Robust Capacity Expansion of Transit Networks*

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Abstract

In this paper we present a methodology to decide capacity expansions for a transit network that finds a robust solution with respect to the uncertainty in demands and travel times. We show that solving for a robust solution is a computationally tractable problem under conditions that are reasonable for a transportation system. For example, the robust problem is tractable for a multicommodity flow problem with a single source and sink per commodity and uncertain demand and travel time represented by bounded convex sets. Preliminary computational results show that the robust solution can reduce the worst case cost by more than 20%, while incurring on a 5% loss in optimality when compared to the optimal solution of a representative scenario.

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1 Introduction

Sizable investments in highway infrastructure are constantly undertaken in order to alleviate congestion and accommodate the increase in traffic demand across the country. For example, an estimated $2.4 billion in highway infrastructure projects are planned just for District 7 (Los Angeles and Ventura Counties) as part of California's Transportation Congestion Relief Plan (TCRP), (Caltrans 2002). Traffic congestion in the United States has a significant economic impact, costing an estimated $69 billion during 2001 (Schrank and Lomax 2003); in addition the demand for vehicular traffic in the country is expected to increase steadily, exceeding 4 trillion vehicle-miles by the year 2020 (Department of Energy 2004). Which projects are executed and in what order can significantly affect how efficiently planners cope and address traffic congestion. Therefore it is essential to develop quantitative methods to aid in planning the capacity expansion of transit networks.

There exists substantial research on capacity expansion (or capacity planning) problems in different domains, such as manufacturing (Eppen et al. 1989, Barahona et al. 2004, Zhang et al. 2004), electric utilities (Murphy and Weiss 1990, Malcolm and Zenios 1994), telecommunications (Balakrishnan et al. 1995, Laguna 1998, Riis and Andersen 2004), inventory management (Hsu 2002), and transportation (Magnanti and Wong 1984, Minoux 1989). This diverse body of work includes some common elements: (1) uncertainty in the problem data is considered and (2) in general terms the problems can be translated to expanding the capacity of a network flow problem. Uncertainty in capacity expansion problems can be traced back to Ferguson and Dantzig (1956). A standard method to represent the uncertainty in the problem is via discrete uncertainty scenarios, an approach that is used in stochastic optimization (Birge and Louveaux 1997), see (Chen et al. 2002, Ahmed et al. 2003) for examples in capacity planning, and in robust optimization as introduced by Mulvey et al. (1995), see (Malcolm and Zenios 1994, Laguna 1998) for examples in capacity planning. Other methods for robust optimization with scenario uncertainty are described in (Gutiérrez et al. 1996) for uncapacitated network design, in (Ferris and Ruszczynski 2000) for routing in a network with failures, and in (Paraskevopoulos et al. 1991) for a manufacturing application. A drawback of uncertainty scenario based methods is the need to solve prohibitively large optimization problems as the number of uncertainty scenarios increases. Methods that can address uncertainty represented by continuous intervals have been developed by exploiting the underlying network flow problem. In particular, when representing the capacity expansions with integer variables the network flow problem is referred to as a network design problem (Minoux 1989). For such integer problems Kouvelis and Yu (1997) introduce robust discrete optimization, which is applicable to a number of discrete network problems. Other combinatorial methods for network design problems with interval uncertainty in objective function coefficients have been investigated in (Averbakh and Berman 2000, Yaman et al. 2001). These approaches to interval uncertainty either can lead to NP-hard formulations for problems whose deterministic version is polynomially solvable or use solution techniques that rely on combinatorial arguments specific to the problem in question, which are not easily generalizable.

In this paper we present a method that finds a solution that is robust (as introduced by Ben-Tal and Nemirovski (1998)) with respect to uncertainty in travel times and demands, and thus can consider the uncertainty due to future value forecasts. We investigate the complexity of this robust optimization approach and find that, as opposed to previous methods to deal with uncertainty in capacity planning, this method is polynomially solvable under reasonable assumptions for transportation networks and very general uncertainty sets. Although we embed our discussion and presentation within a transportation application, the methodology used is general and exploits the underlying network flow problem. Thus the results are applicable in principle to other capacity planning applications. In addition, we present computational results that compare the robust solution to the deterministic optimal solution for nominal data, i.e. data that is representative of the uncertainty set. We find that the robust solution can be significantly better in the worst case and only slightly worse for the nominal data, than the deterministic optimal solution.

The structure of the paper is as follows: in the remainder of the introduction section we describe the robust optimization approach as it pertains to our problem. In Section 2 we present the robust capacity expansion problem and investigate the reason for the difficulty of this problem. In Section 3 we identify the conditions for solvability of the robust capacity expansion problem. We present our computational results in Section 4, and present our concluding remarks in Section 5.

1.1 Robust Optimization methodology

The robust optimization approach was introduced in Ben-Tal and Nemirovski (1998) for convex optimization and in El-Ghaoui et al. (1998) for semidefinite programming. This approach has lead to work on structural design (Ben-Tal and Nemirovski 1997), least-square optimization (El-Ghaoui and Lebret 1997), portfolio optimization problems (Goldfarb and Iyengar 2003, El-Ghaoui et al. 2003), integer programming and network flows (Bertsimas and Sim 2003), and recently supply chain management problems (Bertsimas and Thiele 2003, Ben-Tal et al. 2003a). In particular, the work by Bertsimas and Sim (2003) considers robust solutions for network flow problems with box uncertainty in the cost coefficients.

The robust solution is defined as the solution that achieves the best worst case objective function value. Consider the following optimization problem under uncertainty:

$$\begin{aligned} \min_{u,v} \quad & f(u,v,w) \\ \text{s.t.} \quad & g(u,v,w) \le 0 \ , \end{aligned}$$

where the uncertainty parameter w belongs to a closed bounded and convex uncertainty set $w \in \mathcal{U}$. The robust solution is obtained by solving the following robust counterpart problem (RC):

$$\begin{aligned} z_{RC} = \quad \min_{u,v,\gamma} \quad & \gamma \\ \text{s.t.} \quad & f(u,v,w) \leq \gamma \ \text{ for all } w \in \mathcal{U} \\ & g(u,v,w) \leq 0 \ \text{ for all } w \in \mathcal{U} \ . \end{aligned} \tag{1}$$

An attractive feature of this approach is that the complexity of solving problem (RC) is, for very general cases, the same as the complexity of the original problem. For example, when the original problem is an LP, Ben-Tal and Nemirovski (1999) shows that Problem (1) above is equivalent to an LP when $\mathcal{U}$ is a polyhedron and to a quadratically constrained convex program when $\mathcal{U}$ is a bounded ellipsoidal set. In addition, the size of the resulting problem (RC) is bounded by a polynomial in the original problem's dimensions, which implies a polynomial method for the robust solution.

The robust counterpart for a stochastic problem with recourse, dubbed the adjusted robust counterpart problem (ARC), is introduced in Ben-Tal et al. (2003b). In a problem with recourse, some of the decision variables u are decided a priori, while the rest v can adjust to the outcome of the uncertainty, which yields the following (ARC) problem:

$$\begin{aligned} z_{ARC} = \quad \min_{u,\gamma} \quad & \gamma \\ \text{s.t.} \quad & \text{for all } w \in \mathcal{U} \text{ exists } v \ : \ \begin{cases} f(u,v,w) \leq \gamma \\ g(u,v,w) \leq 0 \ . \end{cases} \end{aligned} \tag{2}$$

Clearly $z_{ARC} \leq z_{RC}$, since selecting one v that is feasible for all $w \in \mathcal{U}$ is a possibility for (ARC). However we do not retain the nice complexity results, as Theorem 3.5 of (Guslitser 2002) shows that the (ARC) problem of an LP with polyhedral uncertainty is NP-hard.

2 The Robust Capacity Expansion Problem

We represent the transportation network using a classic network flow formulation where the system routes the flow to minimize a global measure such as total travel time. We consider a directed network with n nodes and m arcs, represented by an arc-incidence matrix $N \in \Re^{m \times n}$, a vector $u \in \Re^m$ of current arc capacities, and we denote by the variable $x \in \Re^m$ the traffic flow on the system. The demand and supply of this transportation problem are represented by a vector $b \in \Re^n$, and we assume linear transportation costs represented by the non-negative cost vector $c \in \Re^m$. Let capacity expansions on this network be a continuous decision variable $y \in \Re^m$, where expanding arc e incurs in a d_e cost per unit capacity. Given a total budget for investment I, we formulate the deterministic capacity expansion problem, for a given demand b and cost c, as

$$\begin{aligned} z_D(b,c) = \quad \min_{x,y} \quad & c^t x \\ \text{s.t.} \quad & Nx = b \\ & x \leq u + y \\ & d^t y \leq I \\ & x,\ y \geq 0\ . \end{aligned} \tag{3}$$

This basic network flow model can be enhanced, using multiple time periods, non-linear latency cost functions, etc., to be more representative of transportation networks, see Ahuja et al. (1993) for a detailed description of network flows and different modeling alternatives. In this work we concentrate on Problem (3) above and its robust counterpart.

We represent the uncertainty in demand and travel times simply using closed, bounded convex uncertainty sets, i.e. we assume that the demand vector $b \in \mathcal{U}_b$ and that the cost vector $c \in \mathcal{U}_c$. We assume also that the uncertainty in travel times does not create a negative cost arc, i.e. if $c \in \mathcal{U}_c$ then $c_e \geq 0$ for all arcs e. Given uncertainties in b and c, it is natural to separate the decision variables by deciding investment variables y prior to observing the traffic conditions (realizations of b and c), and have the traffic flow x adapt to these conditions minimizing the total travel time. Thus, the problem under uncertainty is a stochastic problem with recourse and the robust capacity expansion problem (RCEP) is the (ARC) Problem (2). Substituting the capacity expansion, Problem (3) in Problem (2) yields the following (RCEP):

$$
\begin{aligned}
z_{ARC} = \quad & \min_{y,\gamma} \quad \gamma \\
& \text{s.t.} \quad d^t y \le I \\
& \qquad\;\; y \ge 0 \\
& \qquad\;\; \text{for all } c \in \mathcal{U}_c,\ b \in \mathcal{U}_b \text{ exists } x \ : \quad \begin{cases} Nx = b \\ 0 \le x \le u + y \\ c^t x \le \gamma \ . \end{cases}
\end{aligned}
\tag{4}
$$

Proposition 1 *The (RCEP) Problem (4) is equivalent to Problem (5) below, in that both problems have the same optimal solution* y^* *and* $z_{ARC} = z_R$*.*

$$
\begin{array}{llllll}
z_R = & \min_y & \max_{c,b} & \min_x & c^t x & \\
& d^t y \le I & c \in \mathcal{U}_c & \text{s.t.} & Nx = b & \\
& y \ge 0 & b \in \mathcal{U}_b & & 0 \le x \le u + y \ . &
\end{array}
\tag{5}
$$

Proof: If $z_{ARC} = \infty$ then for any γ and $y \ge 0$, $d^t y \le I$ there are $b \in \mathcal{U}_b$ and $c \in \mathcal{U}_c$ such that the system $Nx = b,\ 0 \le x \le u + y,\ c^t x \le \gamma$ is infeasible. Therefore, the problem $\min_x\{c^t x \mid Nx = b,\ 0 \le x \le u + y\} > \gamma$, which implies that the objective function of (5) is greater than γ for this y. As $\gamma \to \infty$ this implies $z_R = \infty$. If $z_{ARC} < \infty$, let (y, γ) be a feasible for (4), therefore for every $b \in \mathcal{U}_b$ and $c \in \mathcal{U}_c$ there exists $\bar{x}$ that satisfies $N\bar{x} = b,\ 0 \le \bar{x} \le u + y$, and $c^t \bar{x} \le \gamma$, which implies that the inner-most minimization in (5) is no greater than γ for any c and b, and consequently $z_{ARC} \ge z_R$. Note that $z_R \ge 0$, since $c, x \ge 0$. Let y be feasible for (5) such that its objective function is less than $z_R + \varepsilon$, for some $\varepsilon > 0$. Therefore, for any $b \in \mathcal{U}_b$ and $c \in \mathcal{U}_c$ there exists $\bar{x}$ such that $N\bar{x} = b,\ 0 \le \bar{x} \le u + y$, and $c^t \bar{x} < z_R + \varepsilon$. If $\gamma = z_R + \varepsilon$ then (y, γ) is feasible for (4). Thus $z_{ARC} < \gamma = z_R + \varepsilon$ for any $\varepsilon > 0$, completing the proof. ■

2.1 General uncertainty sets

This robust optimization approach models the uncertainty with the simple assumption that the uncertainty sets $\mathcal{U}_c$ and $\mathcal{U}_b$ are closed, convex, and bounded. These are rather mild assumptions, as they do require a particular uncertainty distribution within the set and such uncertainty sets can represent the confidence intervals with which these uncertain quantities have been estimated, including known dependencies between the uncertain parameters. Throughout the paper we make the following additional assumption on the uncertainty sets

Assumption 1 *For every* $b \in \mathcal{U}_b$ *and* $c \in \mathcal{U}_c$*, the network flow problem* $\min\{\ c^t x \mid Nx = b,\ 0 \le x \le u\ \}$ *is feasible.*

This is a reasonable assumption for a transportation system, since for any demand and cost vectors there always exists a solution that routes the traffic in the system, and there are no unbounded solutions as all arcs have a finite capacity. Note that since $c \ge 0$ for all $c \in \mathcal{U}_c$, the network flow problem is also bounded, thus this LP has an optimal solution.

We now present a few characterizations of closed convex sets for which inclusions can be evaluated efficiently. We describe polyhedral and ellipsoidal sets in $\Re^k$. A polyhedral set in $\Re^k$ formed by the intersection of m hyperplanes is given by $\mathcal{U} = \{x \mid Mx \le g\}$, where M is a $m \times k$ matrix and $g \in \Re^m$. Ellipsoidal uncertainty sets are given by $\mathcal{U} = \{x \mid x = x^0 + \sum_{l=1}^{L} \xi_l x^l,\ \xi \in \mathcal{X}\}$, with $\mathcal{X} = \{\xi \mid \exists w, A\xi + Bw \ge_K d\}$. Where the constraint $a \ge_K b$ represents the conic constraint $a - b \in K$, for some regular cone K.

Below we illustrate the generality of ellipsoidal sets with some examples. Let K^* denote the positive polar of cone K and denote by $e \in \Re^k$ the vector of all ones, $\Re^k_+$ the k dimensional positive orthant, and $\mathcal{L}^{k+1} = \{(x_1, \bar{x}) \in \Re^{k+1} \mid \bar{x} \in \Re^k,\ \|\bar{x}\|_2 \le x_1\}$ the $k+1$ dimensional second order cone, or Lorentz cone.

- $\mathcal{U}$ is an ellipse centered at x^0 with axes $x^1, \ldots, x^L$ if the set $\mathcal{X} = \{\xi \mid \|\xi\|_2 \le 1\}$, which is given by the conic constraints $\mathcal{X} = \{\xi \mid \exists w,\ (w,\xi) \in \mathcal{L}^{L+1},\ -w+1 \in \Re_+\}$.

- $\mathcal{U}$ is a box centered at x^0, with edges in directions $x^1, \ldots, x^L$ if the set $\mathcal{X} = \{\xi \mid \|\xi\|_\infty \leq 1\}$, given by the conic constraints $\mathcal{X} = \{\xi \mid -\xi + e \in \Re^L_+,\ \xi + e \in \Re^L_+\}$.
- $\mathcal{U}$ is the convex combination of discrete uncertainties $x^0, x^0 + x^1, \ldots, x^0 + x^L$ if the set $\mathcal{X} = \{\xi \mid \|\xi\|_1 \leq 1,\ \xi \geq 0\}$, given by the conic constraints $\mathcal{X} = \{\xi \mid \xi \in \Re^L_+,\ -e^t\xi + 1 \in \Re_+\}$.

2.2 Difficulty of solving (RCEP)

Solving (RCEP) seems to be a difficult problem, since it is an instance of the (ARC) Problem (2) which is NP-hard. We investigate whether the structure of the (RCEP) can guarantee a polynomial solution. We begin with defining the worst case cost of investment decision y by $\phi(y)$. Thus, problem (RCEP) is simply $\min\{\phi(y) \mid d^t y \leq I,\ y \geq 0\}$, and the worst case cost is given by

$$\begin{array}{rlll} \phi(y) = & \max_{c,b} & \min_x & c^t x \\ & c \in \mathcal{U}_c & \text{s.t.} & Nx = b \\ & b \in \mathcal{U}_b & & x \leq u + y \\ & & & x \geq 0\ . \end{array} \tag{6}$$

Theorem 1 *Under Assumption 1, $\phi(y)$ is a convex function in y.*

Proof: The assumption implies that the network flow in the inner most minimization problem is feasible and has an optimal solution. Therefore the dual of this LP attains the same objective value. Replacing the inner most minimization problem by its dual yields the following expression for $\phi(y)$:

$$\begin{array}{rlll} \phi(y) = & \max_{c,b} & \max_{\lambda,\pi} & b^t\lambda - (u+y)^t\pi \\ & c \in \mathcal{U}_c & \text{s.t.} & N^t\lambda - \pi \leq c \\ & b \in \mathcal{U}_b & & \pi \geq 0\ , \end{array}$$

which combining maximizations becomes

$$\begin{aligned} \phi(y) = \quad \max_{\lambda,\pi,b,c} \quad & b^t\lambda - (u+y)^t\pi \\ \text{s.t.} \quad & N^t\lambda - \pi \leq c \\ & b \in \mathcal{U}_b,\ c \in \mathcal{U}_c,\ \pi \geq 0\ . \end{aligned} \tag{7}$$

It is straight forward to show from this last expression that $\phi(y)$ is a convex function in y since the maximum of a sum is less than the sum of maximums. ■

Theorem 1 shows that the (RCEP) is the minimization of a convex function over a simplex, thus it can be NP-hard only when evaluating $\phi(y)$ cannot be done in polynomial time. The non-linear term $b^t\lambda$ in the objective of Problem (7) is the challenging aspect of this problem. For example, for deterministic demand, i.e. $\mathcal{U}_b = \{b\}$, the objective becomes linear and computing the value of $\phi(y)$, and thus solving (RCEP), can be done in polynomial time. We study this case in detail in the beginning of the next section.

The following examples, which consider uncertainty in the demand, illustrate that evaluating $\phi(y)$ can indeed be a difficult problem.

Example 1: Consider the network given in Figure 1, with fixed cost vector c and an investment y that yields the capacities on the figure. In this example the only uncertain parameter is the total amount of supply and demand at nodes 1 and 3. This demand and supply pair is parametrized by $\delta \in [-1, 1]$. The minimum cost flow for this example is exactly $4 + 3|\delta|$ and thus it is maximized for $\delta \in \{-1, 1\}$.

Example 2: Consider the network given in Figure 2, where again we have a fixed cost c and an investment y that yields the capacities on the figure. Now, the demands at nodes 2 and 3 are parametrized by $\delta \in [-1, 1]$. The minimum cost flow of this problem has an objective function value of $12+4|\delta|$ and thus it is also maximized for $\delta \in \{-1, 1\}$.

Both examples maximize a convex function to evaluate $\phi(y)$. Although these are simple one dimensional examples, they illustrate the potential difficulty in finding the

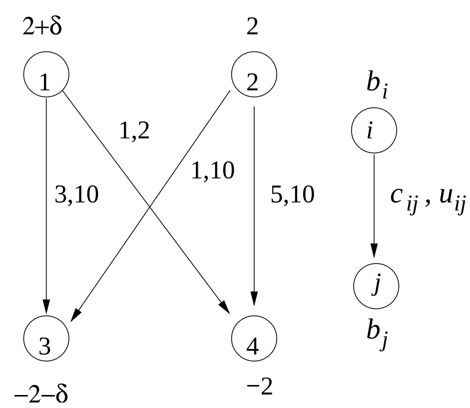

Figure 1: Difficult to evaluate $\phi(y)$. Multiple sources and sinks, $\delta \in [-1, 1]$.

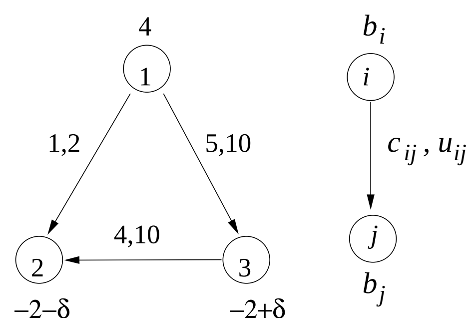

Figure 2: Difficult to evaluate $\phi(y)$. Multiple sink uncertainty, $\delta \in [-1, 1]$.

demand b that defines the worst case. Example 1 is based on what is known as the "more for less paradox": in increasing the supply in node 1 from 1 to 2 (and increasing the demand at node 3 accordingly), we actually reduce the total cost as we replace the expensive flow on $(2, 4)$ with cheaper flow on $(1, 4)$. This stops when the supply at 1 increases above 2 units, since then the arc $(1, 4)$ is saturated and the extra flow is sent on an expensive arc increasing the total cost. A similar phenomenon occurs in Example 2, where we use the capacity of the low cost arc to switch between decreasing the total cost to increasing it. Note that now the total flow sent remains constant.

3 Solving (RCEP)

We now present three cases in which we can obtain tractable solutions for the (RCEP). The first case is when there is fixed demand, in the second case we explore the most general conditions under which the general network flow with uncertain demand is tractable, we also note that these conditions are necessary. Our last case explores conditions under which the multicommodity flow problem with uncertain demand has a tractable (RCEP). It turns out that these conditions are reasonable for transportation systems.

3.1 Case of deterministic demand

In the case of deterministic demand, in other words $\mathcal{U}_b = \{b\}$, the set of feasible flows is fixed since the uncertainty only affects the cost vector. In this case, besides the (RCEP) obtained from (ARC), we can define the following standard robust problem (RC), as in

Problem (1), by also deciding the routing x prior to the realization of the uncertainty:

$$\begin{aligned} z_{RC} = \quad \min_{x,y,\gamma} \quad & \gamma \\ \text{s.t.} \quad & Nx = b \\ & x \leq u + y \\ & d^t y \leq I \\ & x, y \geq 0 \\ & \text{for all } c \in \mathcal{U}_c \;\; c^t x \leq \gamma \ . \end{aligned} \tag{8}$$

Recall that Problem (RC) is a tractable problem when the uncertainty set $\mathcal{U}_c$ is a polyhedral or ellipsoidal set, (Ben-Tal and Nemirovski 1999). In particular, the network flow problem ($I = 0$) with box uncertainty $\mathcal{U}_c = \{c \mid \underline{c} \leq c \leq \underline{c} + \bar{\delta}\}$, is considered in Bertsimas and Sim (2004), which shows that the robust counterpart is

$$\begin{aligned} \min_x \quad & (\underline{c} + \bar{\delta})^t x \\ \text{s.t.} \quad & Nx = b \\ & 0 \leq x \leq u \ . \end{aligned} \tag{9}$$

We now present results showing that Problem (RCEP) is tractable when demand is deterministic for general uncertainty sets $\mathcal{U}_c$. First, let $\mathcal{U}_c$ be a general polyhedral set.

Proposition 2 *If $\mathcal{U}_b = \{b\}$ and $\mathcal{U}_c = \{c \mid Mc \leq g, c \geq 0\}$, then (RCEP) is equivalent to*

$$\begin{aligned} \min_{y,x,w} \quad & g^t w \\ \text{s.t.} \quad & Nx = b \\ & x \leq u + y \\ & x \leq M^t w \\ & d^t y \leq I \\ & x, w, y \geq 0 \end{aligned}$$

Proof: Under Assumption 1 we can represent the worst case cost $\phi(y)$ by Problem (7), which becomes an LP when $\mathcal{U}_b$ is a singleton and $\mathcal{U}_c = \{c \mid Mc \leq g, c \geq 0\}$. Taking the dual of this LP and combining it with the outer minimization in the investment variables y, yields the result. ■

It is surprising to note that for the network flow problem, $I = 0$, with box uncertainty $\mathcal{U}_c = \{c \mid \underline{c} \leq c \leq \underline{c} + \bar{\delta}\}$, the above characterization of (RCEP) is exactly equivalent to the robust counterpart (RC) Problem (9) above. The following theorem shows that this is always true. Problem (RCEP) is equivalent to (RC) for any bounded $\mathcal{U}_c$ and deterministic demand. Therefore (RCEP) is tractable for general uncertainty sets $\mathcal{U}_c$ as it inherits the tractability results of the (RC) problem in this case.

Theorem 2 *If $\mathcal{U}_b = \{b\}$ then (RCEP) is equivalent to (RC)*

Proof: Similar to Proposition 1 we can show that (RC), defined in Problem (8), is equivalent to

$$\begin{array}{rll} z_{RC} = & \min_{y,x} & \max_c \ c^t x \\ & Nx = b & c \in \mathcal{U}_c \\ & x \leq u + y & \\ & d^t y \leq I & \\ & x, y \geq 0 \ . & \end{array}$$

Let $X(y) = \{x \geq 0 \mid Nx = b, \ x \leq u + y\}$. Since the function $f(x, c) = c^t x$ is both linear in x for every feasible c and linear in c for every feasible x, and the convex uncertainty set $\mathcal{U}_c$ is bounded, then a classic result in convex analysis states that for any feasible y

$$\min_{x \in X(y)} \max_{c \in \mathcal{U}_c} c^t x = \max_{c \in \mathcal{U}_c} \min_{x \in X(y)} c^t x \ ,$$

see for example Corollary 37.3.2 in Rockafellar (1997). Substituting this saddle point equivalence in the expression for (RC) above, we obtain Problem (5) with deterministic demand. ■

3.2 Uncertain demand

As pointed out in Examples 1 and 2, evaluating the worst case cost function $\phi(y)$ can be a difficult problem when there is demand uncertainty. We now identify the conditions on the uncertainty set $\mathcal{U}_b$ under which we can evaluate $\phi(y)$ efficiently. The key observation for the result presented below is that when routing flow from a single source to a single sink, the optimal routing sends the flow along the shortest path possible. If the total flow sent increases so does the value of the minimum cost solution, as the extra flow is routed along the shortest path with available capacity from source to sink. In conclusion, in this case there is no "more for less paradox".

A slightly broader case is to consider multiple sinks and a single source, or equivalently multiple sources and a single sink, with demand uncertainty only in a single source and sink pair. We describe the methodology in the case with multiple sinks and single source, and omit the analogous multiple source/single sink case. Let s be the single source, and assume a demand $\bar{b}$ with uncertainty in s and one fixed sink node $t \neq s$. This implies the following demand uncertainty set

$$\mathcal{U}_b = \{b \mid b = \bar{b} + \delta(e_s - e_t),\ \delta \in [0, \bar{\delta}]\}\ , \tag{10}$$

where $e_i \in \Re^m$ is the i-th canonical vector. Note that $\bar{b}_s \geq 0$ and $\bar{b}_i \leq 0$ for any $i \neq s$.

Theorem 3 *Consider a network flow problem with a single source s and an uncertainty set $\mathcal{U}_b$ given by Equation (10). Then $\phi(y)$ is the following convex optimization problem*

$$\begin{aligned} \phi(y) = \quad \max_{\lambda,\pi,c} \quad & \bar{b}^t\lambda + \bar{\delta}(\lambda_s - \lambda_t) - (u+y)^t\pi \\ \text{s.t.} \quad & N^t\lambda - \pi - c \leq 0 \\ & c \in \mathcal{U}_c \\ & \pi \geq 0\ . \end{aligned}$$

Proof: Under the uncertainty set $\mathcal{U}_b$ the definition of $\phi(y)$ becomes

$$\begin{array}{rlll} \phi(y) = & \max_c & \max_\delta & \min_x c^t x \\ & c \in \mathcal{U}_c & \delta \in [0, \bar{\delta}] & Nx = \bar{b} + \delta(e_s - e_t) \\ & & & 0 \le x \le u + y \ . \end{array}$$

The proof is based in showing that the function

$$\begin{array}{rl} \Gamma(\delta) = & \min_x c^t x \\ & Nx = \bar{b} + \delta(e_s - e_t) \\ & 0 \le x \le u + y \end{array}$$

is an non-decreasing function, i.e. if $\delta \le \delta'$ then $\Gamma(\delta) \le \Gamma(\delta')$. If $\Gamma(\delta)$ is non-decreasing it implies that $\phi(y) = \max_{c \in \mathcal{U}_c} \Gamma(\bar{\delta})$, and then simply taking the LP dual of $\Gamma(\bar{\delta})$ we obtain the result. From Assumption 1 we have that function $\Gamma(\delta)$ is finite for all $\delta \in [0, \bar{\delta}]$. Let x' be the optimal solution for $\Gamma(\delta')$. From the flow decomposition theorem (for example Theorem 3.5 in Ahuja et al. (1993)) we have that the arc flow x' can be represented by flow along $s - i$ paths P_{si}, and cycles. In particular, the total flow along the $s - t$ paths $P^1_{st}, \dots, P^k_{st}$ is equal to $\bar{b}_t + \delta'$. We can then remove a total of $\delta' - \delta \ge 0$ units of flow from these paths. Let $x_{\delta' - \delta}$ be the flow removed and recall that $c \ge 0$ for all $c \in \mathcal{U}_c$, then $\tilde{x} = x' - x_{\delta' - \delta}$ is a feasible flow for $\Gamma(\delta)$, which means that $\Gamma(\delta) \le c^t \tilde{x} = c^t x' - c^t x_{\delta' - \delta} \le c^t x' = \Gamma(\delta')$. ■

Remark 1 *The conditions (a) single source s and (b) uncertainty set $\mathcal{U}_b$ given by (10) are necessary and sufficient for $\phi(y)$ to be a convex optimization problem.*

Proof: The theorem proves the sufficiency of these conditions. For the necessity we show that if either of the conditions does not hold, then evaluating $\phi(y)$ is not equivalent to solving a convex problem. Example 1 considers a network with multiple sources and sinks and uncertainty only in a single source and sink pair, i.e. violates only condition (a). Example 2 considers a network with a single source and uncertainty among the sink nodes, i.e. only condition (b) does not hold. Both examples show that evaluating $\phi(y)$ amounts to maximizing a convex function, not a convex optimization problem. ■

We now present two corollaries that show that the robust capacity expansion problem (RCEP) is a tractable problem for a network under these demand uncertainty assumptions and fairly general cost uncertainty sets. We omit the proofs of these corollaries as both of them are proved by taking the dual of the expression for $\phi(y)$ from Theorem 3 after substituting the definition of $\mathcal{U}_c$.

Corollary 1 *Consider a network flow problem with a single source s and an uncertainty set $\mathcal{U}_b$ given by Equation (10). If $\mathcal{U}_c = \{c \mid Mc \leq g,\ c \geq 0\}$, then (RCEP) is equivalent to*

$$\begin{array}{ll} \min_{y,x,w} & g^t w \\ \text{s.t.} & Nx = \bar{b} + \bar{\delta}(e_s - e_t) \\ & x \leq u + y \\ & x \leq M^t w \\ & d^t y \leq I \\ & x, w, y \geq 0\ . \end{array} \qquad \blacksquare$$

Corollary 2 *Consider a network flow problem with a single source s and an uncertainty set $\mathcal{U}_b$ given by Equation (10). Let $\mathcal{U}_c = \{c \mid c = c^0 + \sum_{l=1}^{L} \xi_l c^l,\ \xi \in \mathcal{X}\}$, with $\mathcal{X} = \{\xi \mid \exists w, A\xi + Bw \geq_K d\}$, and let $\mathcal{C} = [c^1, \ldots, c^L]$. Then (RCEP) is equivalent to*

$$\begin{array}{ll} \min_{y,x,z} & (c^0)^t x - d^t z \\ \text{s.t.} & Nx = \bar{b} + \bar{\delta}(e_s - e_t) \\ & x \leq u + y \\ & \mathcal{C}^t x + A^t z = 0 \\ & B^t z = 0 \\ & d^t y \leq I \\ & x, y \geq 0,\ z \geq_{K^*} 0\ . \end{array} \qquad \blacksquare$$

3.3 Multicommodity flow

A relevant network model for transportation problems is the multicommodity flow problem with a single source and single sink for each commodity. Theorem 3 shows that the single commodity network flow problem, with a single source and sink, has a tractable (RCEP). In this section we show that the (RCEP) is also tractable for a multicommodity flow problem where each commodity has a single source and sink. This is also true for the slightly more general conditions of a single source (or sink) per commodity and uncertainty only on a single source - sink pair per commodity. However we omit this result here for clarity of exposition.

Assuming that commodity k has a source s^k and a sink t^k, and the amount to be sent is uncertain, but bounded, we can define the demand uncertainty set by

$$\mathcal{U}_b = \left\{ (b^1, \ldots, b^K) \mid b^k = \delta^k \left(e_{s^k} - e_{t^k} \right), \; \delta^k \in [\delta_l^k, \delta_u^k], \text{ for all } k \in 1, \ldots, K \right\} \;, \tag{11}$$

where we assume that $\delta_l^k \geq 0$ for all $k = 1, \ldots, K$. In other words, the demand uncertainty does not allow a supply node to become a demand node.

Theorem 4 *Consider the multicommodity flow problem, where each commodity has a single source s^k and single sink t^k and that $\mathcal{U}_b$ is given by Equation (11). Then $\phi(y)$ is the following convex optimization problem*

$$\begin{array}{rlll} \phi(y) = & \max_{\lambda,\pi,c} & \sum_{k=1}^{K} \delta_u^k (\lambda_{s^k}^k - \lambda_{t^k}^k) - (u+y)^t \pi & \\ & \text{s.t.} & N^t \lambda^k - \pi - c^k \leq 0 & k = 1, \ldots, K \\ & & (c^1 \ldots c^K) \in \mathcal{U}_c & \\ & & \pi \geq 0 \;. & \end{array}$$

Proof: This proof is analogous to the proof of Theorem 3. Under the uncertainty set $\mathcal{U}_b$ the definition of $\phi(y)$ becomes

$$
\begin{array}{llll}
\phi(y) = & \max_c & \max_{\delta^k \ldots \delta^K} & \min_x \sum_{k=1}^{K} (c^k)^t x^k \\
& (c^1 \ldots c^K) \in \mathcal{U}_c & \delta^k \in [\delta_l^k, \delta_u^k] & N x^k = \delta^k (e_{s^k} - e_{t^k}) \quad k = 1 \ldots K \\
& & & \sum_{k=1}^{K} x^k \leq u + y \\
& & & x^k \geq 0 \qquad\qquad k = 1 \ldots K\ .
\end{array}
$$

Therefore the key in the proof is showing that the function

$$
\begin{array}{ll}
\Gamma(\delta^1 \ldots \delta^K) = & \min_x \sum_{k=1}^{K} (c^k)^t x^k \\
& N x^k = \delta^k (e_{s^k} - e_{t^k}) \quad k = 1 \ldots K \\
& \sum_{k=1}^{K} x^k \leq u + y \\
& x^k \geq 0 \qquad\qquad k = 1 \ldots K
\end{array}
$$

is a non-decreasing function, i.e. if $(\delta^1 \ldots \delta^K) \leq (\delta'^1 \ldots \delta'^K)$ then $\Gamma(\delta^1 \ldots \delta^K) \leq \Gamma(\delta'^1 \ldots \delta'^K)$. If $\Gamma(\delta^1 \ldots \delta^K)$ is non-decreasing it implies that $\phi(y) = \max_{(c^1 \ldots c^K) \in \mathcal{U}_c} \Gamma(\delta_u^1 \ldots \delta_u^K)$, and the result is obtained by taking the LP dual of $\Gamma(\delta_u^1 \ldots \delta_u^K)$. Let $(x'^1 \ldots x'^K)$ be the optimal solution for $\Gamma(\delta'^1 \ldots \delta'^K)$. The flow of commodity k, x'^k can be represented by flow along $s^k - t^k$ paths $P_{s^k t^k}$, and cycles. In particular, the total flow along the $s^k - t^k$ paths is equal to δ'^k. We can then remove a total of $\delta'^k - \delta^k \geq 0$ units of flow from these paths. For each commodity k let $x^k_{\delta' - \delta}$ be the flow removed, define $\tilde{x}^k = x'^k - x^k_{\delta' - \delta}$, and recall that $c^k \geq 0$. By construction the flow $(\tilde{x}^1 \ldots \tilde{x}^K)$ is a feasible flow for $\Gamma(\delta^1 \ldots \delta^K)$. Therefore $\Gamma(\delta^1 \ldots \delta^K) \leq \sum_{k=1}^{K} (c^k)^t \tilde{x}^k = \sum_{k=1}^{K} \left((c^k)^t x'^k - (c^k)^t x^k_{\delta' - \delta} \right) \leq \sum_{k=1}^{K} (c^k)^t x'^k = \Gamma(\delta'^1 \ldots \delta'^K)$. ■

We now present two corollaries that show problem (RCEP) is tractable for the multicommodity flow problem with single source and sink and for fairly general cost uncertainty sets. We omit the proofs of these corollaries as both just require taking the dual of $\phi(y)$ from Theorem 4 after substituting the definition of $\mathcal{U}_c$.

Corollary 3 *Consider the multicommodity flow problem, where each commodity has a single source and sink, and that* $\mathcal{U}_b$ *is given by Equation (11). If* $\mathcal{U}_c = \{(c^1 \dots c^K) \mid M^k c^k \leq g^k, c^k \geq 0\}$*, then the (RCEP) problem is equivalent to the following LP:*

$$
\begin{array}{lll}
\min_{y,x,w} & \displaystyle\sum_{k=1}^{K} (g^k)^t w^k & \\
\text{s.t.} & N x^k = \delta_u^k \left(e_{s^k} - e_{t^k}\right) & k = 1, \dots, K \\
& x^k \leq (M^k)^t w^k & k = 1, \dots, K \\
& \displaystyle\sum_{k=1}^{K} x^k \leq u + y & \\
& d^t y \leq I & \\
& x^k, w^k, y \geq 0 \ . & \blacksquare
\end{array}
$$

Corollary 4 *Consider the multicommodity flow problem, where each commodity has a single source and sink, and that* $\mathcal{U}_b$ *is given by Equation (11). If* $\mathcal{U}_c = \{(c^1 \dots c^K) \mid c^k = c^{k0} + \sum_{l=1}^{L} \xi_l c^{kl},\ \xi \in \mathcal{X}\}$*, with* $\mathcal{X} = \{\xi \mid \exists w, A\xi + Bw \geq_K d\}$*, and let* $\mathcal{C}^k = [c^{1k}, \dots c^{Lk}]$*. Then (RCEP) is equivalent to*

$$
\begin{array}{lll}
\min_{y,x,z} & \displaystyle\sum_{k=1}^{K} (c^{k0})^t x^k - z^t d & \\
\text{s.t.} & N x^k = \delta_u^k \left(e_{s^k} - e_{t^k}\right) & k = 1, \dots, K \\
& \displaystyle\sum_{k=1}^{K} \left(\mathcal{C}^k\right)^t x^k + A^t z = 0 & \\
& B^t z = 0 & \\
& \displaystyle\sum_{k=1}^{K} x^k \leq u + y & \\
& d^t y \leq I & \\
& x^k, y \geq 0,\ z \geq_{K_*} 0 \ . & \blacksquare
\end{array}
$$

4 Computational Experiments

We now present computational experiments that compare the robust solution to the deterministic solution obtained for *nominal data*, i.e. data that is representative of the uncertainty set. These experiments serve to illustrate the conditions under which a robust solution is preferable to a deterministic solution.

For each experiment described in this section, we compute four values: z_D the optimal value of the deterministic solution, z_R the optimal value of the robust solution, z_{wc} the worst case value of the deterministic solution, and z_{ac} the objective value of the robust solution for the nominal data. We obtain $z_D = z_D(\bar{b}, \bar{c})$ as the optimal objective function value of Problem (3) for the nominal data $\bar{b} \in \mathcal{U}_b$ and $\bar{c} \in \mathcal{U}_c$. Let y_D be the optimal investment strategy for the deterministic problem. The value z_R is obtained by solving the appropriate tractable characterization of (RCEP) depending on the form of the uncertainty sets (either Corollary 1 or 3 in the experiments below). Let y_R be the optimal robust investment strategy. The worst case value $z_{wc} = \phi(y_D)$ is obtained from the appropriate tractable characterization of Problem (7) (either Theorem 3 or 4). Finally the cost of the robust solution for the nominal data, z_{ac}, is obtained by solving

$$\begin{aligned} \min \quad & \bar{c}^t x \\ \text{s.t.} \quad & Nx = \bar{b} \\ & 0 \leq x \leq u + y_R \ . \end{aligned}$$

We compare the performance of the robust and deterministic solution through the following two ratios:

$$r_{\text{wc}} = \frac{z_{\text{wc}} - z_{\text{R}}}{z_{\text{R}}} \quad \text{and} \quad r_{\text{ac}} = \frac{z_{\text{ac}} - z_{\text{D}}}{z_{\text{D}}} \ .$$

The quantity r_{wc} is the relative improvement of the robust solution in the worst case and r_{ac} is the relative loss of optimality of the robust solution on the nominal data.

4.1 A 3-node network

Our first example consists of the simple network on three nodes shown in Figure 3. The values on the arcs represent the nominal travel times, all arcs have capacity 1, and $(d_{12}, d_{13}, d_{23}) = (1, 2, 1)$. We consider deterministic demand equal to δ units of flow from 1 to 3, an investment of I, and an uncertainty set on travel times given by $\mathcal{U}_c = \{c \mid 0.5\bar{c} \leq c \leq 1.5\bar{c},\ 4c_{12} + 4c_{23} + c_{13} \leq 9.95\}$. In this experiment, the shortest path between 1 and 3, given by the arc $(1, 3)$, is subject to greater variability than the alternate path, $(1, 2) - (2, 3)$.

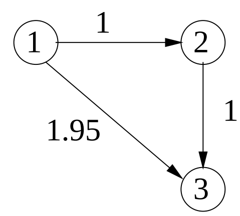

Figure 3: 3-node network.

In Figure 4 we plot r_{wc} and r_{ac} for diverse values of δ and I. Note that the robust solution is able to achieve more than a 20% reduction in the value of the worst case with a smaller than 2% increase in the value for the nominal data. This occurs for cases with a flow δ bigger than 2.25; in addition this worst case benefit improves for larger values of investment budget I. Note that, for a fixed investment I, the benefit of the robust solution in the worst case, r_{wc}, increases and then decreases with δ. Clearly for small flows, close to $\delta = 1$, most of the flow can be sent on either of the existing paths, thus both investment solutions are comparable. As the flow increases however, a larger investment is needed to route flow on the preferred path between 1 and 3. For flows large enough, in the example $I = 2$ for flows larger than $\delta = 2.5$, all the new capacity is installed in the best path, and as the flow keeps increasing the benefit of the robust solution decreases, as it must route flow through the other path.

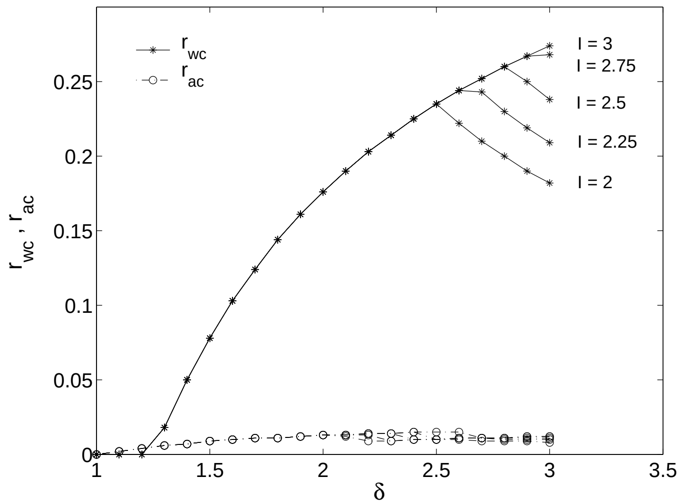

Figure 4: Comparison of robust and deterministic solutions for 3-node network.

4.2 A 21-node network

Our second example considers a larger network, obtained by repeating the triangular network of Experiment 1. The purpose here is to study the scalability of the benefits of the robust solution. The network considers nominal cost values of 1.95 on all diagonal arcs and 1 on the non-diagonal arcs; it also considers a capacity of $u_{ij} = 1$ for all arcs and a rate of investments of 2 for all diagonal arcs and 1 for all non-diagonal arcs. This example considers a deterministic amount of δ flowing from 1 to 21, a total investment of I and an uncertainty set on travel times given by

$$\mathcal{U}_c = \left\{ c \mid 0.5\bar{c} \leq c \leq 1.5\bar{c}, \quad \sum_{(i,j)\ \text{non-diag}} 4c_{ij} + \sum_{(i,j)\ \text{diag}} c_{ij} = 149.25 \right\} .$$

In Figure 6 we plot the ratios r_{wc} and r_{ac} obtained for different values of total flow sent from 1 to 21 and for different total investment budgets I. We note that the robust solution is still better than the deterministic solution in the worst case, in some cases by about 20%, while it is never worse than 2.5% of the deterministic solution on the

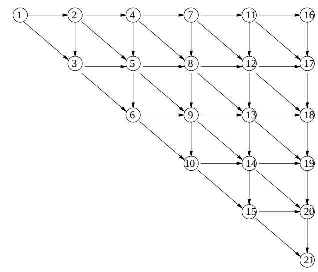

Figure 5: 21-node network.

nominal data. Figure 6 also shows that r_{wc} decreases for flows larger than a certain level and that this drop can be substantial. Similar to the 3-network case, the benefit of the robust solution begins to decrease when the new capacity in the robust solution is committed to the preferred paths and the remaining flow has to be routed through less beneficial paths. For large flows δ, the only feasible investment strategy becomes to expand the bottleneck arcs near the source and sink nodes, as all the paths from 1 to 21 must use the same two pairs of arcs out of 1 and into 21. Thus as δ increases the deterministic and robust solutions converge to this only feasible investment strategy.

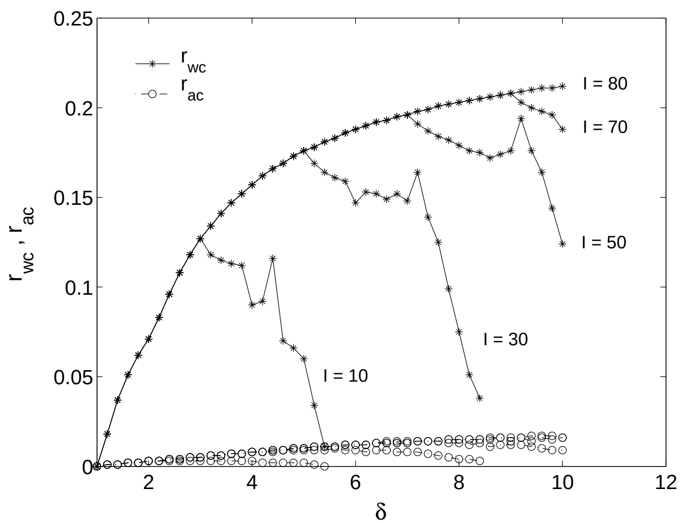

Figure 6: Comparison of robust and deterministic solutions for 21-node network.

4.3 A transportation network

Our last example considers a multicommodity flow problem with cost and demand uncertainty on a planar network. The network is given in Figure 7 with the nominal travel times depicted. This example represents an evening rush hour scenario, where traffic is traveling from work at either node 2 or node 5, to their homes in either nodes 1, 4, or 8. The nominal demand values are 1000 units of flow from node 2 to node 4, 500 units

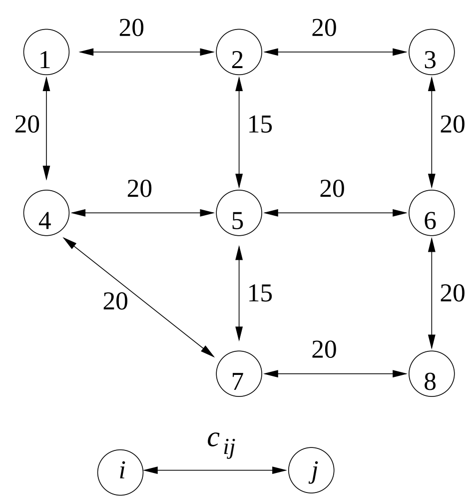

Figure 7: Transportation network problem.

from node 2 to node 1, and 1000 from node 2 to node 8. We also have 500 units of flow from node 5 to node 8. The network has uniform arc capacity of 900, and there is a total budget of 2000 units of arc capacity to distribute. We consider that the travel times on a few arcs are subject to uniform box uncertainty and can vary either up or down by μ%. Let arcs $(2,5),(5,4),(5,7)$, and $(7,8)$ have uncertain travel times. The demand is also under uniform box uncertainty, and all commodities can have their demand/supply vary by μ_b% up or down.

If Figure 8 we present the ratios r_{wc} and r_{ac} for different values of total investment budget I as we vary the amount of uncertainty in travel time μ. This example considers the demand known, i.e. $\mu_b = 0$. We see that as the uncertainty of travel times increases the robust solution becomes more attractive. In fact, for small levels of uncertainty or small investment budgets the deterministic solution would be preferable, as the improvement in the worst case could be even smaller than the worsening for the nominal data. This is not surprising, as for small uncertainty levels the effect of the worse case should be small, and if there is limited budget to invest it should be dedicated to clear needs of the network. However as investment budgets and uncertainty levels increase the robust solution becomes more attractive, reaching more than 15% improvement in the worst case, for less than a 7% overhead for the nominal data, when uncertainty is at least 60% and investment of at least 1500. In Figure 9 we plot the same results for the different

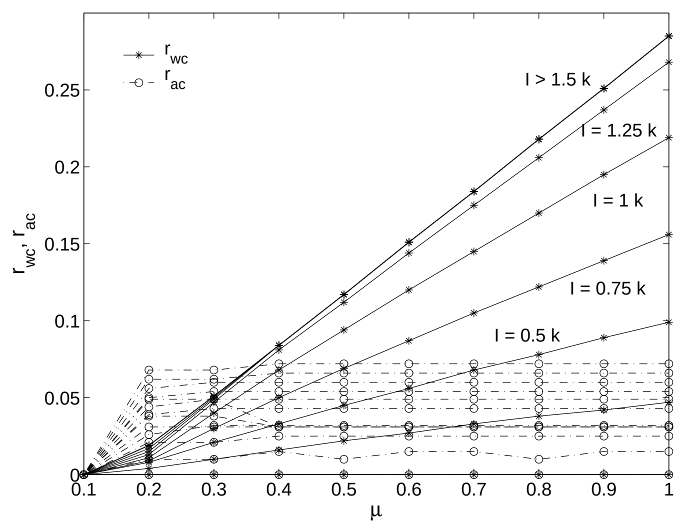

Figure 8: Comparison of robust and deterministic solutions for transportation network as a function of μ, for different I, with $\mu_b = 0$.

values of μ as we vary the investment budget I. Here we observe that we reach the best improvement in r_{wc} starting with $I = 1500$, regardless of the uncertainty in travel time. The same does not happen for r_{ac} which increases as more budget is available until a sharp drop.

In Figure 10 we plot how demand uncertainty affects r_{wc}. For a fixed investment budget $I = 2000$ and different uncertainty sets in travel time, we plot the value of

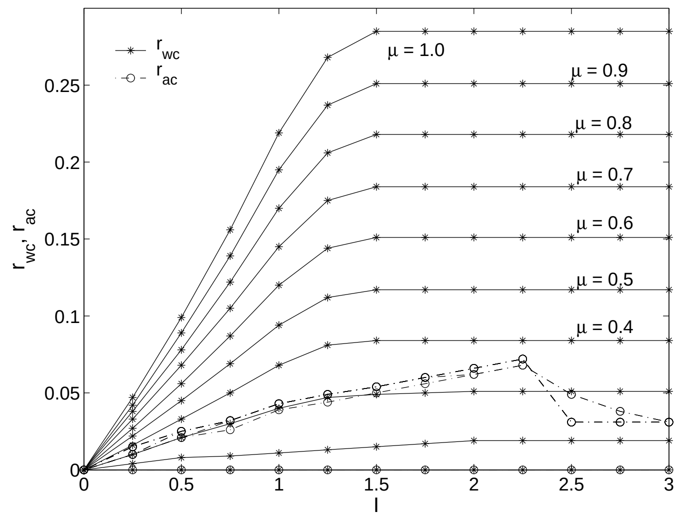

Figure 9: Comparison of robust and deterministic solutions for transportation network as a function of I, for different μ, with $\mu_b = 0$.

ratio r_{wc} as a function of μ_b. Surprisingly we notice that at first the uncertainty in demand increases r_{wc}, for all μ, however for demand uncertainty larger than $\mu_b = 0.2$ there is a decrease in r_{wc} as we increase demand uncertainty. The reason for this drop is explained by Corollary 3, which shows that the robust problem solves an instance with the maximum demand possible. Thus uncertainty in demand amounts to finding a solution that is robust with respect to the cost uncertainty for the largest possible demand. We have seen in our previous examples that the benefits of a robust solution decrease with an increase in flow.

5 Conclusions

The robust capacity expansion problem (RCEP) we consider in this paper, Problem (4), is the basis of an approach to decide capacity expansions for a transit network that finds a robust solution with respect to the uncertainty in demands and travel times. It seems

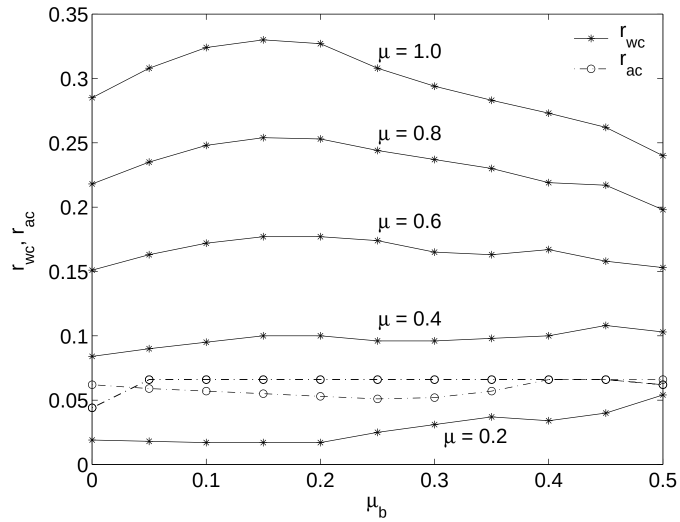

Figure 10: Sensitivity to demand uncertainty μ_b for transportation network, different values of μ and $I = 2000$.

unlikely that problem (RCEP) could be solved efficiently in general, as this problem is a particular instance of the Adjusted Robust Counterpart problem, which is NP-hard. Here we exploit the structure of the capacity expansion problem to show that the (RCEP) is a tractable problem under conditions that are reasonable for a transportation system: if we consider a multicommodity flow problem with a single source and sink per commodity, uncertainty on the demand and travel times is represented by bounded convex sets, all travel times are non-negative, and there exists a feasible way to route any outcome of the uncertain demand.

The computational results obtained indicate that the robust solution is capable of reducing the worst case cost by more than 20% while incurring in about a 5% loss of optimality with respect to the optimal deterministic solution for a nominal uncertainty data. In particular, the robust solution becomes more attractive as the uncertainty in travel times increases and as the budget to decide capacity expansions increases. A result which is intuitive, since a small uncertainty implies that worst case scenarios are similar to the nominal case and if the budget is small, both solutions will tend solve the most pressing problems. The results also show that the greatest benefit of a robust solution is obtained for flow in some medium range, as a network with small amounts of flow is not affected by capacity expansions and a network with large amounts of flow is forced to send flow through less attractive routes. Finally, we note that a robust solution can become less attractive as uncertainty in demand increases, which is reasonable since the robust solution routes the largest amount of flow possible and, as noted above, a robust solution becomes less attractive for large amounts of flow.

The methodology presented here gives a computationally efficient method of obtaining a robust solution for the capacity expansion problem. The methodology is quite general and can be applied in principle to other network design problems, as long as the assumptions needed for the computational tractability of the problem are reasonable, as they are in the transportation setting. For classical network design problems, the additional complication of dealing with a mixed integer program must be addressed. Finally, this work suggests that a robust solution has the potential to be efficient in practice for the capacity expansion of transit networks. Future work should investigate whether the robust solution can be efficient in a realistic example.

References

Ahmed, S., King, A. J. and Parija, G. (2003). A multi-stage stochastic integer programming approach for capacity expansion under uncertainty, *Journal of Global Optimization* **26**(1): 3–24.

Ahuja, R. K., Magnanti, T. L. and Orlin, J. B. (1993). *Network Flows: Theory, Algorithms, and Applications*, Prentice Hall, New Jersey.

Averbakh, I. and Berman, O. (2000). Algorithms for the robust 1-center problem on a tree, *European Journal of Operational Research* **123**(2): 292–302.

Balakrishnan, A., Magnanti, T. and Wong, R. (1995). A decompostiiton algorithm for local access telecommunications network expansion planning, *Operations Research* **43**(1): 58–76.

Barahona, F., Bermon, S., Günlük, O. and Hood, S. (2004). Robust capacity planning in semiconductor manufacturing, *Research Report RC22196*, IBM. http://www.optimization-online.org/DB_HTML/2001/10/379.html.

Ben-Tal, A. and Nemirovski, A. (1997). Robust truss topology design via semidefinite programming, *SIAM Journal on Optimization* **7**(4): 991–1016.

Ben-Tal, A. and Nemirovski, A. (1998). Robust convex optimization, *Mathematics of Operations Research* **23**(4): 769–805.

Ben-Tal, A. and Nemirovski, A. (1999). Robust solutions to uncertain programs, *Operations Research Letters* **25**: 1–13.

Ben-Tal, A., Golany, B., Nemirovski, A. and Vial, J.-P. (2003a). Supplier-retailer flexible commitments contracts: A robust optimization approach, *Cahier de Recherche 2003.06*, Université de Genève, HEC. http://hec.info.unige.ch/recherches_publications/recherche_publications.htm.

Ben-Tal, A., Goryashko, A., Guslitzer, E. and Nemirovski, A. (2003b). Adjustable robust solutions of uncertain linear programs, *Technical Report*, Minerva Optimization Center, Technion. http://iew3.technion.ac.il/Labs/Opt/index.php?4.

Bertsimas, D. and Sim, M. (2003). Robust discrete optimization and network flows, *Mathematical Programming* **98**(1-3): 49–71.

Bertsimas, D. and Sim, M. (2004). The price of robustness, *Operations Research* **52**(1): 35–53.

Bertsimas, D. and Thiele, A. (2003). A robust optimization approach to supply chain management, *Technical report*, MIT, LIDS.

Birge, J. R. and Louveaux, F. (1997). *Introduction to Stochastic Programming*, Springer Verlag, New York.

Caltrans (2002). Traffic congestion relief plan report, district 7, http://www.dot.ca.gov/dist07/tcrp/tcrp_index.shtml.

Chen, Z.-L., Li, S. and Tirupati, D. (2002). A scenario based stochastic programming approach for technology and capacity planning, *Computers and Operations Research* **29**(7): 781–806.

Department of Energy (2004). Annual energy outlook 2004 with projections to 2025, http://www.eia.doe.gov/oiaf/aeo/.

El-Ghaoui, L. and Lebret, H. (1997). Robust solutions to least-square problems to uncertain data matrices, *SIAM J. Matrix Anal. Appl.* **18**: 1035–1064.

El-Ghaoui, L., Oks, M. and Oustry, F. (2003). Worst-case value-at-risk and robust portfolio optimization: A conic programming approach, *Operations Research* **51**(4): 543–556.

El-Ghaoui, L., Oustry, F. and Lebret, H. (1998). Robust solutions to uncertain semidefinite programs, *SIAM J. Optimization*.

Eppen, G., Martin, R. and Schrage, L. (1989). A scenario approach to capacity planning, *Operations Research* **37**(4): 517–527.

Ferguson, A. and Dantzig, G. (1956). The allocation of aircraft to routes - and example of linear programming under uncertain demand, *Management Science* **3**(1): 45–73.

Ferris, M. and Ruszczynski, A. (2000). Robust path choice in networks with failures, *Networks* **35**(3): 181–194.

Goldfarb, D. and Iyengar, G. (2003). Robust portfolio selection problems, *Mathematics of Operations Research* **28**(1): 1–38.

Guslitser, E. (2002). *Uncertainty-immunized solutions in linear programming*, Master's thesis, Minerva Optimization Center, Technion. http://iew3.technion.ac.il/Labs/Opt/index.php?4.

Gutiérrez, G., Kouvelis, P. and Kurawarwala, A. (1996). A robustness approach to uncapacitated network design problems, *European Journal of Operational Research* **94**(2): 362–376.

Hsu, V. N. (2002). Dynamic capacity expansion problem with deferred expansion and age-dependent shortage cost, *M&SOM* **4**(1): 44–54.

Kouvelis, P. and Yu, G. (1997). *Robust discrete optimization and its applications*, Kluwer Academic Publishers, Norwell, MA.

Laguna, M. (1998). Applying robust optimization to capacity expansion of one location in telecommunications with demand uncertainty, *Management Science* **44**(11): S101–S110.

Magnanti, T. and Wong, R. (1984). Network design and transportation planning: Models and algorithms, *Transportation Science* **18**(1): 1–55.

Malcolm, S. and Zenios, S. (1994). Robust optimization for power systems capacity expansion under uncertainty, *J. Opl. Res. Soc.* **45**(9): 1040–1049.

Minoux, M. (1989). Network synthesis and optimum network design problems: Models, solution methods, and applications, *Networks* **19**: 313–360.

Mulvey, J. M., Vanderbei, R. J. and Zenios, S. A. (1995). Robust optimization of large-scale systems, *Operations Research* **43**: 264–281.

Murphy, F. and Weiss, H. (1990). An approach to modeling electric utility capacity expansion planning, *Naval Research Logistics* **37**: 827–845.

Paraskevopoulos, D., Karakitsos, E. and Rustem, B. (1991). Robust capacity planning under uncertainty, *Management Science* **37**(7): 787–800.

Riis, M. and Andersen, K. A. (2004). Multiperiod capacity expansion of a telecommunications connection with uncertain demand, *Comput. Oper. Res.* **31**(9): 1427–1436.

Rockafellar, R. T. (1997). *Convex Analysis*, Princeton University Press, Princeton, New Jersey.

Schrank, D. and Lomax, T. (2003). The 2003 anuual urban mobility report, *Technical report*, Texas Transportation Institute, The Texas A&M University.

Yaman, H., Karaşan, O. and Pinar, M. (2001). The robust spanning tree problem with interval data, *Operations Research Letters* **29**(1): 31–40.

Zhang, F., Roundy, R., Çakanyildrim, M. and Huh, W. T. (2004). Optimal capacity expansion for multi-product, multi-machine manufacturing systems with stochastic demand, *IIE Transactions* **36**(1): 23–36.