Sensitivity analysis in linear optimization: Invariant support set intervals

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Abstract
Sensitivity analysis is one of the most interesting and preoccupying areas in optimization. Many attempts are made to investigate the problem's behavior when the input data changes. Usually variation occurs in the right hand side of the constraints and/or the objective function coefficients. Degeneracy of optimal solutions causes considerable difficulties in sensitivity analysis. In this paper we briefly review three types of sensitivity analysis and consider the question: what is the range of the parameter, where for each parameter value, an optimal solution exists with exactly the same set of positive variables that the current optimal solution has. This problem is coming from managerial requirements. Managers want to know in what range of variation of sources or prices in the market can they keep the installed production lines active while only production levels would change.

Keywords: Parametric Optimization, Sensitivity Analysis, Linear Optimization, Interior Point Method, Optimal Partition.

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1 Introduction

Consider the primal Linear Optimization (LO) problem in the standard form

$$LP \qquad \begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0, \end{aligned}$$

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and its dual

$$LD \qquad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y + s = c \\ & s \geq 0, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $c, x, s \in \mathbb{R}^n$ and $b, y \in \mathbb{R}^m$. Any solution vector $x \geq 0$ satisfying the constraint $Ax = b$ is called a *primal feasible solution* and any vector (y, s) with $s \geq 0$ satisfying $A^T y + s = c$ is called a *dual feasible solution*. For any primal feasible x and dual feasible y it holds that $c^T x \geq b^T y$ (*weak duality property*) and the objective function values of LP and LD are equal if and only if both solution vectors are optimal (*strong duality property*). If x^* is primal-feasible, (y^*, s^*) is dual-feasible and $x^{*T} s^* = 0$, then they are primal-dual optimal solutions. We say (x^*, y^*, s^*) are *strictly complementary optimal* solutions if for all $i \in \{1, 2, \dots, n\}$, either x^*_i or s^*_i is zero but not both. For strictly complementary optimal solutions, we have $x^{*T} s^* = 0$, and $x^* + s^* > 0$. By the Goldman-Tucker Theorem [5], the existence of strictly complementary optimal solutions is guaranteed if both problems LP and LD are feasible. Let $\mathcal{LP}$ and $\mathcal{LD}$ denote the feasible regions of problems LP and LD, respectively. Their optimal solution sets are denoted by $\mathcal{LP}^* = \{x \in \mathcal{LP} \mid x \text{ is optimal}\}$ and $\mathcal{LD}^* = \{(y, s) \in \mathcal{LP} \mid (y, s) \text{ is optimal}\}$, respectively.

The *support* set of a nonnegative vector v is defined as $\sigma(v) = \{i : v_i > 0\}$. The index set $\{1, 2, \dots, n\}$ can be partitioned in two subsets

$$\mathcal{B} = \{i : x_i > 0 \text{ for an optimal solution } x \in \mathcal{LP}^*\},$$

$$\mathcal{N} = \{i : s_i > 0 \text{ for an optimal solution } (y, s) \in \mathcal{LD}^*\}.$$

This partition of the index set $\{1, 2, \dots, n\}$ is known as the *optimal partition*, and denoted by $\pi = (\mathcal{B}, \mathcal{N})$. The uniqueness of the optimal partition is a direct consequence of the convexity of $\mathcal{LP}^*$ and $\mathcal{LD}^*$. It is obvious that for any strictly complementary optimal solutions (x^*, y^*, s^*), the relations $\sigma(x^*) = \mathcal{B}$ and $\sigma(s^*) = \mathcal{N}$ hold.

The Simplex Method invented by Dantzig [2] uses pivoting to solve LO problems. Let x^* and (y^*, s^*) be optimal solutions of problems LP and LD, respectively. A square submatrix A_B of m independent columns of A is called a *basis* with index set B. A primal optimal solution x^* is called a *basic* solution if $x^*_i = 0,\ \forall i \notin B$ and $s^*_j = 0,\ \forall j \in B$. It means that for any basic variable x^*_i in a basic solution, corresponding dual constraint is an active constraint. Any pivoting algorithm gives an optimal *basic* solution. A basic solution x, with exactly m positive variables is called *primal nondegenerate*, otherwise, it is said to be *primal degenerate*. A dual basic solution (y, s) is *dual nondegenerate* if $s_j > 0$ for each nonbasic index j, otherwise the dual basic solution (y, s) is called *dual degenerate*. An optimal basic solution of LP defines an *optimal basic partition* of the index set $\{1, 2, \ldots, n\}$ that separates basic and nonbasic variables. In the case of degeneracy, when there are multiple optimal basic solutions, optimal basic partitions vary from one optimal basic solution to another. It should be mentioned that if the problem LP has a unique nondegenerate optimal solution, then the optimal partition and the optimal basic partition, are identical.

Interior Point Methods (IPMs) solve optimization problems in polynomial time [15, 16]. They start from a feasible (or an infeasible) interior point of the positive orthant and generate an interior solution nearby the optimal set. By using a simple rounding procedure, a strictly complementary optimal solution can be obtained in strongly polynomial time [7]. Any strictly complementary optimal solution provides the optimal partition too.

Let us define the perturbation of the LP's data as follows

$$\begin{aligned} \min \quad & (c + \epsilon_c \triangle c)^T x \\ \text{s.t.} \quad & Ax = \; b + \epsilon_b \triangle b \\ & \quad x \geq 0, \end{aligned} \tag{1}$$

where ϵ_b and ϵ_c are two real parameters, $\triangle b \in \mathbb{R}^m$ and $\triangle c \in \mathbb{R}^n$ are nonzero perturbing direction vectors. In special cases, one of the vectors $\triangle b$ or $\triangle c$ may be zero or all but one of the components of $\triangle b$ and $\triangle c$ are zero. One wants to know what happens to the available (given) optimal solution, if such perturbation occurs. Such questions occurred soon after the simplex method was introduced and the related research area is known as *sensitivity analysis* and *parametric programming* [10, 14]. Most results are valid only under nondegeneracy assumption of optimal solutions and we refer to this study as *classic* sensitivity analysis. When ϵ_b and ϵ_c vary independently, the problem is known as a *multi-parametric programming*. In this paper we study the case when $\epsilon_b = \epsilon_c = \epsilon$.

After the landmark paper of Karmakar [11] that initiated intense research on IPMs, sensitivity analysis became a hot topic again and it regained its importance in post optimality analysis of large scale problems. Several papers are published based on the concept of optimal partition. These investigations focused on finding the range of the parameter ϵ for which the optimal partition remains invariant. In this viewpoint, we do not have to worry about the degeneracy of optimal solutions. Alder and Monteiro [1] and Jansen et. al. [9] started independently this research line for LO problems and Roos et. al. [15] have a comprehensive summary of parametric programming in the context of IPMs for LO problems. Simultaneous perturbation of the right hand side (RHS) and the objective function data when the primal and dual LO problems are in canonical form, is studied by Greenberg [6]. One can find a comprehensive survey of sensitivity analysis of LO problems in [3, 15] and the references therein, but no-simultaneous perturbation of both the RHS and the objective function data is considered.

Recently, Koltai and Terlaky [12] categorized sensitivity analysis in three types for LO problems. Here we briefly review these three types. We prefer to have more descriptive names for them[1].

- **Type I (Basis Invariancy):** Let x^* and (y^*, s^*) be optimal basic solutions of LP and LD, respectively. One wants to find the range of perturbation of parameter ϵ, such that *the given optimal basic solution remains optimal.* This study is in the realm of the simplex method and it is based on the nondegeneracy assumption of the optimal solution. It is worthwhile to mention that when the problem has multiple optimal (and thus degenerate) solutions, then depending on the method used in solving the problem, one gets different optimal basic solutions and consequently different, and thus confusing optimality ranges [8, 12, 15].

- **Type II (Support Set Invariancy):** Let x^* and (y^*, s^*) be any optimal solutions of problems LP and LD, respectively. Further, let $\sigma(x^*) = P$. Thus, the index set $\{1, 2, \ldots, n\}$ can be partitioned as (P, Z), where $Z = \{i : x_i^* = 0\}$. Type II sensitivity analysis deals with *finding an interval of variation of parameter ϵ, such that there is a pair of primal-dual optimal solutions $(x^*(\epsilon), y^*(\epsilon), s^*(\epsilon))$ for any ϵ in this interval, in which the positive variables in x^* remain positive in $x^*(\epsilon)$ and the same holds for zero variables.* It means that the perturbed problem should have an optimal solution with the same support set that x^* has i.e., $\sigma(x^*(\epsilon)) = \sigma(x^*) = P$. If an LO problem has an optimal solution x^* with the support set P, then we say that with support set P this LO problem satisfies

[1] The authors would like to thank H. Greenberg for his suggestion in having descriptive names for these three types of sensitivity analysis.

the *Invariant Support Set* (ISS) property. We also refer to the partition (P, Z) as the *invariant support set partition* of the index set $\{1, 2, \dots, n\}$. Let $\Upsilon_L(\triangle b, \triangle c)$ denote the set of ϵ values with the property that for all $\epsilon \in \Upsilon_L(\triangle b, \triangle c)$, there is a primal optimal solution $x^*(\epsilon)$ of the perturbed LO problem with $\sigma(x^*) = P$. It will be proved that this set is an interval of the real line. We refer to this interval as the *ISS Interval.* The ISS interval $\Upsilon_L(\triangle b, \triangle c)$ can be different from the interval which is obtained from sensitivity analysis of Type I. Koltai and Terlaky [12] only introduced this type of sensitivity analysis but did not present a procedure how to calculate ISS intervals. Recently, Lin and Wen [13] have done Type II sensitivity analysis for a special case, the assignment problem.

In this paper, we give an answer to the question how to determine ISS intervals. We consider this question in three cases, when the perturbation occurs in the RHS of the constraints and/or in the objective function of problem LP.

Observe that one may consider more general cases [2]:

1. $\sigma(x^*(\epsilon)) \subseteq \sigma(x^*)$,
2. $\sigma(x^*(\epsilon)) \supseteq \sigma(x^*)$,
3. Both cases 1 and 2.

An economic interpretation of cases 1 and 2 might be as follows. In case 1, a manager aims not to install new production lines but may decide to uninstall some of the active ones. Case 2 can be interpreted that the manger wants to maintain existing production lines, meanwhile, he is not willing to set up new production lines. Identifying the range of parameter ϵ with the aforementioned goals are the objective of Type II sensitivity analysis in cases 1 and 2. We assert that the methodology presented in this paper answers case 1 as well as case 3. The only difference between these two cases is that the ISS intervals in case 1 are always closed intervals, while these intervals might be an open interval in case 3.

- **Type III (Optimal Partition Invariancy):** In this type of sensitivity analysis, one wants to find the range of variation of parameter ϵ for which *the rate of change of the optimal value function is constant.* A point ϵ, where the slope of the optimal value function

[2]The authors want to appreciate H. Greenberg for mentioning this general classification of Type II sensitivy analysis.

changes is referred to as *break point*. Determining of the rate of changes i.e., the left and right derivatives of the optimal value function at break points is also aimed. It is proved that the optimal value function is a piecewise linear function of the parameter ϵ when the perturbation occurs in either the RHS or the objective function data of problem LP. These intervals, in which the rate of change of the optimal value function is constant, are closed and referred to as *linearity intervals*. There is a one to one correspondence between the linearity intervals, the optimal value function and the optimal partitions of the index sets [1, 4, 6, 15]. The interior of linearity intervals are called *invariancy intervals*, because the optimal partition is invariant on those open intervals.

The case when simultaneous perturbation occurs both in the RHS and the objective function data of problem LP has been studied by Greenberg [6]. He proved that the optimal value function is piecewise quadratic and the optimal partition is invariant on those intervals. Ghaffari et. al. [4] further analyzed the properties of the optimal value function when simultaneous perturbation of the RHS and the objective function happen in Convex Quadratic Optimization (CQO) and specialized the obtained results to LO problems.

In this paper, we investigate Type II sensitivity analysis for LO. When the perturbation occurs in the RHS data, using the given optimal solution, a smaller problem is introduced. To obtain this auxiliary *reduced* problem, we omit the variables that are zero in the given optimal solution. We show that Type II sensitivity analysis of the problem LP and Type III sensitivity analysis for its reduced perturbed problem coincide and the invariancy interval of the reduced problem can be used to identify the ISS interval of the original perturbed problem. In the case when the perturbation occurs in the objective function, it is shown that there is a close relationship between invariancy intervals and the ISS intervals of the original problem. Identifying the ISS interval in simultaneous perturbation of the RHS and objective function data is investigated as well.

The paper is organized as follows. Section 2 is devoted to study the behavior of an LO problem when variation occurs in the RHS data and/or the objective function data of problem LP. The relationship between Type II and Type III sensitivity analysis are studied and we show how the ISS intervals can be determined. The case when perturbation occurs in the objective function data is studied as well as simultaneous perturbation. Some examples are presented in Section 3 to illustrate our methods in $\mathbb{R}^2$. The closing section contains the summary of our findings.

2 Type II Sensitivity Analysis for Linear Optimization

Let a pair of primal-dual optimal solutions (x^*, y^*, s^*) of LP and LD be given and let $\sigma(x^*) = P$. It should be mentioned that x^* is not necessarily a basic nor a strictly complementary optimal solution. Let us consider the perturbed primal and dual LO problems as follows:

$$LP(\triangle b, \triangle c, \epsilon) \qquad \begin{aligned} \min \quad & (c+\epsilon\triangle c)^T x \\ \text{s.t.} \quad & Ax = b + \epsilon\triangle b \\ & x \geq 0, \end{aligned}$$

$$LD(\triangle b, \triangle c, \epsilon) \qquad \begin{aligned} \max \quad & (b+ \epsilon\triangle b)^T y \\ \text{s.t.} \quad & A^T y + s = c + \epsilon\triangle c \\ & s \geq 0, \end{aligned}$$

where $\epsilon \in \mathbb{R}$ and $\triangle b \in \mathbb{R}^m$ and $\triangle c \in \mathbb{R}^n$ are perturbation vectors that do not vanish simultaneously. Let $\pi = (\mathcal{B}, \mathcal{N})$ denote the optimal partition of LP and LD. Having a primal optimal solution x^* with $\sigma(x^*) = P$, a partition (P, Z) of the index set $\{1, 2, \dots, n\}$ is given. We refer to (P, Z) as the ISS partition of the index set $\{1, 2, \dots, n\}$. Simply let $p = |P|$ and $n - p = |Z|$. It is obvious that $P \subseteq \mathcal{B}$ and $\mathcal{N} \subseteq Z$. For optimal basic solutions we have $p \leq m$ and strict inequality holds if x^* is a primal degenerate basic solution. On the other hand, p may be greater than m, in particular if LP is dual degenerate. In this paper, we use the ISS partition (P, Z) of the index set $\{1, 2, \dots, n\}$.

First, we introduce some notation. Let $\mathcal{LP}(\triangle b, \triangle c, \epsilon)$ and $\mathcal{LD}(\triangle b, \triangle c, \epsilon)$ be the feasible sets of problems $LP(\triangle b, \triangle c, \epsilon)$ and $LD(\triangle b, \triangle c, \epsilon)$, respectively. Further, let $\mathcal{LP}^*(\triangle b, \triangle c, \epsilon)$ and $\mathcal{LD}^*(\triangle b, \triangle c, \epsilon)$ denote their optimal solution sets. Let $\Upsilon_L(\triangle b, \triangle c)$ denote the set of all ϵ values such that the problem $LP(\triangle b, \triangle c, \epsilon)$ satisfies the ISS property w.r.t. the ISS partition (P, Z). It is obvious that $\Upsilon_L(\triangle b, \triangle c)$ is nonempty, since $LP(\triangle b, \triangle c, 0) = LP$, that satisfies the ISS property w.r.t. the ISS partition (P, Z). Analogous notations are used when $\triangle b$ or $\triangle c$ is zero. We also use the concept of the *actual invariancy interval* that refers to the interior of the linearity interval that contains zero. In special cases, the actual invariancy interval might be the singleton $\{0\}$.

The following lemma shows that the set $\Upsilon_L(\triangle b, \triangle c)$, where (P, Z) is the given ISS partition, is convex.

Lemma 2.1 *Let problem LP satisfy the ISS property w.r.t. the ISS partition* (P, Z)*. Then* $\Upsilon_L(\triangle b, \triangle c)$ *is a convex set.*

Proof: Let a pair of primal-dual optimal solutions $x^* \in \mathcal{LP}^*$ and $(y^*, s^*) \in \mathcal{LD}^*$, where $\sigma(x^*) = P$ is given. For any two $\epsilon_1, \epsilon_2 \in \Upsilon_L(\triangle b, \triangle c)$, let $(x^*(\epsilon_1), y^*(\epsilon_1), s^*(\epsilon_1))$ and $(x^*(\epsilon_2), y^*(\epsilon_2), s^*(\epsilon_2))$ be given pairs of primal-dual optimal solutions of problems $LP(\triangle b, \triangle c, \epsilon)$ and $LD(\triangle b, \triangle c, \epsilon)$ at ϵ_1 and ϵ_2, respectively. By the assumption we have $\sigma(x^*(\epsilon_1)) = \sigma(x^*(\epsilon_2)) = P$. For any $\epsilon \in (\epsilon_1, \epsilon_2)$, with $\theta = \frac{\epsilon_2 - \epsilon}{\epsilon_2 - \epsilon_1} \in (0, 1)$ we have $\epsilon = \theta\epsilon_1 + (1 - \theta)\epsilon_2$. Let us define

$$x^*(\epsilon) = \theta x^*(\epsilon_1) + (1 - \theta)x^*(\epsilon_2); \tag{2}$$

$$y^*(\epsilon) = \theta y^*(\epsilon_1) + (1 - \theta)y^*(\epsilon_2); \tag{3}$$

$$s^*(\epsilon) = \theta s^*(\epsilon_1) + (1 - \theta)s^*(\epsilon_2). \tag{4}$$

It is easy to verify that $x^*(\epsilon) \in \mathcal{LP}(\triangle b, \triangle c, \epsilon)$ and $(y^*(\epsilon), s^*(\epsilon)) \in \mathcal{LD}(\triangle b, \triangle c, \epsilon)$. Moreover, one can easily conclude the optimality property $x^*(\epsilon)^T s^*(\epsilon) = 0$. We also have $\sigma(x^*(\epsilon)) = P$, i.e., $\epsilon \in \Upsilon_L(\triangle b, \triangle c)$ that completes the proof. □

Lemma 2.1 shows that $\Upsilon_L(\triangle b, \triangle c)$ is a (possibly improper) interval of the real line that contains zero. We refer to $\Upsilon_L(\triangle b, \triangle c)$ as *the ISS interval* of the problem $LP(\triangle b, \triangle c, \epsilon)$ w.r.t. the ISS partition (P, Z). As we will see later, the ISS interval $\Upsilon_L(\triangle b, \triangle c)$ might be open, closed or half-open.

In this section, we develop a computable method to identify the ISS interval for LO problems. We study three cases: perturbation of the RHS data of LP, perturbation of the objective function data of LP, and perturbation of both simultaneously.

2.1 Perturbation of the RHS Data

In this case $\triangle c = 0$ and the primal and dual LO perturbed problems are as follows:

$$LP(\triangle b, 0, \epsilon) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b + \epsilon\triangle b \\ & x \geq 0, \end{array}$$

and

$$LD(\triangle b, 0, \epsilon) \qquad \begin{array}{rl} \max & (b+\epsilon\triangle b)^T y \\ \text{s.t.} & A^T y + s = c \\ & s \geq 0. \end{array}$$

The following lemma shows that the dual optimal solution set $\mathcal{LD}^*(\triangle b, 0, \epsilon)$ is invariant for any $\epsilon \in \Upsilon_L(\triangle b, 0)$.

Lemma 2.2 *For any given $\epsilon \in \Upsilon_L(\triangle b, 0)$, the dual optimal solution set $\mathcal{LD}^*(\triangle b, 0, \epsilon)$ is invariant.*

Proof: It is obvious that $\{0\} \subseteq \Upsilon_L(\triangle b, 0)$. If $\Upsilon_L(\triangle b, 0) = \{0\}$, then the statement obviously holds. Assume that there is a nonzero $\epsilon \in \Upsilon_L(\triangle b, 0)$. Let $(x^*(\epsilon), y^*(\epsilon), s^*(\epsilon)) \in \mathcal{LP}^*(\triangle b, 0, \epsilon) \times \mathcal{LD}^*(\triangle b, 0, \epsilon)$ and $(x^*, y^*, s^*) \in \mathcal{LP}^*(\triangle b, 0, 0) \times \mathcal{LD}^*(\triangle b, 0, 0)$ with $\sigma(x^*) = \sigma(x^*(\epsilon)) = P$ be given. Since $x^*(\epsilon)^T s^* = 0$, then $(x^*(\epsilon), y^*, s^*)$ is also a pair of primal-dual optimal solutions of problems $LP(\triangle b, 0, 0)$ and $LD(\triangle b, 0, 0)$. Analogously, $(x^*, y^*(\epsilon), s^*(\epsilon))$ is another pair of primal-dual optimal solutions of problems $LP(\triangle b, 0, \epsilon)$ and $LD(\triangle b, 0, \epsilon)$. Thus, $\mathcal{LD}^*(\triangle b, 0, 0) = \mathcal{LD}^*(\triangle b, 0, \epsilon)$ and the proof is completed. $\square$

Using Lemma 2.2 we consider (y^*, s^*), the given dual optimal solution of LD, as a dual optimal solution of the perturbed $LD(\triangle b, 0, \epsilon)$ for all $\epsilon \in \Upsilon_L(\triangle b, 0)$. One can prove the linearity (i.e., the convexity and concavity) of the optimal value function of problem $LP(\triangle b, 0, \epsilon)$ on $\overline{\Upsilon}_L(\triangle b, 0)$, where $\overline{\Upsilon}_L(\triangle b, 0)$ denotes the closure of the ISS interval $\Upsilon_L(\triangle b, 0)$.

Corollary 2.3 *The optimal value function of problem $LP(\triangle b, 0, \epsilon)$ is linear on $\overline{\Upsilon}_L(\triangle b, 0)$.*

Proof: Let $(x^*(\epsilon), y^*, s^*)$ be a pair of primal-dual optimal solutions of problems $LP(\triangle b, 0, \epsilon)$ and $LD(\triangle b, 0, \epsilon)$. Lemma 2.2 proved that $(y^*, s^*) \in \mathcal{LD}^*(\triangle b, 0, \epsilon)$ for all $\epsilon \in \Upsilon_L(\triangle b, 0)$. Thus, the optimal value function of problem $LP(\triangle b, 0, \epsilon)$ is

$$\phi(\epsilon) = c^T x^*(\epsilon) = (b + \epsilon\triangle b)^T y^*,$$

that is obviously a linear function of ϵ. $\square$

Now we present a computational method to calculate ϵ_l and ϵ_u, the extreme points of $\overline{\Upsilon}_L(\triangle b, 0)$. Let x^* be a given primal optimal solution of LP, where $\sigma(x^*) = P$. Using the

ISS partition (P, Z) of the index set $\{1, 2, \ldots, n\}$, matrix A and vector c can be partitioned accordingly. Considering ϵ as a parameter and expressing the requirements of Type II sensitivity analysis, the perturbed primal problem $LP(\triangle b, 0, \epsilon)$, can be rewritten as

$$\begin{aligned} \min \quad & c_P^T x_P + c_Z^T\, x_Z \\ \text{s.t.} \quad & A_P x_P + A_Z x_Z = b + \epsilon\triangle b \\ & x_P, x_Z \geq 0. \end{aligned} \tag{5}$$

We need to find the ISS interval $\Upsilon_L(\triangle b, 0)$, i.e., the smallest ($\epsilon_l$) and the largest ($\epsilon_u$) values of ϵ, so that for any $\epsilon \in \Upsilon_L(\triangle b, 0)$ an optimal solution $x^*(\epsilon)$ exists with $\sigma(x^*(\epsilon)) = P$. Type II sensitivity analysis implies that $x_Z^*(\epsilon) = 0$ holds for at least one optimal solution of problem $LP(\triangle b, 0, \epsilon)$ where $\epsilon \in \Upsilon_L(\triangle b, 0)$. One can remove the zero variables from problem (5) to obtain the following *reduced* perturbed problem:

$$\overline{LP}(\triangle b, 0, \epsilon) \qquad \begin{aligned} \min \quad & c_P^T \overline{x}_P \\ \text{s.t.} \quad & A_P \overline{x}_P = b + \epsilon\triangle b \\ & \overline{x}_P \geq 0, \end{aligned}$$

where the bar above variables distinguishes the variables of the reduced problem from the variables of the original problem. Problem $\overline{LP}(\triangle b, 0, \epsilon)$ typically has less variables than problem (5). The dual of $\overline{LP}(\triangle b, 0, \epsilon)$ can be written as

$$\overline{LD}(\triangle b, 0, \epsilon) \qquad \begin{aligned} \max \quad & (b + \epsilon\triangle b\ )^T y \\ \text{s.t.} \quad & A_P^T y + \overline{s}_P = c_P \\ & \overline{s}_P \geq 0. \end{aligned}$$

Having $(x^*, y^*, s^*) \in \mathcal{LP}^* \times \mathcal{LD}^*$ with $\sigma(x^*) = P$, a strictly complementary optimal solution $(x_P^*, y^*, 0)$ of problems $\overline{LP}(\triangle b, 0, 0)$ and $\overline{LD}(\triangle b, 0, 0)$ is known. Thus, the optimal partition of the reduced problem $\overline{LP}(\triangle b, 0, 0)$ is known as well. Let $\overline{\pi} = (P, \emptyset)$ denote this optimal partition. If for a fixed ϵ, we have an optimal solution $\overline{x}_P^*(\epsilon) \in \overline{\mathcal{LP}}^*(\triangle b, 0, \epsilon)$ with $\sigma(\overline{x}_P^*(\epsilon)) = P$, then in the corresponding dual optimal solution, $\overline{s}_P^*(\epsilon) = 0$ holds. In other words, ϵ belongs to the actual invariancy interval of the reduced problem $\overline{LP}(\triangle b, 0, 0)$. By definition, the optimal partition of the reduced problem is invariant on the invariancy interval. Let (ϵ_l, ϵ_u) be the actual invariancy interval of the reduced problem $\overline{LP}(\triangle b, 0, 0)$. Using Theorem IV.73 of [15], ϵ_l and ϵ_u can be derived by solving the following auxiliary LO problems:

$$\epsilon_l = \min_{x_P, \epsilon} \{\epsilon : A_P x_P - \epsilon\triangle b = b, x_P \geq 0\}\,, \tag{6}$$

$$\epsilon_u = \max_{x_P, \epsilon} \{\epsilon : A_P x_P - \epsilon\triangle b = b, x_P \geq 0\}\,. \tag{7}$$

For any given $\epsilon \in (\epsilon_l, \epsilon_u)$, the next proposition provides an optimal solution $x^*(\epsilon)$ of $LP(\triangle b, 0, \epsilon)$ with $\sigma(x^*(\epsilon)) = P$.

Proposition 2.4 *Let $\epsilon \in (\epsilon_l, \epsilon_u)$ be given, where (ϵ_l, ϵ_u) is the actual invariancy interval of the reduced perturbed problem $\overline{LP}(\triangle b, 0, \epsilon)$. Further, let $\bar{x}^*_P(\epsilon)$ be a strictly positive optimal solution of problem $\overline{LP}(\triangle b, 0, \epsilon)$. Then $x^*(\epsilon) = (\bar{x}^*_P(\epsilon), 0) \in \mathbb{R}^n$ is an optimal solution of $LP(\triangle b, 0, \epsilon)$ with $\sigma(x^*(\epsilon)) = P$.*

Proof: Having $(x^*, y^*, s^*) \in \mathcal{LP}^* \times \mathcal{LD}^*$ with $\sigma(x^*) = P$ and $s^*_P = 0$ and given $\bar{x}^*_P(\epsilon)$, we define

$$x^*(\epsilon) = (\bar{x}^*_P(\epsilon), 0), \quad y^*(\epsilon) = y^*, \quad s^*(\epsilon) = s^*.$$

The vectors $x^*(\epsilon)$ and $(y^*(\epsilon), s^*(\epsilon))$ are obviously feasible for problems $LP(\triangle b, 0, \epsilon)$ and $LD(\triangle b, 0, \epsilon)$, respectively. Also, $x^*(\epsilon)^T s^*(\epsilon) = 0$ holds. That proves the optimality of $x^*(\epsilon)$ for problem $LP(\triangle b, 0, \epsilon)$. By definition we have $\sigma(x^*(\epsilon)) = \sigma(\bar{x}^*_P(\epsilon)) = P$, that completes the proof. $\square$

The next theorem is a direct consequence of Proposition 2.4 that identify the ISS interval $\Upsilon_L(\triangle b, 0)$.

Theorem 2.5 *Let ϵ_l and ϵ_u be given by (6) and (7). Then $(\epsilon_l, \epsilon_u) = \boldsymbol{int}(\Upsilon_L(\triangle b, 0))$.*

By the notation used in (6) and (7), if $\epsilon_l = \epsilon_u = 0$, then there is no room to perturb the RHS data of problem $LP(\triangle b, 0, \epsilon)$ in the given direction $\triangle b$ while preserving the ISS property of the problem. In this case, the actual invariancy interval of the reduced problem $\overline{LP}(\triangle b, 0, \epsilon)$ is a singleton $\{0\}$, and the ISS interval $\Upsilon_L(\triangle b, 0) = \{0\}$ is a singleton as well. If the minimization problem (6) is unbounded, it means that the left end of the linearity interval for the reduced problem $\overline{LP}(\triangle b, 0, \epsilon)$ is $-\infty$. Thus, the left end of the ISS interval $\Upsilon_L(\triangle b, 0)$ of the problem $LP(\triangle b, , 0, \epsilon)$ will be $-\infty$ as well. The same discussion is valid for the right end point.

The following corollary presents a relationship between the actual invariancy interval of problem $LP(\triangle b, 0, \epsilon)$ and the ISS interval $\Upsilon_L(\triangle b, 0)$.

Corollary 2.6 *Let $\overline{\Upsilon}_L(\triangle b, 0) = [\epsilon_l, \epsilon_u]$. Then, (ϵ_l, ϵ_u) is a subset of the actual invariancy interval of problem $LP(\triangle b, 0, \epsilon)$.*

Proof: Let (ϵ_1, ϵ_2) be the actual invariancy interval of problem $LP(\triangle b, 0, \epsilon)$. Further, assume that there is an $\epsilon \in (\epsilon_l, \epsilon_u)$ which is not in (ϵ_1, ϵ_2). Without loss of generality, one can consider that ϵ in the left immediate neighboring invariancy interval to (ϵ_1, ϵ_2). Thus, we have $\epsilon_l < \epsilon < \epsilon_1 \leq 0$. Since intervals (ϵ_l, ϵ_1) and $(\epsilon_1, 0)$ are subsets of adjacent invariancy intervals, the optimal value function has different slopes on them, that contradicts with Corollary 2.3. $\square$

It can be easily concluded from Corollary 2.3 and 2.6 that if $\epsilon = 0$ is not an extreme point of the ISS interval $\Upsilon_L(\triangle b, 0)$, then it is not a break point of the optimal value function of problem $LP(\triangle b, 0, \epsilon)$. To investigate the openness of the ISS interval $\Upsilon_L(\triangle b, 0)$, let us consider the case $\epsilon_l < 0 < \epsilon_u$. We know that the optimal partition $\overline{\pi} = (P, \emptyset)$ of problem $\overline{LP}(\triangle b, 0, \epsilon)$ changes at ϵ_l and ϵ_u, that are break points of the optimal value function of the reduced problem $\overline{LP}(\triangle b, 0, \epsilon)$. It means that for all optimal solutions $\overline{x}^*_P(\epsilon_l)$ and $\overline{x}^*_P(\epsilon_u)$, we have

$$\sigma(\overline{x}^*_P(\epsilon_l)) \subset P, \quad \sigma(\overline{x}^*_P(\epsilon_u)) \subset P,$$

and the inclusions are strict. Consequently, ϵ_l and ϵ_u do not belong to $\Upsilon_L(\triangle b, 0)$ and the ISS interval $\Upsilon_L(\triangle b, 0)$ is an open interval. On the other hand, $\epsilon = 0$ might be a break point of the optimal value function of problem $LP(\triangle b, 0, \epsilon)$. The following corollary clarifies that the ISS interval $\Upsilon_L(\triangle b, 0)$ is half-closed/closed in this case. It also shows, in this case the relationship between $\Upsilon_L(\triangle b, 0)$ and the actual invariancy interval of problem $LP(\triangle b, 0, \epsilon)$. It is a direct consequence of Corollaries 2.3 and 2.6.

Corollary 2.7 *Let $\overline{\Upsilon}_L(\triangle b, 0) = [\epsilon_l, \epsilon_u]$ and $\epsilon = 0$ be a break point of the optimal value function of problem $LP(\triangle b, 0, \epsilon)$. Then either $\epsilon_l = 0$ or $\epsilon_u = 0$ or both and exactly one of the following statements holds:*

1. *$(0, \epsilon_u)$ is a subset of the immediate right side invariancy interval of the break point $\epsilon = 0$ and $\Upsilon_L(\triangle b, 0) = [0, \epsilon_u)$;*

2. *$(\epsilon_l, 0)$ is a subset of the immediate left side invariancy interval of the break point $\epsilon = 0$ and $\Upsilon_L(\triangle b, 0) = (\epsilon_u, 0]$;*

3. *$\Upsilon_L(\triangle b, 0) = \{0\}$.*

Remark 2.1 *If the given pair of primal-dual optimal solutions (x^*, y^*, s^*) of problems LP and LD are strictly complementary optimal solutions, then $\mathcal{B} = P$. Though, the optimal partition changes at break points, but it is not necessarily cause to exclude them from the ISS interval.*

It might be instances (see Example 1) that perturbed problems at break point have still an optimal solution with the same support set that the given primal optimal solution has. It means that Type II sensitivity analysis might be different from Type III sensitivity analysis even the given pair of primal-dual optimal solutions are strictly complementary.

Remark 2.2 *If the given optimal solution x^* is a nondegenerate optimal basic solution, then Type I and Type II sensitivity analysis are identical.*

Let us briefly summarize the results obtained so far. Identifying of the ISS interval $\Upsilon_L(\triangle b, 0)$ is equivalent to finding the actual invariancy interval of the reduced problem $\overline{LP}(\triangle b, 0, \epsilon)$. The ISS interval $\Upsilon_L(\triangle b, 0)$ is always a subset of the actual invariancy interval of the problem $LP(\triangle b, 0, \epsilon)$. If the given pair of primal-dual optimal solutions are strictly complementary, then the ISS interval $\Upsilon_L(\triangle b, 0)$ might be the closure of the actual invariancy interval of the original perturbed problem. In this case, if $\epsilon = 0$ is a break point of the optimal value function of problem $LP(\triangle b, 0, \epsilon)$, then by Theorem IV.75 of [15], the ISS interval $\Upsilon_L(\triangle b, 0)$ is the singleton $\{0\}$. It is half-closed if $\epsilon = 0$ is a break point of the optimal value function of problem $LP(\triangle b, 0, \epsilon)$ and the given pair of primal-dual optimal solutions are not strictly complementary. Otherwise, the ISS interval $\Upsilon_L(\triangle b, 0)$ is an open interval. When the problem LP has a unique nondegenerate optimal basic solution, then Type II and Type I sensitivity analysis coincide. We refer to Example 2 in Section 3, for an illustration of these results in $\mathbb{R}^2$.

2.2 Perturbation in the Objective Function Data

Let a pair of primal-dual optimal solutions (x^*, y^*, s^*) of problems LP and LD with $\sigma(x^*) = P$ be given. In this subsection we assume that $\triangle b = 0$ and the perturbed primal and dual LO problems are

$$LP(0, \triangle c, \epsilon) \qquad \begin{array}{rl} \min & (c + \epsilon\triangle c)^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0, \end{array}$$

$$LD(0, \triangle c, \epsilon) \qquad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y + s = c + \epsilon\triangle c \\ & s \geq 0. \end{array}$$

The following lemma states the fact that the primal optimal solution set of the problem $LP(0, \triangle c, \epsilon)$ is invariant on $\Upsilon_L(0, \triangle c)$. Its proof is analogous to the proof of Lemma 2.2.

Lemma 2.8 *For any given $\epsilon \in \Upsilon_L(0, \triangle c)$, the primal optimal solution set $\mathcal{LP}^*(0, \triangle c, \epsilon)$ is invariant.*

Using Lemma 2.8, we may conclude that for any $\epsilon \in \Upsilon_L(0, \triangle c)$, the given $x^* \in \mathcal{LP}^*$ is a primal optimal solution of the problem $LP(0, \triangle c, \epsilon)$. One can also establish the following corollary that shows the linearity of the optimal value function in the ISS interval $\Upsilon_L(0, \triangle c)$. The proof is similar to Corollary 2.3 and skipped.

Corollary 2.9 *The optimal value function of problem $LP(0, \triangle c, \epsilon)$ is linear on $\overline{\Upsilon}_L(0, \triangle c)$.*

It should be mentioned that the reduction procedure used in the previous subsection is not applicable when perturbation occurs in the objective function data of the LO problem, because we may remove some zero variables with varying objective coefficients. From an alternative view point, removing variables from $LP(0, \triangle c, \epsilon)$ is equivalent to delating the corresponding constraints from $LD(0, \triangle c, \epsilon)$. As a consequence, dual feasibility of these constraints would not be considered in the reduced dual problem. However, there is a close relationship between the actual invariancy interval of $LP(0, \triangle c, \epsilon)$ and the ISS interval $\Upsilon_L(0, \triangle c)$. In the sequel, we investigate this relationship that ultimately leads to the identification of the ISS interval $\Upsilon_L(0, \triangle c)$.

There are two possibilities. First, let us consider the case that $\epsilon = 0$ is not a break point of the optimal value function of problem $LP(0, \triangle c, \epsilon)$. The following theorem shows that the linearity interval of the optimal value function of problem $LP(0, \triangle c, \epsilon)$ around the current point coincides with the ISS interval $\Upsilon_L(0, \triangle c)$.

Theorem 2.10 *Let (x^*, y^*, s^*) be a pair of primal-dual optimal solutions of LP and LD, where $\sigma(x^*) = P$. Further, suppose that $\epsilon = 0$ is not a break point of the optimal value function of problem $LP(0, \triangle c, \epsilon)$. Then the ISS interval $\Upsilon_L(0, \triangle c)$ coincides with the closure of the actual invariancy interval.*

Proof: Let $[\epsilon_1, \epsilon_2]$ be the actual linearity interval of $LP(0, \triangle c, \epsilon)$. Then using Theorem IV.75

of [15], ϵ_1 and ϵ_2 can be obtained by solving the following two auxiliary problems

$$\epsilon_1 = \min_{y,s,\epsilon} \left\{ \epsilon : A^T y + s - \epsilon \triangle c = c, s \geq 0, s^T x^* = 0 \right\}, \tag{8}$$

$$\epsilon_2 = \max_{y,s,\epsilon} \left\{ \epsilon : A^T y + s - \epsilon \triangle c = c, s \geq 0, s^T x^* = 0 \right\}. \tag{9}$$

Because $\epsilon = 0$ is not a break point, then $\epsilon_1 < 0 < \epsilon_2$. It follows directly from Corollary IV.57 of [15] that for any given $\epsilon \in (\epsilon_1, \epsilon_2)$, x^* is still an optimal solution of $LP(0, \triangle c, \epsilon)$. It means that the given primal optimal solution x^* is an optimal solution of the perturbed problem $LP(0, \triangle c, \epsilon)$ as well. Further, using Corollary IV.59 of [15] we know that the primal optimal solution sets at the break points ϵ_1 and ϵ_2 are supersets of the primal optimal solution set $\mathcal{LP}^*(0, \triangle c, \epsilon)$, for any $\epsilon \in (\epsilon_l, \epsilon_u)$. Thus, the given primal optimal solution x^* is a primal optimal solution of problems $LP(0, \triangle c, \epsilon_l)$ and $LP(0, \triangle c, \epsilon_u)$ that proves $[\epsilon_1, \epsilon_2] \subseteq \Upsilon_L(0, \triangle c)$.

On the other hand, let us assume to the contrary that there is an $\epsilon \in \Upsilon_L(0, \triangle c)$ with $\epsilon \notin [\epsilon_1, \epsilon_2]$. Without loss of generality, one can assume that $\epsilon < \epsilon_1$ and ϵ belongs to the left immediate linearity interval of the break point ϵ_1. Therefore, the optimal value function has two differen slops in intervals (ϵ, ϵ_1) and (ϵ_1, ϵ_2) that contradicts Corollary 2.9. Thus, $\Upsilon_L(0, \triangle c) \subseteq [\epsilon_1, \epsilon_2]$ that completes the proof. $\square$

Now let us consider that $\epsilon = 0$ is a break point of the optimal value function of $LP(0, \triangle c, \epsilon)$. Considering the facts of Corollary IV.59 of [15], depending on the given optimal solution x^*, the ISS interval $\Upsilon_L(0, \triangle c)$ can be either the left or the right immediate neighboring interval to the break point $\epsilon = 0$ or the singleton $\{0\}$. Thus, to identify the ISS interval $\Upsilon_L(0, \triangle c)$, it is enough to check weather the given optimal solution x^* is still optimal in either the left or the right immediate neighboring invariancy interval. The following theorem summarizes this discussion.

Theorem 2.11 *Let x^* with $\sigma(x^*) = P$ be a given optimal solution of problem LP and $\epsilon = 0$ be a break point of the optimal value function of $LP(0, \triangle c, \epsilon)$. Further, let ϵ_- and ϵ_+ be the left and right immediate neighboring break points of zero (possibly $-\infty$ or ∞). If $x^* \in \mathcal{LP}^*(0, \triangle c, \epsilon)$ at a given $\epsilon \in (0, \epsilon_+]$ then $\Upsilon_L(0, \triangle c) = [0, \epsilon_+]$; and if $x^* \in \mathcal{LP}^*(0, \triangle c, \epsilon)$ at a given $\epsilon \in [\epsilon_-, 0)$ then $\Upsilon_L(0, \triangle c) = [\epsilon_-, 0]$; otherwise, $\Upsilon_L(0, \triangle c) = \{0\}$.*

Remark 2.3 *When $\epsilon = 0$ is a break point of the optimal value function of problem $LP(0, \triangle c, \epsilon)$, according to Theorem 2.11, identifying the ISS interval $\Upsilon_L(0, \triangle c)$ needs the knowledge of the left and right immediate break points of the optimal value function of $LP(0, \triangle c, \epsilon)$ (if exist). To* *calculate these break points, having a dual optimal solution (y^*, s^*) one needs to solve first the minimizing problems*

$$\min_x \left\{\triangle c^T x : Ax = b,\ x \geq 0,\ x^T s^* = 0\right\} = \triangle c^T \tilde{x},$$
$$\max_x \left\{\triangle c^T x : Ax = b,\ x \geq 0,\ x^T s^* = 0\right\} = \triangle c^T \hat{x},$$

to find the left and the right derivatives of the optimal value function at the break point $\epsilon = 0$, respectively (see Lemma IV.67 [15]). Then, solving two auxiliary LO problems

$$\epsilon_- = \min_{\epsilon,y,s} \left\{\epsilon : A^T y + s = c + \epsilon\triangle c,\ s \geq 0,\ s^T \tilde{x} = 0\right\}, \tag{10}$$
$$\epsilon_+ = \max_{\epsilon,y,s} \left\{\epsilon : A^T y + s = c + \epsilon\triangle c,\ s \geq 0,\ s^T \hat{x} = 0\right\}, \tag{11}$$

that identifies the left and the right break points of $\epsilon = 0$, respectively (see Theorem IV.75 [15]). For a given ϵ in these neighboring invariancy intervals, we also need to check the optimality of x^*. If problems (10) and (11) are not unbounded, then optimal solutions of these problems can be used to verify the optimality of x^* in the corresponding linearity intervals. Otherwise, one can do it by solving the dual problem $LD(0, \triangle c, \epsilon)$ for an ϵ in the left and right immediate invariancy intervals. It should be mentioned that unlike $\Upsilon_L(\triangle b, 0)$ that is either an open or half-closed interval if it is not the singleton $\{0\}$, the ISS interval $\Upsilon_L(0, \triangle c)$ is always a closed interval.

2.3 Simultaneous Perturbation

Let us consider that both $\triangle b$ and $\triangle c$ are not zero. The general perturbed primal and dual LO problems $LP(\triangle b, \triangle c, \epsilon)$ and $LD(\triangle b, \triangle c, \epsilon)$ are considered in this subsection. The following theorem presents the relationship between the ISS intervals $\Upsilon_L(\triangle b, \triangle c)$, $\Upsilon_L(\triangle b, 0)$ and $\Upsilon_L(0, \triangle c)$. The result allows us to identify the ISS interval $\Upsilon_L(\triangle b, \triangle c)$ where $\Upsilon_L(\triangle b, 0)$ and $\Upsilon_L(0, \triangle c)$ are known.

Theorem 2.12 *$\Upsilon_L(\triangle b, \triangle c) = \Upsilon_L(\triangle b, 0) \cap \Upsilon_L(0, \triangle c)$.*

Proof: Let (x^*, y^*, s^*) be a pair of primal-dual optimal solutions of problems LP and LD with $\sigma(x^*) = P$. Then, we have

$$Ax^* = b, \qquad x^*_P > 0,\ x^*_Z = 0; \tag{12}$$
$$A^T y^* + s^* = c, \qquad s^*_P = 0,\ s^*_Z \geq 0. \tag{13}$$

First we prove that $\Upsilon_L(\triangle b, 0) \cap \Upsilon_L(0, \triangle c) \subseteq \Upsilon_L(\triangle b, \triangle c)$. Let $\epsilon \in \Upsilon_L(\triangle b, 0)$. Then, there is a primal optimal solution $x^*(\epsilon) \in \mathcal{LP}^*(\triangle b, 0, \epsilon)$ such that

$$Ax^*(\epsilon) = b + \epsilon\triangle b, \qquad x^*_P(\epsilon) > 0,\ x^*_Z(\epsilon) = 0. \tag{14}$$

Further, if $\epsilon \in \Upsilon_L(0, \triangle c)$, then there is a dual optimal solution $(\overline{y}^*(\epsilon), \overline{s}^*(\epsilon)) \in \mathcal{LD}^*(0, \triangle c, \epsilon)$, such that

$$A^T\overline{y}^*(\epsilon) + \overline{s}^*(\epsilon) = c + \epsilon\triangle c, \qquad \overline{s}^*_P(\epsilon) = 0,\ \overline{s}^*_Z(\epsilon) \geq 0. \tag{15}$$

Obviously, (14) and (15) show that $(x^*(\epsilon), \overline{y}^*(\epsilon), \overline{s}^*(\epsilon)) \in \mathcal{LP}^*(\triangle b, \triangle c, \epsilon) \times \mathcal{LD}^*(\triangle b, \triangle c, \epsilon)$ and consequently $\epsilon \in \Upsilon_L(\triangle b, \triangle c)$.

On the other hand, let $\epsilon \in \Upsilon_L(\triangle b, \triangle c)$. Then, there is a pair of primal-dual optimal solutions $(\tilde{x}^*(\epsilon), \tilde{y}^*(\epsilon), \tilde{s}^*(\epsilon)) \in \mathcal{LP}^*(\triangle b, \triangle c, \epsilon) \times \mathcal{LD}^*(\triangle b, \triangle c, \epsilon)$, i.e.,

$$A\tilde{x}^*(\epsilon) = b + \epsilon\triangle b, \qquad \tilde{x}^*_P(\epsilon) > 0,\ \ \tilde{x}^*_Z(\epsilon) = 0; \tag{16}$$

$$A^T\tilde{y}^*(\epsilon) + \tilde{s}^*(\epsilon) = c + \epsilon\triangle c, \qquad \tilde{s}^*_P(\epsilon) = 0,\ \ \tilde{s}^*_Z(\epsilon) \geq 0. \tag{17}$$

Equations (16) and (13) imply $(\tilde{x}^*(\epsilon), y^*, s^*) \in \mathcal{LP}^*(\triangle b, 0, \epsilon) \times \mathcal{LD}^*(\triangle b, 0, \epsilon)$, i.e., $\epsilon \in \Upsilon_L(\triangle b, 0)$. Analogously, from equations (17) and (12) we have $\epsilon \in \Upsilon_L(0, \triangle c)$. The proof is completed. $\square$

Theorem 2.12 shows that one can identify the ISS interval $\Upsilon_L(\triangle b, \triangle c)$ having $\Upsilon_L(\triangle b, 0)$ and $\Upsilon_L(0, \triangle c)$. However, in this case, we need to solve two pairs of LO problems. Meanwhile, it is possible to have only two auxiliary LO problems to determine the ISS interval $\Upsilon_L(\triangle b, \triangle c)$ directly without the necessity of knowing the ISS intervals $\Upsilon_L(\triangle b, 0)$ and $\Upsilon_L(0, \triangle c)$. The following corollary provides these two auxiliary LO problems.

Corollary 2.13 *Let $(x^*, y^*, s^*) \in \mathcal{LP}^* \times \mathcal{LD}^*$ with $\sigma(x^*) = P$ be given. Then, ϵ_l and ϵ_u, the extreme points of $\overline{\Upsilon}_L(\triangle b, \triangle c)$ can be determined by solving the following two LO problems:*

$$\epsilon_l = \min_{\epsilon, x_P, y, s_Z} \{\epsilon : A_P x_P - \epsilon\triangle b = b,\ x_P \geq 0,\quad A_P^T y - \epsilon\triangle c_P = c_P,\ A_Z^T y + s_Z - \epsilon\triangle c_Z = c_Z, s_Z \geq 0\}, \tag{18}$$

$$\epsilon_u = \max_{\epsilon, x_P, y, s_Z} \{\epsilon : A_P x_P - \epsilon\triangle b = b,\ x_P \geq 0,\quad A_P^T y - \epsilon\triangle c_P = c_P,\ A_Z^T y + s_Z - \epsilon\triangle c_Z = c_Z, s_Z \geq 0\}. \tag{19}$$

Proof: We only prove (18), the proof of (19) goes analogously. Let $\epsilon_l(\triangle b)$ and $\epsilon_l(\triangle c)$ be the lower extreme points of $\overline{\Upsilon}_L(\triangle b, 0)$ and $\overline{\Upsilon}_L(0, \triangle c)$, respectively. Thus, we have

$$\epsilon_l(\triangle b) = \min_{x_P, \epsilon} \{\epsilon : A_P x_P - \epsilon \triangle b = b, x_P \geq 0\}, \tag{20}$$

$$\epsilon_l(\triangle c) = \min_{\epsilon, y, s_Z} \{\epsilon : A_P^T y - \epsilon \triangle c_P = c_P,\ A_Z^T y + s_Z - \epsilon \triangle c_Z = c_Z, s_Z \geq 0\}. \tag{21}$$

From Theorem 2.12 we conclude that $\epsilon_l = \max\{\epsilon_l(\triangle b), \epsilon_l(\triangle c)\}$. Since, except ϵ, the variables in problems (20) and (21) are different, one can combine these two problems to obtain the unified auxiliary problem (18). □

Depending on the intersection of $\Upsilon_L(\triangle b, 0)$ and $\Upsilon_L(0, \triangle c)$, the ISS interval $\Upsilon_L(\triangle b, \triangle c)$ might be open, closed or half-open. However, if the given pair of primal-dual optimal solutions (x^*, y^*, s^*) of problems LP and LD are strictly complementary, then $\Upsilon_L(\triangle b, \triangle c)$ is identical to the actual invariancy interval of $LP(\triangle b, \triangle c, \epsilon)$. Since the optimal partition changes at transition points [4], the ISS interval $\Upsilon_L(\triangle b, \triangle c)$ is open in this case.

3 Illustrative Examples

In this section, we apply the methods derived in the previous sections for some simple examples to illustrate Type II sensitivity analysis and the behavior of the optimal value function. We suppose that variation occurs only in the components of either vector b or c, that makes easy to depict the results graphically. Example 1 is an instance showing that Type II and Type III sensitivity analysis are not identical when the given pair of primal-dual optimal solutions are strictly complementary.

Example 1: Consider the primal problem as:

$$\begin{array}{lllllllll} \min & 4x_1 & +6x_2 & +2x_3 & +3x_4 & & & & \\ \text{s.t.} & x_1 + & x_2 & +2x_3 + & x_4 & -x_5 & & = 2 & \\ & x_1 & +2x_2 + & x_3 & & & -x_6 & = 1 & \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq 0. & \end{array} \tag{22}$$

Its dual is

$$
\begin{array}{llllllllll}
\max & 2y_1 + & y_2 & & & & & & & \\
\text{s.t.} & y_1 + & y_2 & +s_1 & & & & & & = 4 \\
 & y_1 & +2y_2 & & +s_2 & & & & & = 6 \\
 & 2y_1 & y_2 & & & +s_3 & & & & = 2 \\
 & y_1 & & & & & +s_4 & & & = 3 \\
 & y_1 & & & & & & -s_5 & & = 0 \\
 & & y_2 & & & & & & -s_6 & = 0 \\
 & & & s_1, & s_2, & s_3, & s_4, & s_5, & s_6 & \geq 0.
\end{array}
\tag{23}
$$

It is easy to verify that (x^*, y^*, s^*) is a pair of primal-dual strictly complementary optimal solutions of problems (22) and (23), where

$$x^* = (0, 0, 1, 0, 0, 0)^T, \quad y^* = (\frac{1}{2}, 1)^T \text{ and } \quad s^* = (\frac{5}{2}, \frac{7}{2}, 0, \frac{5}{2}, \frac{1}{2}, 1)^T.$$

Thus, the optimal partition is $\pi = (\mathcal{B}, \mathcal{N})$, where $\mathcal{B} = \{3\}$ and $\mathcal{N} = \{1, 2, 4, 5, 6\}$. Let $\triangle c = (1, -1, 0, 0, 0, 0)^T$ with $\triangle b = 0$ is the given perturbing direction. Observe that the actual invariancy interval is $(-3, 5)$ while $x^*(-3) = (0, 0, 1, 0, 0, 0)^T$ is still an optimal solution of the perturbed problem for $\epsilon = -3$. Furthermore, $x^*(5) = (0, 0, 1, 0, 0, 0)^T$ is an optimal solution of primal perturbed problem at $\epsilon = 5$. Thus, the ISS interval $\Upsilon(0, \triangle c) = [-3, 5]$, the closure of the actual invariancy interval. It means that Type II and Type III sensitivity analysis are differen while the given primal-dual optimal solutions are strictly complementary.

The following example allows us to show four classes of primal optimal solutions and the results of Type II sensitivity analysis in those cases.

Example 2: Consider the following LO problem in standard form:

$$
\begin{array}{llllll}
\min & & 0 & & & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & & = 1 \\
 & & x_2 & & +x_4 & = 1 \\
 & x_1, & x_2, & x_3, & x_4 & \geq 0.
\end{array}
\tag{24}
$$

It is obvious that all feasible points are optimal. Let $\triangle b = (1, -1)^T$ with $\triangle c = 0$ be the perturbing direction. One can categorize the optimal solutions of problem (24) in four classes (see Figure 1).

- **Case 1. Strictly complementary optimal solution**, such as $x^{(1)} = (0.2, 0.3, 0.5, 0.7)^T$ and $(y^{(1)}, s^{(1)}) = 0$.

THe optimal partition at $\epsilon = 0$ is $\pi = (\mathcal{B}, \mathcal{N})$, where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \emptyset$. For this class of optimal solutions, the ISS interval $\Upsilon_L(\triangle b, 0)$ is a superset of the actual invariancy interval $(-1, 1)$ (see Remark 2.1). Let $\pi(-1) = (\mathcal{B}(-1), \mathcal{N}(-1))$ and $\pi(1) = (\mathcal{B}(1), \mathcal{N}(1))$ denote the optimal partitions at break points $\epsilon = -1$ and $\epsilon = 1$, respectively. It is easy to see that $\mathcal{B}(-1) = \{4\}$ and $\mathcal{B}(1) = \{1, 3\}$. It means that any primal optimal solution at these break points has positive variables less that the given optimal solution $x^{(1)}$ has. Consequently, these break points are excluded from the ISS interval $\Upsilon_L(\triangle b, 0)$ and it coincides to the actual invariancy interval in this concrete instance.

- **Case 2. Primal optimal nondegenerate basic solution**, such as $x^{(2)} = (1, 0, 0, 1)^T$. For the given optimal solution, the reduced perturbed problem is

$$\begin{array}{lll} \min & 0 & \\ \text{s.t.} & x_1 \qquad = 1 + \epsilon & \\ & \qquad x_4 = 1 - \epsilon & \\ & x_1,\, x_4 \geq 0, \;\; \epsilon \text{ is free.} & \end{array}$$

The invariancy interval of this reduced perturbed problem is the interval $(-1, 1)$ and so is the ISS interval $\Upsilon_L(\triangle b, 0)$ of problem (24). For any given $\epsilon \in (-1, 1)$, the optimal solution is moving between 0 and 2 along the x_1 axis (see Figure 2). It should be mentioned that, using Remark 2.2, one can determine this ISS interval by the ratio test as in the simplex method.

- **Case 3. Primal optimal degenerate basic solution**, such as $x^{(3)} = (0, 1, 0, 0)^T$. For the given optimal solution, the reduced perturbed problem is

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & x_2 = 1 + \epsilon \\ & x_2 = 1 - \epsilon \\ & x_2 \geq 0. \end{array}$$

In this case, the solution of the auxiliary LO problems (8) and (9) gives $\epsilon_l = \epsilon_u = 0$, i.e., the ISS interval $\Upsilon_L(\triangle b, 0)$ is the singleton $\{0\}$.

- **Case 4. Primal optimal nonbasic solution**, such as $x^{(4)} = (0.5, 0.5, 0, 0.5)^T$.

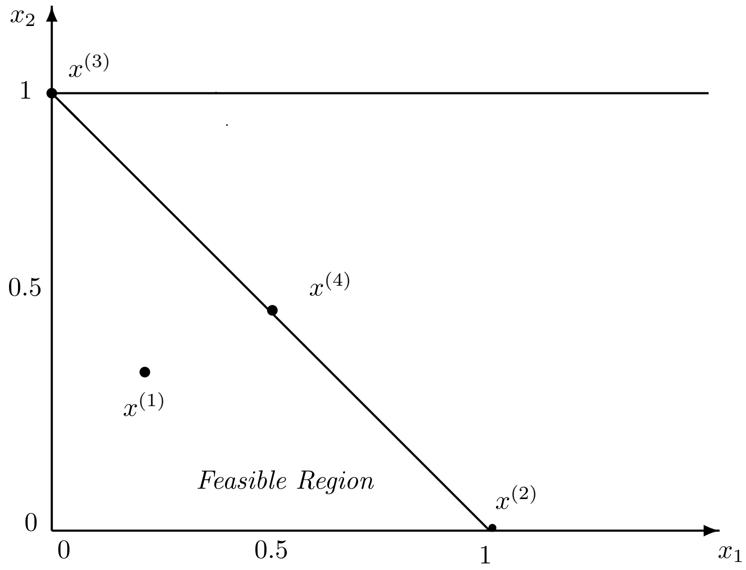

Figure 1: The feasible region (and the primal optimal solution set) of the problem in Example 2.

For this type of optimal solutions, the reduced perturbed problem is

$$\begin{array}{lllll} \min & & 0 & & \\ \text{s.t.} & x_1 & +x_2 & & = 1+\epsilon \\ & & x_2 & +x_4 & = 1-\epsilon \\ & x_1, & x_2, & x_4 & \geq 0, \end{array} \tag{25}$$

and the two auxiliary LO problems to find the ISS interval $\Upsilon_L(\triangle b, 0)$ are

$$\begin{array}{llllll} \min(\max) & \epsilon & & & & \\ \text{s.t.} & x_1 & +x_2 & & -\epsilon & = 1 \\ & & x_2 & +x_4 & +\epsilon & = 1 \\ & x_1, & x_2, & x_4 & & \geq 0 \\ & \epsilon \text{ is free.} & & & & \end{array}$$

By solving these problems, one can identify the ISS interval $\Upsilon_L(\triangle b, 0)$ as $(-1, 1)$. With $\epsilon = 0.5 \in (-1, 1)$, the vector $x^*_P(0.5) = (x_1, x_2, x_4)^T = (1.2, 0.3, 0.2)^T$ is an optimal solution of the reduced problem (25) (it may be obtained by an IPM). As in Proposition 2.4, it can be the base to construct an optimal solution $x^*(0.5) = (1.2, 0.3, 0, 0.2)^T$ for the original problem (24) (see Figure 2).

The following example illustrates the possible cases for the given primal optimal solution when the perturbation happens only in the objective function data of the problem LP.

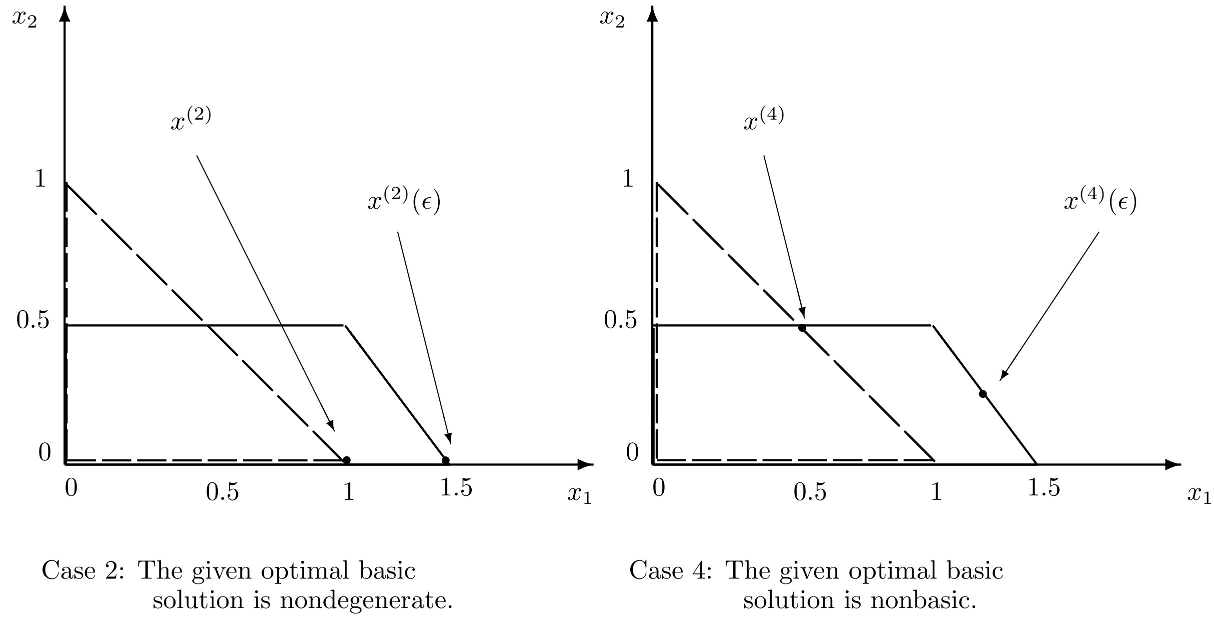

Figure 2: Optimal solutions after perturbation with $\epsilon = 0.5$ in Example 2 (Trapezoid regions show feasible sets of perturbed problem for $\epsilon = 0.5$).

Example 3: Consider the minimization problem

$$\begin{array}{llllllll} \min & -x_1 & -x_2 & & & & & \\ \text{s.t.} & x_1 & +x_2 & +x_3 & & & & = 10 \\ & x_1 & & & +x_4 & & & = 6 \\ & & x_2 & & & +x_5 & & = 8 \\ & 2x_1 & +x_2 & & & & +x_6 & = 16 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq 0, \end{array} \tag{26}$$

that has multiple optimal solutions (see Figure 3). The feasible set of this problem is the region on the plane $x_1 + x_2 + x_3 = 10$, that is enclosed by the solid line. The bold line depicts the primal optimal solution set in this figure. Let $\triangle c = (-1, 0, -1, 0, 0, 0)^T$ with $\triangle b = 0$ be the given perturbation vector. Observe that, $\epsilon = 0$ is a break point of the optimal value function of problem (26). The left and right (immediately neighboring to zero) linearity intervals are $(-\infty, 0]$ and $[0, 1]$, respectively. Observe the the optimal value function is

$$f(\epsilon) = \begin{cases} -10 - 2\epsilon & \epsilon < 0, \\ -10 - 6\epsilon & 0 \leq \epsilon \leq 1, \\ -10\epsilon & \epsilon > 1. \end{cases}$$

One might categorize the optimal solutions of problem (26) in three classes:

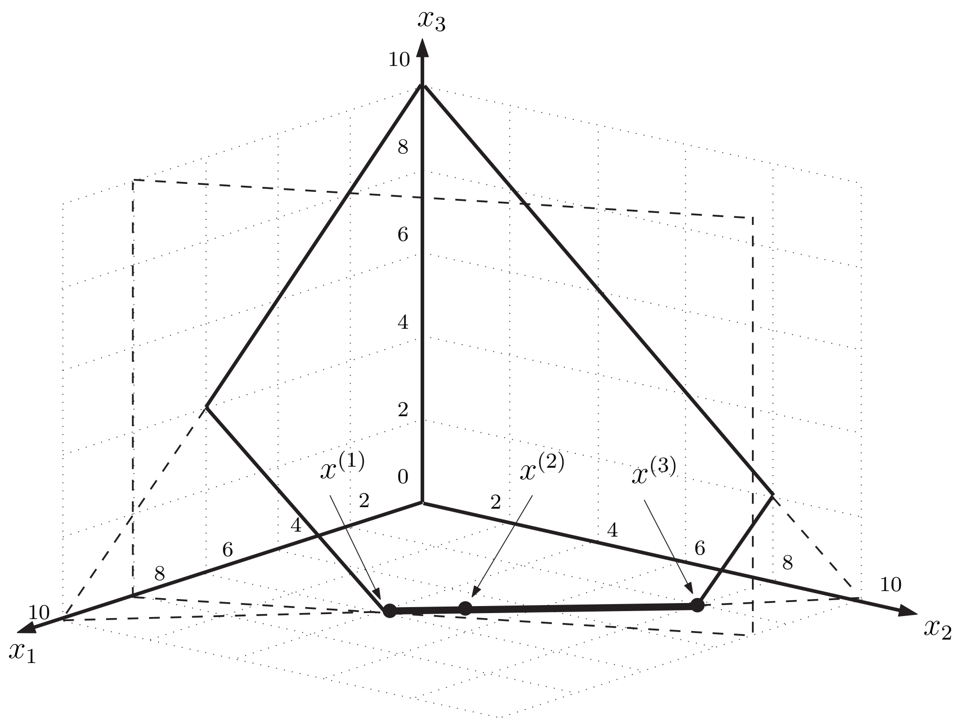

Figure 3: The feasible region and the primal optimal solutions set in Example 3.

- **Case 1. Primal optimal degenerate basic solutions**, such as $x^{(1)} = (6, 4, 0, 0, 4, 0)^T$. For the given optimal solution, the ISS interval is $\Upsilon_L(0, \triangle c) = [0, 1]$, that coincides with the right immediate neighboring linearity interval.

- **Case 2. Primal optimal nonbasic (strictly complementary) solutions**, such as $x^{(2)} = (5, 5, 0, 1, 3, 1)^T$. It is easy to see that $x^{(2)}$ is a strictly complementary optimal solution with $y = (-1, 0, 0, 0)^T$ and $s = (0, 0, 1, 0, 0, 0)^T$. Thus, the ISS partition (P, Z) is its optimal partition at the break point $\epsilon = 0$. By Theorem IV.76 of [15], the solution of the two auxiliary LO problems (8) and (9) leads to $\epsilon_l = \epsilon_u = 0$. Further, optimal partitions on the left and right invariancy intervals of the break point differ from (P, Z). Thus, the ISS interval $\Upsilon_L(0, \triangle c) = \{0\}$.

- **Case 3. Primal optimal nondegenerate basic solutions**, such as $x^{(3)} = (2, 8, 0, 4, 0, 4)^T$. For the given primal optimal solution $\Upsilon_L(0, \triangle c) = (-\infty, 0]$, and this is the left immediate neighboring linearity interval.

Let us have the perturbation vector $\triangle c = (0, 0, 0.25, 0, 0, 0)^T$ with $\triangle b = 0$. In this case, the optimal value function is

$$f(\epsilon) = \begin{cases} \frac{5}{2}\epsilon & \epsilon < 4, \\ 10 & \epsilon \geq 4, \end{cases}$$

and $\epsilon = 0$ is not a break point of the optimal value function. Thus, for any given optimal solution $\Upsilon_L(0, \triangle c) = (-\infty, 4]$. By Theorem 2.10, this coincides with the actual linearity interval of the optimal value function.

4 Conclusions

Type II sensitivity analysis for LO is studied and concluded that Type II sensitivity analysis requires the identification of ISS intervals. We have developed computational procedures to calculate the ISS intervals in case of LO problems. We investigated the case when perturbation occurs in b or c, and also when both b and c are perturbed simultaneously. When perturbation occurs is vector b, identifying the ISS interval converts to solving two smaller subproblems. It is proved that when the given optimal solution is basic and nondegenerate, then Type II and Type I sensitivity analysis coincide. Moreover, when the given pair of primal-dual optimal solutions are strictly complementary optimal solutions, while Type II and Type III sensitivity analysis are not necessarily identical, but the ISS interval is at most equal to the closure of the actual invariancy interval. It is shown that in this case, depending on the given optimal solution, the ISS interval might be open/half-open or the singleton $\{0\}$. If perturbation happens in vector c, it is shown that the ISS interval is always a closed interval that coincides with one of the linearity intervals, or with the given break point as a singleton if the given optimal solution is strictly complementary. It is also proved that, in case of simultaneous perturbation, the ISS interval coincides with the intersection of the ISS intervals when perturbation happens only in b or c. A feature of our method to identify the ISS intervals is that we need to solve only two auxiliary LO problems that typically have smaller size than the original problems. It is worthwhile to mention that all these auxiliary LO problems can be solved in polynomial time by an IPM. We illustrated the results by some simple examples.

Since convex quadratic optimization is a natural generalization of LO, the question about the computability of Type II sensitivity analysis can be asked there too. In spite of the straightforwardness of this question the answer seems to be far from trivial. How to calculate ISS intervals in case of convex quadratic optimization is the subject of further research. We also interested in establishing a methodology to identify the ISS interval when the Case 2 of Type II sensitivity analysis is considered.

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