Mathematical Programming manuscript No.
(will be inserted by the editor)

Alexandre Belloni · Robert M. Freund

SYMMETRY POINTS OF A CONVEX SET:

Basic Properties and Computational Complexity

June, 2004

Abstract. Given a convex body $S \subset \mathbb{R}^n$ and a point $x \in S$, let $\mathrm{sym}(x,S)$ denote the symmetry value of x in S:

$$\mathrm{sym}(x,S) := \max\{\alpha \geq 0 : x + \alpha(x - y) \in S \text{ for every } y \in S\} \ ,$$

which essentially measures how symmetric S is about the point x, and define

$$\mathrm{sym}(S) := \max_{x \in S} \ \mathrm{sym}(x,S) \ .$$

We call x^* a symmetry point of S if x^* achieves the above supremum. These symmetry measures are all invariant under invertible affine transformation and/or change in norm, and so are of interest in the study of the geometry of convex sets. Furthermore, these measures arise naturally in the complexity theory of interior-point methods. In this study we demonstrate various properties of $\mathrm{sym}(x,S)$ such as under operations over S, or as a function of x for a fixed S. Several relations with convex geometry quantities like volume, distance and diameter, cross-ratio distance are proved. Set approximation results are also shown. Furthermore, we provide a characterization of symmetry points x^*. When S is polyhedral and is given as the intersection of halfspaces $S = \{x \in \mathbb{R}^n : Ax \leq b\}$, then x^* and $\mathrm{sym}(S)$ can be computed by solving $m+1$ linear programs of size $m \times n$. We also present an interior-point algorithm that, given an approximate analytic center x^a of S, will compute an approximation of $\mathrm{sym}(S)$ to any given relative tolerance ϵ in no more than

$$\left\lceil 10m^{1.5} \ln\left(\frac{10m}{\epsilon}\right) \right\rceil$$

iterations of Newton's method.

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1. Introduction

There is a variety of measures of symmetry (or asymmetry) for convex sets that have been studied over the years, see Grünbaum [4] for example. Herein we study some mathematical properties of the symmetry measure of Minkowski [9], which in all likelihood was the first and most useful such symmetry measure. Given a closed convex set S and a point $x \in S$, define the symmetry of S about x as follows:

$$\mathrm{sym}(x,S) := \max\{\alpha \geq 0 : x + \alpha(x - y) \in S \text{ for every } y \in S\} \ , \qquad (1)$$

Alexandre Belloni: MIT Operations Research Center, E40-149, 77 Massachusetts Ave., Cambridge, Massachusetts 02142, USA. e-mail: `belloni@mit.edu`

Robert M. Freund: MIT Sloan School of Management, 50 Memorial Drive, Cambridge, Massachusetts 02139-4307, USA. e-mail: `rfreund@mit.edu`

which intuitively states that $\mathrm{sym}(x,S)$ is the largest scalar α such that every point $y \in S$ can be reflected through x by the factor α and still lie in S. The symmetry value of S then is:

$$\mathrm{sym}(S) := \max_{x \in S} \ \mathrm{sym}(x,S) \ , \tag{2}$$

and x^* is a *symmetry point* of S if x^* achieves the above supremum (also called a "critical point" in [4],[6] and [9]). S is *symmetric* if $\mathrm{sym}(S) = 1$.

Symmetric convex sets play an important role in convexity theory. Consider the Löwner-John theorem, which states that every convex body $S \subset \mathbb{R}^n$ can be α-rounded for some $\alpha \le n$, whereby $\mathrm{sym}(S) \ge \frac{1}{n}$; however, when S is symmetric, then S can in fact be $\sqrt{n}$-rounded, see [5]. This leads to the question of what, if anything, we can say about bounds on the value α for an α-rounding of S when $\mathrm{sym}(S) < 1$. There are many other geometric properties of convex bodies S that are also connected to $\mathrm{sym}(S)$. Points in S with high symmetry must be far from ∂S (in a relative measure). Furthermore, if $\mathrm{sym}(S)$ is large, then points in S with high symmetry value must be close to another (in a relative measure). Also, there are inter-relationships between "central" properties of symmetry points, the centroid, and the analytic center of S. Notice that $\mathrm{sym}(x,S)$ and $\mathrm{sym}(S)$ are invariant under invertible affine transformation and change in norm. Finally, the relevance of $\mathrm{sym}(x,S)$ has been revived in the complexity theory of interior-point methods for convex optimization, see Nesterov and Nemirovski [10] and Renegar [12].

An outline of this paper is as follows. Section 2 contains general properties of the symmetry values as a function of x and S. In section 3, the geometry of the symmetry function is highlighted through many inequalities involving distances, volumes, and set approximation concepts. Section 4 aims mainly to characterize the symmetry points for general convex sets. Lastly, section 5 is dedicated to develop the complexity of computing the symmetry of a polytope S polyhedra given by the intersection of halfspaces.

Notation. Let $S \subset \mathbb{R}^n$ denote a convex set and $\langle \cdot, \cdot \rangle$ will be the conventional inner product in the appropriate Euclidean space. $\mathrm{int} S$ denotes the interior of S. Using traditional convex analysis notation, we let $\mathrm{aff}(S)$ be the minimal affine subspace that contains S and let $S^\perp$ be its orthogonal subspace complement. The polar of S is defined as $S^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \le 1 \text{ for all } x \in S\}$. Given a convex function $f(\cdot)$, for $x \in \mathrm{dom} f(\cdot)$ the subdifferential of $f(\cdot)$ is defined as $\partial f(x) := \{s \in \mathbb{R}^n : f(y) \ge f(x) + \langle s, y - x \rangle \text{ for all } y \in \mathrm{dom} f(\cdot)\}$.

2. General Properties of $\mathrm{sym}(x,S)$ and $\mathrm{sym}(S)$

We make the following assumption:

Assumption A: S is a convex body, i.e., S is a nonempty closed bounded convex set with a nonempty interior.

When S is convex but is not bounded or closed, then certain properties of $\mathrm{sym}(S)$ break down; we refer the interested reader to Section A for a discussion.

We assume that S has an interior as a matter of convenience, as one can always work with the affine hull of S or its subspace translation with no loss of generality, but at considerable notational and expositional expense.

Notice that the definition of $\mathrm{sym}(x,S)$ given in (1) is equivalent to the following "set-containment" definition:

$$\mathrm{sym}(x,S) = \sup\{\alpha \geq 0 : \alpha(x-S) \subseteq (S-x)\} \ . \tag{3}$$

Intuition suggests that $\mathrm{sym}(x,S)$ inherits many nice properties from the convexity of S, as shown in the following:

Theorem 1. *Under Assumption A,* $\mathrm{sym}(\cdot,S) : S \to [0,1]$ *is a continuous quasiconcave function.*

The proof of this theorem uses the following lemma:

Lemma 1. *Suppose that S is a convex body in a Euclidean space and $x \in intS$ and $\alpha \geq 0$. Then $\alpha < \mathrm{sym}(x,S)$ if and only if $\alpha(x-S) \subseteq int(S-x)$.*

Proof: ($\Rightarrow$) The case $\alpha = 0$ is trivial. For positive values of α, since $x \in \mathrm{int}S$, $\alpha_1 < \alpha_2$ implies $\alpha_1(x-S) \subset \alpha_2 \mathrm{int}(x-S)$.

($\Leftarrow$) If the set inequality holds for a fixed α, since $C = x - \alpha(S-x)$ is a compact set, $dist(C, \partial S) > 0$. Thus one could increase α and the set inequality would still be valid which implies that $\alpha < \mathrm{sym}(x,S)$. $\square$

Proof of Theorem 1: Consider a sequence $x^k \to x$ and the corresponding sequence of real numbers $\alpha^k = \mathrm{sym}(x^k,S)$. By definition, for every k by Lemma 1,

$$x^k - \alpha^k(S-x^k) = (1+\alpha^k)(x^k - x) + x - \alpha^k(S-x) \subset S$$
$$x^k - \alpha^k(S-x^k) = (1+\alpha^k)(x^k - x) + x - \alpha^k(S-x) \nsubseteq \mathrm{int}S$$

In particular, since these relations hold for any subsequence, consider two subsequences $\{n_k\}$ and $\{m_k\}$, such that $\alpha_{n_k} \to \limsup \alpha_k$ and $\alpha_{m_k} \to \liminf \alpha_k$. Since α^k is bounded, $\|(1+\alpha^k)(x^k - x)\| \to 0$. Then, $x - (\limsup \alpha^k)(S-x) \cap \partial S \neq \emptyset$, and $x - (\liminf \alpha^k)(S-x) \cap \partial S \neq \emptyset$. Thus Lemma 1 implies that $\liminf \alpha^k = \limsup \alpha^k = \mathrm{sym}(x,S)$, which proves continuity.

To prove the quasiconcavity property, let $x^1, x^2 \in S$, and $\alpha_1 = \mathrm{sym}(x^1,S)$, $\alpha_2 = \mathrm{sym}(x^2,S)$. Without loss of generality, suppose $\alpha_1 \leq \alpha_2$. By definition, we have $x^1 - \alpha_1 S \subseteq S - \alpha_1 x^1$, and by Lemma 1, $x^2 - \alpha_1 S \subseteq S - \alpha_1 x^2$.

Then, for any $\gamma \in (0,1)$

$$\gamma w^1 + (1-\gamma)w^2 - \alpha_1 S \subseteq S - \alpha_1(\gamma w^1 + (1-\gamma)w^2),$$

which implies that $\mathrm{sym}(\gamma w^1 + (1-\gamma)w^2, S) \geq \alpha_1$. $\square$

For symmetric convex sets, it is possible to prove a stronger statement, but first we prove the following proposition.

Proposition 1. *Under Assumption A, let S be a symmetric set centered at the origin, and let $\|\cdot\|_S$ denote the norm induced by S. Then, for every $x \in S$,*

$$\mathrm{sym}(x,S) = \frac{1-\|x\|_S}{1+\|x\|_S}.$$

Proof: First observe that for any $y \in S$, $\|y\|_S \leq 1$. Second, let $x \in S$ and $\|x\|_S = t$. Consider any chord of S that intersects x, and let p, q be the end points of this chord. Notice that $\|p\|_S = \|q\|_S = 1$ and using the triangle inequality,

$$\|p-x\|_S \leq \|x\|_S + \|p\|_S \quad \text{and} \quad \|q\|_S \leq \|q-x\|_S + \|x\|_S$$

Thus,

$$\frac{\|q-x\|_S}{\|p-x\|_S} \geq \frac{\|q\|_S - \|x\|_S}{\|x\|_S + \|p\|_S} = \frac{1-\|x\|_S}{1+\|x\|_S}$$

□

Theorem 2. *Under Assumption A, let S be a symmetric set centered at the origin. Then $\mathrm{sym}(\cdot,S)$ is a logconcave function in S.*

Proof: Let g be a twice continuously differentiable convex function with bounded domain and image equal to the interval $[0,1]$. Then

$$\begin{aligned}
\nabla \ln\left(\frac{1-g(x)}{1+g(x)}\right) &= \frac{-2\nabla g(x)}{(1-g(x)^2)} \\
\nabla^2 \ln\left(\frac{1-g(x)}{1+g(x)}\right) &= \frac{-2\nabla^2 g(x)(1-g(x)^2)}{((1-g(x)^2)^2} - \frac{4(1-g(x))\nabla g(x)\nabla g(x)^T}{((1-g(x)^2)^2}
\end{aligned} \tag{4}$$

which is semi definite negative proving logconcavity of $\frac{1-g}{1+g}$.

Now, we can build a sequence $\{g_k\}_{k\geq 1}$ of convex functions twice differentiable converging pointwise to $\|\cdot\|_S$. Due to convexity, it is converging uniformly. Finally, since the space of logconcave functions is closed under uniform convergence, logconcavity will hold in the limit. □

It is curious that $\mathrm{sym}(\cdot,S)$ is not a concave function. To see this, consider $S = [0,1] \subset \mathbb{R}$; then a trivial computation yields $\mathrm{sym}(x,S) = \min\left\{\frac{x}{(1-x)}; \frac{(1-x)}{x}\right\}$, which is not concave on S and is not differentiable at $x = \frac{1}{2}$. It is an open question whether in general $\frac{1}{\mathrm{sym}(\cdot,S)}$ is convex or $\ln(\mathrm{sym}(\cdot,S))$ is concave.

Proposition 2. *Let $S,T \subset \mathbb{R}^n$ be convex bodies, and let $x \in S$ and $y \in T$. Then:*

1. *(Superminimality under intersection) If $x \in S \cap T$,*

$$\mathrm{sym}(x, S\cap T) \geq \min\{\mathrm{sym}(x,S), \mathrm{sym}(x,T)\} \tag{5}$$

2. *(Superminimality under Minkowski sums)*

$$\mathrm{sym}(x+y, S+T) \geq \min\{\mathrm{sym}(x,S), \mathrm{sym}(y,T)\} \tag{6}$$

3. *(Invariance under polarity)*

$$\mathrm{sym}(0, S - x) = \mathrm{sym}(0, (S - x)^\circ) \tag{7}$$

4. *(Minimality under Cartesian product)*

$$\mathrm{sym}((x, y), S \times T) = \min\{\mathrm{sym}(x, S), \mathrm{sym}(y, T)\} \tag{8}$$

5. *(Lower bound under affine transformation) Let $A(\cdot)$ be an affine transformation. Then*

$$\mathrm{sym}(A(x), A(S)) \geq \mathrm{sym}(x, S) \tag{9}$$

with equality if $A(\cdot)$ is invertible.

Proof: Without loss of generality, we can translate the sets and suppose that $x = 0$. Let $\alpha = \min\{\mathrm{sym}(0, S), \mathrm{sym}(0, T)\}$. Then $-\alpha S \subset S$, $-\alpha T \subset T$ which implies

$$-\alpha(S \cap T) = -\alpha S \cap -\alpha T \subset S \cap T,$$

and (5) is proved.

To prove (6), again, without loss of generality, we can translate both sets and suppose that $x = y = 0$, and define $\alpha = \mathrm{sym}(0, S)$ and $\beta = \mathrm{sym}(0, T)$. By definition, $-\alpha S \subset S$ and $-\beta T \subset T$. Then it follows trivially that

$$-\alpha S - \beta T \subset (S + T)$$

Replacing α and β by the minimum between them, the result follows.

In order to prove (7), we can assume $x = 0$, then

$$\mathrm{sym}(0, S) = \alpha \Rightarrow -\alpha S \subseteq S.$$

Assuming $\mathrm{sym}(0, S^\circ) < \alpha$, there exist $\bar{y} \in S^\circ$ such that $-\alpha\bar{y} \notin S^\circ$. Thus, there exists $x \in S$, $-\alpha\bar{y}^T x > 1$. However, $-\alpha x \in -\alpha S \subseteq S$, then

$$-\alpha\bar{y}^T x = \bar{y}^T(-\alpha x) \leq 1 \text{ , since } \bar{y} \in S^\circ,$$

which is a contradiction. Thus

$$\mathrm{sym}(0, S) \leq \mathrm{sym}(0, S^\circ) \leq \mathrm{sym}(0, S^{\circ\circ}) = \mathrm{sym}(0, S).$$

Equality (8) is left as a simple exercise.

To prove inequality (9), we can assume that $A(\cdot)$ is a linear operator and that $x = 0$ (since $\mathrm{sym}(x, S)$ is invariant under translation), and suppose that $\alpha < \mathrm{sym}(x, S)$. Then, $-\alpha S \subseteq S$ which implies that $A(-\alpha S) \subseteq A(S)$. Since $A(\cdot)$ is a linear operator, $A(-\alpha S) = -\alpha A(S) \subseteq A(S)$. It is straightforward to show that quality holds in (9) when $A(\cdot)$ is invertible. □

Remark 1. Unlike the case of affine transformation, $\mathrm{sym}(x, S)$ is not invariant under projective transformation.

3. Geometric Properties

This section reveals the geometry behind this symmetry measure. Not surprisingly, the set-containment definition motivated most of the results. Also, this section quantifies the dependance of some classical results for symmetric sets on the symmetric assumption itself. As expected, our extensions are sharp for symmetric sets and "deteriorates" as we consider points with small symmetry values.

We start with two theorems that connect $\mathrm{sym}(x,S)$ to bounds on the n-dimensional volume of the intersection of S with a halfspace cut through x, and with the $(n-1)$-dimensional volume of the intersection of S with a hyperplane passing through S. Let $v \in \mathbb{R}^n$, $v \neq 0$ be given, and for all $x \in S$ define $H(x) := \{z \in S : v^T z = v^T x\}$ and $H^+(x) := \{z \in S : v^T z \leq v^T x\}$. Also let $\mathsf{Vol}_n(\cdot)$ denotes the volume measure on $\mathbb{R}^n$. We have:

Theorem 3. *Under Assumption A, if $x \in S$, then*

$$\frac{\mathrm{sym}(x,S)^n}{1+\mathrm{sym}(x,S)^n} \leq \frac{\mathsf{Vol}_n(H^+(x))}{\mathsf{Vol}_n(S)} \leq \frac{1}{1+\mathrm{sym}(x,S)^n} \ . \tag{10}$$

Proof: Without loss of generality, assume x to be the origin and $\alpha = \mathrm{sym}(x,S)$. Define $K_1 = H^+(x)$ and $K_2 = S \backslash K_1$. Clearly, $\mathsf{Vol}_n(K_1) + \mathsf{Vol}_n(K_2) = \mathsf{Vol}_n(S)$. The key observation is that

$$-\alpha K_2 \subset K_1 \ \text{ and } \ -\alpha K_1 \subset K_2$$

Thus

$$\mathsf{Vol}_n(S) \geq \mathsf{Vol}_n(K_1) + \mathsf{Vol}_n(-\alpha K_1) = \mathsf{Vol}_n(K_1)(1+\alpha^n)$$

which proves the second inequality.

Finally, the first inequality follows trivially from

$$\mathsf{Vol}_n(S) = \mathsf{Vol}_n(K_1) + \mathsf{Vol}_n(K_2) \leq \frac{\mathsf{Vol}_n(K_1)}{\alpha^n} + \mathsf{Vol}_n(K_1)$$

$\square$

For the next theorem, define the function $f(x) = \mathsf{Vol}_{n-1}(H(x))^{1/(n-1)}$ for all $x \in S$. It follows from the Brunn-Minkowski inequality that $f(\cdot)$ is concave (see [1]).

Theorem 4. *Under Assumption A, for every point $x \in S$,*

$$\frac{f(x)}{\max_{y \in S} f(y)} \geq \frac{2\mathrm{sym}(x,S)}{1+\mathrm{sym}(x,S)}$$

Proof: Let $\alpha = \mathrm{sym}(x,S)$ and y^* be the closest point to x^* such that $y^* \in \arg\max_y f(y)$. Note that $x + \alpha(x - H(y^*)) \subset H(x + \alpha(x - y^*))$. Thus, using the Brunn-Minkowski inequality,

$$\begin{aligned} f(x) &\geq \frac{\alpha}{1+\alpha} f(y^*) + \frac{1}{1+\alpha} f(x + \alpha(x - y^*)) \\ &\geq \frac{\alpha}{1+\alpha} f(y^*) + \frac{\alpha}{1+\alpha} f(y^*) \\ \frac{f(x)}{f(y^*)} &\geq \frac{2\alpha}{1+\alpha} \end{aligned} \tag{11}$$

□

If S is a symmetric convex body, then it is a straightforward exercise to show that the symmetry point of S is unique. Roughly speaking, if two points in a convex body have high symmetry values, they cannot be too far apart. The next theorem quantifies the relation between the symmetry of a pair of points and the distance between them. Given $x, y \in S$ with $x \neq y$, let $p(x,y), q(x,y)$ be the pair of endpoints of the largest cord in S passing through x and y, namely:

$$\begin{array}{ll} p(x,y) = x + s(x-y) \in \partial S & \text{where } s \text{ is a maximal scalar} \\ q(x,y) = y + t(y-x) \in \partial S & \text{where } t \text{ is a maximal scalar .} \end{array} \tag{12}$$

Theorem 5. *Under Assumption A, let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. For any $x, y \in S$ satisfying $x \neq y$, let $\alpha = \mathrm{sym}(x,S)$ and $\beta = \mathrm{sym}(y,S)$. Then:*

$$\|x - y\| \leq \left(\frac{1 - \alpha\beta}{1 + \alpha + \beta + \alpha\beta} \right) \|p(x,y) - q(x,y)\| \ . \tag{13}$$

Proof: Consider the line which pass through x and y. By definition of $\mathrm{sym}(x,S)$, $x - \alpha(y - x) \in x - \alpha(S - x) \subset S$. Now, observe that

$$\mathrm{sym}(x - \alpha(y-x), x - \alpha(S - x)) = \mathrm{sym}(-\alpha y, -\alpha S) = \mathrm{sym}(y,S) = \beta.$$

This implies that

$$x - \alpha(y-x) - \beta(x - \alpha(S-x) - (x - \alpha(y-x))) = x - \alpha(y-x) - \beta\alpha(y-x) + \beta\alpha(S-x) \subset S$$

which implies that $x - (\alpha + \alpha\beta)(y - x) \in S$.

Repeating the argument, we obtain $x - \left((\alpha + \alpha\beta)\sum_{i=1}^{\infty} \alpha^i\beta^i\right)(y-x) \in S$. A similar argument will show that $y + \left((\beta + \alpha\beta\sum_{i=0}^{\infty} \alpha^i\beta^i\right)(y-x) \in S$.

Thus

$$\begin{aligned} \frac{\|p(x,y) - q(x,y)\|}{\|x-y\|} &\geq 1 + (\alpha + \beta + 2\alpha\beta)\sum_{i=0}^{\infty} \alpha^i\beta^i \\ &= 1 + \frac{\alpha + \beta + 2\alpha\beta}{1 - \alpha\beta} \\ &= \frac{1 + \alpha + \beta + \alpha\beta}{1 - \alpha\beta} \end{aligned} \tag{14}$$

□

Another relative measure of distance is the "cross-ratio distance" with respect to S. Let $x, y \in S$, $x \neq y$, be given and let s, t be as defined in (12). The cross-ratio distance is defined as:

$$d_S(x,y) := \frac{(1+t+s)}{ts} \ .$$

We have:

Theorem 6. *Under Assumption A, for any $x, y \in S$, $x \neq y$, let s, t be as defined in (12). Then*

$$d_S(x,y) \leq \frac{1}{\mathrm{sym}(x,S) \cdot \mathrm{sym}(y,S)} - 1 \ .$$

Proof: Let $\alpha = \mathrm{sym}(x,S)$ and $\beta = \mathrm{sym}(y,S)$. By definition of symmetry, $t \geq \alpha(1+s)$ and $s \geq \beta(1+t)$. Then

$$\begin{aligned} d_S(x,y) = \frac{(1+t+s)}{ts} &\leq \frac{(1+t+s)}{\alpha(1+s)\beta(1+t)} \\ &= \frac{1}{\alpha\beta}\frac{(1+t+s)}{(1+s+t+st)} = \left(\frac{1}{\alpha\beta}\right)\frac{1}{1+\frac{1}{d_S(x,y)}} \end{aligned} \qquad (15)$$

Thus $d_S(x,y) \leq \frac{1}{\alpha\beta} - 1$. □

We now examine the approximation of a convex sets by another convex set. We say that P is a β-approximation of S if there exists a point $x \in S$ such that $\beta P \subset S - x \subset P$. In the case when P is an ellipsoid centered at the origin, then the statement "P is a β-approximation of S" is equivalent to "βP provides a $\frac{1}{\beta}$-rounding of S."

Theorem 7. *Under Assumption A, let P be a convex body that is a β-approximation of S, and suppose that $\mathrm{sym}(P) = \alpha$. Then, $\mathrm{sym}(S) \geq \beta\alpha$.*

Proof: By definition we have $\beta P \subset S - x \subset P$ for some $x \in S$. Since $\mathrm{sym}(\cdot,\cdot)$ is invariant under translations, we can assume that $x = 0$. And because $\mathrm{sym}(C)$ is invariant under nonzero scalings of C, we have

$$-\alpha\beta S \subset -\alpha\beta P \subset \beta P \subset S \ .$$

□

Theorem 8. *Under Assumption A, suppose that $x \in int\, S$. Then there exists an ellipsoid E centered at x such that*

$$E \subseteq S \subseteq \frac{\sqrt{n}}{\mathrm{sym}(x,S)} E \ . \qquad (16)$$

Proof: Suppose without loss of generality that $x = 0$ (otherwise we translate S), and let $\alpha = \mathrm{sym}(0,S)$. Clearly, $-\alpha S \subset S$, and $\alpha S \subset S$. Consider a $\sqrt{n}$-rounding E of $S \cap (-S)$. Then $\alpha S \subset S \cap (-S) \subset \sqrt{n}E \subset \sqrt{n}S$. □

Theorem 9. *Let $x^L \in S$ be the center of a Löwner-John pair. Then, x^L is guarantee to provide an $\sqrt{\frac{n}{\mathrm{sym}(x^L,S)}}$-rounding for S.*

Proof: We can write $S = \mathbf{conv}\ (\{v\}_{v\in\partial S})$, and construct an Löwner-John pair as a solution to the following optimization problem

$$\begin{array}{ll} \min_{Q,c} & -\ln\det Q \\ \text{s.t.} & (v-c)^TQ(v-c) \leq 1, \ \text{for } v \in \partial S \\ & Q \succcurlyeq 0 \end{array} \tag{17}$$

The KKT conditions of this problem are necessary and sufficient (since it can be cast as an SDP with a SOC constraint), and are

$$\begin{array}{l} -Q^{-1} + \sum_{v\in\partial S}\lambda_v(v-c)(v-c)^T = 0 \\ \sum_{v\in\partial S}\lambda_v 2Q(v-c) = 0 \\ \lambda \geq 0 \\ (v-c)^TQ(v-c) \leq 1 \\ \lambda_v(v-c)^TQ(v-c) = \lambda_v \\ Q \succcurlyeq 0 \end{array}$$

Let $E^o = \{x \in \mathbb{R}^n : (x-c)^TQ(x-c) \leq 1\}$, and $E^I = \{x \in \mathbb{R}^n : (x-c)^TQ(x-c) \leq \frac{\alpha}{n}\}$, where $\alpha = \mathrm{sym}(c,S)$. By construction, $S \subseteq E^o$. It is left to prove that $E^I \subseteq S$. It is equivalent to show that for every $b \in \mathbb{R}^n$,

$$\max_{x\in E^I} b^Tx \leq \max_{y \in S} b^Ty$$

We will need the following lemmas,

Lemma 2. *Let $w_1,\ldots,w_k$ be numbers, and for any $p \in \mathbb{R}^k_+$, $e^Tp = 1$, define $\mu = p^Tw$ and $\sigma^2 = \sum_{i=1}^k p_i(w_i-\mu)^2$. Then, $(w_{max}-\mu)(\mu - w_{min}) \geq \sigma^2$.*

Clearly, $\sum_{i=1}^k p_i(w_{max}-w_i)(w_i - w_{min}) \geq 0$. Thus,

$$\begin{array}{l} \mu w_{max} + \mu w_{min} - \sum_{i=1}^k w_i^2 - w_{min}w_{max} \geq 0 \\ \mu w_{max} + \mu w_{min} - \mu^2 - w_{min}w_{max} \geq \sum_{i=1}^k p_iw_i^2 - \mu^2 = \sigma^2 \\ (w_{max}-\mu)(\mu - w_{min}) \geq \sigma^2 \end{array}$$

proving the lemma.

Lemma 3. *Given $\{y^i\}_{i=1}^k \subset \mathbb{R}^k$, and $p \in \mathbb{R}^k_+$, $e^Tp = 1$, such that $\sum_{i=1}^n p_iy^i = 0$, $\|y^i\|_2 \leq 1$ for every $i = 1,\ldots,k$, and $\frac{1}{n}\sum_{i=1}^k p_iy^iy^{i^T} = I$. Then for any $b \in \mathbb{R}^n$ with $\|b\|_2 = 1$,*

$$\max_{i=1,\ldots,k} b^Ty^i \geq \sqrt{\frac{\mathrm{sym}(0, \mathbf{conv}\ (\{y^i\}_{i=1}^k))}{n}}$$

Fix $b \in \mathbb{R}^n$, and define

$$w_i = b^T y^i, \mu = \sum_{i=1}^k p_i w_i = \sum_{i=1}^k p_i b^T y^i = b^T \sum_{i=1}^k p_i y^i = 0$$

$$\sigma^2 = \sum_{i=1}^k p_i (w_i - \mu)^2 = \sum_{i=1}^k p_i w_i^2 = \sum_{i=1}^k p_i b^T y_i y_i^T b = \frac{1}{n} b^T b = \frac{1}{n}$$

Using the previous lemma, $(w_{max} - \mu)(\mu - w_{min}) \geq \sigma^2 = \frac{1}{n}$. So, $w_{max}(-w_{min}) \geq \frac{1}{n}$.

Note that if $\alpha = \mathrm{sym}(0, \mathbf{conv}\ (\{y^i\}_{i=1}^k))$, we have

$$\max_i b^T y^i \leq -\min_i b^T y^i \leq \frac{1}{\alpha} \max_i b^T y^i.$$

Thus,

$$\begin{array}{rl} \frac{1}{n} & \leq \max_i w_i (-\min_i w_i) \\ & \leq \max_i w_i \frac{1}{\alpha} \max_i w_i \\ & = \dfrac{[\max_i w_i]^2}{\alpha} \end{array}$$

obtaining that $\max_i b^T y^i \geq \sqrt{\frac{\alpha}{n}}$, and proving the lemma.

To use the lemma we will approximate S by the convex hull of a finite number of point of its boundary. Due to the continuity of the symmetry function and, by the theorem of the maximum, the continuity of the objective function as we vary our approximation of S in (17) we can do it without any concern to take limits. So we assume that $S = \mathbf{conv}\ (\{v^i\}_{i=1}^k)$.

Defining $y^i = Q^{1/2}(v^i - c)$ and $p_i = \frac{\lambda_i}{n}$, we have $\|y^i\|_2 \leq 1$ and $p \geq 0$. Using the KKT conditions,

$$\begin{array}{rl} Q^{-1} & = \sum_{i=1}^k \lambda_i (v^i - c)(v^i - c)^T \\ I & = \sum_{i=1}^k \lambda_i Q^{1/2} (v^i - c)(v^i - c)^T Q^{1/2} \\ n = tr(I) & = \sum_{i=1}^k \lambda_i tr\left(Q^{1/2}(v_i - c)(v_i - c)^T Q^{1/2}\right) \\ & = \sum_{i=1}^k \lambda_i tr\left((v_i - c)^T Q^{1/2} Q^{1/2} (v_i - c)\right) \\ & = \sum_{i=1}^k \lambda_i (v_i - c)^T Q^{1/2} Q^{1/2} (v_i - c) \\ & = \sum_{i=1}^k \lambda_i = e^T \lambda \end{array}$$

So, $e^T p = \frac{e^T \lambda}{n} = 1$ and

$$\sum_{i=1}^k p_i y^i y^{iT} = \sum_{i=1}^k \frac{\lambda_i}{n} Q^{1/2}(v_i - c)(v_i - c)^T Q^{1/2} = \frac{1}{n} I$$

Consider $\bar{b} = \frac{Q^{-1/2} b}{\sqrt{b^T Q^{1/2} b}}$ for an arbitrary $b \in \mathbb{R}^n$ (note that $\|\bar{b}\| = 1$). We have that

$$\max_{x \in E^I} b^T x = \max_{(x-c)Q(x-c) \leq \frac{\alpha}{n}} b^T y = b^T c + \sqrt{\tfrac{\alpha}{n}}\sqrt{b^T Q^{-1} b}$$

Applying the second lemma,

$$\begin{aligned} b^T c + \sqrt{\tfrac{\alpha}{n}}\sqrt{b^T Q^{-1} b} &\leq b^T c + \sqrt{b^T Q^{-1} b}(\max_i \bar{b}^T y^i) \\ &= b^T c + \max_i b^T v^i - b^T c \\ &= \max_i b^T v^i \\ &= \max_x \{b^T x : x \in S\} \end{aligned}$$

Thus, $E^I \subseteq S$. □

Remark 2. Note that Theorem 8 is valid for every point in S and Theorem 9 focuses on the center of Löwner-John pais. If one restricts x to be a symmetry points, we conjecture that the bound can be improved by a factor to get a $\left(\sqrt{\frac{n}{\mathrm{sym}(S)}}\right)$-rounding.

Theorem 10. *Under Assumption A, let $\|\cdot\|$ be any norm on $\mathbb{R}^n$, and let $B(x,r)$ denote the ball centered at x with radius r. Suppose that*

$$B(x,r) \subset S \subset P \subset S + B(0,\delta).$$

Then

$$\mathrm{sym}(x,S) \geq \mathrm{sym}(x,P)\left(1 - \frac{\delta}{r}\right).$$

Proof:

Let $\alpha = \mathrm{sym}(x,P)$. Consider any cord of P that passes through x, which divides it into two segments. Assume that the length of one segment is Δ, then the other segment must have length at most Δ/α. Then, the first segment of this cord in S must have length at least $\Delta - \delta$, while the second have length at most Δ/α. Since it holds for any segment,

$$\begin{aligned} \mathrm{sym}(x,S) &\geq \tfrac{\Delta-\delta}{\Delta/\alpha} = \alpha\left(1 - \tfrac{\delta}{\Delta}\right) \\ &\geq \alpha\left(1 - \tfrac{\delta}{r}\right) \end{aligned} \tag{18}$$

where the last inequality follows because $\Delta \geq r$. □

Remark 3. Instead of using a norm $\|\cdot\|$ in Theorem 10, we could instead consider a convex body B that is symmetric about the origin, and replace "$B(x,r)$" by "$x + rB$" and "$B(0,\delta)$" by "δB" in the supposition of the theorem.

As (10),(4), and (16) might be seen, the next two results also illustrate how one can nicely relax the symmetry assumption and still obtain similar results, that is, we do not expect that these results completely break down as we "continously" relax the symmetry assumption stated in their original version. We are going to prove an extension of Theorem 2.4.1 of Dudley [2], which relies on the Brunn-Minkowski inequality in the symmetric case.

Theorem 11. *Let $S \subset \mathbb{R}^n$ be a compact convex set which contains the origin in its interior, and let $\alpha = \mathrm{sym}(0,S)$. Let $f(\cdot)$ be a nonnegative quasiconcave even function that is Lebesgue integrable. Then for $0 \leq \beta \leq 1$ and any $y \in \mathbb{R}^n$,*

$$\int_S f(x+\beta y)dx \geq \alpha^n \int_S f\left(x + \frac{y}{\alpha}\right)dx \ . \tag{19}$$

Proof: We refer to [2] for the case with $\alpha = 1$. Then, assume that $f(\cdot)$ is an indicator function of a set K. This implies that K is convex and $\mathrm{sym}(0,K) = 1$. In fact,

$$\begin{array}{rl} \int_S f(x+\beta y)dx \geq \int_{S\cap -S} f(x+\beta y)dx \geq \int_{S\cap -S} f(x+y)dx & \\ = \mathsf{Vol}_n((S\cap -S)\cap(K-y)) = \mathsf{Vol}_n((S\cap -S)\cap \alpha(\frac{K-y}{\alpha})) & \\ \geq \mathsf{Vol}_n(\alpha S \cap \alpha(K - \frac{y}{\alpha})) = \alpha^n \mathsf{Vol}_n(S\cap(K-\frac{y}{\alpha})) & \end{array} \tag{20}$$

where the second inequality holds simply because $\alpha S \subset S \cap -S$ and $K \subset \frac{K}{\alpha}$. Thus it holds for simple quasiconcave even functions, and using standard arguments of dominated and monotone convergence, it will hold for all nonnegative quasiconcave even Lebesgue-integrable functions. □

The following corollary shows the usefulness of Theorem 11 in probability theory. We note that the density function of an uniform or an n-dimensional Gaussian random variable with mean $\mu = 0$ satisfies the functional conditions of Theorem 11.

Corollary 1. *Let X be a random variable in $\mathbb{R}^n$ whose density function $f(\cdot)$ is an even quasiconcave function. In addition, let Y be an arbitrary random variable independent of X. If $S \subset \mathbb{R}^n$ is a compact convex set which contains the origin in its interior, then*

$$P(X+\beta Y \in S) \geq \alpha^n P\left(X + \frac{Y}{\alpha} \in S\right) \ . \tag{21}$$

Proof: The key observation is that α does not depend on y, thus

$$\begin{array}{rl} P(X+\beta Y \in S) = \int\int_{S-\beta y} f(x)dxdP(y) = \int\int_S f(x-\beta y)dxdP(y) & \\ \geq \alpha^n \int\int_S f(x - \frac{y}{\alpha})dxdP(y) = \alpha^n P(X + \frac{Y}{\alpha} \in S) & \end{array} \tag{22}$$

□

We end this section with a comment on a question posed by Hammer in [6]: what is the upper bound on the difference between $\mathrm{sym}(S)$ and $\mathrm{sym}(x^c, S)$, where x^c is the centroid (center of mass) of S? It is well known that $\mathrm{sym}(x^c,S) \geq 1/n$, see [6], and it follows trivially from the Löwner-John theorem that $\mathrm{sym}(S) \geq 1/n$ as well. Now let S be the Euclidean half-ball: $S := \{x \in \mathbb{R}^n : \langle x, x\rangle \leq 1, x_1 \geq 0\}$. It is an easy exercise to show that the unique symmetry point of S is $x^* = (\sqrt{2}-1)e^1$ and that $\mathrm{sym}(S) = \frac{\sqrt{2}}{2}$, and so in this case $\mathrm{sym}(S)$ is a constant independent of the dimension n. On the other hand, $\mathrm{sym}(x^c, S) = \Omega(\frac{1}{\sqrt{n}})$ (see [1]), and so for this class of instances the symmetry of the centroid is substantially less than the symmetry of the set for large n. For an arbitrary convex body S, note that in the extreme cases where $\mathrm{sym}(S) = 1$ or $\mathrm{sym}(S) = 1/n$ the difference between $\mathrm{sym}(S)$ and $\mathrm{sym}(x^c, S)$ is zero; we conjecture that tight bounds on this difference are only small when $\mathrm{sym}(S)$ is either very close to 1 or very close to $1/n$.

4. Characterization of Symmetry Points via the Normal Cone

Let $\mathrm{Sopt}(S)$ denote the set of symmetry points of the convex body S. In this section we provide a characterization of $\mathrm{Sopt}(S)$ through the set-containment definition of the $\mathrm{sym}(x, S)$ based on (3) and (2):

$$\begin{array}{rll} \mathrm{sym}(S) = \max\limits_{x,\alpha} & & \alpha \\ & s.t. & \alpha(x - S) \subseteq (S - x) \\ & & \alpha \geq 0 \ . \end{array} \tag{23}$$

For any given $x \in S$ let $\alpha = \mathrm{sym}(x, S)$. Motivated by the set-containment definition of $\mathrm{sym}(x, S)$ in (3), let $V(x)$ denote those points $v \in \partial S$ that are also elements of the set $x + \alpha(x - S)$. We call these points the "touching points" of x in S. More formally, we have:

$$V(x) := \partial S \cap (x + \alpha(x - S)) \quad \text{where} \quad \alpha = \mathrm{sym}(x, S) \ . \tag{24}$$

Let $N_S(y)$ denote the normal cone map for points $y \in S$. We assemble the union of all normal cone vectors of all of the touching points of x and call the resulting set the "support vectors" of x:

$$SV(x) = \{s \in \mathbb{R}^n : s \in N_S(v) \text{ for some } v \in V(x), \|s\| = 1\} \ . \tag{25}$$

The following theorem essentially states that $x^* \in S$ is a symmetry point of S if and only if the origin is in the convex hull of the support vectors of x:

Theorem 12. *Under Assumption A, let $x^* \in S$. The following statements are equivalent:*

(i) $x^* \in \mathit{Sopt}(S)$
(ii) $0 \in \mathbf{conv}\ SV(x^*)$.

The proof of this theorem is based on the following construction.

Lemma 4. *The function $f(\cdot)$ defined as*

$$f(x) = \sup_{y \in \partial S} \sup_{s \in N_S(y), \|s\|=1} \langle s, x - y \rangle. \tag{26}$$

satisfies $f(x) = 0$ for $x \in \partial S$, $f(x) > 0$ for $x \notin S$, and $f(x) < 0$ for $x \in \mathit{int} S$.

Proof: As the supremum of affine functionals, f is convex. For $x \in \partial S$, $f(x) \geq 0$. For any $(y, s) \in \partial S \times N_S(y)$, $\langle s, x - y \rangle \leq 0$, for all $x \in S$ by definition of the normal cone. For $x \in \text{int}S$, there exists $\delta > 0$such that $B(x, \delta) \subset S$. Since all s used in $f(\cdot)$ have norm one, $f(x) < \delta/2$. Finally, for $x \notin S$, there exists a supporting hyperplane s of S at $y \in \partial S$ that strictly separate x and S, that is, $s \in N_S(y)$. This implies that $f(x) > 0$. □

Proof of Theorem 12. Assume (i). The symmetry points are the solution for the following mathematical programming problem

$$\begin{aligned} \text{sym}(S) = \max_{x,\alpha} \ & \alpha \\ s.t. \ \ & f(x - \alpha(v - x)) \leq 0 \ \text{ for all } v \in S \end{aligned} \tag{27}$$

Note that the necessary conditions for this problem implies that

$$0 \in \sum_{v \in V(x^*)} \lambda_v \partial f(v)$$

for some nonzero, nonnegative λ. Since $\partial f(v) = \textbf{conv} \ \{s \in N_S(v) : \|s\| = 1\}$, it implies (ii).

Assume (ii). First note that for any $v \in V(x^*)$, $0 \notin \partial f(v)$ (otherwise f would be nonnegative). Thus, $0 \in \textbf{conv} \ SV(x^*)$ implies that $\text{int}(\textbf{cone} \ SV(x^*))^o = \emptyset$. Thus, for any $d \in \mathbb{R}^n$, $\langle d, s \rangle \geq 0$ for some $s \in SV(x^*)$. That is, $v = x^* - \alpha^*(w - x^*) \in V(x^*)$ for some $w \in \partial S$, $s \in \partial f(v)$. Thus,

$$\begin{aligned} f(x^* + d - \alpha^*(w - x^* - d)) &\geq f(x^* - \alpha(w - x^*)) + (1 + \alpha^*)\langle s, d \rangle \\ &\geq f(x^* - \alpha(w - x^*)) = 0 \end{aligned} \tag{28}$$

This implies that $K = x^* + d - \alpha^*(S - x^* - d) \nsubseteq \text{int}S$. If $K \nsubseteq S$, then $sym(x^* + d, S) < \alpha^*$, otherwise, if $K \subset S$, $\text{sym}(x^* + d, S) = \alpha^*$ by Lemma 1. □

We close this section with some properties of the set of symmetry points $\text{Sopt}(S)$. Note that $\text{Sopt}(S)$ is not necessarily a singleton. To see how multiple symmetry points can arise, consider $S := \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1, 0 \leq x_3 \leq 1\}$, which is the cross product of a 2-dimensional simplex and a unit interval. Therefore $\text{sym}(S) = \min\{\frac{1}{2}, 1\} = \frac{1}{2}$ and $\text{Sopt}(S) = \{x \in \mathbb{R}^3 : x_1 = x_2 = \frac{1}{2}, x_3 \in [\frac{1}{3}, \frac{2}{3}]\}$.

Proposition 3. *Under Assumption A, Sopt(S) is compact convex set with no interior. If S is a strictly convex set, then Sopt(S) is a singleton.*

Proof: Convexity follows directly from the quasiconcavity of $\text{sym}(\cdot, S)$.
Suppose that $\exists \hat{x} \in \text{intSopt}(S)$, this implies that exists $\delta > 0$ such that for all $x \in B(\hat{x}, \delta) \subset \text{Sopt}(S)$, $\text{sym}(x, S) = \alpha$.

$$\alpha(\hat{x} + \delta d - S) \subseteq S - (\hat{x} + \delta d) \,, \forall d, \|d\| \leq 1$$

$$\alpha(\hat{x} - S) + B(0, \delta(1 + \alpha)) \subseteq S - \hat{x}$$

using Lemma 1, $\alpha < \text{sym}(\hat{x}, S)$, a contradiction.

For last statement, suppose $\exists\{x_1, x_2\} \subset \mathrm{Sopt}(S)$, let α be the optimal symmetry value. Since any strict convex combination of elements of S must lie in the interior of S, for any $\gamma \in (0,1)$,

$$(\gamma x_1 + (1-\gamma)x_2) - \alpha(S - (\gamma x_1 + (1-\gamma)x_2)) \subseteq \text{ int } S$$

using Lemma 1, $\mathrm{sym}(\gamma x_1 + (1-\gamma)x_2, S) > \alpha$. □

Remark 4. In [8], Klee proved the following notable relation between $\mathrm{sym}(S)$ and the dimension of $\mathrm{Sopt}(S)$:

$$\frac{1}{\mathrm{sym}(S)} + \dim(\mathrm{Sopt}(S)) \leq n \ ,$$

which immediately implies that multiple symmetry points can only exist in dimensions $n \geq 3$.

5. Computing a Symmetry Point of S when S is Polyhedral

Our interest lies in computing an ε-approximate symmetry point of S, which is a point $x \in S$ that satisfies:

$$\mathrm{sym}(x, S) \geq (1-\varepsilon)\mathrm{sym}(S) \ .$$

5.1. Polyhedra Represented by Linear Inequalities

In this section, we assume that S is given as the intersection of m inequalities, i.e., $S := \{x \in \mathbb{R}^n : Ax \leq b\}$. We present two methods for computing an ε-approximate symmetry point of S. The first method is based on approximately solving a single linear program with $m^2 + m$ inequalities. For such a method, an interior-point algorithm would require $O(m^6)$ operations per Newton step, which is clearly unattractive. Our second method involves solving $m+1$ linear programs each of which involves m linear inequalities in n unrestricted variables. This method is more complicated to evaluate, but is clearly more attractive should one want to compute an ε-approximate symmetry point in practice.

5.1.1. First Approach Let $\bar{x} \in S$ be given, and let $\alpha \leq \mathrm{sym}(\bar{x}, S)$. Then from the definition of $\mathrm{sym}(\cdot, S)$ in (1) we have:

$$A(\bar{x} + v) \leq b \quad \Rightarrow \quad A(\bar{x} - \alpha v) \leq b \ ,$$

which we restate as:

$$Av \leq b - A\bar{x} \quad \Rightarrow \quad -\alpha A_{i\cdot} v \leq b_i - A_{i\cdot}\bar{x} \ , i = 1, \ldots, m \ . \tag{29}$$

Now apply a theorem of the alternative to each of the $i = 1, \ldots, m$ implications (29). Then (29) is true if and only if there exists an $m \times m$ matrix Λ of multipliers that satisfies:

$$\begin{aligned} \Lambda A &= -\alpha A \\ \Lambda(b - A\bar{x}) &\leq b - A\bar{x} \\ \Lambda &\geq 0 \ . \end{aligned} \tag{30}$$

Here "$\Lambda \geq 0$" is componentwise for all m^2 components of Λ. This means that $\mathrm{sym}(\bar{x}, S) \geq \alpha$ if and only if (30) has a feasible solution. This also implies that $\mathrm{sym}(S)$ is the optimal objective value of the following optimization problem:

$$\begin{aligned} \max_{x,\Lambda,\alpha} \ & \alpha \\ \text{s.t.} \ & \Lambda A = -\alpha A \\ & \Lambda(b - Ax) \leq b - Ax \\ & \Lambda \geq 0 \ , \end{aligned} \tag{31}$$

and any solution $(x^*, \Lambda^*, \alpha^*)$ of (31) satisfies $\mathrm{sym}(S) = \alpha^*$ and $x^* \in \mathrm{Sopt}(S)$. Notice that (31) is not a linear program. To convert it to a linear program, we make the following change of variables:

$$\gamma = \frac{1}{\alpha} \ , \ \Pi = \frac{1}{\alpha}\Lambda \ , \ y = \frac{1+\alpha}{\alpha}x \ ,$$

which can be used to transform (31) to the following linear program:

$$\begin{aligned} \min_{y,\Pi,\gamma} \ & \gamma \\ \text{s.t.} \ & \Pi A = -A \\ & \Pi b + Ay - b\gamma \leq 0 \\ & \Pi \geq 0 \ . \end{aligned} \tag{32}$$

If (y^*, Π^*, γ^*) is a solution of (32), then $\alpha^* := 1/\gamma^* = \mathrm{sym}(S)$ and $x^* := \frac{1}{1+\gamma^*}y^* \in \mathrm{Sopt}(S)$. Notice that (32) has $m^2 + m$ inequalities and mn equations. Suppose we know an approximate analytic center x^a of S. Then it is possible to develop an interior-point method approach to solving (32) using information from x^a, and one can prove that a suitable interior-point method will compute an ε-approximate symmetry point of S in $O\left(m \ln\left(\frac{m}{\varepsilon}\right)\right)$ iterations of Newton's method. However, due to the $m^2 + m$ inequalities, each Newton step requires $O(m^6)$ operations, which is clearly unattractive.

5.1.2. Second Approach The motivation for this approach comes from the dual problem associated with (32):

$$\begin{aligned} \max_{\Psi,\sigma,\Phi} \ & -\mathrm{tr}(A\Psi) \\ \text{s.t.} \ & b\sigma^T - A\Psi - \Phi = 0 \\ & A^T\sigma = 0 \\ & b^T\sigma = 1 \\ & \sigma \geq 0, \Phi \geq 0 \ , \end{aligned} \tag{33}$$

where $\mathrm{tr}(M)$ denotes the trace of a square matrix M. Notice that if $\sigma > 0$, then the first set of constraints implies that $\frac{\Psi_{\cdot i}}{\sigma_i} \in S$, and the objective function can be rewritten as $-1 + \sum_{i=1}^m \sigma_i \left(b_i - A_{i\cdot}\frac{\Psi_{\cdot i}}{\sigma_i}\right)$.

Define the following scalar quantities $\delta_i^*, i = 1, \ldots, m$:

$$\begin{array}{rl} \delta_i^* := \max\limits_x & -A_{i\cdot}x \\ \text{s.t.} & Ax \le b \ , \end{array} \tag{34}$$

and notice that $b_i + \delta_i^*$ is the range of $A_i x$ over $x \in S$ if the i^{th} constraint is not strictly redundant on S. We compute $\delta_i^*, i = 1, \ldots, m$ by solving m linear programs whose feasible region is exactly S. The following proposition shows that if we know $\delta_i^*, i = 1, \ldots, m$, then computing $\text{sym}(x, S)$ for any $x \in S$ is a simple min-ratio test.

Proposition 4. *Let $S = \{x \in \mathbb{R}^n : Ax \le b\}$ be given. For each $x \in S$,*

$$\text{sym}(x, S) = \min_{i=1,\ldots,m} \left\{ \frac{b_i - A_{i\cdot}x}{\delta_i^* + A_{i\cdot}x} \right\} \ .$$

Proof: Let $\alpha = \text{sym}(x, S)$. Then for all $y \in S$, $x + \alpha(x - y) \in S$, so

$$A_{i\cdot}x + \alpha A_{i\cdot}x + \alpha(-A_{i\cdot}y) \le b_i \ , i = 1, \ldots, m \ .$$

This implies that

$$A_{i\cdot}x + \alpha A_{i\cdot}x + \alpha\delta_i^* \le b_i \ , i = 1, \ldots, m \ ,$$

whereby

$$\alpha \le \min_{i=1,\ldots,m} \left\{ \frac{b_i - A_{i\cdot}x}{\delta_i^* + A_{i\cdot}x} \right\} \ .$$

On the other hand, let $\gamma := \min\limits_{i=1,\ldots,m} \left\{ \frac{b_i - A_{i\cdot}x}{\delta_i^* + A_{i\cdot}x} \right\}$. Then for all $y \in S$ and $i = 1, \ldots, m$ we have:

$$b_i - A_{i\cdot}x \ge \gamma(\delta_i^* + A_{i\cdot}x) \ge \gamma(-A_{i\cdot}y + A_{i\cdot}x), \ i = 1, \ldots, m \ .$$

Thus

$$A_{i\cdot}x + \gamma A_{i\cdot}x + \gamma(-A_{i\cdot}y) \le b_i \ ;$$

therefore

$$A_{i\cdot}(x + \gamma(x - y)) \le b_i$$

which implies that $\alpha \ge \gamma$. Thus $\alpha = \gamma$. $\square$

Proposition 4 can be used to create another single linear program to compute $\text{sym}(S)$ as follows. Let $\delta^* := (\delta_1^*, \ldots, \delta_m^*)$ and consider the following linear program that uses δ^* in the data:

$$\begin{array}{rl} \max\limits_{x,\check{\theta}} & \check{\theta} \\ \text{s.t.} & Ax + \check{\theta}(\delta^* + b) \le b \ . \end{array} \tag{35}$$

Proposition 5. *Let $(x^*, \check{\theta}^*)$ be an optimal solution of the linear program (35). Then x^* is a symmetry point of S and* $\text{sym}(S) = \frac{\check{\theta}^*}{1-\check{\theta}^*}$.

Proof: Suppose that $(x, \check{\theta})$ is a feasible solution of (35). Then $\dfrac{1}{\check{\theta}} \geq \dfrac{\delta_i^* + b_i}{b_i - A_i.x}$, whereby

$$\frac{1-\check{\theta}}{\check{\theta}} = \frac{1}{\check{\theta}} - 1 \geq \frac{\delta_i^* + A_i.x}{b_i - A_i.x} \ ,$$

and so

$$\frac{b_i - A_i.x}{\delta_i^* + A_i.x} \geq \frac{\check{\theta}}{1-\check{\theta}} \ , i = 1, \ldots, m \ .$$

It then follows from Proposition 4 that $\mathrm{sym}(x, S) \geq \dfrac{\check{\theta}}{1-\check{\theta}}$, which implies that $\mathrm{sym}(S) \geq \frac{\check{\theta}^*}{1-\check{\theta}^*}$. The proof of the reverse inequality follows similarly. □

This yields the following "exact" method for computing $\mathrm{sym}(S)$ and a symmetry point x^*:

Exact Method:

Step 1 For $i = 1, \ldots, m$ solve the linear program (34) for δ_i^*.
Step 2 Let $\delta^* := (\delta_1^*, \ldots, \delta_m^*)$. Solve the linear program (35) for an optimal solution $(x^*, \check{\theta}^*)$. Then $x^* \in \mathrm{Sopt}(S)$ and $\mathrm{sym}(S) = \frac{\check{\theta}^*}{1-\check{\theta}^*}$.

This method involves the exact solution of $m+1$ linear programs. The first m linear programs can actually be solved in parallel, and their optimal objective values are used in the data for the $(m+1)^{\mathrm{st}}$ linear program. The first m linear programs each have m inequalities in n unrestricted unknowns. The last linear program has m inequalities and $n+1$ unrestricted unknowns, and could be reduced to n unknowns using variable elimination if so desired.

From a complexity perspective, it is desirable to consider solving the $m+1$ linear programs for a feasible and near-optimal solution. Ordinarily, this would be easy to analyze. But in this case, the approximately optimal solution to the m linear programs (34) will then yield imprecise *input* data for the linear program (35). Nevertheless, one can construct an inexact method with an appropriately good complexity bound. Below is a description of such a method.

Inexact Method:

Step 1 For $i = 1, \ldots, m$, approximately solve the linear program (34), stopping each linear program when a feasible solution $\bar{x}$ is computed for which the duality gap $\bar{g}$ satisfies $\bar{g} \leq \frac{\varepsilon(b_i - A_i.\bar{x})}{4.1}$. Set $\bar{\delta}_i \leftarrow -A_i.\bar{x}$.
Step 2 Let $\bar{\delta} := (\bar{\delta}_1, \ldots, \bar{\delta}_m)$. Approximately solve the linear program

$$\begin{array}{ll} \max\limits_{x,\theta} & \theta \\ \text{s.t.} & Ax + \theta(\bar{\delta} + b) \leq b, \end{array} \tag{36}$$

stopping when a feasible solution $(\bar{x}, \bar{\theta})$ is computed for which the duality gap $\bar{g}$ satisfies $\bar{\theta} \geq (\bar{\theta} + \bar{g})(1 - \frac{\varepsilon}{4.1})$. Then $\bar{x}$ will be an ε-approximate symmetry point of S and and $\frac{\bar{\theta}(1-\varepsilon)}{1-\bar{\theta}} \leq \mathrm{sym}(S) \leq \frac{\bar{\theta}}{1-\bar{\theta}}$.

Notice that this method requires that the LP solver computes primal and dual feasible points (or simply primal feasible points and the duality gap) at each of its iterations; such a requirement is satisfied, for example, by a standard feasible interior-point method, see Appendix B.

In order to prove a complexity bound for the Inexact Method, we will assume that S is bounded and has an interior, and that an approximate analytic center x^a of the system $Ax \le b$ has already been computed; for details also see Appendix B.

Theorem 13. *Let $\varepsilon \in (0, 1/10)$ be given. Suppose that $n \ge 2$ and x^a is a $\beta = \frac{1}{8}$-approximate analytic center of S. Then starting with x^a and using a standard feasible interior-point method to solve each of the linear programs in Steps 1 and 2, the Inexact Method will compute an ε-approximate symmetry point of S in no more than*

$$\left\lceil 10m^{1.5} \ln\left(\frac{10m}{\varepsilon}\right)\right\rceil$$

total iterations of Newton's method.

We now proceed to assemble the steps of the proof of Theorem 13. The following two propositions will be used to show that the method indeed computes an ε-approximate symmetry point of S.

Proposition 6. *Let $\varepsilon \in (0, 1/10)$ be given, set $\tilde{\varepsilon} := \varepsilon/4.1$, and suppose that Step 1 of the Inexact Method is executed. Then $\bar{\delta} = (\bar{\delta}_1, \dots, \bar{\delta}_m)$ satisfies $\delta_i^* - \tilde{\varepsilon}(b_i + \delta_i^*) \le \bar{\delta}_i \le \delta_i^*$, $i = 1, \dots, m$. Furthermore, for any given $x \in S$, let $\theta := \min_i \left\{\frac{b_i - A_i.x}{\bar{\delta}_i + b_i}\right\}$. Then*

$$\mathrm{sym}(x, S) \in \left[\frac{\theta}{1-\theta}\left(1 - \frac{2\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right), \frac{\theta}{1-\theta}\right] .$$

Proof: For a given $i = 1, \dots, m$ let $\bar{g}$ denote the duality gap computed in the stopping criterion of Step 1 of the Inexact Method. Then

$$\delta_i^* \ge \bar{\delta}_i \ge \delta_i^* - \bar{g} \ge \delta_i^* - \tilde{\varepsilon}(b_i + \delta_i^*) , \tag{37}$$

which also implies

$$(1 - \tilde{\varepsilon})(b_i + \delta_i^*) \le (b_i + \bar{\delta}_i) \le (b_i + \delta_i^*) . \tag{38}$$

For a given $x \in S$ let $\alpha := \mathrm{sym}(x, S)$ and $\check{\theta} := \min_i \left\{\frac{b_i - A_i.x}{\delta_i^* + b_i}\right\}$. Then from Proposition 4 we have

$$\alpha = \min_i \left\{\frac{b_i - A_i.x}{\delta_i^* + A_i.x}\right\} = \frac{\check{\theta}}{1 - \check{\theta}} . \tag{39}$$

Notice that $\bar{\delta}_i \leq \delta_i^*$ for all i, whereby $\theta \geq \check{\theta}$, which implies that $\alpha = \frac{\check{\theta}}{1-\check{\theta}} \leq \frac{\theta}{1-\theta}$. We also see from (39) that $\check{\theta} \leq 1/2$. Next notice that (38) implies that $\check{\theta} \geq \theta(1-\tilde{\varepsilon})$. Therefore

$$\begin{aligned}
\alpha = \frac{\check{\theta}}{1-\check{\theta}} &\geq \frac{\theta(1-\tilde{\varepsilon})}{1-\check{\theta}} = \frac{\theta(1-\tilde{\varepsilon})}{1-\theta}\frac{1-\theta}{1-\check{\theta}} \\
&= \frac{\theta(1-\tilde{\varepsilon})}{1-\theta}\left(1+\frac{\check{\theta}-\theta}{1-\check{\theta}}\right) \geq \frac{\theta(1-\tilde{\varepsilon})}{1-\theta}\left(1+\frac{\check{\theta}-\frac{1}{1-\tilde{\varepsilon}}\check{\theta}}{1-\check{\theta}}\right) \\
&= \frac{\theta(1-\tilde{\varepsilon})}{1-\theta}\left(1+\frac{\check{\theta}(\frac{-\tilde{\varepsilon}}{1-\tilde{\varepsilon}})}{1-\check{\theta}}\right) \geq \frac{\theta(1-\tilde{\varepsilon})}{1-\theta}\left(1-\frac{\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right) \\
&\geq \frac{\theta}{1-\theta}\left(1-\frac{2\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right) \ ,
\end{aligned} \tag{40}$$

where the next-to-last inequality follows from $\check{\theta} \in [0,1/2]$. □

Proposition 7. *Let $\varepsilon \in (0,1/10]$ be given, set $\tilde{\varepsilon} := \varepsilon/4.1$, and suppose that Steps 1 and 2 of the Inexact Method are executed, with output $(\bar{x},\bar{\theta})$. Then*

$$\mathrm{sym}(\bar{x},S) \geq \mathrm{sym}(S)\left(1-\frac{4\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right) \geq (1-\varepsilon)\mathrm{sym}(S) \ .$$

Proof: Let θ^* denote the optimal objective value of (36), and notice that $\bar{\delta} \leq \delta^*$ implies that $\theta^* \geq \check{\theta}^*$. Let $\bar{g}$ be computed in Step 2 of the method. It follows from the stopping criterion in Step 2 that

$$\bar{\theta} \geq (\bar{\theta}+\bar{g})(1-\tilde{\varepsilon}) \geq (\theta^*)(1-\tilde{\varepsilon}) \geq (\check{\theta}^*)(1-\tilde{\varepsilon}) \ . \tag{41}$$

From Proposition 6 we have

$$\begin{aligned}
\mathrm{sym}(\bar{x},S) &\geq \frac{\bar{\theta}}{1-\bar{\theta}}\left(1-\frac{2\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right) \geq \frac{\check{\theta}^*(1-\tilde{\varepsilon})}{1-\check{\theta}^*(1-\tilde{\varepsilon})}\left(1-\frac{2\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right) \\
&= \frac{\check{\theta}^*(1-\tilde{\varepsilon})}{1-\check{\theta}^*}\left(1-\frac{2\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right)\frac{1-\check{\theta}^*}{1-\check{\theta}^*(1-\tilde{\varepsilon})} \\
&\geq \mathrm{sym}(S)(1-\tilde{\varepsilon})\left(1-\frac{2\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right)\left(\frac{1/2}{1-1/2+(1/2)\tilde{\varepsilon}}\right) \\
&= \mathrm{sym}(S)(1-\tilde{\varepsilon})\left(1-\frac{2\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right)\left(1-\frac{\tilde{\varepsilon}}{1+\tilde{\varepsilon}}\right) \\
&\geq \mathrm{sym}(S)\left(1-\tfrac{4\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\right) \geq \mathrm{sym}(S)(1-\varepsilon) \ ,
\end{aligned} \tag{42}$$

where the middle inequality uses the fact that $\check{\theta}^* \in [0, 1/2]$, and the final inequality uses the fact that $\epsilon \in (0, 1/10]$. □

The correctness of the Inexact Method now follows directly from Propositions 6 and 7. It remains to prove the complexity bound of Theorem 13, which will be accomplished with the help of the following two propositions.

Proposition 8. *Let $\varepsilon \in (0, 1/10)$ be given, and set $\tilde{\varepsilon} := \varepsilon/4.1$. Suppose that x^a is a $\beta = \frac{1}{8}$-approximate analytic center of S. Then starting with x^a, the stopping criterion of each linear program in Step 1 will be reached in no more than*

$$\left\lceil (2 + 4\sqrt{m}) \ln \left(\frac{42m}{\varepsilon} \right) \right\rceil$$

iterations of Newton's method.

Proof: Step 1 is used to approximately solve each of the linear programs (34) for $i = 1, \ldots, m$. Let us fix a given i. Appendix B describes the generic complexity of a standard feasible path-following interior-point method, which we will apply to (34). A triplet (x, s, z) together with a path parameter μ is a β-approximate solution for μ for the linear program (34) if the following system is satisfied

$$\begin{array}{l} Ax + s = b, \ s > 0 \\ A^T z = -A_{i\cdot} \\ \| \frac{1}{u} S z - e \| \le \beta \ . \end{array} \tag{43}$$

Now let x^a denote the given $\beta = \frac{1}{8}$-approximate analytic center of the system $Ax \le b$. Then there exists (or it is easy to compute) multipliers z^a together with slacks s^a that satisfy the following system:

$$\begin{array}{l} Ax^a + s^a = b, \ s^a > 0 \\ A^T z^a = 0 \\ \| S^a z^a - e \| \le \frac{1}{8} \ . \end{array} \tag{44}$$

Define:

$$(x^0, s^0, z^0, \mu^0) = (x^a, s^a, 8 s_i^a z - e^i, 8 s_i^a) \ , \tag{45}$$

where e^i is the i^{th} unit vector in $\mathbb{R}^m$. It is then straightforward to show that (45) is a (1/4)-approximate solution of (43) for the parameter μ^0, so we can start the interior-point method with (45). We next bound the value of the parameter μ when the stopping criterion is achieved. Let $(\bar{x}, \bar{s}, \bar{z}, \bar{\mu})$ denote the values of (x, s, z, μ) when the algorithm stops. To keep the analysis simple, we assume that the stopping criterion is met exactly. We therefore have from (51) that:

$$(5/4) m \bar{\mu} \ge \bar{g} = \tilde{\varepsilon}(b_i - A_{i\cdot} \bar{x}) = \tilde{\varepsilon} \bar{s}_i \ ,$$

which leads to the ratio bound:

$$\frac{\mu^0}{\bar{\mu}} \le \frac{8 m s_i^a}{(4/5) \tilde{\varepsilon} \bar{s}_i} \ .$$

However, noting that

$$\bar{s}_i = b_i - A_i\bar{x} \geq b_i + \delta_i^* - \bar{g} = b_i + \delta_i^* - \tilde{\varepsilon}\bar{s}_i \geq b_i - A_i x^a - \tilde{\varepsilon}\bar{s}_i = s_i^a - \tilde{\varepsilon}\bar{s}_i \ ,$$

we obtain $s_i^a \leq (1+\tilde{\varepsilon})\bar{s}_i$, and substituting this into the ratio bound yields:

$$\frac{\mu^0}{\bar{\mu}} \leq \frac{10m(1+\tilde{\varepsilon})}{\tilde{\varepsilon}} \leq \frac{42m}{\varepsilon} \ ,$$

using $\varepsilon \leq 1/10$ and $\tilde{\varepsilon} = \varepsilon/4.1$. This then yields via Theorem 14 the bound of

$$\left\lceil (2+4\sqrt{m}) \ln\left(\frac{42m}{\varepsilon}\right)\right\rceil$$

iterations of Newton's method. □

Proposition 9. *Let $\varepsilon \in (0, 1/10)$ be given,$m \geq 3$ and set $\tilde{\varepsilon} := \varepsilon/4.1$. Suppose that x^a is a $\beta = \frac{1}{8}$-approximate analytic center of S. Then starting with x^a, the stopping criterion of the linear program in Step 2 will be reached in no more than*

$$\left\lceil (2+4\sqrt{m}) \ln\left(\frac{6m}{\varepsilon}\right)\right\rceil$$

iterations of Newton's method.

Proof: We proceed in a similar fashion as the proof of Proposition 8, making use of the generic interior-point results described in Appendix B. A quadruplet (x, θ, s, z) together with a path parameter μ is a β-approximate solution for μ for the linear program (36) if the following system is satisfied

$$\begin{array}{l} Ax + (\bar{\delta}+b)\theta + s = b, \ \ s > 0 \\ A^T z = 0 \\ (\bar{\delta}+b)^T z = 1 \\ \|\frac{1}{u} Sz - e\| \leq \beta \ . \end{array} \tag{46}$$

Now let x^a denote the given $\beta = \frac{1}{8}$-approximate analytic center of the system $Ax \leq b$, which together with multipliers z^a and slacks s^a, satisfy (44). Define:

$$(x^0, \theta^0, s^0, z^0, \mu^0) = \left(x^a, 0, s^a, \frac{z^a}{(\bar{\delta}+b)^T z^a}, \frac{1}{(\bar{\delta}+b)^T z^a}\right) \ . \tag{47}$$

It is then straightforward to show that (47) is a (1/4)-approximate solution of (46) for the parameter μ^0, so we can start the interior-point method with (47). We now bound the value μ^0. It follows from (44) and $m \geq 3$ that

$$(s^a)^T z^a = e^T(S^a z^a - e + e) \geq -\frac{1}{8}\sqrt{m} + m \geq \frac{9m}{10} \ .$$

Therefore

$$\begin{aligned}
\frac{1}{\mu^0} &= (b+\bar{\delta})^T z^a \\
&\geq (b+\delta^*)^T z^a (1-\tilde{\varepsilon}) \qquad \text{(from (38))} \\
&\geq (s^a)^T z^a (1-\tilde{\varepsilon}) \\
&\geq \frac{9m(1-\tilde{\varepsilon})}{10} ,
\end{aligned}$$

whereby $\mu^0 \leq \frac{10}{9m(1-\tilde{\varepsilon})}$.

We next bound the value of the parameter μ when the stopping criterion is achieved. Let $(\bar{x}, \bar{\theta}, \bar{s}, \bar{z}, \bar{\mu})$ denote the values of (x, θ, s, z, μ) when the algorithm stops. To keep the analysis simple, we assume that the stopping criterion is met exactly. We therefore have from (51) that:

$$\bar{\theta} = (\bar{\theta} + \bar{g})(1-\tilde{\varepsilon}) \leq (\bar{\theta} + (5/4)m\bar{\mu})(1-\tilde{\varepsilon}) \ .$$

This implies that

$$\begin{aligned}
\bar{\mu} &\geq \frac{(4/5)\bar{\theta}\tilde{\varepsilon}}{m(1-\tilde{\varepsilon})} \\
&\geq \frac{(4/5)\check{\theta}^*\tilde{\varepsilon}}{m} && \text{(from (41))} \\
&= \frac{(4/5)\tilde{\varepsilon}}{m}\left(\frac{\mathrm{sym}(S)}{1+\mathrm{sym}(S)}\right) && \text{(from Proposition 5)} \\
&\geq \frac{(4/5)\tilde{\varepsilon}}{m \times (n+1)} && (\text{since } \mathrm{sym}(S) \geq \tfrac{1}{n}) \\
&\geq \frac{(4/5)\tilde{\varepsilon}}{m^2} && (\text{since } m \geq n+1) \ .
\end{aligned}$$

We now have:

$$\frac{\mu^0}{\bar{\mu}} \leq \frac{m^2}{(4/5)\tilde{\varepsilon}} \frac{10}{9m(1-\tilde{\varepsilon})} = \frac{50m}{36\tilde{\varepsilon}(1-\tilde{\varepsilon})} \leq \frac{6m}{\varepsilon} \ ,$$

using $\varepsilon \leq 1/10$ and $\tilde{\varepsilon} = \varepsilon/4.1$. This then yields via Theorem 14 the bound of

$$\left\lceil (2+4\sqrt{m}) \ln\left(\frac{6m}{\varepsilon}\right) \right\rceil$$

iterations of Newton's method. □

Proof of complexity bound of Theorem 13: From Propositions 8 and 9 it follows that the total number of Newton steps computed by the Inexact Method is bounded from above by:

$$m\left\lceil (2+4\sqrt{m})\ln\left(\frac{42m}{\varepsilon}\right)\right\rceil + \left\lceil (2+4\sqrt{m})\ln\left(\frac{6m}{\varepsilon}\right)\right\rceil \leq \left\lceil 10m^{1.5}\ln\left(\frac{10m}{\varepsilon}\right)\right\rceil$$

since $m \geq n+1 \geq 3$ and $\varepsilon < 1/10$. □

Acknowledgements. We thank Arkadi Nemirovski for his insightful comments on the subject of symmetry, and for contributing the half-ball example which we used at the end of Section 3.

Appendix

A. $\mathrm{sym}(x,S)$ and $\mathrm{sym}(S)$ under Relaxed Assumptions

In all sections of the paper we assumed that the set S is a convex body, that is, a convex compact set with non empty interior. This section is dedicated to justify this assumption discussing the extreme cases for completeness.

Remark 5. If one does not restrict the set S and the point x, some extreme cases may occur. It is clear that $\mathrm{sym}(x,S) = -\infty$, for all $x \notin S$ (this case includes $S = \emptyset$). On the other hand, if S is any affine subspace, $\mathrm{sym}(x,S) = \infty$ for all $x \in S$. Finally, if the recession cone of S is not a subspace, then $\mathrm{sym}(x,S) = 0$ for all $x \in S$.

There is one "extreme" case not treated in the previous remark. Although it is not hard, it is stated as a Lemma.

Lemma 5. *Suppose that $S = P + H$, where H is a subspace and P is a bounded convex set in $H^\perp$, and $x \in S$, then the symmetry of S with respect to x is completely defined by P.*

Proof: Without loss of generality, we can assume that $x = 0$ since symmetry is invariant under translation. Trivially, $-\alpha S \subseteq S$ iff $-\alpha(P+H) \subseteq (P+H)$. Since P and H lie in orthogonal spaces, for each $x \in S$, there exist a unique $(u,v) \in P \times H$ such that $x = u + v$. Since $-\alpha H = H$, $-\alpha x \in S$ iff $-\alpha u \in P$. □

Regarding the symmetry points, it is interesting to observe that at least one such point always exists. This is always the case independently of S being closed, open or neither. On the other hand, even if $\mathrm{sym}(0,S) = 1$, one cannot claim that $-\mathrm{sym}(0,S)S \subseteq S$ in general[1].

In fact, for any $G \subseteq \partial S$ such that $G \cup \mathrm{int}S$ is convex, and for any $x \in \mathrm{int}S$, one has $\mathrm{sym}(x,\mathrm{int}S) = \mathrm{sym}(x, G\cup \mathrm{int}S) = \mathrm{sym}(x,\bar{S})$.

[1] For example, consider the interval $S = [-1,1)$, its symmetry value is one, attained at 0, but $(-1,1] \nsubseteq [-1,1)$.

B. A Standard Interior Point Method for Linear Programming.

Consider the following linear programming problem in "dual" form, where M is an $m \times k$ matrix:

$$
\begin{array}{rl}
P: & \mathrm{VAL} := \max_x \ c^T x \\
& \quad \text{s.t. } Mx + s = f \\
& \qquad s \geq 0 \\
& \qquad x \in \mathbb{R}^n, \quad s \in \mathbb{R}^m
\end{array}
\tag{48}
$$

For $\beta \in (0, 1)$, a β-approximate analytic center of the primal feasibility system $Mx \leq g$ is a feasible solution x^a of P (together with its slack vector $s^a = g - Mx^a$) for which there exists dual multipliers z^a that satisfy:

$$
\begin{array}{l}
Mx^a + s^a = f, \ s^a > 0 \\
M^T z^a = 0 \\
\|S^a z^a - e\| \leq \beta \ ,
\end{array}
\tag{49}
$$

where S is the diagonal matrix whose diagonal entries correspond to the components of s, and e is the vector of ones: $e = (1, \ldots, 1)$.

We say that $(\bar{x}, \bar{s}, \bar{z})$, together with barrier parameter $\bar{\mu} > 0$, is a β-approximate solution for the parameter $\bar{\mu}$ if the following system is satisfied:

$$
\begin{array}{l}
M\bar{x} + \bar{s} = f, \ \bar{s} > 0 \\
M^T \bar{z} = c \\
\|\frac{1}{\bar{u}} \bar{Z}\bar{s} - e\| \leq \beta \ .
\end{array}
\tag{50}
$$

The duality gap associated with the variables $(\bar{x}, \bar{s}, \bar{z})$ is $\bar{g} := f^T \bar{z} - c^T \bar{x} = \bar{s}^T \bar{z}$. It follows from (50) that this gap must satisfy:

$$
m\bar{\mu}(1 - \beta) \leq \bar{g} \leq m\bar{\mu}(1 + \beta) \ , \tag{51}
$$

hence

$$
\mathrm{VAL} \leq c^T \bar{x} + m\bar{\mu}(1 + \beta) \ . \tag{52}
$$

A standard interior-point method for solving P uses Newton's method to compute successive β-approximate solutions for a decreasing sequence of values of μ. Following [13] or [14], one can prove the following result about the efficiency of the method.

Theorem 14. *Suppose that $\beta = 1/4$ and that (x^0, s^0, z^0) is a given β-approximate solution for the barrier parameter $\mu^0 > 0$, and we wish to compute a β-approximate solution $(\bar{x}, \bar{s}, \bar{z})$ for the barrier parameter $\bar{\mu} \in (0, \mu^0)$. Then such a solution can be computed in at most*

$$
\left\lceil (2 + 4\sqrt{m}) \ln\left(\frac{\mu^0}{\bar{\mu}}\right) \right\rceil
$$

iterations of Newton method, and will satisfy $\bar{g} := f^T \bar{z} - c^T \bar{x} \leq (5/4)m\bar{\mu}$ and $\mathrm{VAL} \leq c^T \bar{x} + (5/4)m\bar{\mu}$. □

C. Polyhedra Represented by the Convex Hull of Points

In this case, m points are given, $\{w^i\}_{i=1}^m \subset \mathbb{R}^n$, and our set is defined by

$$S = \left\{ \sum_{i=1}^m \lambda_i w^i \in \mathbb{R}^n : e^T\lambda = 1, \lambda \geq 0 \right\}$$

In order to verify if $\mathrm{sym}(x, S) \geq \alpha$, we need to check if

$$(1+\alpha)x - \alpha w^i \in \mathbf{conv}\ \{w^j : j = 1, \ldots, m\} \text{ for every } i = 1, \ldots, m, \tag{53}$$

which can be achieved by solving a system of linear inequalities since x and α is fixed.

To solve the global problem, solve the following LP

$$\begin{array}{lll} \max\limits_{\alpha,y,\lambda,\mu} & \alpha & \\ \text{s.t.} & -\alpha w^i + y = z^i & \text{for } i = 1, \ldots, m \\ & z^i = \sum\limits_{k=1}^m \lambda_k^i w^k & \text{for } i = 1, \ldots, m \\ & y = \sum\limits_{k=1}^m \mu_k w^k & \\ & e^T\lambda^i = 1, \lambda^i \geq 0, & \text{for } i = 1, \ldots, m \\ & e^T\mu = 1 + \alpha, \mu \geq 0 & \end{array} \tag{54}$$

A suitable initial point for this problem is $\alpha = 0$, $z^i = y = \frac{1}{m}\sum_{k=1}^m w^k, \lambda^i = \mu = \frac{1}{m}e$.

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