ON COST MATRICES WITH TWO AND THREE DISTINCT VALUES OF HAMILTONIAN PATHS AND CYCLES

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Abstract. Polynomially testable characterization of cost matrices associated with a complete digraph on n nodes such that all the Hamiltonian cycles (tours) have the same cost is well known. Tarasov [25] obtained a characterization of cost matrices where tour costs take two distinct values. We provide a simple alternative characterization of such cost matrices that can be tested in $O(n^2)$ time. We also provide analogous results where tours are replaced by Hamiltonian paths. When the cost matrix is skew-symmetric, we provide polynomially testable characterizations such that the tour costs take three distinct values. Corresponding results for the case of Hamiltonian paths are also given. Using these results, special instances of the asymmetric travelling salesman problem (ATSP) are identified that are solvable in polynomial time and that have improved ϵ-approximation schemes. In particular, we observe that the 3/2 performance guarantee of the Christofides algorithm extends to all metric Hamiltonian symmetric matrices. Further, we identify special classes of ATSP for which polynomial ϵ-approximation algorithms are available for $\epsilon \in \{3/2, 4/3, 4\tau, \frac{3\tau^2}{2}, \frac{4+\delta}{3}\}$ where $\tau > 1/2$ and $\delta \geq 0$

Key Words: Combinatorial optimization, Graphs, Hamiltonian cycles, Hamiltonian paths, Approximation algorithms.
AMS Subject Classification: 90C27,90C57,90C35

1. Introduction

Let G be a directed graph with node set $V(G) = \{1, 2, \ldots, n\}$ and arc set $E(G)$. For each arc $(i, j) \in E(G)$ a cost c_{ij} is prescribed. Let C be the cost matrix associated with G such that the $(i, j)^{th}$ element of C is c_{ij} if $(i, j) \in E(G)$ and ∞ if $(i, j) \notin E(G)$. If G is an undirected graph then the matrix C is symmetric. For any Hamiltonian cycle (tour) H of G the cost of H corresponding to C is given by $C(H) = \sum_{(i,j)\in H} c_{ij}$. Cost matrix C is said to be a *k distinct cost tour matrix (DTC(k) matrix)* if and only if the number of distinct values of costs of tours in G is exactly k. In particular, C is a $DTC(1)$ matrix if and only if every tour in G has the same cost.

It is easy to see that for any digraph G and any arbitrary mappings $a, b : V(G) \to \Re$, if a cost matrix C associated with G is defined as

$$c_{ij} = a_i + b_j \text{ for all } (i, j) \in E(G) \tag{1.1}$$

then all tours in G have the same value. Interestingly, Gabovich [6] proved that this condition totally characterizes the class of DTC(1) matrices when G is a complete digraph. He attributes the result for the undirected case to [19] and [23]. For future reference we summarize the characterization of DTC(1) matrices for a complete digraph below.

Theorem 1.1. [6] *If G is a complete digraph with associated cost matrix C, then the following statements are equivalent.*

1. *All tours in G have the same cost β with respect to C.*
2. *There exist $\{a_i, b_i : i = 1, 2, \ldots, n\}$ such that $c_{ij} = a_i + b_j$ for all $(i, j) \in E(G)$ and $\sum_{i=1}^{n}(a_i + b_i) = \beta$.*

(Note that by a complete digraph we mean a digraph in which each two vertices are joined by two oppositely oriented arcs.) It is easy to see that if C is symmetric, we can choose $a_i = b_i$ for all i and if C is skew-symmetric we can choose $a_i = -b_i$. Independent proofs of Theorem 1.1 are reported in [5, 7, 12]. Independent proofs for the undirected case are also reported in [16, 22]. Condition (2) in Theorem 1.1 can be tested in $O(n^2)$ time.

A cost matrix C associated with G is said to be a k *distinct cost Hamiltonian path matrix (DPC(k) matrix)* if and only if the number of distinct values of costs of Hamiltonian paths in G is exactly k. Thus if C is a $DPC(1)$ matrix then every Hamiltonian path in G has the same cost. The structure of a DPC(1) matrix associated with a complete digraph is much simpler compared to that of a DTC(1) matrix.

Lemma 1.2. [12] *A cost matrix C associated with a complete digraph G is a DPC(1) matrix if and only if it is a constant matrix, (i.e. all the non-diagonal elements of C are identical).*

In view of Theorem 1.1 and Lemma 1.2, a natural question is to identify the structure of DTC(k) and DPC(k) matrices associated with complete digraphs for $k \geq 2$. Tarasov [25] provided an elegant characterization of DTC(2) matrices. His proof is inductive in nature with a base case for $n = 5$ the validity of which is established by complete enumeration using a computer. In this paper we provide an alternative characterization of DTC(2) matrices associated with complete digraphs. Our characterization is simple and can be tested in $O(n^2)$ time. Further, our proof does not use complete enumeration. We also provide a complete characterization of DPC(2) matrices associated with complete digraphs and establish a relationship between DTC(2) and DPC(2) matrices.

Tarasov [25] also obtained a characterization of cost matrices associated with the assignment problem with three distinct objective function values. However, no corresponding results are known for Hamiltonian cycles. We give a complete characterization of DTC(3) and DPC(3) skew-symmetric cost matrices associated with complete digraphs. It is also shown that there are no skew symmetric DPC(2) matrices of size greater than 3.

Given a digraph G and an associated cost matrix C, the well known *travelling salesman problem (TSP)* is to find a tour H in G such that its cost $C(H)$ is as small as possible. If the graph G is undirected or equivalently the matrix C is symmetric, the resulting TSP is called symmetric travelling salesman problem (STSP). Thus STSP is a special case of TSP. To emphasize the fact we are considering a directed graph, we some times refer to TSP as asymmetric travelling salesman problem (ATSP).

The general TSP is well known to be NP-hard. It is NP-hard even to find an ϵ-approximate solution for this problem for any $\epsilon > 0$ [18]. However, there are several special cases of TSP that are solvable in polynomial time [11] and several special cases that are solvable using polynomial ϵ-approximation algorithms. The books edited by Lawler et al [18] and Gutin and Punnen [8] provide the state of the art on the topic. Polynomial solvability of an instance of TSP and existence of a polynomial ϵ-approximation algorithm for it depend on the properties of the associated cost matrix. Note that some polynomially solvable classes of TSP are characterized in terms of the structure of the underlying (di)graph. However, any such characterization can be rephrased in terms of properties of the cost matrix. For a comprehensive study on polynomially solvable cases of TSP, we refer to [11].

The simplest of all polynomially solvable cases of TSP is the constant TSP, where the cost matrix is a DTC(1) matrix. We show that the characterizations of DTC(2) and DTC(3) cost matrices discussed above identify new polynomially solvable cases of the TSP. In addition, we show that these also help us in solving some instances of ATSP as STSP. It is well known that any ATSP on n nodes can be formulated as an STSP on $2n$ nodes. However, we show that for special ATSP, our results make it possible to find equivalent STSP of the same size.

For any tour $H = (u_1, u_2, \ldots, u_n, u_1)$, we denote its *reversal* $(u_n, u_{n-1}, \ldots, u_1, u_n)$ by H^*. We say that a digraph G is symmetrical if and only if for any arc (i, j) in $E(G)$, the arc (j, i) is also in $E(G)$; and we say that G is Hamiltonian symmetrical if and only if for any tour H in G, its reversal H^* is also in G. Thus, every symmetrical digraph is Hamiltonian symmetrical. A cost matrix C associated with a Hamiltonian symmetrical digraph G is said to be *Hamiltonian symmetrical* [9] if and only if $C(H) = C(H^*)$ for every tour H in G. Recently, Halskau [9] showed that cost matrix C associated with a complete digraph G is Hamiltonian symmetrical if and only if there exist mappings $a, b : V(G) \to \Re$ such that

$$c_{ij} = a_i + b_j + d_{ij}, \tag{1.2}$$

where $D = (d_{ij})$ is a symmetric matrix of same size as C. He also showed that this condition can be tested easily in $O(n^2)$ time. From equation (1.2), it can be seen that

$$C(H) = D(H) + \alpha \text{ for all tours } H \in G. \tag{1.3}$$

Thus, as observed by Halskau [9], solving ATSP with cost matrix C is equivalent to solving STSP with cost matrix D. If $\alpha \neq 0$ the transformation given by equation (1.2) (and hence equation (1.3)) does not preserve ϵ-optimality. We construct a simple transformation from ATSP to STSP that characterizes Hamiltonian symmetrical matrices associated with symmetrical digraphs. This transformation preserves ϵ-optimality as well as τ-triangular inequality [1] and range inequality [17]. Thus known performance guarantees of various approximation algorithms for the STSP extend to the more general class of Hamiltonian symmetric TSPs. In particular, for metric ATSP with a Hamiltonian symmetric cost matrix, we observe that a 3/2-approximate solution can be obtained using the Christofides algorithm [18]. It is an open question to find a polynomial ϵ-approximation scheme for the metric ATSP for any constant ϵ [14]. The best known performance ratio for a polynomial approximation algorithm for such an instance of ATSP is $4/3 \log_3 n \approx 0.842 \log_2 n$ [13]. When C is Hamiltonian symmetrical and satisfies the weak τ-triangle inequality (see Section 4), we observe that the performance ratio becomes $\min\{4\tau, \frac{3}{2}\tau^2\}$ for $\tau \geq 1$ and when C satisfies the weak range inequality (see Section 4) the performance ratio becomes $\frac{4+\delta}{3}$ for $\delta \geq 0$.

Kabadi and Punnen [12] introduced a special class of graphs called SC-Hamiltonian graphs, that includes complete (di)graphs, complete bipartite (di)graphs etc. for which Theorem 1.1 continues to hold. We observe that the transformation from ATSP to STSP discussed above extends easily to all symmetrical, SC-Hamiltonian digraphs. We also consider relationship between ATSP and STSP G is a complete digraph and $|C(H) - C(H^*)| = 0$ or α for some positive number α. Using the notion of DTC(2) and skew-symmetric DTC(3) matrices we identify special classes of ATSP on n nodes for which an optimal solution can be obtained by solving $n/2$ symmetric TSPs on n nodes.

The major contributions of the paper are summarized below.

- A simple alternative characterization of DTC(2)matrices associated with complete digraphs is given along with a simple proof. This further enhance the knowledge of structural properties of this class of matrices.

- Complete polynomially testable characterizations of DPC(2) matrices, skew-symmetric DPC(3) matrices, and skew-symmetric DTC(3) matrices associated with complete digraphs are given.
- New special cases of ATSP are identified that can be solved in polynomial time. Further, special classes of ATSP are identified for which polynomial ϵ-approximation algorithms are available for $\epsilon \in \{3/2, 4/3, 4\tau, 3/2\tau^2, \frac{4+\delta}{3}\}$ where $\tau > 1/2$ and $\delta \geq 0$.

It would be interesting to find simple proofs for our characterization of skew-symmetric DTC(3) and DPC(3) matrices associated with complete digraphs. Further, it is a challenging problem to identify polynomially testable characterizations of general DTC(k) and DPC(k) matrices (if exists) for a given $k \geq 3$.

The paper is organized as follows. In Section 2 we discuss our characterization of DTC(2) and DPC(2) matrices associated with complete digraphs. Section 3 deals with our characterization of DTC(3) and DPC(3) skew-symmetric matrices associated with complete digraphs. Special ATSPs are considered in Section 4 and concluding remarks are given in Section 5.

We conclude this section by introducing some notations and tour construction schemes. For any (di)graph G, its vertex set is denoted by V(G) and its edge set is denoted by $E(G)$. Unless otherwise specified, throughout the rest of this paper we assume that G is a complete digraph and thus, all the non-diagonal elements of the associated cost matrix C are finite. For any cost matrix C associated with G and any subgraph H of G, we denote its cost $\sum_{ij\in H} c_{ij}$ by $C(H)$. By elements of a cost matrix we mean its non-diagonal elements with finite values. Let $H = (u_1, u_2, \ldots, u_n, u_1)$ be a tour in G. We describe below four schemes to construct new tours from H. These constructions are used extensively in the subsequent sections.

Scheme 1 (*Ordered 3-exchange*): Let (i, j) be a given arc not in H. Without loss of generality, we assume that $i = u_1$. Let $j = u_r$ for some $2 < r < n$. Choose some integer ℓ such that $r \leq \ell \leq n$. Then the new tour obtained by this construction is given by $H' = (u_1, u_r, u_{r+1}, \ldots, u_\ell, u_2, u_3, \ldots, u_{r-1}, u_{\ell+1}, u_{\ell+2}, u_n, u_1)$.

Scheme 2 (*Arc reversal*): Let (i, j) be an arc in H. Without loss of generality, we assume that $i = u_1$ and $j = u_2$. Then the new tour obtained by this construction is given by $\bar{H} = (u_2, u_1, u_3, u_4, \ldots, u_{n-1}, u_n, u_2)$.

Scheme 3 (*Inverse arc reversal*): Let (i, j) be an arc in H. Without loss of generality, we assume that $i = u_1$ and $j = u_2$. Then the new tour obtained by this construction is given by $\hat{H} = (u_1, u_2, u_n, u_{n-1}, \ldots, u_4, u_3, u_1)$.

Scheme 4 (*Path reversal*): Consider a path $(u_r, u_{r+1}, \ldots, u_s)$ in H for some $1 \leq r, s, \leq n$. (Here, indices of u are taken modulo n.) Then the new tour obtained by this construction is given by $\tilde{H} = (u_s, u_{s-1}, \ldots, u_r, u_{s+1}, \ldots u_{r-1}, u_s)$. Note that in Scheme 4, if we choose $u_r = i$ and $s = r + 1$ we get Scheme 2 and if we set $u_r = j$ and $s = r - 1$ we get Scheme 3.

2. DTC(2) and DPC(2) Matrices For Complete Digraphs

In this section we discuss our characterizations of DTC(2) and DPC(2) matrices associated with complete digraphs. Thus, *throughout this section, all non-diagonal elements of cost matrices considered are finite.*

For any cost matrix C and any $r \in \{1, 2, \ldots, n\}$, define $a_r = 0$, $b_r = 0$, $a_i = c_{ir}$ for $i \neq r$, and $b_i = c_{ri}$ for $i \neq r$. Define the matrix $\hat{C} = (\hat{c}_{ij})_{n \times n}$, as $\hat{c}_{ij} = c_{ij} - a_i - b_j$. We call $\hat{C}$ the *r-reduced matrix of C*. For the r-reduced matrix $\hat{C}$, it can be seen that $\hat{c}_{rj} = \hat{c}_{jr} = 0$, for $j \in \{1, 2, \ldots, n\}$, $j \neq r$. We call the $(n-1) \times (n-1)$ submatrix C^0 of $\hat{C}$ obtained by deleting its r^{th} row and r^{th} column the *r-reduced submatrix of C*. For convenience, we refer to the n-reduced matrix and the n-reduced submatrix of C as simply *the reduced matrix* and *the reduced submatrix* of C, respectively.

To motivate the study of DTC(2) and DPC(2) matrices, let us start with an example. Consider the cost matrix C given below and its reduced matrix $\hat{C}$.

$$C = \begin{bmatrix} \infty & 6 & 4 & 5 & 9 & 1 & 5 \\ 5 & \infty & 5 & 6 & 10 & 2 & 6 \\ 3 & 5 & \infty & 4 & 8 & 0 & 3 \\ 6 & 8 & 6 & \infty & 11 & 3 & 7 \\ 11 & 13 & 11 & 12 & \infty & 8 & 12 \\ 8 & 10 & 8 & 9 & 13 & \infty & 9 \\ 5 & 7 & 4 & 6 & 10 & 2 & \infty \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} \infty & -6 & -5 & -6 & -6 & -6 & 0 \\ -6 & \infty & -5 & -6 & -6 & -6 & 0 \\ -5 & -5 & \infty & -5 & -5 & -5 & 0 \\ -6 & -6 & -5 & \infty & -6 & -6 & 0 \\ -6 & -6 & -5 & -6 & \infty & -6 & 0 \\ -6 & -6 & -5 & -6 & -6 & \infty & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \infty \end{bmatrix}$$

It is difficult to see that the tour costs corresponding to the cost matrix C above have only two distinct values. Now look at the reduced matrix $\hat{C}$. Its reduced submatrix contains only two distinct elements -5 and -6. This is no accident and we will show that the reduced submatrix of a DTC(2) matrix always contains exactly two distinct elements. However, this is not enough to characterize DTC(2) matrices. Obviously, if these two distinct elements are distributed arbitrarily, the corresponding tour costs could have a large number distinct values. So the question is, in what patterns are the two distinct elements distributed within the reduced submatrix and the answer to this yields the required characterization. The following facts are easy to verify.

Fact 1 For any $r \in \{1, 2, \ldots, n\}$, the r-reduced matrix of a skew-symmetric, $n \times n$ matrix is a skew-symmetric matrix; and the r-reduced matrix of a symmetric, $n \times n$ matrix is a symmetric matrix.

Fact 2 For any positive integer k, if we subtract a constant from all entries in any row or column of a DTC(k) matrix then the resulting matrix is also a DTC(k) matrix.

Fact 3 For any positive integer k and any $r \in \{1, 2, \ldots, n\}$, an $n \times n$ matrix C is a DTC(k) matrix if and only if its r-reduced matrix $\hat{C}$ is a DTC(k) matrix.

We now give a characterization of DTC(k) matrices in terms of DPC(k) matrices.

Theorem 2.1. *A cost matrix C is a DTC(k) matrix if and only if for any $r \in \{1, 2, \ldots, n\}$, its r-reduced submatrix is a DPC(k) matrix.*

Proof. For any $n \times n$ cost matrix C and any $r \in \{1, 2, \ldots, n\}$, let $\hat{C}$ and C^0 be the r-reduced matrix and the r-reduced submatrix of C, respectively. Let G^0 be the digraph obtained from G by deleting node r. Then C^0 is a cost matrix associated with complete digraph G^0. Since row r and column r of $\hat{C}$ have all zero non-diagonal entries, any Hamiltonian path P in G^0 can be extended to a Hamiltonian cycle H in G such that $C^0(P) = \hat{C}(H)$ and conversely, for any Hamiltonian cycle H

in G, the Hamiltonian path P in G^0 obtained by deleting the node r has $\hat{C}(H) = C^0(P)$. Hence, $\hat{C}$ is a DTC(k) matrix if and only if C^0 is a DPC(k) matrix. The result follows in view of Fact 3. □

The following corollary of Theorem 1.1, Lemma 1.2 and Theorem 2.1 can be easily verified and will be useful in this section and the next.

Corollary 2.2. *Let C be an $n \times n$, skew-symmetric cost matrix associated with a complete digraph G. Then C is a DPC(1) matrix if and only if all its non-diagonal elements are zero. If every non-diagonal element of C is either 0 or $\pm x$ for some positive value x, then C is a DTC(1) matrix if and only if there exists $S \subseteq N = \{1, 2, \ldots, n\}$ such that for any $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$,*

$$c_{ij} = \begin{cases} -\alpha & \text{if } i \notin S \text{ and } j \in S \\ \alpha & \text{if } i \in S \text{ and } j \notin S \\ 0 & \text{Otherwise.} \end{cases}$$

for $\alpha \in \{-x, x\}$.

Our main results in this section are polynomially testable characterizations of DTC(2) and DPC(2) matrices. Theorem 2.1 provides a characterization of DTC(2) matrices in terms of DPC(2) matrices. We shall give a characterization of DPC(2) matrices that can be verified in polynomial time. For this, we need a characterization of a special class of DTC(2) matrices with only two types of non-diagonal entries α and β and such that the two distinct values of tour costs are $\{n\alpha, (n-1)\alpha + \beta\}$ or $\{n\beta, (n-1)\beta + \alpha\}$. Such a DTC(2) matrix is called an *elementary DTC(2) matrix.*

Theorem 2.3. *A cost matrix C is an elementary DTC(2) matrix if and only if the following three conditions are satisfied.*

(1) The non-diagonal elements of C have only two distinct values, α and β.
(2) C is not a DTC(1) matrix.
(3) Either the set of arcs with cost α or the set of arcs with cost β is one of the following three types: (i) $\{(i,j),(j,i)\}$ for some pair i,j of nodes; (ii) $\{(i,u) : u \in S\}$ where $i \in N$, $S \subset V(G) \setminus \{i\}$ and $S \neq \emptyset$; (iii)$\{(u,i) : u \in S\}$ where $i \in N$, $S \subset V(G) \setminus \{i\}$ and $S \neq \emptyset$.

Proof. If a cost matrix C satisfies the conditions of the theorem, then it is easy to verify that the set of distinct tour costs is either $\{n\alpha, (n-1)\alpha + \beta\}$ or $\{n\beta, (n-1)\beta + \alpha\}$.

Conversely, suppose C is an elementary DTC(2) matrix with non-diagonal elements of values α and β, and neither the set of arcs of cost α nor the set of arcs with cost β is of the above three types. Then, since C is not a DTC(1) matrix, there exists a pair of arcs $(i,j),(u,v)$ of cost α and a pair of arcs $(x,y),(s,t)$ of cost β with $i \neq u$, $j \neq v$, $x \neq s$, and $y \neq t$. It can be verified that there exists a tour containing both the arcs (i,j) and (u,v) and a tour containing both the arcs (x,y) and (s,t), contradicting the fact that D is an elementary DTC(2) matrix. □

Lemma 2.4. *If C is a DPC(2) matrix and H is any tour in G, then arcs of H have at most two distinct costs.*

Proof. If possible, let H contain three arcs, say e, f and g, with distinct costs. By deleting these arcs one at a time, we get three Hamiltonian paths in G with distinct costs. This contradicts the fact that C is a DPC(2) matrix and hence the result follows. □

Using Lemma 2.4, we now prove a stronger result.

Lemma 2.5. *If C is a DPC(2) matrix and $n > 3$, then the arcs of G have exactly two distinct costs.*

Proof. Obviously, arcs of G have at least two distinct costs. If possible let there be more than two distinct costs. Then it is possible to find a tour H containing arcs with at least two distinct costs, say α and β. By Lemma 2.4 arcs of H have exactly two distinct costs, α and β. Let (i, j) be an arc of G not in H with cost γ where $\gamma \neq \alpha, \beta$. Using Scheme 1 of the previous section, with $u_1 = i$ and $\ell = n$, generate tour H'. Clearly, H' contains at least one and at most three arcs of cost γ. By Lemma 2.4, all other arcs of H' have cost precisely one of α or β. Let us assume it is α. Then, H contains at most 3 arcs of cost β. Thus cost of H is $(n-x)\alpha + x\beta$ and cost of H' is $(n-y)\alpha + y\gamma$ for some x, y such that $1 \leq x, y \leq 3$. Since $n > 3$, it can be verified that the collection of all Hamiltonian paths, obtained from H and H' by deleting an arc, have at least three distinct costs, contradicting the fact that C is a DPC(2) matrix.

□

If $n = 3$ it is possible to have a DPC(2) matrix with more than two distinct elements. For example consider the matrix

$$C^* = \begin{bmatrix} \infty & 0 & 1 \\ 1 & \infty & 2 \\ 0 & 1 & \infty \end{bmatrix}$$

All the Hamiltonian paths in the complete digraph G on three nodes with C^* as the cost matrix have cost either 0 or 2; yet C^* has non-diagonal elements with three distinct values. However, Lemma 2.5 can be shown to hold even for n=3 if C is a symmetric matrix.

Theorem 2.6. *For any integer $n > 3$, an $n \times n$ matrix C associated with a complete digraph G is a DPC(2) matrix if and only if it satisfies one of the following two conditions.*

(1) C is a DTC(1) matrix and non-diagonal elements of C have exactly two distinct values.
(2) C is an elementary DTC(2) matrix.

Proof. Suppose C satisfies conditions (1) or (2). We must show that C is a DPC(2) matrix. If Condition (1) is satisfied then all the tours in G have the same cost, say δ, and C contains elements of value, say α and β, only. Then the Hamiltonian paths in G have costs $\delta - \alpha$ and $\delta - \beta$ and hence C is a DPC(2) matrix. Suppose Condition (1) is not satisfied but Condition (2) is satisfied. Thus arcs of G have exactly two distinct costs, say α and β, and the set of distinct values of costs of tours in G is $\{n\alpha, (n-1)\alpha + \beta\}$ or $\{n\beta, (n-1)\beta + \alpha\}$. Thus the set of distinct values of costs of Hamiltonian paths in G is $\{(n-1)\alpha, (n-2)\alpha + \beta\}$, or $\{(n-1)\beta, (n-2)\beta + \alpha\}$. Thus C is a DPC(2) matrix.

Conversely, assume that C is a DPC(2) matrix. By Lemma 2.5 the elements of C must be either α or β for some α and β. Suppose C does not satisfy any of the conditions (1) and (2). Then there exist two tours H^1 and H^2 in G with costs $C(H^1) = x\alpha + (n-x)\beta$ and $C(H^2) = y\alpha + (n-y)\beta$ such that $\{0, 1\} \neq \{x, y\} \neq \{n-1, n\}$. Without loss of generality, let $y < x$. If $x = n$ and $y = 0$. Choose an arc (i, j) in G of cost β and using Scheme 1 given in the previous section, generate tour H' from the tour H^1 with $i = u_1$, $j = u_r$ and $\ell = n$. The cost of the tour H' is $z\alpha + (n-z)\beta$ for some $n - 3 \leq z < n$. It is easy to see that by removing from each of H^1, H^2 and H' an arc of each type, we get Hamiltonian paths of at least three distinct costs. We thus have a contradiction.

We thus have to consider the following cases: (i) $y = 0$ and $1 < x < n$, (ii) $1 \leq y < x - 1$ and (iii) $1 \leq y = x - 1 < n - 1$. In each of these cases, by removing from each of H^1 and H^2 an arc of each type, we get Hamiltonian paths of at least three distinct costs. This contradicts the fact that

C is a DPC(2) matrix and hence the result follows.

□

In the case of symmetric matrices, Theorem 2.6 can be shown to hold even for the case $n = 3$. We get the following interesting corollary of Theorem 2.6, Fact 1, Theorem 2.1 and Corollary 2.2.

Corollary 2.7. *There is no DPC(2) skew-symmetric matrix of size $n \geq 4$, and no DTC(2) skew-symmetric matrix of size $n \geq 5$.*

Proof. Let C be an $n \times n$, skew-symmetric matrix for some $n \geq 4$. It follows from Corollary 2.2 that C cannot be a DTC(1) matrix with exactly two distinct types of elements, and it follows from Theorem 2.3 that it cannot be an elementary DTC(2) matrix. Hence, by Theorem 2.6, C cannot be a DPC(2) matrix. The second part of this corollary now follows from Fact 1 and Theorem 2.1.

□

It can be verified that there are skew-symmetric DPC(2) matrices for $n = 3$.

Theorem 2.6 gives a characterization of the class of DPC(2) matrices in terms of the classes of DPC(1) and elementary DTC(2) matrices. A complete characterization of the class of DPC(1) matrices is given in Lemma 1.2 and a complete characterization of elementary DTC(2) matrices is given in Theorem 2.3. Thus we have a complete characterization of DPC(2) matrices and hence by Theorem 2.1, we have a complete characterization of the class of DTC(2) matrices. We now observe that using our characterizations we can recognize DTC(2) and DPC(2) matrices in strongly polynomial time.

Given an $n \times n$ matrix C, we can check if it is a DTC(2) matrix as follows. Construct the reduced matrix $\hat{C}$ of C, obtain the $(n-1) \times (n-1)$ reduced submatrix C^0 by deleting row n and column n and verify if C^0 satisfies the conditions (i) or (ii) of Theorem 2.6 (which can be easily done in $O(n^2)$ time). By Theorem 2.1, C^0 satisfies one of these conditions if and only if C is a DTC(2) matrix. If C^0 satisfies one of the conditions of Theorem 2.6, then it is possible to construct two Hamiltonian paths in G^0 of distinct costs in $O(n^2)$ time. These paths can be extended to tours in G of distinct costs with respect to C. The smaller of these tours is an optimal solution to the corresponding instance of TSP on G. All these computations can be performed in $O(n^2)$ time.

It may be noted that if C is a DTC(2) matrix then any tour with objective function value greater than or equal to the average value of all tours is an optimal solution to the corresponding instance of TSP. It is known that several well known heuristics for TSP produce solutions with objective function value no worse than the average cost of all tours [21]. Thus any such heuristic guarantees an optimal solution when the cost matrix of a given instance TSP is a DTC(2) matrix.

3. Skew-Symmetric DTC(3) and DPC(3) Matrices For Complete Digraphs

Tarasov [25] gave a complete characterization of distance matrices for the assignment problem with three distinct objective function values. No corresponding results for Hamiltonian cycles are available. In this section we provide complete characterization of such skew-symmetric cost matrices. As we show in Section 4, skew-symmetric matrices are useful in converting some special ATSPs into STSPs.

We first prove several Lemmas which allow us to present the proof of our main result in a simple way. *Throughout this section, we assume that the digraph G under consideration is a complete digraph and the associated cost matrix C is skew-symmetric.* Our first aim is to establish that a skew-symmetric DPC(3) matrix will not contain more than three distinct elements.

The following Lemma is easy to verify.

Lemma 3.1. *For a skew-symmetric cost matrix C, if there exists a Hamiltonian path (tour) of cost α, then its reversal has a cost of $-\alpha$. Hence, if C is a DPC(k) (DTC(k)) matrix for some $k \geq 3$, then the corresponding distinct costs of Hamiltonian paths (tours) will be $\{\pm\alpha_1, \pm\alpha_2, \ldots \pm \alpha_s\}$ if k is even, and $\{0, \pm\alpha_1, \pm\alpha_2, \ldots \pm\alpha_s\}$ if k is odd, for some $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_s$, where $s = \lfloor \frac{k}{2} \rfloor$. In particular, if C is a DPC(3) (DTC(3)) matrix, then the corresponding distinct costs of Hamiltonian paths (tours) will be $\{0, \pm\alpha_1\}$ for some $\alpha_1 > 0$.*

Lemma 3.2. *If C is a DPC(3) matrix, then no tour in G contains more than three arcs of distinct costs.*

Proof. Let H be a tour in G containing four arcs with distinct costs w, x, y, z. Then G contains Hamiltonian paths of distinct values $C(H) - w, C(H) - x, C(H) - y, C(H) - z$. This contradicts the fact that C is a DPC(3) matrix.

□

Lemma 3.3. *If C is DPC(3) and $n \geq 5$ then no tour in G contains arcs of costs $x, -x$ and y for any distinct non-zero $x, -x, y$.*

The proof of this lemma is little bit long and we relegate this to the appendix.

Lemma 3.4. *If C is DPC(3) and $n \geq 5$ then no tour in G contains arcs of costs x, y and z for any distinct non-zero real numbers x, y, z.*

Proof. If possible let G contain a tour H with arcs having costs x, y and z. Without loss of generality, let us assume that $H = (1, 2, \ldots, n, 1)$. From Lemma 3.2 H does not contain any arc of cost other than x, y and z.

Case 1 : H contains a pair of non-adjacent arcs $(a, a+1)$ and $(b, b+1)$ with same cost : Let the cost of each of these arcs be x. If there exists some arc in H of cost y or z that is not in $\{(a-1, a), (a+1, a+2)\}$ then set $(i, j) = (a, a+1)$. Else, since $n \geq 5$, there exists some arc in H of cost y or z that is not in $\{(b-1, b), (b+1, b+2)\}$ and therefore set $(i, j) = (b, b+1)$. Construct a tour $\bar{H}$ from H using the Scheme 2 and the chosen arc (i, j). The tour $\bar{H}$ contains arcs of costs $\{-x,\ x,\ y\}$ or $\{-x,\ x, z\}$. But this contradicts Lemma 3.3.

The condition of Case 1 is always satisfied if H contains three arcs of same cost or $n \geq 7$.

Case 2 : $n = 6$ and the tour H contains exactly 2 arcs of each of the costs $x,\ y,\ z$: By deleting different arcs from H we get Hamiltonian paths of three different costs $2x + 2y + z$, $2x + y + 2z$ and $x + 2y + 2z$. Without loss of generality, let us assume that $x < y < z$. Then Lemma 3.1 implies that $2x + y + 2z = 0$ and $x + 2y + 2z = -(2x + 2y + z)$. But this implies that $x = -z$ and we have a contradiction.

Case 3 : $n = 5$ and the tour H contains exactly 2 arcs of cost x that are adjacent and exactly two arcs of cost y that are adjacent: By deleting different arcs from H we get Hamiltonian paths of three different costs $2x+2y$, $2x+y+z$ and $x+2y+z$. Without loss of generality, let us assume that $x < y$.

Subcase 1: $x < y < z$: In this case, Lemma 3.1 implies that $2x + y + z = 0$ and $x + 2y + z = -(2x + 2y)$. These imply that $y > 0$, $x = -3y$ and $z = 5y$, and therefore the distinct costs of Hamiltonian paths are $\{-4y, 0, 4y\}$. Suppose the arcs of cost x are $(1, 2)$ and $(2, 3)$ and the arcs of cost y are $(3,4)$ and $(4,5)$. Construct a tour $\hat{H}$ from H using Scheme 3 with arc $(2,3)$ as arc (i,j). Then by Lemma 3.2 it follows that $\hat{H}$ contains only arcs of costs x, $-y$ and $-z$. Deleting from $\hat{H}$ an arc of cost $-y$, we get a Hamiltonian path of cost no more than $-8y$, contradicting the fact that C is a DPC(3) matrix.

Subcase 2: $x < z < y$ *:* In this case, Lemma 3.1 implies that $2x + 2y = 0$, which implies that $x = -y$. We thus have a contradiction.
Subcase 3: $z < x < y$*:* In this case consider the reversal H^* of H which satisfies the condition of Subcase 1, a contradiction. This completes the proof of the lemma.

□

Lemma 3.5. *If C is DPC(3) then no tour in G contains arcs of cost $0, x$ and y, where x and y are non-zero values with $|x| \neq |y|$.*

Proof. Suppose G contains a tour H with arcs of costs 0, x and y. From Lemma 3.2 no arc in H can have cost other than $0, x$ or y. By deleting different arcs from H we can generate Hamiltonian paths of distinct costs $C(H), C(H) - x, C(H) - y$. Without loss of generality, let us assume that $x < y$.

If $0 < x < y$ then $C(H) - y < C(H) - x < C(H)$. Then Lemma 3.1 implies that $C(H) - x = 0$ or $C(H) = x$, which is impossible.
If $x < y < 0$, then Lemma 3.1 implies that $C(H) - y = 0$ or $C(H) = y$, which is again impossible.
If $x < 0 < y$, then Lemma 3.1 implies that $C(H) = 0$ and $C(H) - y = -y$ equals $-C(H) + x = x$, a contradiction. This proves the result.

□

Lemma 3.6. *If C is a DPC(3) skew-symmetric matrix with more than three distinct values of non-diagonal elements and $n \geq 5$, then no tour in G contains arcs of cost $0, x$ and $-x$ for any positive value x.*

Proof. Suppose G contains a tour H with arcs with costs 0, x and $-x$ for some positive value x. Lemma 3.2 implies that no arc in H can have cost other than $0, x$ or $-x$. We can generate from H Hamiltonian paths of distinct costs $C(H) - x, C(H)$ and $C(H) + x$. Lemma 3.1 therefore implies that $C(H) = 0$, and therefore, H contains equal number, (say α), of arcs of values x and $-x$. Since C contains more than three distinct values of non-diagonal elements, there exists an arc (s,t) in G of some value $y \notin \{-x, 0, x\}$. It is easy to see that we can construct from H using Scheme 1, either arc (s,t) or arc (t,s) as arc (i,j) and a proper choice of ℓ, a Hamiltonian path containing either arcs of costs x, $-x, y$ or arcs of costs $0, x, y$. In either case, we arrive at a contradiction using Lemma 3.3 or Lemma 3.5. This proves the lemma.

□

Theorem 3.7. *If C is an $n \times n$, skew-symmetric DPC(3) matrix where $n \geq 5$, then the number of distinct values of non-diagonal elements of C is no more than three.*

Proof. If possible let C be an $n \times n$, skew-symmetric DPC(3) matrix with $n \geq 5$ and more than three distinct values of non-diagonal elements. Then it will contain non-diagonal elements of four distinct values of the type $\{-x, x, -y, y\}$ for some positive, distinct values x and y. It is easy to see that G contains a tour H with at least two arcs of distinct costs.

If H contains three arcs of distinct costs, we have a contradiction by one of the lemmas 3.3, 3.5, 3.5 and 3.6.

Suppose H contains exactly two arcs of distinct costs, say α and β. Since $n \geq 5$, at least three of the arcs will have same cost, say α. If $\beta \neq -\alpha$ we can construct a tour H_1 as in the proof of Lemma 3.4 containing arcs of costs $\alpha, -\alpha, \beta$. If $\beta = -\alpha$, then again we can construct a tour in G containing $\alpha, -\alpha, \gamma$ for some γ such that $|\gamma| \neq |\alpha|$. In either case we obtain a contradiction to Lemma 3.3. This proves the theorem.

□

We shall now give polynomially testable characterizations of DPC(3) and DTC(3) matrices. For this, we need a characterization of a special class of skew-symmetric DTC(3) matrices with non-diagonal elements $0,\ x,\ -x$ for some positive x such that (i) the three distinct values of tour costs are $\{-x,\ 0,\ x\}$ and (ii) every tour of cost x contains precisely one arc of cost x and the remaining arcs of cost 0. We call such a DTC(3) matrix an *elementary DTC(3) matrix*. The following property of tours associated with a DTC(3) matrix is useful in characterizing this class of matrices.

Lemma 3.8. *If C is an elementary DTC(3) matrix with non-diagonal elements $0, x$ and $-x$ for $x \neq 0$, then no tour in G has two adjacent arcs of cost x or two adjacent arcs of cost $-x$.*

Proof. If possible, let H be a tour in G where at least two arcs of cost x that are adjacent in H. Without loss of generality assume that H contains the path $1-2-3$ and $c_{12} = c_{23} = x$. Let s and t be two nodes in G such that arcs $(s, 1)$ and $(3, t)$ are in H. Let $P(t, s)$ denote the path from t to s in H and let $C(P(s, t)) = \theta$. Since C is DTC(3), $C(H) = 0$. Let H_1 be the tour obtained from H by reversing the path $1-2-3$ (Scheme 4). Then

$$C(H) = 2x + \theta + c_{s1} + c_{3t} = 0 \tag{3.1}$$

and

$$C(H_1) = -2x + \theta + c_{1t} + c_{s3} = 0 \tag{3.2}$$

From (3.1) and (3.2) we have

$$4x = c_{1t} + c_{s3} - c_{s1} - c_{3t}. \tag{3.3}$$

Since elements of C are $x, -x$ or 0, From (3.3) we have $c_{1t} = c_{s3} = x$ and $c_{s1} = c_{3t} = -x$ and hence from (3.1) $\theta = 0$. Now construct a tour H_2 from H by reversing arc $(1, 2)$ (Scheme 2) we have

$$C(H_2) = -2x + c_{13} + c_{s2} = 0. \tag{3.4}$$

From (3.4), $c_{13} = c_{s2} = x$. Now the tour $s-2-3-1-t-P(t, s)$ have cost $2x$, a contradiction to the fact that C is an elementary DTC(3) matrix.

If at least two arcs of cost $-x$ are adjacent in H then using the reversal of H in place of H in the above argument we get a contradiction. This completes the proof.

□

An immediate consequence of Lemma 3.8 is that if the cost matrix C is an elementary DTC(3) matrix then for any node i of G, either all arcs coming into i have cost in $\{0, x\}$ or all arcs coming into node i have cost from $\{0, -x\}$. This property is crucial to the proof of our characterization of elementary DTC(3) matrices. Thus we summarize the matrix version of this property in the following corollary.

Corollary 3.9. *If C is an elementary DTC(3) matrix, then no row (or column) contains both x and $-x$.*

Theorem 3.10. *A skew-symmetric cost matrix C with non-diagonal elements $0,\ x,\ -x$ for some $x > 0$ is an elementary DTC(3) matrix if and only if there exists some $r \in N$ and a non-empty, proper subset S of $N - \{r\}$ such that for any $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$,*

$$c_{ij} = \begin{cases} \alpha & \text{if } i \in S \text{ and } j = r \\ -\alpha & \text{if } i = r \text{ and } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

for $\alpha \in \{x, -x\}$.

Proof. Let C be a skew-symmetric with non-diagonal elements $0,\ x,\ -x$ for some $x > 0$.

If C satisfies the condition of the theorem then consider any tour H in G. Let the arcs in H incident to node r be $\{(i, r), (r, j)\}$. If $\{i, j\} \subseteq S$ or $\{i, j\} \cap S = \emptyset$, then $C(H) = 0$. If $|\{i, j\} \cap S| = 1$, then for some $\alpha \in \{x, -x\}$, $C(H) = \alpha$ and the tour contains precisely one arc of cost α and the other arcs of cost 0. Thus, C is an elementary DTC(3) matrix.

Conversely, suppose C is an elementary DTC(3) matrix. It follows from Corollary 3.9 that for each $i \in V(G)$, the cost of each outgoing arc of node i belongs to $\{0, \alpha\}$ and the cost of each incoming arc of node i belongs to $\{0, -\alpha\}$ for some $\alpha \in \{x, -x\}$. Thus the rows of C (and hence columns of C as well) can be renumbered such that

$$C = \begin{pmatrix} O_1 & O_2 & A \\ O_3 & O_4 & O_5 \\ -A & O_6 & O_7 \end{pmatrix}$$

where O_2, O_3, O_5, o_6 are matrices with all entries zero, O_1, O_4, O_7 are matrices with all non-diagonal entries zero, and A is a matrix with all entries $0, x$ or all entries $0, -x$ and with no row or column with all zero entries. For definiteness, we assume without loss of generality that entries of A are $0, x$. Let the columns of O_1 be indexed by set $S_1 = \{1, 2, \ldots, p\}$, columns of O_2 by set $S_2 = \{p+1, p+2, \ldots, p+q\}$ and columns of A by set $S_3 = \{p+q+1, p+q+2, \ldots, n\}$. If $S_2 = \emptyset$ then the matrix A has at least one zero element. Otherwise by subtracting x from each row in S_1 and adding x to each column in S_1 we can reduce all the non-diagonal elements of the matrix to zero and hence C is a DTC(1) matrix, a contradiction. If A has exactly one row or one column then C is of the required type. Thus A has at least two rows and columns.

Case 1 *A contains at least one zero entry:* Then there exist $i \in S_1$, $j \in S_3$ such that $c_{ij} = 0$ and there exist $z \in S_1$ and $t \in S_3$ such that $z \neq i, t \neq j$ and $c_{it} = c_{zj} = x$. Now construct a tour containing the path $z - j - i - t$. If $C(H) = x$ or $-x$ we have a contradiction to the fact that C is an elementary DTC(3) matrix. Thus $C(H) = 0$. Now construct the tour $\hat{H}$ from H by reversing arc (i, j) (Scheme 2). Since $c_{zi} = c_{jt} = 0$, $C(\hat{H}) = 2x$, a contradiction.

Case 2 *All entries of A are x:* In this case $S_2 \neq \emptyset$. Choose $i \in S_2$ and $t, z \in S_3$; $t \neq z$. Then $c_{1,i} = c_{zi} = 0$ and $c_{1,t} = c_{2,z} = x$. Construct a tour H in G containing the path $2 - z - i - 1 - t$. If $C(H) = x$ or $-x$ we have a contradiction. Thus $C(H) = 0$. Construct a tour $\hat{H}$ from H by reversing the path $z - i - 1$. Then $C(\hat{H}) = 2x$, a contradiction. This completes the proof. □

Theorem 3.11. *Let C be a skew-symmetric $n \times n$ cost matrix corresponding to the complete digraph G on node set $N = \{1, 2, \ldots, n\}$ where $n \geq 5$. Then C is a DPC(3) matrix if and only if one of the following holds:*
(i) C is a DTC(1) matrix with non-diagonal elements $\{0, x, -x\}$ for some $x > 0$.
(ii) C is an elementary DTC(3) matrix.

Proof. If C satisfies any one of the two conditions of the theorem, then it can be readily verified that the distinct values of costs of Hamiltonian paths in G are $\{x, -x, 0\}$.

Conversely suppose C is a skew-symmetric DPC(3) matrix with $n \geq 5$. Then C contains at least two distinct elements and by Theorem 3.7 it does not contain more than three distinct elements. Thus the distinct elements of C are either of the form $x, -x$ or of the form $0, x, -x$ for some $x > 0$, and cost of each tour and Hamiltonian path in G is of the form px for some integer p.

If C is a DTC(1) matrix then by Corollary 2.2 if follows that C must contain non-diagonal elements of values $x, -x, 0$. By Corollary 2.7, C cannot be a DTC(2) matrix.

Suppose C is a DTC(k) matrix for some $k \geq 3$. Choose a tour H in G of cost px with largest possible value of p. The tour H must contain at least one arc of cost x.

Case 1: $p = n$: The tour H contains only arcs of values x. Hence, there exist a Hamiltonian path of cost $(n-1)x$. Choose any arc (i, j) in H and construct a new tour $\bar{H}$ using Scheme 2. The new tour has arcs of cost $x, -x$ and has a total cost of $\bar{p}x$ for some $(n-2) \geq \bar{p} \geq (n-2) \geq -1$, (since $n \geq 5$). Hence, there exist Hamiltonian paths of costs $(\bar{p}-1)x, (\bar{p}+1)x$. If $0 \leq \bar{p} < n-2$, then we have Hamiltonian paths of four distinct costs $-(n-1)x, -(\bar{p}+1)x, (\bar{p}+1)x, (n-1)x$. If $\bar{p} = n-2$ or -1 then we have Hamiltonian paths of four distinct costs $-(n-1)x, -(\bar{p}-1)x, (\bar{p}-1)x, (n-1)x$. In either case, we have contradiction to the fact that C is a DPC(3) matrix.

Case 2: $2 \leq p < n$: In this case, there exist Hamiltonian paths of four distinct costs four distinct costs $-(p+\alpha)x, -(p-1)x, (p-1)x, (p+\alpha)x$, where $\alpha = 1$ if the tour H contains arcs of cost $-x$ and $\alpha = 0$ if it contains arcs of cost 0. We thus have a contradiction.

Case 3 : $p = 1$: By Lemma 3.1, C must be a DTC(3) matrix with distinct values of tour costs $0, \pm x$. If there exist tours of cost x containing arcs of costs $-x$ and 0, then we get Hamiltonian paths of costs $\pm x, \pm 2x$, contradicting the fact that C is a DPC(3) matrix. Suppose a tour of cost x contains an arc of cost $-x$. Then, there exist Hamiltonian paths of costs $-2x,\ 2x$. In this case, if any tour of cost 0 contains arc of cost x or $-x$ then we get Hamiltonian paths of costs $-x,\ x$, contradicting the fact that C is a DPC(3) matrix. Hence, every tour of cost 0 contains all arcs of cost 0. Choose an arc (i, j) of cost 0 and contract it in G to a pseudonode 0. Then all the arcs in the resultant digraph have cost 0. Since C is skew-symmetric, this implies that all the non-diagonal elements of C are zero, a contradiction. Hence, every tour in G of cost x contains only one arc of cost x and all other arcs of cost 0. Thus, C is an elementary DTC(3) matrix. This proves the theorem.

□

Corollary 3.12. *For any integer $n \geq 6$, an $n \times n$, skew-symmetric matrix C is a DTC(3) matrix if and only if there exists an $r \in \{1, 2, \ldots, n\}$ such that the r-reduced submatrix of C is an elementary DTC(3) matrix.*

Proof. If an r-reduced submatrix of C is an elementary DTC(3) matrix then it is a DPC(3) matrix and by Theorem 2.1 that C is a DTC(3) matrix.

Conversely, suppose C is a DTC(3) matrix. Let $\hat{C}$ and C^0 be its reduced, (i.e. n-reduced) matrix and submatrix, respectively. Then by Theorem 2.1, C^0 is a DPC(3) matrix. Hence, it satisfies condition (i) or condition (ii) of Theorem 3.11. If C^0 is an elementary DTC(3) matrix, then we have the desired result with $r = n$. Else, suppose C^0 is a DTC(1) matrix with non-diagonal elements $\{0, x, -x\}$ for some $x > 0$. Then C^0 has the structure specified in Corollary 2.2 for some proper, non-empty subset S of $\{1, 2, \ldots, n-1\}$. It is easy to see that for any $r \in \{1, 2, \ldots, n-1\}$, the r-deduced matrix of $\hat{C}$, (which is the same as the r-reduced matrix of C), is an elementary DTC(3) matrix. This proves the result.

□

4. Special Asymmetric TSPs

In this section we consider some special asymmetric TSP's that can be solved as one or more symmetric TSP's of same size. The digraph G in this section is not necessarily complete. Two cost matrices C and D associated with the same digraph G are said to be *tour value equivalent* if and only if $C(H) = D(H)$ for every tour H in G. Clearly, C and D are tour value equivalent if and only if $C - D$ is a DTC(1) matrix with all tours costs equal to zero.

For arbitrary graphs, testing tour value equivalence of two cost matrices is NP-hard. To see this, suppose a polynomial time oracle exist which with input C, D and the associated digraph G tells us "yes" if the matrices are tour value equivalent and "no" if they are not. For a given digraph G, let C and D be cost matrices associated with it with all finite entries of value zero and one, respectively. Invoke the oracle with C, D and G as input. If the oracle answers "yes" then G has no Hamiltonian tours else G contains at least one tour. Since testing Hamiltonicity of a digraph is NP-hard, testing tour value equivalence is also NP-hard. It may be noted that, given two cost matrices C and D and a digraph G, the decision problem: "Are C and D tour value equivalent?" is not known to be in NP. But this problem belongs to the class co-NP.

In spite of this negative result, for a large class of (di)graphs, called SC-Hamiltonian graphs [12], tour value equivalence can be tested in polynomial time. A digraph G, not necessarily complete, is said to be *separable constant Hamiltonian (SC-Hamiltonian)* if and only if it is Hamiltonian and for any DTC(1) matrix associated with it, there exist mappings $a, b : V(G) \to \Re$ such that $c_{ij} = a_i + b_j$ for all (i, j) in $E(G)$. An undirected graph G is said to be *SC-Hamiltonian* if and only if it is Hamiltonian and for any associated DTC(1) matrix C, there exists a mapping $a : V(G) \to \Re$ such that $c_{ij} = a_i + a_j$ for all (i, j) in $E(G)$. Obviously, testing if a given (di)graph is SC-Hamiltonian is NP-hard. Kabadi and Punnen [12] identified a large class of graphs and digraphs that are SC-Hamiltonian. This class includes complete (di)graphs, complete bipartite (di)graphs etc.

Theorem 4.1. *Two distance matrices C and D associated with an SC-Hamiltonian digraph are tour value equivalent if and only if there exist mappings $a, b : V(G) \to \Re$ such that $c_{ij} - d_{ij} = a_i + b_j$ for all (i, j) in $E(G)$, and $\sum_{i=1}^{n}(a_i + b_i) = 0$.*

Proof. By definition, C and D are tour value equivalent if and only if $C(H) = D(H)$ for every tour H in G which is true if and only if $(C - D)(H) = 0$ for every tout H in G. Let $Q = C - D$. Since G is SC-Hamiltonian, $Q(H) = 0$ for every tour H in G if and only if there exist mappings $a, b : V(G) \to \Re$ such that $q_{ij} = a_i + b_j$ and $\sum_{i=1}^{n}(a_i + b_i) = 0$.

□

Since a_i's and b_j's of Theorem 4.1 can be obtained in $O(n^2)$ time, the tour value equivalence of two cost matrices associated with an SC-Hamiltonian graph can be verified in $O(n^2)$ time.

It is easy to show that tour value equivalence is reflexive, symmetric and transitive and hence an equivalence relation. Thus tour value equivalence partitions the space of cost matrices of a digraph

G into equivalence classes. It can be verified that each of these equivalence classes is a convex set.

Let C be a cost matrix associated with a symmetrical digraph G. Recall from Section 1 that C is *Hamiltonian symmetrical* if and only if $C(H) = C(H^*)$ for every tour H in G where H^* is the reversal of H. Since $C(H^*) = C^T(H)$, C is Hamiltonian symmetrical if and only if C and C^T are tour cost equivalent. Recently, Halskau [9] showed that when G is complete, C is Hamiltonian symmetrical if and only if there exists mappings $a, b : V(G) \to \Re$ such that $c_{ij} = a_i + b_j + d_{ij}$ where $D = (d_{ij})$ is a symmetric matrix. It can be verified that this characterization extends to all symmetrical, SC-Hamiltonian, digraphs. Halskau [9] provided other characterizations of Hamiltonian symmetrical cost matrices for a complete digraph as given in the following theorem.

Theorem 4.2. [9] *Let C be any $n \times n$ cost matrix associated with a complete digraph G. Then the following statements are equivalent:*

(1) C is Hamiltonian symmetrical
(2) $C = K + D$, where K is a DTC(1) matrix and D is a symmetric matrix.
(3) $S^k(C)$ is symmetrical for any node k where $S^k(C)$ is the savings matrix associated with C and the $(i,j)^{th}$ element of $S^k(C)$ is $c_{ik} + c_{kj} - c_{ij}$.
(4) $c_{ij} - c_{ji} = \frac{1}{n}((R_i(C) - K_i(C)) - (R_j(C) - K_j(C))\ \forall i, j, i \neq j$ where $R_i(C)$ is the ith row sum of C and $K_i(C)$ is the ith column sum on C.

It may be noted that $S^k(C)$, the savings matrix associated with C, is the negative of k-reduced matrix of C discussed in Section 2. We now give another simple characterization of Hamiltonian symmetrical matrices.

Theorem 4.3. *Let C be a cost matrix associated with a symmetrical digraph G. Then C is Hamiltonian symmetrical if and only if C and $A = \frac{1}{2}(C + C^T)$ are tour value equivalent. If, in addition, G is SC-Hamiltonian then Hamiltonian symmetry of C can be tested in $O(n^2)$ time.*

Proof. Note that C is Hamiltonian symmetrical if and only if $C(H) = C^T(H)$ for all tours H in G. Since any matrix C can be written as $C = 1/2(C + C^T) + 1/2(C - C^T)$, the proof of the first part of the theorem follows. Since for symmetrical, SC-Hamiltonian graphs tour value equivalence can be tested in $O(n^2)$ time, the proof of the second part follows.

□

The characterization above is valid for all symmetrical digraphs. But for digraphs that are not SC-Hamiltonian, verification of the condition above is difficult. In fact for arbitrary symmetrical digraphs, it can be shown that this verification is NP-hard. The characterization of Theorem 4.3 has important applications in approximation algorithms.

The arc weights of G are said to satisfy the τ-triangle inequality if and only if for any three nodes i, j and k of G, $\tau(c_{ij} + c_{jk}) \geq c_{ik}$ [1]. When $\tau = 1$, τ-triangle inequality reduces to the triangle inequality. For $1/2 \leq \tau < 1$, τ-triangle inequality is a restriction of the triangle inequality and for $\tau > 1$ it is a relaxation of the triangle inequality. We now consider a further relaxation of the τ-triangle inequality. A matrix C satisfies *weak τ-triangle inequality* if and only if, for any triplet (i, j, k) with $i \neq j \neq k$,

$$\tau(c_{ij} + c_{ji} + c_{jk} + c_{kj}) \geq c_{ki} + c_{ik} \tag{4.1}$$

In the above definition, if $\tau = 1$ we say that C satisfies *weak triangle inequality.*

Lemma 4.4. *Let C be a cost matrix and $A = (C + C^T)/2$.*

(1) The matrix A satisfies τ-triangle inequality if and only if the matrix C satisfies weak τ-triangle inequality.
(2) If C satisfies τ-triangle inequality then it satisfies weak τ-triangle inequality.
(3) If C is Hamiltonian symmetrical, then C satisfies weak triangle inequality if and only if C satisfies triangle inequality.

Proof. Consider the triplet (i, j, k) corresponding to three nodes of G. Then, A satisfies τ-triangle inequality if and only

$$\tau(a_{ij} + a_{jk}) \geq a_{ik} \Leftrightarrow \tau(\frac{c_{ij} + c_{ji}}{2} + \frac{c_{jk} + c_{kj}}{2}) \geq \frac{c_{ik} + c_{ki}}{2}$$
$$\Leftrightarrow \tau(c_{ij} + c_{ji} + c_{jk} + c_{kj}) \geq c_{ki} + c_{ik}.$$

This completes the proof of part (1). If C satisfies τ-triangle inequality, then

$$\tau(c_{ij} + c_{jk}) \geq c_{ik} \tag{4.2}$$

and

$$\tau(c_{kj} + c_{ji}) \geq c_{ki}. \tag{4.3}$$

Adding inequalities (4.2) and (4.3), we get the proof of (2).
Let us now prove part (3). Since C is Hamiltonian symmetrical, we have $C = A + X$ where $X = (C - C^T)/2$ is a DTC(1) skew symmetric matrix. Thus there exist $a_1, a_2, \ldots, a_n$ such that $x_{ij} = a_i - a_j$ (See the discussion after Theorem 1.1.) and hence the elements of X satisfy the triangle *equality*. Now suppose C satisfies weak triangle inequality. Then by part (1) A satisfies triangle inequality. Thus $A + X$ (and hence C) satisfies triangle inequality. The converse of part (3) follows from part (2) of the lemma.

□

It may be noted that from Lemma 4.4, if C satisfies τ-triangle inequality, then $A = (C + C^T)/2$ also satisfies τ-triangle inequality. Further, part (3) of the above lemma says for Hamiltonian symmetrical matrices, τ-triangular inequality and weak τ-triangular inequality are equivalent if $\tau = 1$. But we like to point out that for $\tau \neq 1$ this equivalence need not hold even for Hamiltonian symmetrical matrices.

The best known performance bound for a polynomial time ϵ-approximation algorithm for the metric asymmetric TSP is $\epsilon = 4/3 \log_3 n \approx 0.842 \log_2 n$ [13]. When C is Hamiltonian symmetrical and satisfies triangle inequality, we can obtain 3/2-approximate solution for the ATSP by applying Christofides algorithm on the cost matrix $(C + C^T)/2$. Thus Lemma 4.4 and Theorem 4.3 extend the applicability Christofides bound beyond the class of symmetric matrices. It may be noted that Lemma 4.4 need not hold for the symmetric matrix D obtained by Halskau [9] if we want to use D in place of A. When the edge costs satisfy the τ-triangle inequality, Bender and Chekuri [2] obtained a 4τ-approximation algorithm and Andreae and Bandelt [1] obtained a $(3\tau^2/2 + \tau/2)$-approximation algorithm for the STSP. Thus in view of Theorem 4.3 and Lemma 4.4 these results extend to Hamiltonian symmetric matrices satisfying weak τ-triangle inequality.

Another application of Theorem 4.3 is when the arc costs satisfy weak range inequality. i.e.

$$\max_{ij}\{c_{ij} + c_{ji}\} \leq \tau \min_{ij}\{c_{ij} + c_{ji}\} \tag{4.4}$$

where $\tau > 1$. The concept of weak range inequality is a generalization of the range inequality considered by Kumar and Rangan which is given by

$$\max_{ij} c_{ij} \leq (2+\epsilon) \min_{ij} c_{ij}. \tag{4.5}$$

Kumar and Rangan [17] showed that the Christofides algorithm produces a 4/3-optimal solution when $\epsilon = 0$ and the cycle cover algorithm [4] produces a $\frac{4+\epsilon}{3}$ solution for all $\epsilon \geq 0$.

Lemma 4.5. *Let C be a cost matrix and $A = (C + C^T)/2$.*

(1) The matrix A satisfies range inequality if and only if C satisfies weak range inequality.
(2) If C satisfies range inequality, then it satisfies weak range inequality.

Proof. The proof of part (1) follows from the definition. Proof of part (2) follows from the inequality:

$$\max_{ij}\{c_{ij} + c_{ji}\} \leq 2\max_{ij}\{c_{ij}\} \leq 2\tau \min_{ij}\{c_{ij}\} \leq \tau \min_{ij}\{c_{ij} + c_{ji}\}.$$

□

Thus by applying the cycle cover algorithm [4] on the cost matrix $A = (C + C^T)/2$ we get a $(2+\epsilon)$-approximate solution for ATSP when the cost matrix C is Hamiltonian symmetrical and satisfies the weak range inequality for $\tau = 2 + \epsilon$. Again, it may be noted that not all symmetric matrices D obtained in [9] satisfy the weak range inequality even if C satisfies the weak range inequality.

For the maximization version of the TSP, the best known polynomial time approximation algorithm has a performance ratio of 2/3 for the ATSP [13] and 3/4 bound for the STSP [3]. Thus from Theorem 4.3, the maximization TSP when C is Hamiltonian symmetrical can be approximated by the 3/4-approximation scheme given in [24] for the STSP. In addition, if C satisfies the triangle inequality this bound can be improved to 7/8 by using the algorithm of Hassin and Rubinstein [10] on the matrix $(C + C^T)/2$. Several polynomially solvable cases of the symmetric TSP can be exploited (for both maximization and minimization versions) to solve the ATSP with cost matrix C whenever $1/2(C + C^T)$ satisfies the required conditions [3, 11].

The discussions above exploits properties of DTC(1) matrices. The next theorem takes advantage of our characterization of skew-symmetric DTC(3) matrices. We first state a lemma that can be easily proved.

Lemma 4.6. *Let C be an $n \times n$ cost matrix associated with asymmetrical digraph G. Then $D = C - C^T$ is a cost matrix associated with G and for any $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k$ for some positive integer k, the following statements are equivalent.*

(1) For any tour H in G, $|C(H) - C(H^*)| = \alpha_i$ for some $i \in \{1, 2, \ldots, k\}$ and for any $i \in \{1, 2, \ldots, k\}$ there exists a tour H in G such that $|C(H) - C(H^*)| = \alpha_i$.
(2) The set of distinct values of costs of tours in G with respect to cost matrix D is precisely $\{\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_k\}$. Thus the number of distinct tour values in G with respect to D is $2k$ if $\alpha_1 > 0$ and $(2k - 1)$ if $\alpha_1 = 0$.

Theorem 4.7. *Let C be an $n \times n$ cost matrix associated with the complete digraph G on node set $N = \{1, 2, \ldots, n\}$ such that for some $x > 0$ any tour H in G satisfies $|C(H) - C^T(H)| = 0$ or $\pm x$. Then a minimum (maximum) cost tour in G can be identified by solving at the most $n/2$ symmetric TSPs on n nodes.*

Proof. Suppose C satisfies the condition of the theorem. Let $D = C - C^T$. Then by Lemma 4.6, D is a DTC(3) matrix with tour costs $0, \pm x$. By Corollary 3.12 it follows that there exist distinct $r, \ell \in N$ and $S \subset N - \{r, \ell\}$ such that the non-diagonal elements of the r-reduced matrix D^0 of D satisfy

$$d^0_{ij} = \begin{cases} \alpha & \text{if } i \in S \text{ and } j = \ell \\ -\alpha & \text{if } i = \ell \text{ and } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

for $\alpha \in \{x, -x\}$. Since D is skew-symmetric, $D(H) = D^0(H)$ for any tour H in G. Also $D^0(H) = \pm x$ if and only if the subpath $j - \ell - i$ of H satisfies $|\{i, j\} \cap S| = 1$ and these are precisely the tours that satisfy $|C(H) - C^T(H)| = x$. We call such a tour a type I tour and the remaining tours, type II tours. Thus for every type II tour H, $C(H) - C^T(H) = 0$. Let $A = \frac{C+C^T}{2}$. Then for every type I tour H, $A(H) = C(H) \pm x/2$ and for every type II tour $A(H) = C(H)$.

Suppose we want to find a tour $\bar{H}$ in G such that $C(\bar{H})$ is minimum. Find a tour $\hat{H}$ such that $A(\hat{H})$ is minimum. If the tour is of type I, then the tour $\bar{H} \in \{\hat{H}, \hat{H}^*\}$ such that $C(\bar{H}) = A(\hat{H}) - x/2$ is the desired tour. (It may be recalled that $\hat{H}^*$ is the reverse of $\hat{H}$.) If the tour $\hat{H}$ is of type II then we need to find a minimum cost type I tour. This can be done as follows.

For each $u \in S$, define the matrix A^u with non diagonal elements as follows:

$$a^u_{ij} = \begin{cases} a_{ij} - M/n & \text{if } i = \ell, j \in N - S - \{\ell\} \text{ or } i \in N - S - \{\ell\}, j = \ell \\ -M & \text{if } i = \ell, j = u \text{ or } i = u, j = \ell \\ a_{ij} & \text{otherwise.} \end{cases}$$

Let H^u be the minimum cost tour in G with respect to cost matrix A^u. Then it is easy to see that H^u is a minimum cost type I tour with respect to A containing the arc (ℓ, u). Choose v such that $A(H^v) = \min\{A(H^u) : u \in S\}$. Then, H^v is a type I tour with minimum cost with respect to A. If $C(\hat{H}) \leq A(H^v) - x/2$ then $\hat{H}$ is a minimum cost tour in G with respect to C. Else $\tilde{H} \in \{H^v, H^{v*}\}$ with $C(\tilde{H}) = A(H^v) - x/2$ is a minimum cost tour in G with respect to C.

The maximization case can be proved analogously.

□

5. Conclusion

We have obtained an alternative characterization DTC(2) matrices and our proof of validity is relatively easy. Complete characterization of DTC(2) matrices, skew-symmetric DTC(3) matrices, and skew-symmetric DPC(3) are given. These characterizations leads to new polynomially solvable special cases of the TSP. Our characterization of skew-symmetric matrices can be used to solve ATSP with special structures as a sequence of at most $n/2$ closely related STSPs. We also identified special classes of ATSPs for which polynomial ϵ-approximation schemes exist for constant ϵ.

An interesting and challenging question is to study characterization of general DTC(k) and DPC(k) matrices for $k \geq 3$ and we leave this question open.

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Appendix

Proof of Lemma 3.3:

Proof. Without loss of generality assume $x > 0$. Suppose there exists a tour H in G containing arcs of costs $x, -x$, and y. Renumber the nodes in G if necessary to make $H = (1, 2, \ldots, n, 1)$. From

Lemma 3.2 all arcs of H have weight $x, -x$, or y. Let $C(H) = \theta$. Then by removing different arcs from the tour we get Hamiltonian paths of distinct costs $\theta - x$, $\theta + x$ and $\theta - y$. It follows from Lemma 3.1 that one of these Hamiltonian paths have cost zero and hence θ equals x, $-x$, or y. We consider three mutually exclusive and exhaustive cases.

Case 1: $-x < y < x$**:** In this case, by Lemma 3.1, $\theta = y$ and $\theta - x (= y - x) = -(\theta + x)(= -(y + x))$, which implies that $y = 0$, a contradiction.

Case 2: $x < y$**:** In this case, by Lemma 3.1, $\theta = x$, $y = 3x$ and the three distinct costs of Hamiltonian paths are $\{-2x, 0, 2x\}$. For each $z \in \{-x, x, y\}$, let $n(z)$ = number of arcs in the tour H of cost z. Then $n(-x) = 3n(y) + n(x) - 1$. We now consider three different subcases.

Subcase 1: $n(y) \geq 2$ and there exists a pair of non-adjacent arcs in H, one of cost x and the other of cost y: From such a pair of arcs, choose the one, say arc (i, j), of cost y. Construct a tour $\hat{H}$ from H using the arc (i, j) and Scheme 3. Then the tour $\hat{H}$ contains at least one arc of cost y, at least $3 + n(x)$ arcs of cost x and at least one and at most $2 + n(x)$ arcs of cost $-x$. Hence, each arc in $\hat{H}$ has cost x, $-x$ or y and the cost $\hat{\theta}$ of the new tour satisfies $\hat{\theta} \geq 4x$. The cost of the Hamiltonian path, obtained from $\hat{H}$ by deleting an arc of cost x, is at least $3x$ and therefore, is not in the set $\{-2x, 0, 2x\}$, contradicting the fact that C is a DPC(3) matrix.

Subcase 2: $n(y) = 2$ and the unique arc in H of cost x is adjacent to both the arcs in H of cost y: Let us assume without loss of generality that the arcs in H of cost y are $(1, 2)$ and $(3, 4)$. The arc in H of cost x is then $(2, 3)$. Construct a tour $\bar{H}$ from H using the arc $(1, 2)$ as (i, j) and Scheme 2. The tour $\bar{H}$ contains arcs of costs $-y$, y and $-x$. Using this tour and case (i) above, we get a contradiction.

Subcase 3: $n(y) = 1$: In this case, $n(-x) = 2 + n(x) \geq 3$. Let us assume without loss of generality that the arc in H of cost y is $(1, 2)$. We consider five possibilities designated as subcase 3.1, subcase 3.2,..., subcase 3.5.

Subcase 3.1: At least one neighbor and at least one non-neighbor in H of the arc $(1, 2)$ has cost x: Construct a tour $\hat{H}$ from H using the arc $(1, 2)$ as (i, j) and Scheme 3. Then the tour $\hat{H}$ contains at least one arc of cost y, at least $1 + n(x)$ arcs of cost x and at least one and at most $1 + n(x)$ arcs of cost $-x$. Hence, each arc in $\hat{H}$ has cost x, $-x$ or y and the cost $\hat{\theta}$ of the new tour satisfies $\hat{\theta}) \geq 3x$. Repeating Case 2 with the tour $\hat{H}$ we get a contradiction.

Subcase 3.2: $n(x) = 2$ and both the neighbors in H of the arc $(1, 2)$ have cost x: In this case, $n = 7$ and arc $(3, 4)$ has cost $-x$. Construct tour $\bar{H}$ from H using the arc $(3, 4)$ as (i, j) and Scheme 2. Then the tour $\bar{H}$ contains at least one arc of cost y, at least 2 arcs of cost x and at least 2 arcs of cost $-x$, and at least one neighbor and at least one non-neighbor in $\bar{H}$ of the arc $(1, 2)$ have cost x. Hence, each arc in $\hat{H}$ has cost x, $-x$ or y. If the cost $\bar{\theta}$ of the new tour is not x, then by repeating Case 2 with this new tour, we arrive at a contradiction. Else, by repeating the Subcase 3.2 with the new tour we arrive at a contradiction.

Subcase 3.3: $n(x) = 2$ and none of the neighbors in H of the arc $(1, 2)$ has cost x: Suppose there exists an arc $(s, t) = (a, b)$ (other than the arc $(2, 1)$) that is not in H and has some cost $z \notin \{-x, x\}$. If the set $\{(a, a + 1), (b - 1, b)\}$ contains all the arcs in H of cost x or if it contains the arc $(1, 2)$

then in stead of the arc (a, b) consider the arc (b, a) and denote in by (s, t) and its cost by z. The set $\{(b, b+1), a-1, a)\}$ will then not contain the arc $(1, 2)$ nor will it contain all the arcs in H of cost x. Construct tour H' from H using the Scheme 1 with arc (s, t) as (i, j) and choosing ℓ such that H' contains the arc $(1, 2)$ and at least one arc of cost x. Then the tour H' contains at least one arc of each of the costs $-x,\ x,\ y,\ z$. Hence, $z = y$ and the tour H' contains at least two arcs of cost y and at least one arc of each of the costs $-x$ and x. Repeating Subcase 1 or Subcase 2 of Case 2 with H' we then arrive at a contradiction. Hence, the only arcs in G of cost other than $-x$ and x are $(1, 2)$ of cost y and arc $(2, 1)$ of cost $-y$.

If there exists a tour H_2 in G containing $(1, 2)$ and all other arcs of cost x, then by deleting an arc of value x from H_2 we get a Hamiltonian path of value $(n+1)x \geq 6x$, contradicting the fact that C is a DPC(3) matrix. If there exists a tour H_3 in G containing $(1, 2)$ and all other arcs of cost $-x$, then by deleting arc $(1, 2)$ from H_3 we get a Hamiltonian path of value $(1-n)x \leq -4x$, a contradiction. Thus every tour in G containing the arc $(1, 2)$ contains at least one arc of each of the costs x and $-x$. If the cost of any such tour is not x, then by repeating Case 2 with this tour, we arrive at a contradiction. Else, let G^0 be the graph obtained by contracting the arc $(1, 2)$ in G to a pseudonode 0. Then every Hamiltonian tour in G^0 has cost $-2x$. Let C^0 be the distance matrix associated with G^0 with rows/ columns arranged in order $(0, 3, \ldots, n)$. Then all the non-diagonal elements of C^0 are $\pm x$. Let $S^1 = \{i : 3 \leq i < n : c^0_{in} = x\}$ and $S^2 = \{i : 3 \leq i < n : c^0_{in} = -x\}$. Let D be the reduced submatrix matrix of C^0. Then by Theorem 2.1 and Corollary 2.2, all the non-diagonal elements of D must equal 0. This implies that $c^0_{ij} = 2x\ \forall\ i \in S^1,\ j \in S^2$; $c^0_{ij} = -2x\ \forall\ i \in S^2,\ j \in S^1$; and $c^0_{ij} = 0 \forall\ i \in S^1,\ j \in S^1$ and $\forall\ i \in S^2,\ j \in S^2$. But this contradicts the fact that all the non-diagonal elements of C^0 are $\pm x$.

Subcase 3.4 : $n(y) = 1$, $n(x) = 1$ and the arcs in H of costs x and y are neighbors: In this case, $n = 5$. Suppose the arc $(2, 3)$ has cost x. The tour $H^1 = (1, 2, 4, 3, 5, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that each of the arcs $(2, 4)$ and $(3, 5)$ has cost $-x$. Now, the tour $H^2 = (1, 2, 3, 5, 4, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that the cost of the arc $(4, 1)$ is $-3x = -y$. The tour H^2 thus contains arcs with four distinct costs which contradicts Lemma 3.2.

Now, suppose the arc $(5, 1)$ has cost x. The tour $H^1 = (1, 2, 3, 5, 4, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that each of the arcs $(3, 5)$ and $(4, 1)$ has cost $-x$. Now, the tour $H^2 = (1, 2, 4, 3, 5, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that the cost of the arc $(2, 4)$ is $-3x = -y$. The tour H^2 thus contains arcs with four distinct costs, a contradiction.

Subcase 3.5 : $n(y) = 1$, $n(x) = 1$ and the arcs in H of costs x and y are non-neighbors: In this case too, $n = 5$. Suppose the arc $(3, 4)$ has cost x. The tour $H^1 = (1, 2, 5, 4, 3, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that each of the arcs $(2, 5)$ and $(3, 1)$ has cost $-x$. Similarly, the tour $H^2 = (1, 2, 5, 3, 4, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that each of the arcs $(5, 3)$ and $(4, 1)$ has cost $-x$. Now, the tour $H^3 = (1, 2, 3, 5, 4, 1)$ has at least one arc of each of the costs $-x$, x and y and its total cost is $3x$, a contradiction.

Now, Suppose the arc $(4, 5)$ has cost x. The tour $H^1 = (1, 2, 5, 4, 3, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that each of the arcs $(2, 5)$ and $(3, 1)$ has cost $-x$. Similarly, the tour $H^2 = (1, 2, 5, 4, 3, 1)$ has at least one arc of each of the costs $-x$, x and y. Hence, its total cost must be x, which implies that each of the arcs $(2, 4)$

and $(5, 3)$ has cost $-x$. Now, the tour $H^3 = (1, 2, 4, 3, 5, 1)$ has at least one arc of each of the costs $-x$, x and y and its total cost is $3x$, a contradiction.

Case 3: $y < -x$: In this case, by considering the reversal H^* of H and repeating Case 2, we arrive at a contradiction.
This completes the proof. □

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