* covering property [7] *(or its absence when the corresponding constant is zero) of F either at the point (x°, y°) (constants (25), (26)) or in its neighborhood (constant (27)). It is well known (see e.g.* [25, 33]*) that the covering in a neighborhood (or* linear openness*) is equivalent to the metric (or pseudo) regularity property* [11, 13, 14, 30] *(and to the Aubin property* [2, 33] *of the inverse mapping). Thus the regularity in part (v) of Definition 3 is actually the metric regularity. Besides the "covering" collection of constants (25)–(27) one can define the corresponding "regularity" one in a similar way.*

The relations between (25)–(27) and (1)–(3) are given by the following theorem.

Theorem 2. *Define $\Omega_1 = \operatorname{gph}(F)$, $\Omega_2 = X \times \{y^\circ\}$. The following assertions hold.*
(i) $\theta_\rho[\Omega_1, \Omega_2](x^\circ, y^\circ) \le \min(\theta_\rho[F](x^\circ, y^\circ)/2, \rho) \le \theta_{2\rho}[\Omega_1, \Omega_2](x^\circ, y^\circ)$ *for any $\rho > 0$.*
(ii) $\theta[\Omega_1, \Omega_2](x^\circ, y^\circ) \le \min(\theta[F](x^\circ, y^\circ)/2, 1) \le 2\theta[\Omega_1, \Omega_2](x^\circ, y^\circ)$.
(iii) $\hat\theta[\Omega_1, \Omega_2](x^\circ, y^\circ) \le \min(\hat\theta[F](x^\circ, y^\circ)/2, 1) \le 2\hat\theta[\Omega_1, \Omega_2](x^\circ, y^\circ)$.

Proof. (i) Let $\rho > 0$ and $0 \le r < \theta_\rho[\Omega_1, \Omega_2](x^\circ, y^\circ)$. Due to (1) this means that

$$(\Omega_1 - (a_1, b_1)) \cap (\Omega_2 - (a_2, b_2)) \cap B_\rho(x^\circ, y^\circ) \ne \emptyset$$

for any $a_1, b_1, a_2, b_2 \in B_r$. Since

$$\Omega_2 - (a_2, b_2) = X \times \{y^\circ - b_2\},$$

the last condition implies that for any $b_1, b_2 \in B_r$ one has $y^\circ - b_2 \in B_\rho(y^\circ)$ and

$$F^{-1}(y^\circ + b_1 - b_2) \cap B_\rho(x^\circ) \ne \emptyset.$$

The first inclusion immediately yields $r \le \rho$ while the second condition leads to the relation

$$B_{2r}(y^\circ) \subset F(B_\rho(x^\circ)),$$

which due to (25) means that $\theta_\rho[F](x^\circ, y^\circ) \ge 2r$. This proves the first inequality in (i).

Let $0 \le r \le \rho$ and $2r < \theta_\rho[F](x^\circ, y^\circ)$, i.e. $B_{2r}(y^\circ) \subset F(B_\rho(x^\circ))$. Then

$$B_{2r}(y^\circ) \subset F(B_{2\rho}(x^\circ + a_1))$$

for any $a_1 \in B_r$. Consequently

$$F^{-1}(y^\circ + b_1 - b_2) \cap B_{2\rho}(x^\circ + a_1) \ne \emptyset$$

for any $a_1, b_1, b_2 \in B_r$. The last condition can be rewritten as

$$\Omega_1 \cap (\Omega_2 - (0, b_2 - b_1)) \cap B_{2\rho}(x^\circ + a_1) \times \{y^\circ + b_1 - b_2\} \ne \emptyset$$

or

$$(\Omega_1 - (a_1, b_1)) \cap (\Omega_2 - (a_2, b_2)) \cap B_{2\rho}(x^\circ) \times \{y^\circ - b_2\} \ne \emptyset$$

for any $a_1, b_1, a_2, b_2 \in B_r$. This means that $\theta_{2\rho}[\Omega_1, \Omega_2](x^\circ, y^\circ) \geq r$, which proves the second inequality in (i).

Inequalities (ii) and (iii) follow from (i). □

Corollary 2.1. *Let Ω_1, Ω_2 be as in Theorem 2. The following assertions hold.*

(i) *F is extremal at $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ is extremal at (x°, y°).*
(ii) *F is locally extremal at $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ is locally extremal at (x°, y°).*
(iii) *F is stationary at $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ is stationary at (x°, y°).*
(iv) *F is weakly stationary at $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ is weakly stationary at (x°, y°).*
(v) *F is regular at $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ is regular at (x°, y°).*

Thus, all the properties of the multifunction F defined in Definition 3 can be deduced from the corresponding properties of the set system. The contrary is also true and the concepts in Definitions 1 and 3 are in a sense equivalent.

Let us consider again a set system $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ $(n > 1)$ with $x^\circ \in \cap_{i=1}^n \Omega_i$.

Theorem 3. *Define $F : X \rightrightarrows X^n$: $F(x) = (\Omega_1 - x) \times (\Omega_2 - x) \times \ldots \times (\Omega_n - x)$, $x \in X$. The following assertions hold.*

(i) *$\theta_\rho[\Omega_1, \ldots \Omega_n](x^\circ) = \theta_\rho[F](x^\circ, 0, \ldots, 0)$ for any $\rho > 0$.*
(ii) *$\theta[\Omega_1, \ldots \Omega_n](x^\circ) = \theta[F](x^\circ, 0, \ldots, 0)$.*
(iii) *$\hat{\theta}[\Omega_1, \ldots \Omega_n](x^\circ) = \hat{\theta}[F](x^\circ, 0, \ldots, 0)$.*

Proof. Due to the definition of F condition $x^\circ \in \cap_{i=1}^n \Omega_i$ is equivalent to the inclusion $(0, \ldots, 0) \in F(x^\circ)$, and condition

$$B_r(0, \ldots, 0) \subset F(B_\rho(x^\circ))$$

means that for any $a_i \in B_r$, $i = 1, 2, \ldots, n$ there exists an $x \in B_\rho(x^\circ)$ such that $a_i \in \Omega_i - x$, $i = 1, 2, \ldots, n$. This is equivalent to

$$\Big(\bigcap_{i=1}^n (\Omega_i - a_i)\Big) \bigcap B_\rho(x^\circ) \neq \emptyset.$$

Due to the definitions (25) and (1) this proves (i). (ii) and (iii) follow immediately. □

Corollary 3.1. *Let $F : X \rightrightarrows X^n$ be as in Theorem 3. The following assertions hold.*

(i) *$\{\Omega_1, \ldots \Omega_n\}$ is extremal at $x^\circ \Leftrightarrow F$ is extremal at $(x^\circ, 0, \ldots, 0)$.*
(ii) *$\{\Omega_1, \ldots \Omega_n\}$ is locally extremal at $x^\circ \Leftrightarrow F$ is locally extremal at $(x^\circ, 0, \ldots, 0)$.*
(iii) *$\{\Omega_1, \ldots \Omega_n\}$ is stationary at $x^\circ \Leftrightarrow F$ is stationary at $(x^\circ, 0, \ldots, 0)$.*
(iv) *$\{\Omega_1, \ldots \Omega_n\}$ is weakly stationary at $x^\circ \Leftrightarrow F$ is weakly stationary at $(x^\circ, 0, \ldots, 0)$.*
(v) *$\{\Omega_1, \ldots \Omega_n\}$ is regular at $x^\circ \Leftrightarrow F$ is regular at $(x^\circ, 0, \ldots, 0)$.*

Remark 15. *The multifunction F defined in Theorem 3 was used in* [13] *and other papers when investigating properties of set systems. Due to Proposition 22 (ii) and Corollaries 1.1 and 3.1 the "strong metric inequality" (18) is equivalent to the regularity of F at $(x^\circ, 0, \ldots, 0)$. It is not difficult to show that the metric inequality (17) is equivalent to the inverse mapping F^{-1} being* Lipschitz upper semicontinuous [32] *at* 0.

9. Dual Criteria

The stationarity and regularity properties of set systems were defined above in terms of primal space elements. They admit some dual characterizations in terms of "normal" elements.

Let X^* denote the space (topologically) dual to X and $\langle\cdot,\cdot\rangle$ be the bilinear form defining duality between X and X^*.

Recall that the *(Fréchet) normal cone* to a set Ω at $x^\circ \in \Omega$ is defined as

$$N(x^\circ|\Omega) = \left\{ x^* \in X^* :\ \limsup_{x \xrightarrow{\Omega} x^\circ} \frac{\langle x^*, x - x^\circ\rangle}{\|x - x^\circ\|} \leq 0 \right\}. \tag{28}$$

This convex cone is a natural (and one of the simplest) generalization of the normal cone in the sense of convex analysis.

Let us define one more constant for the system of closed sets $\Omega_1, \Omega_2, \ldots, \Omega_n$:

$$\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = \lim_{\delta \to +0} \inf \left\{ \left(\left\| \sum_{i=1}^n x_i^* \right\| / \sum_{i=1}^n \|x_i^*\| \right)_\infty : \right.$$
$$\left. x_i^* \in N(x_i|\Omega_i),\ x_i \in \Omega_i \cap B_\delta(x^\circ),\ i = 1, \ldots, n \right\}. \tag{29}$$

One more "extended" division operation $(\cdot,\cdot)_\infty$ is used here. It differs from the $(\cdot,\cdot)_\circ$ operation, which was used in (15), (16), in the fourth rule definition: 4) $(0/0)_\infty = \infty$. This allows us to exclude the case $x_1^* = x_2^* = \cdots = x_n^* = 0$ when calculating (29). If this is the only case ($x^\circ \in \text{int} \cap_{i=1}^n \Omega_i$) one automatically gets $\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = \infty$.

Note that the normal cones are "calculated" in (29) not at x° but at the points from its neighborhood the size of which then tends to zero. This is an example of a "fuzzy" definition.

The definition (29) can be simplified a little if to make use of the *strict δ-normal cone* [16, 17, 20] ($\delta \geq 0$) to a closed set Ω at $x^\circ \in \Omega$:

$$\hat{N}_\delta(x^\circ|\Omega) = \bigcup \{N(x|\Omega) :\ x \in \Omega \cap B_\delta(x^\circ)\}. \tag{30}$$

(29) takes the following form:

$$\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = \lim_{\delta \to +0} \inf \left\{ \left(\left\| \sum_{i=1}^n x_i^* \right\| / \sum_{i=1}^n \|x_i^*\| \right)_\infty : \right.$$
$$\left. x_i^* \in \hat{N}_\delta(x^\circ|\Omega_i),\ i = 1, \ldots, n, \right\}. \tag{31}$$

Obviously $N(x^\circ|\Omega) \subset \hat{N}_\delta(x^\circ|\Omega)$ for any $\delta \geq 0$. Contrary to (28) the cone (30) is nonconvex in general.

One can use also the *limiting normal cone* [22, 23] based on (28), (30):

$$\bar{N}(x^\circ|\,\Omega) \;=\; \bigcap_{\delta>0} \mathrm{cl}^*\hat{N}_\delta(x^\circ|\,\Omega), \tag{32}$$

where the symbol cl^* denotes weak* sequential closure of a set (a collection of the limits of all weakly* convergent sequences of elements of this set) in the dual space. In other words, $x^* \in \bar{N}(x^\circ|\Omega) \Leftrightarrow$ there exist sequences $\{x_k\} \subset \Omega$, $\{x_k^*\} \subset X^*$ such that $x_k^* \in N(x_k|\Omega)$, $k = 1, 2, \ldots$, and $x_k \to x^\circ$, $x_k^* \stackrel{w^*}{\to} x^*$ when $k \to \infty$. The analog of (29), (31) is defined as

$$\bar{\eta}[\Omega_1, \ldots, \Omega_n](x^\circ) = \inf\left\{ \left(\left\| \sum_{i=1}^n x_i^* \right\| / \sum_{i=1}^n \|x_i^*\| \right)_\infty : \right. \\ \left. x_i^* \in \bar{N}(x^\circ|\Omega_i),\ i = 1, \ldots, n, \right\}. \tag{33}$$

Remark 16. *Of course,* $\bar{\eta}[\Omega_1, \ldots, \Omega_n](x^\circ) = \eta[\Omega_1, \ldots, \Omega_n](x^\circ)$ *if* $\dim X < \infty$*. To be able to apply (32) and (33) in infinite dimensions one needs to impose additional compactness-type assumptions guaranteeing nontriviality of the limits in the weak* topology (see.* [27]*).*

The condition $\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$ plays a crucial role when characterizing weak stationarity of the set system.

(29) can be rewritten as

$$\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = \lim_{\delta\to+0} \inf\left\{ \left\| \sum_{i=1}^n x_i^* \right\| : \right. \\ \left. x_i^* \in \hat{N}_\delta(x^\circ|\Omega_i),\ i = 1, \ldots, n,\ \sum_{i=1}^n \|x_i^*\| = 1 \right\}. \tag{34}$$

Proposition 32. $\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$ *if and only if for any* $\delta > 0$ *there exist elements*

$$\omega_i \in \Omega_i \cap B_\delta(x^\circ),\ x_i^* \in N(\omega_i|\Omega_i),\ i = 1, 2, \ldots, n,$$

such that

$$\sum_{i=1}^n \|x_i^*\| = 1, \quad \left\| \sum_{i=1}^n x_i^* \right\| < \delta.$$

Remark 17. *Adopting the terminology from* [15, 18, 19, 22] *(based on* [8]*) the condition* $\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$ *(and its representation in Proposition 32) can be addressed to as the* generalized Euler equation*. It can be considered as (some kind of) a* separation property *for nonconvex sets. Note that it is formulated in a "fuzzy" form.*

The next theorem proved in [21] gives the relations between (31) and (3) and dual conditions of weak stationarity and regularity of set systems.

Theorem 4. *The following inequality holds:*

$$\hat{\theta}[\Omega_1,\ldots,\Omega_n](x^\circ) \le \eta[\Omega_1,\ldots,\Omega_n](x^\circ). \tag{35}$$

If X is Asplund and $\hat{\theta}[\Omega_1,\ldots,\Omega_n](x^\circ) < 1$ then

$$\eta[\Omega_1,\ldots,\Omega_n](x^\circ) \le \frac{\hat{\theta}[\Omega_1,\ldots,\Omega_n](x^\circ)}{1-\hat{\theta}[\Omega_1,\ldots,\Omega_n](x^\circ)}. \tag{36}$$

The first part of the theorem is elementary. The proof of the second part (the Asplund space case) is based on the application of the two fundamental results of variational analysis: the *Ekeland variational principle* [9] and the *sum rule* due to M. Fabian [10]. Recall that a Banach space is called Asplund (see [31, 36]) if any continuous convex function on it is Fréchet differentiable on a dense set of points. Asplund spaces form a rather broad subclass of Banach spaces. It contains e.g. all reflexive spaces and all spaces that admit equivalent norms, Fréchet differentiable at all nonnull points.

Corollary 4.1. *If $\eta[\Omega_1,\ldots,\Omega_n](x^\circ) = 0$ then the system of sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ is weakly stationary at x°.*

If X is Asplund then the Extended extremal principle *is true:*

(i) *The system $\Omega_1, \Omega_2, \ldots, \Omega_n$ is weakly stationary at x° if and only if $\eta[\Omega_1,\ldots,\Omega_n](x^\circ) = 0$.*

Remark 18. *Due to Proposition 1 it follows from the second part of Corollary 4.1 that in the Asplund space setting the equality $\eta[\Omega_1,\ldots,\Omega_n](x^\circ) = 0$ (the generalized Euler equation) is a necessary condition of local extremality of the set system. This result first proved in* [22] *(in a slightly weaker form) is currently known as the* Extremal principle [24, 26]*. The stronger condition (i) in Corollary 4.1 is called the* Extended extremal principle [20]*. Taking into account the extremal characterizations of Asplund spaces in* [26] *one can conclude that asplundity of the space is not only sufficient but also necessary for the Extended extremal principle to be valid (see* [20]*).*

Theorem 5. *The following assertions are equivalent:*

(i) *X is an Asplund space.*
(ii) *The Extremal principle is valid in X.*
(iii) *The Extended extremal principle is valid in X.*

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References

[1] Asimov, L., and Ellis, A. J. *Convexity Theory and Applications in Functional Analysis.* London Mathematical Society Monographs No. 16. Academic Press, London, 1980.

[2] Aubin, J.-P., and Ekeland, I. *Applied Nonlinear Analysis.* Wiley, New York, 1984.

[3] Bernard, F., and Thibault, L. Prox-regularity of functions and sets in Banach spaces. *Set-Valued Analysis 12* (2004), 25–47.

[4] Borwein, J. M., and Jofré, A. A nonconvex separation property in Banach spaces. *Math. Meth. Oper. Res. 48* (1998), 169–179.

[5] Bounkhel, M., and Thibault, L. On various notions of regularity of sets in nonsmooth analysis. *Nonlin. Anal. 48* (2002), 223–246.

[6] Clarke, F. H. *Optimization and Nonsmooth Analysis*. Wiley, New York, 1983.

[7] Dmitruk, A. V., Milyutin, A. A., and Osmolovsky, N. P. Lyusternik's theorem and the theory of extrema. *Russ. Math. Surv. 35* (1980), 11–51.

[8] Dubovitskii, A. Y., and Milyutin, A. A. Extremum problems in the presence of restrictions. *U.S.S.R. Comp. Maths. Math. Phys. 5* (1965), 1–80.

[9] Ekeland, I. On the variational principle. *J. Math. Anal. Appl. 47* (1974), 324–353.

[10] Fabian, M. Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss. *Acta Univ. Carolinae 30* (1989), 51–56.

[11] Ioffe, A. D. Regular points of Lipschitz functions. *Trans. Amer. Math. Soc. 251* (1979), 61–69.

[12] Ioffe, A. D. Approximate subdifferentials and applications. 3: the metric theory. *Mathematika 36* (1989), 1–38.

[13] Ioffe, A. D. Metric regularity and subdifferential calculus. *Rus. Math. Surveys 55* (2000), 501–558.

[14] Klatte, D., and Kummer, B. *Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications*, vol. 60 of *Nonconvex Optimization and Its Applications*. Kluwer, Dordrecht-Boston-London, 2002.

[15] Kruger, A. Y. Generalized differentials of nonsmooth functions and necessary conditions for an extremum. *Siberian Math. J. 26* (1985), 370–379.

[16] Kruger, A. Y. On calculus of strict ε-semidifferentials. *Dokl. Akad. Nauk Belarus 40*, 4 (1996), 34–39. in Russian.

[17] Kruger, A. Y. Strict ε-semidifferentials and extremality conditions. *Dokl. Akad. Nauk Belarus 41*, 3 (1997), 21–26. in Russian.

[18] Kruger, A. Y. On extremality of sets systems. *Dokl. Nat. Akad. Nauk Belarus 42*, 1 (1998), 24–28. in Russian.

[19] Kruger, A. Y. Strict (ε, δ)-semidifferentials and extremality conditions. *Optimization 51* (2002), 539–554.

[20] Kruger, A. Y. On Fréchet subdifferentials. *J. Math. Sci. 116* (2003), 3325–3358.

[21] Kruger, A. Y. Weak stationarity: eliminating the gap between necessary and sufficient conditions. *Optimization 53* (2004), 147–164.

[22] Kruger, A. Y., and Mordukhovich, B. S. Extremal points and the Euler equation in nonsmooth optimization. *Dokl. Akad. Nauk BSSR 24*, 8 (1980), 684–687. in Russian.

[23] Mordukhovich, B. S. *Approximation Methods in Problems of Optimization and Control*. Nauka, Moscow, 1988. in Russian.

[24] Mordukhovich, B. S. The extremal principle and its applications to optimization and economics. In *Optimization and Related Topics*, A. Rubinov and B. Glover, Eds., Applied Optimization, Vol. 47. Kluwer, 2001, pp. 343–369.

[25] Mordukhovich, B. S., and Shao, Y. Differential characterizations of covering, metric regularity, and Lipschitzian properties of multifunctions between Banach spaces. *Nonlin. Anal. 25* (1995), 1401–1424.

[26] Mordukhovich, B. S., and Shao, Y. Extremal characterizations of Asplund spaces. *Proc. Amer. Math. Soc. 124* (1996), 197–205.

[27] Mordukhovich, B. S., and Shao, Y. Nonsmooth sequential analysis in Asplund spaces. *Trans. Amer. Math. Soc. 348* (1996), 1235–1280.

[28] Ng, K. F., and Yang, W. H. Regularities and their relations to error bounds. *Math. Program., Ser. A 99* (2004), 521–538.

[29] Ngai, H., and Théra, M. Metric inequality, subdifferential calculus and applications. *Set-Valued Analysis 9* (2001), 187–216.

[30] Penot, J.-P. Metric regularity, openess and Lipschitz behavior of multifunctions. *Nonlin. Anal.: Theory, Meth. Appl. 13* (1989), 629–643.

[31] Phelps, R. R. *Convex Functions, Monotone Operators and Differentiability, 2nd edition.* Lecture Notes in Mathematics, Vol. 1364. Springer, New York, 1993.

[32] Robinson, S. M. Some continuity properties of polyhedral multifunctions. *Math. Progr. Study 14* (1981), 206–214.

[33] Rockafellar, R. T., and Wets, R. J.-B. *Variational Analysis.* Springer, New York, 1998.

[34] Rubinov, A. M. *Abstract Convexity and Global Optimization.* Kluwer, 2000.

[35] Vulikh, B. C. *Introduction to the Theory of Partially Ordered Vector Spaces.* Wolters-Noordhoff, Groningen, 1967.

[36] Yost, D. Asplund spaces for beginners. *Acta Univ. Carol., Math. Phys. 34*, 2 (1993), 159–177.

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