Numerical experiments with an interior-exterior point method for nonlinear programming

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Abstract

The paper presents an algorithm for solving nonlinear programming problems. The algorithm is based on the combination of interior and exterior point methods. The latter is also known as the primal-dual nonlinear rescaling method. The paper shows that in certain cases when the interior point method (IPM) fails to achieve the solution with the high level of accuracy, the use of the exterior point method (EPM) can remedy this situation. The result is demonstrated by solving problems from COPS and CUTE problem sets using nonlinear programming solver LOQO that is modified to include the exterior point method subroutine.

Keywords. Interior point method, exterior point method, primal-dual, nonlinear rescaling.

1 Introduction.

This paper considers a method for solving the following optimization problem

$$\begin{aligned} &\min f(x), \\ &\text{s.t. } h(x) \geq 0, \end{aligned} \tag{1}$$

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where $h(x) = (h_1(x), \dots, h_m(x))$ is a vector function. We assume that $f : \mathbb{R}^n \to \mathbb{R}^1$ and all $h_i : \mathbb{R}^n \to \mathbb{R}^1$, $i = 1, \dots, m$ are twice continuously differentiable functions.

To solve this problem we use a method based on the combination of interior and exterior point methods. We describe these methods in the following sections. This section explains both interior and exterior point methods in the context of their development. It illustrates how the two methods are related and why their integration is a reasonable approach for solving nonlinear programming problems.

In the past two decades interior point methods have proven to be efficient and are widely used for solving linear and nonlinear programming problems.

The interior point methods are closely related to the sequential unconstrained minimization technique (SUMT) developed by Fiacco and McCormick [4] for solving constrained optimization problem with inequality constraints. The sequential unconstrained minimization technique is based on a sequence of unconstrained minimizations of the classical log-barrier function followed by the barrier parameter update.

Among all variations of interior point methods related to SUMT, the primal-dual interior point method is the most efficient. At each step it solves the primal-dual system equivalent to the optimality criteria for the minimization of the classical log-barrier function. Since the late 1980s the primal-dual interior point method has become the most popular algorithm for large scale linear programming. It has solid a theoretical foundation and is computationally efficient. The developed theory and numerical experiments revealed that the primal-dual interior point method shows excellent performance for large scale practical problems [9, 22]. The performance of the primal-dual interior point method on nonlinear programming problems is robust. The algorithm shows solid global convergence properties.

The primal-dual interior point method overcame well-known difficulties associated with the sequential unconstrained minimization technique. Of particular importance is that the efficiency of SUMT is compromised by the singularity of the classical barrier function and its derivatives at the solution, which makes it difficult to use methods of unconstrained minimization effectively. The primal-dual interior point method suffers the least of any other variation of the interior point methods from the ill-conditioning. Nevertheless, for certain problems even the primal-dual interior point method fails to achieve the desired level of accuracy.

The problems associated with the sequential unconstrained minimization technique encouraged the optimization community to look for alternatives. Thus in the early 1980s Polyak [12] suggested a different approach for solving constrained optimization problems with inequality constraints. He developed the theory of modified barrier functions (MBF) and the corresponding modified barrier function methods . Independently, in 1970s Kort and Bertsekas [8] introduced the exponential multipliers method. Both modified barrier function and exponential multipliers methods are particular cases of the nonlinear rescaling principle [13, 15].

The nonlinear rescaling principle consists of a sequence of unconstrained minimizations of the Lagrangian for the equivalent problem followed by the Lagrange multipliers update. Later, keeping in mind the theoretical and numerical success related to the primal-dual interior point methods, there was motivation to develop the primal-dual method based on the nonlinear rescaling theory, which is an exterior point method (EPM). Instead of performing a sequence of unconstrained minimizations, the exterior point method solves the primal-dual system by Newton's method [7, 14, 16]. In general, the trajectory of the exterior point method approaches the solution outside the feasible set.

The exterior point method allows for simultaneous computation of the primal and the dual approximations. Furthermore, the EPM has interesting local convergence properties. Under the standard second order optimality conditions the exterior point method converges with a linear rate under the fixed barrier parameter [16]. If the barrier parameter decreases at each step, the rate of convergence of the exterior point method is superlinear [7]. Locally the exterior point method has a trajectory similar to that of Newton's method applied to the Lagrange system of equations that correspond to the active constraints [7].

Taking into account the robustness and the global convergence properties of the interior point method and the fast local convergence properties of the exterior point method, these two methods can potentially augment each other and result in a more robust combination: an interior-exterior point method (IEPM). The main idea of this paper is to develop an algorithm based on such a combination and test it on a variety of problems. We incorporated the exterior point method into the general nonconvex nonlinear programming solver LOQO, which is based on the primal-dual interior point method.

The interior point algorithm for nonconvex nonlinear programs, implemented in LOQO, has been described and studied in [1, 19, 20, 21]. The appropriate choice of a filter or a merit function together with regularization of a Hessian of the Lagrangian [19] contributes to the global convergence of the interior point algorithm. In some cases, however, the interior point method experiences numerical problems when approaching the solution. In this paper we propose to switch to the exterior point method, which has fast local convergence properties [7, 16], when the numerical problems occur.

The structure of the matrices for the Newton directions in the interior and exterior point methods are identical. Therefore the sparse linear algebra developed by Vanderbei [18] for LOQO, can be used in both methods.

The paper is organized as follows. In the next section we describe briefly the interior point algorithm implemented in LOQO and its connection to the sequential unconstrained minimization technique. In section 3 we discuss the exterior point method in connection to the nonlinear rescaling principle. Section 4 describes the interior-exterior point method (IEPM) and presents the numerical results for testing IEPM on COPS [2] and CUTE [3] problem sets. Section 5 contains the discussion of numerical results and concluding remarks.

2 The interior point method.

We will focus on the minimization problem with inequality constraints (1). Equality constraints can be included in the formulation, but we ignore them to simplify the presentation. Let us consider the following optimization problem. Applying the classical log-barrier function to problem (1) we obtain

$$B(x, \mu) = f(x) - \mu \sum_{i=1}^{m} \ln h_i(x),$$

where $\mu > 0$ is a barrier parameter.

Sequential unconstrained minimization technique replaces a constrained optimization problem with a sequence of unconstrained optimization problems. Assuming that $\log t = -\infty, t \leq 0$ we obtain

$$x(\mu) = argmin\{B(x, \mu) | x \in \mathbb{R}^n\} \tag{2}$$

Solving problem (2) sequentially for a monotoneously decreasing sequence $\{\mu^k\}$ such that $\lim_{k\to\infty} \mu^k = 0$ gives a sequence $\{x(\mu^k)\}$ yielding $h(x(\mu^k)) > 0$ and $\lim_{k\to\infty} f(x(\mu^k)) = f(x^*)$, where x^* is the solution of the problem (1).

To find the minimum of $B(x, \mu)$ in x is equivalent to solving the system

$$\nabla_x B(x, \mu) = \nabla f(x) - \mu \sum_{i=1}^{m} \frac{\nabla h_i(x)}{h_i(x)} = 0 \tag{3}$$

Let $x(\mu) : \nabla B(x(\mu), \mu) = 0$ and $y_i(\mu) = \mu / h_i(x(\mu))$, $i = 1, \ldots, m$. Therefore the pair $(x(\mu), y(\mu))$ is the solution of the following primal-dual system of equations

$$\nabla L(x, y) = \nabla f(x) - \sum_{i=1}^{m} y_i \nabla h_i(x) = 0, \tag{4}$$

$$y_i h_i(x) = \mu, \quad i = 1, \ldots, m, \tag{5}$$

where $L(x, y) = f(x) - \sum_{i=1}^{m} y_i h_i(x)$ is the Lagrangian of the problem (1).

The primal or primal-dual interior point methods perform one Newton step towards the solution of systems (3) or (4)-(5) respectively followed by changing the barrier parameter μ. Therefore there are similarities and differences between the sequential unconstrained minimization technique and interior point methods. The methods are related to each other because they both rely on the primal-dual central path $(x(\mu), y(\mu))$ introduced by Fiacco and McCormick in the 1960s. In the late 1980s and early 1990s, when interior point methods emerged as popular methods in optimization it became evident that the central path is also the main component of IPM developments [10]. The key difference between interior point methods and the sequential unconstrained minimization technique is in the role Newton's method plays in their frameworks. In the sequential unconstrained minimization technique Newton's method is used for the unconstrained minimizations, which result in approximations of the central path. After each minimization the barrier parameter is decreased. Interior point methods usually perform just one Newton step for the system similar to (4)-(5) toward the central path followed by the barrier parameter update. For efficiency of interior point methods it is critical to keep their trajectory in the intersection of an interior of the feasible set and the Newton area, the area where Newton's method is well defined [17]. The size of this intersection generally depends on the value of the barrier parameter. Whereas for the sequential unconstrained minimization technique the rate of change of the barrier parameter is not a key issue, an uncontrolled change of the barrier parameter could compromise the efficiency of interior point methods.

The interest in the central path, the classical barrier function and the corresponding methods has been revived in the course of linear programming development [9, 22], especially after the recognition of the role that Newton's method plays in interior point methods [6]. IPMs have proven to be efficient and widely used for LP. The success of interior point methods in linear programming sparked the interest to applying the methods for nonlinear programming.

Let us briefly describe the interior point method implemented in LOQO. By introducing nonnegative slack variables $w = (w_1, \ldots, w_m)$, the problem (2) can be replaced by the following problem

$$\begin{aligned} &\min f(x) - \mu \sum_{i=1}^{m} \log w_i, \\ &\text{s.t. } h(x) - w = 0, \end{aligned} \tag{6}$$

where $\mu > 0$ is a barrier parameter. The solution of this problem satisfies the following primal-dual system

$$\begin{aligned} \nabla f(x) - \nabla h(x)^T y &= 0, \\ -\mu e + WYe &= 0, \\ h(x) - w &= 0, \end{aligned} \tag{7}$$

where $y = (y_1, \ldots, y_m)$ is a vector of the Lagrange multipliers or dual variables for problem (6), $\nabla h(x)$ is the Jacobian of vector function $h(x)$, Y and W are diagonal matrices with elements y_i and w_i respectively and $e = (1, \ldots, 1) \in \mathbb{R}^m$.

Applying Newton's method to the system (7) leads to the following linear system for the Newton directions

$$\begin{bmatrix} \nabla_{xx}^2 L(x,y) & 0 & -\nabla h(x)^T \\ 0 & Y & W \\ \nabla h(x) & -I & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta w \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\nabla f(x) + \nabla h(x)^T y \\ \mu e - WYe \\ -h(x) + w \end{bmatrix},$$

where $\nabla_{xx}^2 L(x,y) = \nabla^2 f(x) - \sum_{i=1}^m y_i \nabla^2 h_i(x)$ is the Hessian of the Lagrangian of problem (1). After eliminating Δw from this system we obtain the following reduced system

$$\begin{bmatrix} -\nabla_{xx}^2 L(x,y) & \nabla h(x)^T \\ \nabla h(x) & WY^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \sigma \\ \rho + WY^{-1}\gamma \end{bmatrix}, \tag{8}$$

where

$$\sigma = \nabla f(x) - \nabla h(x)^T y,$$

$$\gamma = \mu W^{-1} e - y,$$

$$\rho = w - h(x).$$

Then we can find Δw by the following formula

$$\Delta w = WY^{-1}(\gamma - \Delta y).$$

One step of the IPM algorithm $(x, w, y) \to (\hat{x}, \hat{w}, \hat{y})$ is as follows

$$\hat{x} = x + \alpha \Delta x,$$

$$\hat{w} = w + \alpha \Delta w,$$

$$\hat{y} = y + \alpha \Delta y,$$

where α is a steplength chosen according to a merit function [19] or a filter method [1, 5] and to keep the slacks w_i and Lagrange multipliers y_i positive.

If the Hessian $\nabla^2_{xx}L(x, y)$ is not positive definite the algorithm replaces it with the regularized Hessian

$$R_\lambda(x, y) = \nabla^2_{xx}L(x, y) + \lambda I, \quad \lambda \geq 0,$$

where I is the identity matrix in $\mathbb{R}^{n,n}$. The regularization prevents convergence to a local maximum. Parameter λ is chosen big enough to guarantee that the regularized Hessian $\hat{H}(x, y)$ is positive definite. The interior point method generates a sequence $\{x^k, w^k, y^k\}$ as described above for a sequence of positive barrier parameters $\{\mu^k\}$ converging to zero.

The detailed description of the algorithm can be found in [1, 19, 20, 21]. We draw the reader's attention, however, to the fact that the sequence of slack variables $\{w^k\}$ is required to stay positive throughout the computation. Starting with a strictly positive vector w^0, the interior point algorithm chooses the steplength small enough to keep the slack variables and Lagrange multipliers positive and, thus, prevents the trajectory of the method from hitting the boundary of the feasible set. When there is a risk for the slacks to become zero, the algorithm reduces the steplength.

Sometimes, however, the steplength becomes too small, which compromises the convergence of the algorithm. When this happens, we switch to the exterior point method. The trajectory of the exterior point method is allowed to leave the feasible set. Therefore it is not necessary to keep the slack variables positive. Usually, the trajectory of the exterior point method approaches solution outside of the feasible set.

3 The exterior point method.

The exterior point methods are related to the nonlinear rescaling principle the same way as the interior point methods are related to the sequential unconstrained minimization technique. The exterior point method is also known as the primal-dual nonlinear rescaling method and described in [7, 16]. Here we just review its basic principles.

Let $-\infty \leq t_0 < 0 < t_1 \leq \infty$. We consider a class Ψ of twice continuously differential functions $\psi : (t_0, t_1) \rightarrow \mathbb{R}$ that satisfy the following properties

1^0. $\psi(0) = 0$.

2^0. $\psi'(t) > 0$.

3^0. $\psi'(0) = 1$.

4^0. $\psi''(t) < 0$.

5^0. there is $a > 0$ such that $\psi(t) \leq -at^2$, $t \leq 0$.

6^0. a) $\psi'(t) \leq bt^{-1}$, b) $-\psi''(t) \leq ct^{-2}$, $t > 0$, $b > 0$, $c > 0$.

Let us consider a few transformations $\psi \in \Psi$.

1. Exponential transformation [8]

$$\psi_1(t) = 1 - e^{-t}.$$

2. Logarithmic modified barrier function [12]

$$\psi_2(t) = \log(t + 1).$$

3. Hyperbolic modified barrier function [12]

$$\psi_3(t) = \frac{t}{1 + t}.$$

The exponential transformation $\psi_1(t)$ leads to the exponential multipliers method while the logarithmic and hyperbolic transformations lead to the modified barrier function method. In this paper we use the logarithmic modified barrier function $\psi(t) = \psi_2(t) = \log(t + 1)$ to conduct numerical experiments.

We transform the constraints of problem (1) into an equivalent set of constraints using functions $\psi \in \Psi$.

For any given transformation $\psi \in \Psi$ and any barrier parameter $\mu > 0$ due to $1^0 - 3^0$ the following problem is equivalent to problem (1)

$$\begin{aligned} &\min f(x), \\ &\text{s.t. } \mu\psi\left(\mu^{-1}h_i(x)\right) \geq 0, \quad i = 1, \ldots, m. \end{aligned} \tag{9}$$

The classical Lagrangian $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^1_{++} \to \mathbb{R}^1$ for the equivalent problem (9) that is given by formula

$$\mathcal{L}(x, y, \mu) = f(x) - \mu \sum_{i=1}^{m} y_i \psi(\mu^{-1} h_i(x)).$$

is the main tool for the nonlinear rescaling method. One step of the nonlinear rescaling method maps the given approximation (x, y) to the next $(\hat{x}, \hat{y})$ by the following formulas

$$\hat{x} = argmin \left\{ \mathcal{L}(x, y, \mu) \,|\, x \in \mathbb{R}^n \right\}, \tag{10}$$

$$\hat{y}_i = \psi' \left(\mu^{-1} h_i(\hat{x}) \right) y_i, \quad i = 1, \ldots, m. \tag{11}$$

The Lagrangian for the equivalent problem $\mathcal{L}(x, y, \mu)$ plays in nonlinear rescaling theory a similar role to that the classical barrier function $B(x, \mu)$ plays in the sequential unconstrained minimization technique. But unlike the classical barrier function, the Lagrangian for the equivalent problem $\mathcal{L}(x, y, \mu)$ in addition to the barrier parameter also depends on the Lagrange multipliers associated with each constraint. When based on the logarithmic modified barrier function $\psi_2(t)$, the Lagrangian for the equivalent problem retains the most important properties of the classical barrier function, e.g. self-concordance [11]. This gives similar complexity results for the method with the fixed Lagrange multipliers and decreasing barrier parameter [12]. At the same time, the nonlinear rescaling principle eliminates the main problems of the sequential unconstrained minimization technique associated with the singularity of the classical barrier function $B(x, \mu)$ and its derivatives at the solution. In particular, the nonlinear rescaling method keeps stable the Newton area for unconstrained minimization and exhibits the “hot start” phenomenon [12] under the standard second order optimality conditions: from some point along the trajectory the primal approximation remains in the area where Newton’s method is well defined after each Lagrange multipliers update.

Figure 1 demonstrates the sequential unconstrained minimization technique and the nonlinear rescaling principle for the following problem

$$\min x^2,$$

s.t.

$$x \geq 1, \quad x \geq 0.$$

The solution of this problem is $x^* = 1$. The area where Newton’s method is well defined for minimization of the classical log-barrier function shrinks to a point near the solution while the Newton area for the minimization of the Lagrangian $\mathcal{L}(x, y, \mu)$ for the equivalent problem stays stable.

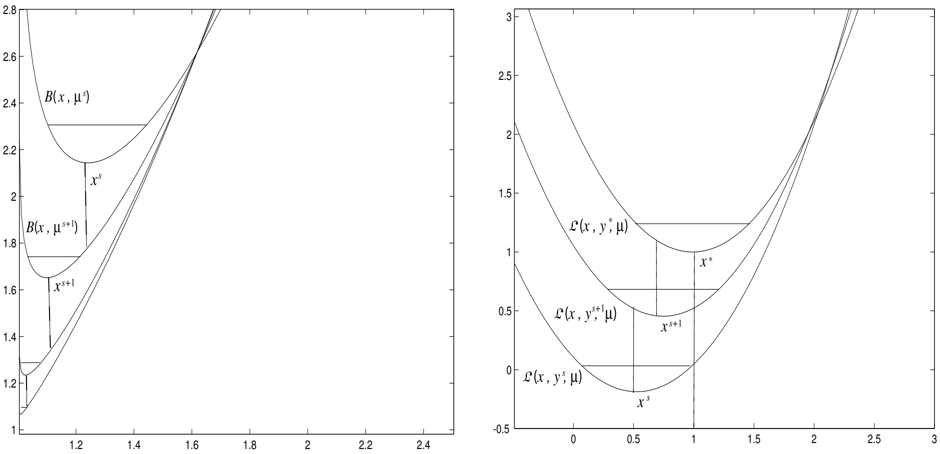
The exterior point method was developed to avoid unconstrained minimization at each step. One step of the nonlinear rescaling method (10)-(11) is equivalent to solving the for $(\hat{x}, \hat{y})$ the following primal-dual system

$$\nabla_x \mathcal{L}(\hat{x}, \hat{y}, \mu) = \nabla f(\hat{x}) - \sum_{i=1}^{m} \psi' \left(\mu^{-1} h_i(\hat{x})\right) y_i \nabla h_i(\hat{x}) = 0, \tag{12}$$

$$\hat{y}_i = \psi' \left(\mu^{-1} h_i(\hat{x})\right) y_i, \quad i = 1, \ldots, m. \tag{13}$$

After replacing the terms $\psi' \left(\mu^{-1} h_i(\hat{x})\right) y_i$ in (12) by $\hat{y}_i$, $i = 1, \ldots, m$, we obtain another equivalent nonlinear system

$$\nabla_x L(\hat{x}, \hat{y}) = \nabla f(\hat{x}) - \sum_{i=1}^{m} \hat{y}_i \nabla h_i(\hat{x}) = 0, \tag{14}$$



sequential unconstrained minimization technique

nonlinear rescaling

Figure 1: Newton areas for minimization

$$\hat{y} - \Psi'\left(\mu^{-1}h(\hat{x})\right) y = 0, \tag{15}$$

where $\Psi'\left(\mu^{-1}h(\hat{x})\right) = diag\left(\psi'\left(\mu^{-1}h_i(\hat{x})\right)\right)_{i=1}^{m}$.

Assuming that $\hat{x} = x + \Delta x$, $\hat{y} = y + \Delta y$, and by linearizing (14)-(15) we obtain the following system for finding the primal-dual Newton direction $(\Delta x, \Delta y)$

$$\begin{bmatrix} \nabla^2_{xx} L(x,y) & -\nabla h^T(x) \\ -\mu^{-1}\Psi''(\cdot)\, Y \nabla h(x) & I \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\nabla_x L(x,y) \\ \bar{y} - y \end{bmatrix},$$

where $\nabla^2_{xx} L(x,y)$ is the Hessian of the Lagrangian $L(x,y)$, $\bar{y} = \Psi'\left(\mu^{-1}h(x)\right) y$, $\Psi''(\cdot) = \Psi''\left(\mu^{-1}h(x)\right) = diag\left(\psi''\left(\mu^{-1}h_i(x)\right)\right)_{i=1}^{m}$, I is identical matrix in $\mathbb{R}^{m,m}$ and $Y = diag\,(y_i)_{i=1}^{m}$. After negating the first equation and multiplying the second one by $-\mu[\Psi''(\cdot)\, Y]^{-1}$ we obtain the following system

$$\begin{bmatrix} -\nabla^2_{xx} L(x,y) & \nabla h^T(x) \\ \nabla h(x) & -\mu[\Psi''(\cdot)\, Y]^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla_x L(\cdot) \\ -\mu[\Psi''(\cdot)\, Y]^{-1}(\bar{y} - y) \end{bmatrix}. \tag{16}$$

In particular, for the transformation $\psi(t) = \log(t+1)$ used for the numerical experiments in this paper, system (16) becomes

$$\begin{bmatrix} -\nabla^2_{xx} L(x,y) & \nabla h^T(x) \\ \nabla h(x) & \mu^{-1}[H(x)+\mu I]^2 Y^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla_x L(\cdot) \\ \mu^{-1}[H(x)+\mu I]^2 Y^{-1}(\bar{y}-y) \end{bmatrix},$$

where $H(x) = diag\,(h_i(x))_{i=1}^m$.

One step of the exterior point method updates the current primal-dual approximation as follows

$$\hat{x} = x + \Delta x, \tag{17}$$

$$\hat{y} = y + \Delta y. \tag{18}$$

To avoid convergence to a local maximum we regularize the Hessian $\nabla^2_{xx} L(x,y)$ as before

$$R_\lambda(x,y) = \nabla^2_{xx} L(x,y) + \lambda I, \quad \lambda \ge 0.$$

The matrix in system (16) has the same structure and sparsity pattern as those in system (8). Therefore we can use the sparse numerical linear algebra technology developed in [18] for LOQO.

Being efficient in the neighborhood of the primal-dual solution, method (16)-(18) may not converge globally. To control the convergence we chose *a priori* a factor $0 < q < 1$ and introduce the merit function

$$\nu(x,y) = \max\left\{ \|\nabla_x L(x,y)\|, \, -\min_{1\le i\le m} h_i(x), \, -\min_{1\le i\le m} y_i, \, \sum_{i=1}^m |y_i h_i(x)| \right\}, \tag{19}$$

measuring the violation of the KKT conditions. It is shown in [16] that under the standard second order optimality conditions for any *a priori* chosen factor $0 < q < 1$ there is a neighborhood of the solution where the method (16)-(18) converges linearly with this factor $0 < q < 1$. Also, it is shown in [7] that if functions $f(x)$ and $h_i(x)$ are smooth enough then the merit function converges to zero with the same rate.

We design the exterior point algorithm as follows. If one step of method (16)-(18) does not reduce the value of the merit function $\nu(x, y)$ by the desired factor $0 < q < 1$, we assume that the switch from the interior point method to method (16)-(18) is premature. In this case the trajectory of the exterior point method is beyond the area of the linear convergence with this factor $0 < q < 1$. When it happens, the algorithm follows the trajectory of the nonlinear rescaling method (10)-(11). We use the primal direction Δx obtained from system (16) for the first step of the minimizations (10). It is shown in [7] that this direction is descending for the Lagrangian $\mathcal{L}(x, y, \mu)$, therefore the algorithm does not lose the computational work involved in solving the system (16). Newton's method with a steplength and the regularized Hessian is used for the unconstrained minimization of the Lagrangian (10) followed by the Lagrange multipliers update (11). The detailed description of this algorithm one can find in [7, 16]. Here we mention only that such exterior point method is a globally convergent algorithm to a local minimum for a wide class of problems [16]. Therefore, if the switch to the exterior point method occurs while the trajectory of the algorithm is outside of the area of linear convergence of EPM with given factor $0 < q < 1$, the nonlinear rescaling method (10)-(11) will bring the trajectory to this area.

4 The interior-exterior point method: Numerical results.

The main difference between the interior and exterior point methods is their "driving force" of convergence. The former requires the decrease to zero of the barrier parameter $\mu > 0$. The latter converges due to the information carried by the vector of the Lagrange multipliers y. The interior point method, which has global convergence properties, exhibits robust behavior bringing its trajectory to the neighborhood of the solution. Under the standard second order optimality conditions the exterior point method converges in the neighborhood of the solution with a linear rate under the fixed barrier parameter [16]. If the barrier parameter is decreased, the exterior point method converges locally with the superlinear rate [7]. Therefore, the robustness of the interior point method and the local convergence properties of the exterior point method encourage us to consider the combination of the methods. The methods can augment each other. Indeed, the interior point method can bring the trajectory to the area of a superlinear convergence of the exterior point method, while the exterior point method can improve the convergence in case the interior point method experiences numerical problems.

It is important to properly define the switching rule between the interior and exterior point methods. The ideal approach would be to characterize the area of convergence of the exterior point method and to find a way to verify whether the primal-dual trajectory is in this convergence area. This is the subject of future research. Currently, we are using the following simple consideration for the switching criteria.

The interior point method implemented in LOQO produced strong results on COPS [2] and CUTE [3] problem sets. It solved 86.8% of the COPS problems and 85.1% of the CUTE problems with a default accuracy setting of 8 digits of agreement between primal and dual objective functions and a primal-dual infeasibility measure of `1e-6`. So we decided to switch to the exterior point method only if the interior point method stops making progress.

Although there is a chance of a false detection that the interior point method is not making progress, the switching rule is still attractive due to its simplicity. It allows us to test the hypothesis that the exterior point method is capable of solving the problems that the interior point method cannot solve alone and thus the combination of these two methods, the interior-exterior point method (IEPM) can solve more problems than the interior or exterior point methods can individually.

To define the switching rule formally we use the merit function $\nu(x, y)$ (19) that controls convergence of the algorithm. The switching rule from the interior point method to the exterior point method is conservative. If for seven consecutive iterations the interior point method does not reduce the value of merit function $\nu(x, y)$ and the steplength α becomes less than the requested accuracy of the solution, then the algorithm switches to the exterior point method. In other words, we detect the "stalling" situations when the algorithm fails to evolve. Also, the algorithm switches to the exterior point method if for one third of the iteration limit the interior point method does not reduce the best achieved value of the merit function $\nu(x, y)$, even if the steplength is bigger than the requested accuracy. In other words, the algorithm detects if there is no overall progress for a large number of iterations. The interior-exterior point algorithm and the switching rule are formally described in Figure 2.

Testing the interior-exterior point method on the COPS set has resulted in an increase of the number of solved problems from 59 to 64 out of 68. It means reducing by more than half the number of unsolved problems. The additional problems that the interior-exterior point method solved are *Hanging chain* for $n_h = 200$, *Minimal surface with obstacle* for $n_y = 100$, *linear tangent particle steering* for $n_h = 100$, $n_h = 200$ and $n_h = 400$. The statistics for the solutions of these problems are shown in Table 1. We show the solution time in seconds, values of both primal and dual functions, both primal and dual infeasibility and a number of iterations. If the iteration number exceeds 500, we call the problem not solved.

Step 1: **Initialization:**
An initial primal approximation $x^0 \in \mathbb{R}^n$ is given.
Initial slacks $w^0 \in \mathbb{R}^n_{++}$ and Lagrange multipliers $y^0 \in \mathbb{R}^m_{++}$ are given.
An accuracy parameter $\varepsilon > 0$ is given.
Set $(x, w, y) := (x^0, w^0, y^0)$, $rec := \nu(x, y)$, $it := 0$, $cnt_1 := 0$, $cnt_2 := 0$.
Step 2: If $r \leq \varepsilon$, Stop, **Output**: x, y.
Step 3: Find new $(\hat{x}, \hat{w}, \hat{y}) = IPMSTEP(x, w, y)$, $it := it + 1$.
Step 4: If $\nu(\hat{x}, \hat{y}) \leq \varepsilon$, Stop, **Output**: $\hat{x}$, $\hat{y}$.
Step 5: If $it = itlim$, Stop, **Output**: Solution is not found.
Step 6: If $\nu(\hat{x}, \hat{y}) \leq 0.99\nu(x, y)$ or $steplength \geq \varepsilon$, Set $cnt_1 := 0$, Goto Step 8.
Step 7: Set $cnt_1 := cnt_1 + 1$.
Step 8: If $\nu(\hat{x}, \hat{y}) \leq 0.99rec$, Set $rec := \nu(\hat{x}, \hat{y})$, $cnt_2 := 0$, Goto Step 10.
Step 9: Set $cnt_2 := cnt_2 + 1$.
Step 10: If $cnt_1 < 7$ and $3 * cnt_2 < itlim$, Set $(x, w, y) := (\hat{x}, \hat{w}, \hat{y})$, Goto Step 3.
Step 11: Find new $(\hat{x}, \hat{w}, \hat{y}) = EPMSTEP(x, w, y)$, $it := it + 1$.
Step 12: If $\nu(\hat{x}, \hat{w}, \hat{y}) \leq \varepsilon$, Stop, **Output**: $\hat{x}$, $\hat{y}$.
Step 13: If $it = itlim$, Stop, **Output**: Solution is not found.
Step 14: Goto Step 11.

Figure 2: IEPM

Testing the method on the CUTE set increased the number of the solved problems from 967 to 1003 out of 1135. If we increase the level of accuracy of the solution to 12 digits of agreement between primal and dual functions and infeasibility measure `1e-12`, the interior point method solves 879 problems and the interior-exterior point method solves 936, an improvement of 57 additional solved problems. It is appropriate to mention that the interior-exterior point method actually solved 60 problems that have not been solved by the interior point method. On the other hand the interior-exterior point method did not solve 3 problems that the interior point method solved alone.

Tables 2-9 present the behavior of the algorithms. The tables show the statistics of the problem, the primal and dual objective values, the primal and dual infeasibility and the number of digits of agreement between the primal and the dual functions, which characterizes the primal-dual gap.

	chain	Min surface	linear tangent particle steering		
	$n_h = 200$	$n_y = 100$	$n_h = 100$	$n_h = 200$	$n_h = 400$
time (sec)	44.01	90.81	7.89	11.29	24.07
pr. val =	5.068917340	2.506949256	0.554595401	0.554577016	0.554572413
dual val =	5.068917342	2.506949256	0.554595401	0.554577016	0.554572413
pr. inf =	3.5e-10	4.9e-09	5.7e-09	3.3e-11	1.3e-11
dual inf =	5.7e-11	4.1e-12	5.6e-10	1.4e-10	1.2e-10
steps	218	171	278	157	163

Table 1: COPS

Tables 2-4 demonstrate the superiority of the interior-exterior point method over individual performance of the interior and exterior point methods. It took 150 iterations of the interior point method to solve the problem *trigger* (Table 2). The exterior point method by itself reached the iterations limit due to its slow progress towards the solution (Table 3). However, the interior-exterior point method solves the problem *trigger* in 55 iterations (Table 4).

Tables 5-8 show the behavior of the interior-exterior point method for some problems. Table 5 shows the performance of the interior-exterior point method for problem *hager1*. The interior point method had numerical difficulties reducing the dual infeasibility below `1e-4` while the exterior point method achieved the desired level of accuracy in one iteration. Table 6 shows the performance of the interior-exterior point method for problem *spanhyd*. The interior point method could not obtain the desired primal-dual gap before the exterior point method attained it. The behavior of the interior-exterior point method for problem *ubh1* is shown in Table 7. Again, the interior point method failed to improve the dual infeasibility beyond the level of `1e-4`, the exterior point method passed this threshold. Table 8 shows the ability of the exterior point method to achieve very accurate solutions. To demonstrate this we set the accuracy level to 12 digits of agreement between the primal and dual functions and `1e-12` infeasibility tolerance.

It is common for the interior-exterior point method to exhibit higher primal infeasibility after switching to the exterior point method. It reflects the effect of "open boundaries," when the trajectory of the algorithm goes outside of the feasible set to overcome the numerical problems in the neighborhood of the solution. Afterwards, however, the primal and dual infeasibility, as well as the duality gap, decrease with, at least, a linear rate.

Tables 5-8 show that the interior point method performs several steps without any improvement before the exterior point method is used. These "stalling" situations are simple to detect, however, in some cases the interior point method does not make progress without exhibiting "stalling". Such a behavior of the interior point method occurs for the difficult nonconvex nonlinear problems. The interior point method struggles to approach the local minima and eventually succeeds in many cases. Using the exterior point method prematurely can worsen the behavior of the algorithm. The trajectory of the algorithm at this point can be far from the area of fast convergence of the exterior point method therefore the algorithm exceeds the limit of iterations. This is why the interior-exterior point method did not solve several problems from the CUTE set, which the interior point method could solve by itself. Nevertheless, using the exterior point method in LOQO has demonstrated an improvement.

Table 9 shows the performance of the interior-exterior point method for the *steering* problem from the COPS set with $n_h = 400$. This problem appeared to be difficult for the interior point method, while for the exterior point method it took 8 iterations to obtain the solution with high accuracy. This particular problem could be solved by the exterior point method alone in 8 iterations. However, we used the same "conservative" switching rule for all the tested problems. The interior point method implemented in LOQO is the robust algorithm and we advocate the strategy of giving the interior point method the chance to converge by itself.

5 Concluding remarks.

The extensive numerical testing of the interior-exterior point method has shown that the interior point method and the exterior point method are capable of augmenting each other. Their combined performance is better than either method can achieve individually.

One reason for the appearing of numerical problems that the interior point method experience near the solution is related to the essential need of the algorithm to keep slack variables positive and the ill-condition of the system for finding the Newton directions. At the solution the values of slack variables, corresponding to the active constraints are zero. Therefore, when the trajectory of the interior point method approaches the solution the Newton directions must be computed very precisely to avoid premature annulling of the slacks. However, the closer the trajectory of the interior point method gets to the solution, the greater are the numerical errors. These errors occur because of the ill-condition of the system for Newton directions (8). As a result, the steplength, which keeps the slacks positive, becomes very small and numerical problems occur. On the other hand, the exterior point method is less likely to exhibit such behavior. There is no need for the exterior point method to keep slacks positive. The method allows the trajectory to leave the interior of the feasible set. Also, the system for Newton directions is better conditioned under the standard second order optimality conditions [7, 16].

The interior and exterior point methods stem from different ideologies. The interior point method is closely related to the sequential unconstrained minimization technique with the classical log-barrier function, while the exterior point method is based on the nonlinear rescaling theory. However, both

variables: non-neg 0, free 6, bdd 0, total 6
constraints: eq 6, ineq 0, ranged 0, total 6

	primal		dual		Sig
Iter	Obj Value	Infeas	Obj Value	Infeas	Fig
interior point method...					
1	0.000000e+00	1.1e+01	0.000000e+00	1.1e+01	60
2	0.000000e+00	1.4e-01	1.403131e+00	6.2e-02	
3	0.000000e+00	1.2e-01	5.146799e-01	5.4e-02	
4	0.000000e+00	4.0e-02	2.080966e-01	5.0e-02	1
	....				
24	0.000000e+00	1.4e-04	2.105166e-09	7.6e-05	9
	....				
44	0.000000e+00	4.9e-08	2.984260e-09	1.9e-04	9
	....				
64	0.000000e+00	5.3e-09	5.966625e-10	2.9e-05	9
	....				
84	0.000000e+00	7.9e-10	9.923364e-11	1.1e-05	10
	....				
104	0.000000e+00	8.3e-11	3.319541e-11	1.2e-06	10
	....				
124	0.000000e+00	8.1e-11	3.317923e-11	1.2e-06	10
	....				
148	0.000000e+00	7.9e-11	3.315807e-11	1.1e-06	10
149	0.000000e+00	7.9e-11	3.315777e-11	1.1e-06	10
150	0.000000e+00	6.2e-12	3.334547e-11	8.8e-07	10
Solution time: 0.36 sec					

Table 2: trigger

variables: non-neg 0, free 6, bdd 0, total 6
constraints: eq 6, ineq 0, ranged 0, total 6

	primal		dual		Sig
Iter	Obj Value	Infeas	Obj Value	Infeas	Fig
exterior point method...					
1	0.000000e+00	4.9e+01	0.000000e+00	0.0e+00	60
2	0.000000e+00	4.7e-01	4.060030e-12	1.4e-10	11
3	0.000000e+00	4.6e-01	9.338127e+02	4.3e+00	
	....				
50	0.000000e+00	9.9e-04	6.527877e-03	1.9e+00	2
	....				
100	0.000000e+00	6.4e-04	2.725498e-03	1.5e+00	3
	....				
200	0.000000e+00	3.6e-04	8.279044e-04	1.2e+00	3
	....				
300	0.000000e+00	2.4e-04	3.829583e-04	1.0e+00	3
	....				
400	0.000000e+00	1.8e-04	2.185300e-04	9.2e-01	4
	....				
500	0.000000e+00	1.5e-04	1.382180e-04	8.6e-01	4
ITERATIONS LIMIT					

Table 3: trigger

```
variables:  non-neg   0,   free   6,   bdd      0,   total   6
constraints:  eq      6,   ineq   0,   ranged   0,   total   6
```

Iter	primal Obj Value	primal Infeas	dual Obj Value	dual Infeas	Sig Fig
interior point method...					
1	0.000000e+00	1.1e+01	0.000000e+00	1.1e+01	60
2	0.000000e+00	1.4e-01	1.403131e+00	6.2e-02	
3	0.000000e+00	1.2e-01	5.146799e-01	5.4e-02	
4	0.000000e+00	4.0e-02	2.080966e-01	5.0e-02	1
...	...	...	...	...	...
24	0.000000e+00	1.4e-04	2.105166e-09	7.6e-05	9
...	...	...	...	...	...
51	0.000000e+00	3.8e-08	1.490365e-09	9.0e-05	9
52	0.000000e+00	3.7e-08	1.263796e-09	8.5e-05	9
53	0.000000e+00	3.7e-08	1.209599e-09	8.5e-05	9
exterior point method...					
54	0.000000e+00	1.9e-05	2.875654e-11	3.3e-07	11
55	0.000000e+00	4.7e-10	2.877929e-11	5.2e-13	11
Solution time: 0.08 sec					

Table 4: trigger

the interior and exterior point methods are more efficient than their "parent" methods, which are based on sequential unconstrained minimization. Both the interior and exterior point methods solve the primal-dual systems by Newton's method. The systems for finding Newton directions have the same sparsity pattern but different properties. In particular, the system in the exterior point method is well-conditioned under the standard second order optimality conditions. This fact contributes to better local convergence properties of the exterior point method. Moreover, in the neighborhood of the solution the exterior point method is equivalent to Newton's method for solving the Lagrange system of equations that corresponds to the active constraints [7]. The latter system does not have complementarity constraints. Therefore one can expect robust and efficient behavior of the exterior point method as it approaches the solution. On the other hand, the interior point method is more efficient on early stages of the computations when it brings the trajectory to the neighborhood of the primal-dual solution.

```
variables:   non-neg      0,   free   10000,   bdd      0,   total   10000
constraints:  eq      5000,   ineq       0,   ranged   0,   total    5000
```

Iter	primal		dual		Sig
	Obj Value	Infeas	Obj Value	Infeas	Fig
interior point method...					
1	0.000000e+00	1.0e+00	0.000000e+00	1.0e+02	60
2	2.539833e-01	4.6e-01	2.053126e+03	4.6e+01	
3	4.681954e-01	3.2e-01	1.566904e+03	3.2e+01	
4	5.135067e-01	3.0e-01	1.405465e+03	3.0e+01	
5	1.502854e+00	1.5e-01	-5.364568e+02	1.5e+01	
6	1.706352e+00	1.1e-02	-1.165211e+01	1.1e+00	
7	1.577373e+00	4.7e-03	-4.258698e+00	4.7e-01	
8	1.442700e+00	3.3e-03	-2.323425e+00	3.3e-01	
9	1.021528e+00	1.8e-03	3.351165e+00	1.8e-01	
10	9.198403e-01	4.3e-04	1.496132e+00	4.3e-02	1
11	8.823541e-01	6.7e-05	9.673408e-01	6.7e-03	1
12	8.807931e-01	4.0e-06	8.808010e-01	4.0e-04	5
13	8.807978e-01	2.1e-07	8.807982e-01	1.7e-03	7
14	8.807975e-01	1.0e-08	8.807975e-01	8.1e-03	8
15	8.807975e-01	5.2e-10	8.807975e-01	6.2e-04	9
16	8.807975e-01	2.6e-11	8.807975e-01	5.2e-04	11
17	8.807975e-01	1.3e-12	8.807975e-01	5.0e-04	12
18	8.807975e-01	6.5e-14	8.807975e-01	5.0e-04	13
19	8.807975e-01	7.5e-15	8.807975e-01	5.0e-04	15
20	8.807975e-01	6.8e-15	8.807975e-01	5.1e-04	15
21	8.807975e-01	6.9e-15	8.807975e-01	5.1e-04	15
22	8.807975e-01	6.8e-15	8.807975e-01	5.0e-04	15
23	8.807975e-01	6.9e-15	8.807975e-01	5.0e-04	15
24	8.807975e-01	6.9e-15	8.807975e-01	5.0e-04	14
25	8.807975e-01	6.9e-15	8.807975e-01	5.0e-04	14
26	8.807975e-01	6.9e-15	8.807975e-01	5.0e-04	15
exterior point method...					
27	8.807971e-01	6.1e-10	8.807971e-01	1.3e-10	10
Solution time: 9.17 sec					

Table 5: hager1

The numerical results obtained for the interior-exterior point method confirm that the exterior point method, if used in the final stage of the computations, allowed us to obtain solutions with high level of accuracy for 21% and 55% of unsolved CUTE and COPS problems respectively. The obtained results show that the interior and the exterior point methods can complement and augment each other and their combination can overcome some difficulties faced by these methods individually.

Important issues remain for future research. First, it is interesting to

```
variables:  non-neg    0,   free   0,   bdd      72,   total   72
constraints:  eq      32,   ineq   0,   ranged    0,   total   32
```

Iter	primal Obj Value	primal Infeas	dual Obj Value	dual Infeas	Sig Fig
interior point method...					
1	1.559264e+09	2.8e-03	-6.315201e+08	1.6e+00	
2	1.558885e+09	2.8e-03	-6.327803e+08	1.6e+00	
3	1.558993e+09	2.8e-03	-6.454882e+08	1.6e+00	
4	1.557228e+09	2.8e-03	-6.723412e+08	1.6e+00	
5	1.554499e+09	2.8e-03	-7.093762e+08	1.6e+00	
6	1.545694e+09	2.8e-03	-7.906524e+08	1.6e+00	
7	1.526434e+09	2.7e-03	-9.122066e+08	1.6e+00	
8	1.450789e+09	2.7e-03	-1.283512e+09	1.7e+00	
9	1.280181e+09	2.5e-03	-1.918232e+09	1.7e+00	
10	6.522237e+08	1.8e-03	-2.609382e+09	1.7e+00	
11	1.857241e+08	9.4e-04	-1.427218e+09	1.2e+00	
12	5.991890e+06	1.6e-04	-2.968070e+08	2.4e-01	
13	6.007828e+04	1.6e-05	-5.676497e+07	2.4e-02	
14	5.914511e+03	5.0e-06	-1.810637e+07	7.4e-03	
15	1.634570e+03	2.2e-06	-1.481901e+07	4.4e-03	
16	2.524663e+02	2.7e-07	-1.788272e+06	4.2e-04	
17	2.397756e+02	2.1e-08	-9.697436e+04	2.3e-05	
18	2.397381e+02	1.1e-09	-4.695920e+03	1.2e-06	
19	2.397380e+02	5.5e-11	-8.151921e+00	5.8e-08	
20	2.397380e+02	2.8e-12	2.273012e+02	3.5e-09	1
21	2.397380e+02	1.4e-13	2.391289e+02	8.9e-10	3
22	2.397380e+02	7.6e-15	2.397205e+02	6.8e-10	4
23	2.397380e+02	5.8e-16	2.397360e+02	1.2e-09	5
24	2.397380e+02	7.0e-16	2.397360e+02	1.2e-09	5
25	2.397380e+02	6.8e-16	2.397360e+02	1.2e-09	5
26	2.397380e+02	7.4e-16	2.397360e+02	1.2e-09	5
27	2.397380e+02	7.3e-16	2.397360e+02	1.2e-09	5
28	2.397380e+02	7.4e-16	2.397360e+02	1.2e-09	5
29	2.397380e+02	7.6e-16	2.397360e+02	1.2e-09	5
30	2.397380e+02	7.8e-16	2.397360e+02	1.2e-09	5
31	2.397380e+02	7.8e-16	2.397360e+02	1.2e-09	5
exterior point method...					
32	2.397380e+02	4.6e-09	2.397379e+02	2.1e-10	7
33	2.397380e+02	1.0e-12	2.397380e+02	5.2e-13	8
Solution time: 0.23 sec					

Table 6: spanhyd

```
variables:  non-neg       0,   free  11994,   bdd      6003,   total   17997
constraints:  eq      12000,   ineq       0,   ranged      0,   total   12000
```

Iter	primal Obj Value	primal Infeas	dual Obj Value	dual Infeas	Sig Fig
interior point method...					
1	0.000000e+00	1.0e+00	-1.200600e+04	1.1e+02	
2	8.822437e-03	1.0e+00	-6.424551e+03	1.1e+02	
3	3.481630e-02	1.0e+00	-3.026802e+03	1.1e+02	
4	1.300931e-01	1.0e+00	6.367401e+02	1.1e+02	
5	3.761932e-01	1.0e+00	3.436357e+03	1.1e+02	
6	8.771693e-01	9.9e-01	5.995151e+03	1.1e+02	
7	1.585928e+00	9.9e-01	8.197131e+03	1.1e+02	
8	2.495221e+00	9.8e-01	1.032752e+04	1.1e+02	
9	3.419941e+00	9.8e-01	1.218742e+04	1.1e+02	
10	4.666799e+00	9.7e-01	1.437898e+04	1.1e+02	
...					
40	3.196855e+01	8.2e-01	4.206661e+04	8.9e+01	
...					
90	8.017669e+01	4.8e-01	4.759204e+04	5.2e+01	
...					
140	1.493845e+02	9.6e-02	1.294405e+04	1.0e+01	
...					
190	2.668066e+00	3.3e-06	3.310610e+00	3.6e-04	1
...					
240	1.550585e+00	4.1e-15	1.550585e+00	1.2e-03	9
...					
290	1.541456e+00	4.0e-15	1.541456e+00	7.5e-04	11
...					
340	1.533541e+00	4.1e-15	1.533541e+00	8.4e-04	13
341	1.533396e+00	4.1e-15	1.533396e+00	8.2e-04	13
342	1.533237e+00	4.0e-15	1.533237e+00	8.1e-04	13
343	1.533092e+00	4.1e-15	1.533092e+00	8.1e-04	13
344	1.532933e+00	4.0e-15	1.532933e+00	8.0e-04	13
345	1.532789e+00	4.1e-15	1.532789e+00	7.9e-04	13
346	1.532630e+00	4.0e-15	1.532630e+00	8.0e-04	13
347	1.532486e+00	4.1e-15	1.532486e+00	8.0e-04	13
348	1.532328e+00	4.1e-15	1.532328e+00	8.5e-04	13
349	1.532185e+00	4.2e-15	1.532185e+00	8.2e-04	13
350	1.532027e+00	4.1e-15	1.532027e+00	8.6e-04	13
351	1.531884e+00	4.1e-15	1.531884e+00	8.3e-04	13
352	1.531726e+00	4.0e-15	1.531726e+00	8.4e-04	13
exterior point method...					
353	1.116001e+00	2.2e-04	1.116000e+00	2.7e-09	6
354	1.116001e+00	1.8e-08	1.116001e+00	1.1e-09	11
Solution time: 279.90 sec					

Table 7: ubh1

```
variables:   non-neg    198,   free   20002,   bdd      0,   total   20200
constraints:  eq      9996,   ineq       0,   ranged   0,   total    9996
```

Iter	primal Obj Value	primal Infeas	dual Obj Value	dual Infeas	Sig Fig
interior point method...					
1	1.010000e+04	2.8e+00	1.029800e+04	2.0e+00	2
2	4.682848e+03	1.5e+00	1.823216e+04	1.0e+00	
3	4.369838e+03	1.4e+00	2.879688e+04	1.0e+00	
4	4.412801e+03	1.4e+00	5.728435e+04	9.9e-01	
5	6.319269e+03	1.4e+00	1.720124e+05	9.6e-01	
6	1.794380e+04	1.3e+00	3.801494e+05	9.0e-01	
7	5.442827e+04	1.2e+00	6.399589e+05	8.2e-01	
8	2.830191e+05	8.6e-01	1.209450e+06	6.0e-01	
9	7.138364e+05	5.2e-01	1.606252e+06	3.7e-01	
10	1.506710e+06	1.3e-01	1.813354e+06	8.9e-02	1
11	1.774080e+06	1.7e-02	1.819506e+06	1.2e-02	2
12	1.815949e+06	9.4e-04	1.818467e+06	6.7e-04	3
13	1.818270e+06	4.8e-05	1.818393e+06	3.4e-05	4
14	1.818386e+06	2.4e-06	1.818393e+06	1.7e-06	5
15	1.818392e+06	1.2e-07	1.818393e+06	8.6e-08	7
16	1.818393e+06	6.7e-09	1.818393e+06	4.7e-09	8
17	1.818393e+06	4.1e-10	1.818393e+06	2.9e-10	9
18	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
19	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
20	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
21	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
22	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
23	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
24	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
25	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
26	1.818393e+06	3.7e-11	1.818393e+06	2.6e-11	10
exterior point method...					
27	1.818393e+06	7.8e-03	1.818393e+06	1.0e-12	10
28	1.818393e+06	1.3e-06	1.818393e+06	1.4e-12	13
29	1.818393e+06	7.1e-11	1.818393e+06	3.6e-14	16
30	1.818393e+06	4.8e-13	1.818393e+06	2.9e-14	17
Solution time: 64.05 sec					

Table 8: aug2dc

```
variables:   non-neg      1,   free   1598,   bdd      401,   total   2000
constraints:  eq      1601,   ineq      0,   ranged     0,   total   1601
```

	primal		dual		Sig
Iter	Obj Value	Infeas	Obj Value	Infeas	Fig
1	0.000000e+00	2.0e+00	-1.259779e+03	2.7e+02	
2	-1.339301e-03	1.8e+00	-2.946247e+03	9.8e+00	
3	9.256311e-02	1.7e+00	-6.361842e+03	8.4e-01	
4	1.218290e-01	1.6e+00	-1.108174e+04	1.1e+00	
5	3.033407e-01	6.8e-01	-5.094542e+03	1.1e+00	
6	3.878316e-01	3.3e-01	-2.158297e+03	3.6e-01	
7	4.059846e-01	2.5e-01	1.096162e+03	3.4e-01	
8	4.139702e-01	2.4e-01	9.871021e+03	3.5e-01	
9	4.282002e-01	2.3e-01	2.872757e+04	2.5e-01	
10	4.388559e-01	2.2e-01	5.131689e+04	2.2e-01	
11	4.513439e-01	2.2e-01	7.843205e+04	1.9e-01	
12	4.563174e-01	2.1e-01	9.806644e+04	1.6e-01	
	....	....	....	....	
22	4.949472e-01	1.9e-01	5.858241e+05	5.0e-02	
	....	....	....	....	
42	5.082092e-01	1.7e-01	4.364400e+06	1.5e-02	
	....	....	....	....	
62	5.115278e-01	1.7e-01	1.542371e+07	8.2e-03	
	....	....	....	....	
82	5.134261e-01	1.6e-01	3.275524e+07	6.9e-03	
	....	....	....	....	
102	5.154642e-01	1.5e-01	5.452473e+07	5.9e-03	
	....	....	....	....	
122	5.200465e-01	1.5e-01	7.754123e+07	5.4e-03	
	....	....	....	....	
142	5.244719e-01	1.5e-01	1.140656e+08	4.6e-03	
	....	....	....	....	
152	5.274492e-01	1.4e-01	1.299771e+08	4.4e-03	
153	5.283641e-01	1.4e-01	1.343009e+08	4.4e-03	
154	5.287349e-01	1.4e-01	1.359851e+08	4.4e-03	
155	5.288309e-01	1.4e-01	1.364162e+08	4.4e-03	
156	5.290971e-01	1.4e-01	1.375800e+08	4.3e-03	
157	5.301784e-01	1.4e-01	1.422321e+08	4.5e-03	
exterior point method...					
158	4.468802e-01	4.2e-02	-1.884222e+00	1.4e+00	
159	5.060117e-01	6.7e-02	5.607121e-01	8.3e-01	1
160	5.460477e-01	9.0e-03	5.546379e-01	3.1e-01	2
161	5.543196e-01	2.2e-04	5.545523e-01	3.0e-03	4
162	5.545707e-01	1.4e-06	5.545720e-01	2.0e-06	6
163	5.545724e-01	1.3e-11	5.545724e-01	1.2e-10	10
164	5.545724e-01	7.1e-15	5.545724e-01	6.2e-15	16
Solution time: 24.61 sec					

Table 9: steering

characterize the area of linear and superlinear convergence of the exterior point method. Such analysis would contribute to better characterization of the switching rule between the two methods. Second, it is important to understand better the global convergence properties of the interior point method. It would validate that the interior point method brings its trajectory in the neighborhood of the primal-dual solution. Finally, there are still problems that the interior-exterior point method could not solve. So further improvement of the method is essential.

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