# Adjustable Robust Optimization Models for Nonlinear Multi-Period Optimization

Akiko Takeda *  Shunsuke Taguchi†  Reha H. Tütüncü‡

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**Abstract**

We study multi-period nonlinear optimization problems whose parameters are uncertain. We assume that uncertain parameters are revealed in stages and model them using the adjustable robust optimization approach. For problems with polytopic uncertainty, we show that quasi-convexity of the optimal value function of certain subproblems is sufficient for the reducibility of the resulting robust optimization problem to a single-level deterministic problem. We relate this sufficient condition to the quasi cone-convexity of the feasible set mapping for adjustable variables and provide several examples satisfying these conditions.

## 1 Introduction

Uncertainty is an inevitable feature of many decision-making environments. On a regular basis engineers, economists, investment professionals, and others need to make decisions to optimize a system with incomplete information and considerable uncertainty. Robust optimization (RO) is a term that is used to describe both modeling strategies and solution methods for optimization problems that are defined by uncertain inputs [3, 4]. The objective of robust optimization models and algorithms is to obtain solutions that are guaranteed to perform well (in terms of feasibility or near-optimality) for all, or at least most, possible realizations of the uncertain input parameters.

Standard robust optimization formulations assume that the uncertain parameters will not be observed until after all the decision variables are determined and therefore do not allow for recourse actions that may be based on realized values of some of these parameters.

*Dept. of Mathematical & Computing Sciences, Tokyo Institute of Technology, Tokyo, Japan takeda@is.titech.ac.jp

†Dept. of Mathematical & Computing Sciences, Tokyo Institute of Technology, Tokyo, Japan taguchi0@is.titech.ac.jp

‡Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA reha@cmu.edu.

This is not always the case for uncertain optimization problems. In particular, multi-period decision models involve uncertain parameters some of which are revealed during the decision process and a subset of the decision variables can be chosen after these parameters are observed in a way to correct the sub-optimality of the decisions made with less information in earlier stages. *Adjustable robust optimization* (ARO) formulations model these decision environments, allowing recourse action. These models are related to the two-stage (or multi-stage) stochastic programming formulations with recourse.

ARO models were recently introduced in [6, 5] for uncertain linear programming problems. Consider, for example, the two-stage linear optimization problem given below whose first-stage decision variables ($\boldsymbol{x}^1$) need to be determined now, while the second-stage decision variables ($\boldsymbol{x}^2$) can be chosen after the uncertain parameters of the problem ($A^1$, $A^2$, and $\boldsymbol{b}$) are realized:

$$\min_{\boldsymbol{x}^1,\boldsymbol{x}^2}\{\boldsymbol{c}^\top\boldsymbol{x}^1 : A^1\boldsymbol{x}^1 + A^2\boldsymbol{x}^2 \leq \boldsymbol{b}\}. \tag{1}$$

Let $\mathcal{U}$ denote the *uncertainty set* for parameters $A^1$, $A^2$, and $\boldsymbol{b}$, i.e., the set of all potentially realizable values of these uncertain parameters. The standard robust optimization formulation for this problem seeks to find vectors $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ that optimize the objective function and satisfy the constraints of the problem for all possible realizations of the constraint coefficients. In this formulation, both sets of variables must be chosen before the uncertain parameters can be observed and therefore cannot depend on these parameters. Consequently, the standard robust counterpart of this problem can be written as follows:

$$\min_{\boldsymbol{x}^1}\{\boldsymbol{c}^\top\boldsymbol{x}^1 : \exists\boldsymbol{x}^2\ \forall(A^1, A^2, \boldsymbol{b}) \in \mathcal{U} : A^1\boldsymbol{x}^1 + A^2\boldsymbol{x}^2 \leq \boldsymbol{b}\}. \tag{2}$$

In contrast, the adjustable robust optimization formulation allows the choice of the second-period variables $\boldsymbol{x}^2$ to depend on the realized values of the uncertain parameters. As a result, the adjustable robust counterpart problem is given as follows:

$$\min_{\boldsymbol{x}^1}\{\boldsymbol{c}^\top\boldsymbol{x}^1 : \forall(A^1, A^2, \boldsymbol{b}) \in \mathcal{U},\ \exists\boldsymbol{x}^2 = \boldsymbol{x}^2(A^1, A^2, \boldsymbol{b}) : A^1\boldsymbol{x}^1 + A^2\boldsymbol{x}^2 \leq \boldsymbol{b}\}. \tag{3}$$

Clearly, the feasible set of the second problem is larger than that of the first problem in general and therefore the model is more flexible. ARO models can be especially useful when robust counterparts are unnecessarily conservative. The price to pay for this additional modeling flexibility appears to be the increased difficulty of the resulting ARO formulations. Even for problems where the robust counterpart is tractable, it can happen that the ARO formulation leads to an NP-hard problem. One of the factors that contribute to the added difficulty in ARO models is the fact that the feasible set of the recourse actions (second-period decisions) depends on both the first-period decisions and the realization of the uncertain parameters. Consequently, the pioneering study of Ben-Tal et al. [5] on this subject considers several simplifying assumptions either on the uncertainty set, or on the dependence structure of recourse actions to uncertain parameters.

Adjustable robust optimization models result from natural formulations of multi-stage decision problems with uncertain parameters and the development of efficient solution techniques for such problems represents the next frontier in robust optimization research. In this article, we contribute to this research by developing tractable ARO formulations for a class of multi-period optimization problems with nonlinear constraints and objective functions. After considering the simple case of finite uncertainty sets, we focus on polytopic uncertainty sets defined as a convex hull of a finite set of points. We investigate sufficient conditions under which the ARO problem reduces to a single deterministic optimization problem. In particular, we show that when the feasible sets of the second-period problem satisfy a certain quasi-convexity property such a reduction is possible. We provide examples exhibiting this property.

The rest of this article is organized as follows. In Section 2 we discuss adjustable robust optimization models for two-period optimization problems with discrete and polytopic uncertainty sets and derive a sufficient condition for the tractability of these problems. In Section 3 we relate the quasi cone-convexity of the mapping that defines feasible sets for adjustable variables to the sufficient condition introduced in the previous section. In Section 4 we provide several example feasible set mappings that satisfy the quasi cone-convexity property. We conclude in Section 5 by extending some of these ideas to problems with three or more periods.

## 2 Adjustable Robust Optimization Models

In this section we consider a two-period decision-making environment. We let $\boldsymbol{u}$ and $\boldsymbol{v}$ represent the first and second-period decision variables, respectively, and $U$ and $V$ represent their feasible sets. We let $\boldsymbol{p}$ denote a vector of parameters for the problem. The objective is to choose feasible vectors $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$ such that the objective function, denoted by $f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})$ is minimized:

$$\inf_{\boldsymbol{u} \in U, \boldsymbol{v} \in V} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}). \tag{4}$$

When the vector $\boldsymbol{p}$ is known and the feasible set $V$ for the second-period decision variables is independent of $\boldsymbol{u}$, the first-period decisions, this problem can be solved as a deterministic, single-period problem. We consider an environment where the parameter vector $\boldsymbol{p}$ is uncertain but is known to belong to an uncertainty set $P$. We assume that these parameters, possibly determined by events that take place between two periods, will be realized and observed after the first-period decision are made but before the second-period decisions need to be made. Furthermore, we assume that the feasible set $V$ for the second-period decisions depends on the choice of $\boldsymbol{u}$ as well as the observed values of the parameters $\boldsymbol{p}$ and therefore, is denoted by $V(\boldsymbol{u}, \boldsymbol{p})$, or equivalently, by $V_{\boldsymbol{u}}(\boldsymbol{p})$ in the remainder of the article.

As we mentioned in the Introduction, an example of this framework with a linear objective function and linear constraints as in (1) is considered in [6, 5]. For this problem, in addition to the standard robust counterpart (RC) problem (2), Ben-Tal *et al.* introduce and study the so-called *adjustable robust counterpart* (ARC) problem given in (3). It is easy to see that the ARC is more flexible (has a larger feasible set) than the RC. Ben-Tal *et al.* argue that the ARC is also more difficult in general than the corresponding RC and give examples of problems whose robust counterparts are tractable while their ARC formulations are NP-hard. They also note two special cases: one where the ARC is equivalent to the RC, and therefore is easy when the RC is, and another where the ARC is a simple linear program. The first case arises when the uncertainty is assumed to be *constraint-wise*. The assumption of constraint-wise uncertainty is discussed in detail in [5, 6] and indicates that uncertain parameters appearing in a particular constraint of the problem do not appear in any of the remaining constraints. In fact, under the assumption of constraint-wise uncertainty, Guslitser shows that the ARC and RC are equivalent even for nonlinear convex programming problems [6]. The second case, namely the case where the ARC is a linear program arises if the matrix $A^2$ in (1) is certain and the uncertainty set for the matrix vector pair $(A^1, \boldsymbol{b})$ is given as the convex hull of a finite set. We will explore similar uncertainty sets below, but for nonlinear optimization problems.

## 2.1 Min-max-min Representation of the ARC Problem

For problem (4) with $V = V(\boldsymbol{u}, \boldsymbol{p})$, the adjustable robust counterpart problem is obtained as follows:

$$\inf_{\boldsymbol{u} \in U, t} \{t : \forall \boldsymbol{p} \in P \;\; \exists \boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p}) : f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) \leq t\}. \tag{5}$$

We sometimes find it more convenient to work with the following representation of the ARC problem:

$$\inf_{\boldsymbol{u} \in U} \sup_{\boldsymbol{p} \in P} \inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}). \tag{6}$$

Using the convention that $\inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) = \infty$ when $V(\boldsymbol{u}, \boldsymbol{p}) = \emptyset$ with some $\boldsymbol{u} \in U$ and $\boldsymbol{p} \in P$, the equivalence of problems (5) and (6) is easy to show:

**Proposition 1.** *The adjustable robust counterpart problem (5) and the min-max-min problem (6) are equivalent.*

*Proof:* As we discussed above, the ARC problem (5) was proposed in [5] where $\boldsymbol{u}$ is called a non-adjustable vector variable and $\boldsymbol{v}$ is called an adjustable vector variable. One of the following two cases must hold:

(a) there exists $\boldsymbol{u} \in U$ such that $V(\boldsymbol{u}, \boldsymbol{p}) \neq \emptyset$ for $\forall \boldsymbol{p} \in P$,

(b) for $\forall \boldsymbol{u} \in U$, there exists $\boldsymbol{p} \in P$ such that $V(\boldsymbol{u}, \boldsymbol{p}) = \emptyset$.

We'll show that problems (5) and (6) have identical optimal values in both cases.

For (a) we assume that there exists $\boldsymbol{u} \in U$ such that $V(\boldsymbol{u}, \boldsymbol{p}) \neq \emptyset$ for $\forall \boldsymbol{p} \in P$. Define the subset $U^{(a)}$ of $U$ as

$$U^{(a)} \equiv \{\boldsymbol{u} \in U : V(\boldsymbol{u}, \boldsymbol{p}) \neq \emptyset \text{ for } \forall \boldsymbol{p} \in P\}.$$

By our assumption, $U^{(a)}$ is nonempty. Next we show that (6) is equivalent to

$$\inf_{\boldsymbol{u}\in U^{(a)}} \sup_{\boldsymbol{p}\in P} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}). \tag{7}$$

It is obvious that the optimal value of (6) is less than that of (7) because of $U^{(a)} \subseteq U$, so it is enough to show that the optimal solution $\boldsymbol{u}^* \in U$ of (6) must lie in $U^{(a)}$ for the equivalence of (6) and (7). Indeed, if we suppose that $\boldsymbol{u}^* \notin U^{(a)}$, there must exist $\boldsymbol{p} \in P$ such that $V(\boldsymbol{u}^*,\boldsymbol{p}) = \emptyset$ and optimal value (6) must be $\infty$. This contradicts the fact that $\sup_{\boldsymbol{p}\in P} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p})$ is finite when $\boldsymbol{u} \in U^{(a)}$. Therefore $\boldsymbol{u}^* \in U^{(a)}$ and the equivalence of (6) and (7) is shown.
Next we show that (7) is equivalent to (5). To "normalize" the problem–this is the term used by Ben Tal *et al.* [5] for problems with linear objective functions with no uncertainty–we introduce an artificial variable $t$ to represent the objective function of (7) and impose the constraint $t \geq \inf_{\boldsymbol{v}(\boldsymbol{p})\in V(\boldsymbol{u},\boldsymbol{p})} f\left(\boldsymbol{u},\boldsymbol{v}(\boldsymbol{p}),\boldsymbol{p}\right), \ \ \forall \boldsymbol{p} \in P, \forall \boldsymbol{u} \in U^{(a)}$. Then,

$$\begin{aligned}
&\inf_{\boldsymbol{u}\in U^{(a)}} \sup_{\boldsymbol{p}\in P} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}) \\
\iff &\left|\begin{array}{ll} \displaystyle\inf_{\boldsymbol{u}\in U^{(a)}} & t \\ \text{s.t.} & \displaystyle\inf_{\boldsymbol{v}(\boldsymbol{p})\in V(\boldsymbol{u},\boldsymbol{p})} f\left(\boldsymbol{u},\boldsymbol{v}(\boldsymbol{p}),\boldsymbol{p}\right) \leq t, \ \ \forall \boldsymbol{p} \in P \end{array}\right. \\
\iff &\inf_{\boldsymbol{u}\in U^{(a)},t} \left\{t : \forall \boldsymbol{p} \in P \ \ \exists \boldsymbol{v} \in V\left(\boldsymbol{u},\boldsymbol{p}\right) : f\left(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}\right) \leq t\right\}, \\
\iff &\inf_{\boldsymbol{u}\in U,t} \left\{t : \forall \boldsymbol{p} \in P \ \ \exists \boldsymbol{v} \in V\left(\boldsymbol{u},\boldsymbol{p}\right) : f\left(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}\right) \leq t\right\},
\end{aligned}$$

and we find that (5) and (6) are equivalent.

In case (b), the ARC problem (5) has no feasible solutions and therefore the optimal value of (5) is $\infty$. Similarly, we observe that for all $\boldsymbol{u} \in U$, the optimal value of $\sup_{\boldsymbol{p}\in P} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p})$ is also $\infty$. Therefore, both problems (5) and (6) attain the same optimal value $\infty$. ∎

In the next two subsections, we explore ARC problems for the cases of a discrete uncertainty set and of a polytopic uncertainty set, i.e., a set given as the convex hull of a finite number of points.

## 2.2 Adjustable Robust Counterpart with Discrete Uncertainty Sets

We first consider the case that $P$ consists of a finite number of elements: $P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k\}$. From problem (5), we see that for every $\boldsymbol{p} \in P$, there is a corresponding variable $\boldsymbol{v}$ satisfying constraints of (5). We introduce new variables $\boldsymbol{v}_i$ to represent the second-period decision variables corresponding to each element $\boldsymbol{p}_i, i \in \{1, \ldots, k\}$ of the uncertainty set and transform (5) into an equivalent single-level optimization problem.

$$
\begin{array}{rll}
\inf\limits_{\boldsymbol{u},\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,t} & t & \\
\text{s.t.} & f\left(\boldsymbol{u},\boldsymbol{v}_i,\boldsymbol{p}_i\right) \le t, & (i=1,\ldots,k) \\
& \boldsymbol{u} \in U, & \\
& \boldsymbol{v}_i \in V\left(\boldsymbol{u},\boldsymbol{p}_i\right), & (i=1,\ldots,k).
\end{array}
\tag{8}
$$

Despite the increase in the number of variables through duplication, this single-level, deterministic optimization problem is a tractable problem for many classes of functions $f$ and sets $V(\boldsymbol{u},\boldsymbol{p})$. As an example, we consider the following set up:

$$
\begin{array}{rcl}
f\left(\boldsymbol{u},\boldsymbol{v}_i,\boldsymbol{p}_i\right) & \equiv & f_0\left(\boldsymbol{u},\boldsymbol{v}_i,\boldsymbol{p}_i\right) \\
U & \equiv & \{\boldsymbol{u} : g_\ell(\boldsymbol{u}) \le 0, \;\; \ell=1,\ldots,m_1\}, \\
V\left(\boldsymbol{u},\boldsymbol{p}_i\right) & \equiv & \{\boldsymbol{v}_i : f_\ell\left(\boldsymbol{u},\boldsymbol{v}_i,\boldsymbol{p}_i\right) \le 0, \;\; \ell=1,\ldots,m_2\}
\end{array}
\tag{9}
$$

where

$$
\begin{array}{rl}
f_\ell\left(\boldsymbol{u},\boldsymbol{v}_i,\boldsymbol{p}_i\right) = f_\ell\left(\boldsymbol{w}_i,\boldsymbol{p}_i\right) = & \boldsymbol{w}_i^\top Q_\ell(\boldsymbol{p}_i)\boldsymbol{w}_i + \boldsymbol{q}_\ell(\boldsymbol{p}_i)^\top \boldsymbol{w}_i + b_\ell(\boldsymbol{p}_i) \\
= & \boldsymbol{w}_i^\top Q_{i\ell}\boldsymbol{w}_i + \boldsymbol{q}_{i\ell}^\top \boldsymbol{w}_i + b_{i\ell}, \\
g_\ell\left(\boldsymbol{u}\right) = & \boldsymbol{u}^\top R_\ell \boldsymbol{u} + \boldsymbol{r}_\ell^\top \boldsymbol{u} + d_\ell, \\
\boldsymbol{w}_i = & (\boldsymbol{u},\boldsymbol{v}_i)^\top.
\end{array}
$$

Above, we can use arbitrary functions $Q_\ell(\boldsymbol{p})$, $\boldsymbol{q}_\ell(\boldsymbol{p})$ and $b_\ell(\boldsymbol{p})$ of the uncertain parameter vector $\boldsymbol{p} \in P$ as long as the images of these functions are in the appropriate spaces. Using (9) we rewrite problem (8) as follows:

$$
\begin{array}{rll}
\min\limits_{\boldsymbol{u},\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,t} & t & \\
\text{s.t.} & \boldsymbol{w}_i^\top Q_{i0}\boldsymbol{w}_i + \boldsymbol{q}_{i0}^\top \boldsymbol{w}_i + b_{i0} \le t, & (i=1,\ldots,k) \\
& \boldsymbol{u}^\top R_\ell \boldsymbol{u} + \boldsymbol{r}_\ell^\top \boldsymbol{u} + d_\ell \le 0, & (\ell=1,\ldots,m_1) \\
& \boldsymbol{w}_i^\top Q_{i\ell}\boldsymbol{w}_i + \boldsymbol{q}_{i\ell}^\top \boldsymbol{w}_i + b_{i\ell} \le 0, & (i=1,\ldots,k, \ell=1,\ldots,m_2).
\end{array}
\tag{10}
$$

This is a quadratically constrained optimization problem. If all the matrices $Q_{i\ell}$ as well as $R_\ell$ are positive-semi definite, then the feasible set is convex, the problem can be reformulated as a second-order cone programming problem and can be solved easily using existing methods and software.

## 2.3 Adjustable Robust Counterpart with Polytopic Uncertainty Sets

In this subsection we consider uncertainty sets of the form $conv(P)$ where $P = \{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_k\}$ and $conv(P)$ denotes the convex hull of $P$. Using this uncertainty set we consider the following adjustable robust optimization problem:

$$
\inf_{\boldsymbol{u}\in U} \sup_{\boldsymbol{p}\in conv(P)} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}). \tag{11}
$$

We are interested in characterizing tractable instances of this problem. In particular, we would like to identify conditions under which

$$\sup_{\boldsymbol{p}\in conv(P)} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}) = \max_{\boldsymbol{p}\in P} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}), \tag{12}$$

so that the ARC problem can be reduced to a single-level deterministic optimization problem as in the previous subsection. For this purpose, we first focus on the inner max-min problem in (11). Let us first define:

$$g_{\boldsymbol{u}}(\boldsymbol{p}) := \inf_{\boldsymbol{v}\in V_{\boldsymbol{u}}(\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}). \tag{13}$$

Recall that $V_{\boldsymbol{u}}(\boldsymbol{p}) = V(\boldsymbol{u},\boldsymbol{p})$ with given $\boldsymbol{u} \in U$. Then, the inner max-min problem is:

$$\sup_{\boldsymbol{p}\in conv(P)} \inf_{\boldsymbol{v}\in V_{\boldsymbol{u}}(\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}) = \sup_{\boldsymbol{p}\in conv(P)} g_{\boldsymbol{u}}(\boldsymbol{p}).$$

A sufficient condition for (12) to hold is that with the given $\boldsymbol{u} \in U$, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a *quasi-convex* function in $\boldsymbol{p} \in conv(P)$, that is,

$$g_{\boldsymbol{u}}(\lambda\boldsymbol{p}_1 + (1-\lambda)\boldsymbol{p}_2) \leq \max\{g_{\boldsymbol{u}}(\boldsymbol{p}_1), g_{\boldsymbol{u}}(\boldsymbol{p}_2)\}$$

holds for any $\boldsymbol{p}_1, \boldsymbol{p}_2 \in conv(P)$ and $\lambda \in (0,1)$. Equivalently, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is *quasi-convex* if all its level sets are convex sets. We state and prove the following simple result:

**Proposition 2.** *If $g_{\boldsymbol{u}}(\boldsymbol{p})$ defined in (13) is a quasi-convex function in $\boldsymbol{p} \in conv(P)$, then*

$$\max_{\boldsymbol{p}\in conv(P)} g_{\boldsymbol{u}}(\boldsymbol{p}) = \max_{\boldsymbol{p}\in P} g_{\boldsymbol{u}}(\boldsymbol{p}).$$

*Proof:* We will actually show that $\sup_{\boldsymbol{p}\in conv(P)} g_{\boldsymbol{u}}(\boldsymbol{p}) = \max_{\boldsymbol{p}\in P} g_{\boldsymbol{u}}(\boldsymbol{p})$. Since the maximization on the right-hand-side is over the finite set $P \subset conv(P)$, the sup is necessarily achieved and the replacement of sup with max is justified.

Since $P \subset conv(P)$, it follows that $\sup_{\boldsymbol{p}\in conv(P)} g_{\boldsymbol{u}}(\boldsymbol{p}) \geq \max_{\boldsymbol{p}\in P} g_{\boldsymbol{u}}(\boldsymbol{p})$. To show the reverse inequality consider any $\boldsymbol{p}_0 \in conv(P)$. Then, there exists $\lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1$ such that $\boldsymbol{p}_0 = \sum_{i=1}^k \lambda_i \boldsymbol{p}_i$. From the quasi-convexity of the function $g_{\boldsymbol{u}}(\boldsymbol{p})$, we immediately obtain that

$$g_{\boldsymbol{u}}(\boldsymbol{p}_0) = g_{\boldsymbol{u}}(\sum_{i=1}^k \lambda_i \boldsymbol{p}_i) \leq \max\{g_{\boldsymbol{u}}(\boldsymbol{p}_1), \ldots, g_{\boldsymbol{u}}(\boldsymbol{p}_k)\}.$$

Taking the supremum over all $\boldsymbol{p}_0 \in conv(P)$ on the left-hand-side we obtain

$$\sup_{\boldsymbol{p}\in conv(P)} g_{\boldsymbol{u}}(\boldsymbol{p}) \leq \max_{\boldsymbol{p}\in P} g_{\boldsymbol{u}}(\boldsymbol{p}),$$

concluding the proof of the proposition. ∎

Therefore, when $g_{\boldsymbol{u}}(\boldsymbol{p})$ is quasi-convex, $conv(P)$ can be replaced by $P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k\}$ in (11), and the problem reduces to the single-level optimization problem (8) with finitely many constraints. In the next section, we will identify necessary and sufficient conditions on the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ that lead to quasi-convex $g_{\boldsymbol{u}}(\boldsymbol{p})$.

**Remark 1.** *In the remainder of the paper we consider a "normalized" version of problem (11), and assume that the objective function of the inner-most minimization problem is linear in* $\boldsymbol{v}$ *and is independent of the first period decision variables* $\boldsymbol{u}$ *and the uncertain parameters* $\boldsymbol{p}$*. This assumption can be made without loss of generality as indicated by the following simple transformation: For given* $\boldsymbol{u} \in U$ *and* $\boldsymbol{p} \in conv(P)$, $g_{\boldsymbol{u}}(\boldsymbol{p}) = \inf_{v \in V_{\boldsymbol{u}}(\boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})$ *is equivalent to:*

$$\begin{array}{ll} \inf\limits_{\boldsymbol{v}, v_0} & v_0 \\ s.t. & f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) \leq v_0 \\ & \boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p}). \end{array}$$

*Defining* $\tilde{V}_{\boldsymbol{u}}(\boldsymbol{p}) = \{\tilde{\boldsymbol{v}} = (\boldsymbol{v}, v_0) : \boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p}), f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) \leq v_0\}$ *we observe that*

$$g_{\boldsymbol{u}}(\boldsymbol{p}) := \inf_{\tilde{\boldsymbol{v}} \in \tilde{V}_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \tilde{\boldsymbol{v}}, \tag{14}$$

*with* $\boldsymbol{c} = [0 \ \ldots 0 \ 1]$*. The normalized form of* $g_{\boldsymbol{u}}(\boldsymbol{p})$ *is useful in the succeeding discussion.*

# 3 Quasi-convex Maps and Functions

We argued in the previous section that the quasi-convexity of the function $g_{\boldsymbol{u}}(\boldsymbol{p}) = \inf_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \boldsymbol{v}$ is a sufficient condition for the reducibility of the ARC problem (11) to a single-period optimization problem. Clearly, convexity properties of this function are related to the structure of the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ for $\boldsymbol{u} \in U$ and $\boldsymbol{p} \in P$. In what follows, we describe a necessary and sufficient condition on the set-valued mapping $V_{\boldsymbol{u}}(\boldsymbol{p})$ for $g_{\boldsymbol{u}}(\boldsymbol{p})$ to be a *quasi-convex* function in $\boldsymbol{p}$. We also consider explicit descriptions of the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ through constraints and investigate conditions on these constraint functions so that the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfy the necessary and sufficient condition mentioned in the previous sentence.

## 3.1 Quasi-convex Maps $V_{\boldsymbol{u}}(\boldsymbol{p})$

For a given real topological vector space $W$, let $2^W$ denote its power set. Given $\boldsymbol{u} \in U$ and an appropriate choice of $W$, $V_{\boldsymbol{u}}(\boldsymbol{p})$ can be considered as a set-valued map $V_{\boldsymbol{u}} : conv(P) \to 2^W$. We also write $V_{\boldsymbol{u}} : conv(P) \rightsquigarrow W$. Let $Q$ be a closed convex cone in $W$ and define a relation $\leq_Q$ in $W$ by the closed convex cone $Q$: for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in W$, $\boldsymbol{v}_1 \leq_Q \boldsymbol{v}_2 \Leftrightarrow \boldsymbol{v}_2 - \boldsymbol{v}_1 \in Q$.

**Definition 1.** *A set-valued map* $V_{\boldsymbol{u}} : conv(P) \rightsquigarrow W$ *is said to be quasi Q-convex (see [2, 7]) if*

$$\begin{array}{l} \textit{for } \forall \boldsymbol{p}_1, \forall \boldsymbol{p}_2 \in conv(P), \forall \boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1), \forall \boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2), \forall \alpha \in (0,1), \\ \textit{if } \boldsymbol{w} \in W \textit{ satisfies } \boldsymbol{v}_1 \leq_Q \boldsymbol{w}, \boldsymbol{v}_2 \leq_Q \boldsymbol{w}, \\ \textit{then } \exists \boldsymbol{v}' \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) \textit{ s.t. } \boldsymbol{v}' \leq_Q \boldsymbol{w}. \end{array} \tag{15}$$

Consider $Q = \{\boldsymbol{q} : \boldsymbol{c}^\top \boldsymbol{q} \geq 0\}$ defined using the coefficient vector $\boldsymbol{c}$ of the objective function.

**Proposition 3.** *Assume that $V_{\boldsymbol{u}}(\boldsymbol{p})$ is closed, bounded, and nonempty for $\forall \boldsymbol{u} \in U$ and $\forall \boldsymbol{p} \in P$. Then, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$ if and only if the set-valued map $V_{\boldsymbol{u}}(\boldsymbol{p})$ is quasi $Q$-convex with $Q = \{\boldsymbol{q} : \boldsymbol{c}^\top \boldsymbol{q} \geq 0\}$.*

*Proof:* We observe that quasi $Q$-convexity of the map $V_{\boldsymbol{u}}(\boldsymbol{p})$ is sufficient to guarantee that the function $g_{\boldsymbol{u}}(\boldsymbol{p})$ is quasi-convex function in $\boldsymbol{p}$. Indeed, for $\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in conv(P)$, choose

$$\begin{array}{l} \boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1) \text{ s.t. } g_{\boldsymbol{u}}(\boldsymbol{p}_1) = \boldsymbol{c}^\top \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2) \text{ s.t. } g_{\boldsymbol{u}}(\boldsymbol{p}_2) = \boldsymbol{c}^\top \boldsymbol{v}_2. \end{array}$$

Such $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ exist since $V_{\boldsymbol{u}}(\boldsymbol{p}_i)$ are assumed to be closed and bounded. Define

$$\bar{\boldsymbol{w}} = \text{argmax}_{\boldsymbol{v}_i} \{\boldsymbol{c}^\top \boldsymbol{v}_1, \boldsymbol{c}^\top \boldsymbol{v}_2\},$$

which indicates that $\boldsymbol{v}_1 \leq_Q \bar{\boldsymbol{w}}$ and $\boldsymbol{v}_2 \leq_Q \bar{\boldsymbol{w}}$. When $V_{\boldsymbol{u}}$ is quasi $Q$-convex, from (15), we have that for any $\alpha \in (0,1)$, there exists $\boldsymbol{v}' \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$ such that $\boldsymbol{v}' \leq_Q \bar{\boldsymbol{w}}$.

Then, using the above $\boldsymbol{v}'$ and $\bar{\boldsymbol{w}}$, we obtain

$$\begin{aligned} g_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) &= \inf_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)} \boldsymbol{c}^\top \boldsymbol{v} \\ &\leq \boldsymbol{c}^\top \boldsymbol{v}' \\ &\leq \boldsymbol{c}^\top \bar{\boldsymbol{w}} \\ &= \max\{\boldsymbol{c}^\top \boldsymbol{v}_1, \boldsymbol{c}^\top \boldsymbol{v}_2\} \\ &= \max\{g_{\boldsymbol{u}}(\boldsymbol{p}_1), g_{\boldsymbol{u}}(\boldsymbol{p}_2)\}. \end{aligned}$$

The second inequality follows from

$$\boldsymbol{v}' \leq_Q \bar{\boldsymbol{w}} \;\;\Rightarrow\;\; \bar{\boldsymbol{w}} - \boldsymbol{v}' \in Q \;\;\Rightarrow\;\; \boldsymbol{c}^\top(\bar{\boldsymbol{w}} - \boldsymbol{v}') \geq 0.$$

Therefore, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$.

Next, we show that (15) is also necessary for $g_{\boldsymbol{u}}(\boldsymbol{p})$ to be a quasi-convex function in $\boldsymbol{p}$. We suppose that (15) is not satisfied, and then show that $g_{\boldsymbol{u}}(\boldsymbol{p})$ cannot be a quasi-convex function in $\boldsymbol{p}$.

If (15) is not satisfied, there must exist $\bar{\boldsymbol{p}}_1, \bar{\boldsymbol{p}}_2 \in conv(P), \bar{\boldsymbol{v}}_1 \in V_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_1), \bar{\boldsymbol{v}}_2 \in V_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_2), \bar{\alpha} \in (0,1)$ such that

$$\begin{array}{c} \text{for some } \bar{\boldsymbol{w}} \in W \text{ s.t. } \bar{\boldsymbol{v}}_1 \leq_Q \bar{\boldsymbol{w}}, \bar{\boldsymbol{v}}_2 \leq_Q \bar{\boldsymbol{w}}, \\ \boldsymbol{v}' >_Q \bar{\boldsymbol{w}} \text{ for } \forall \boldsymbol{v}' \in V_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2). \end{array}$$

From the definition (14) of $g_{\boldsymbol{u}}(\boldsymbol{p})$,

$$\begin{array}{l} g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_1) \leq \boldsymbol{c}^\top \bar{\boldsymbol{v}}_1 \leq \boldsymbol{c}^\top \bar{\boldsymbol{w}} \\ g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_2) \leq \boldsymbol{c}^\top \bar{\boldsymbol{v}}_2 \leq \boldsymbol{c}^\top \bar{\boldsymbol{w}}. \end{array} \tag{16}$$

Since $\forall \boldsymbol{v}' \in V_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2)$ satisfies $\boldsymbol{v}' >_Q \bar{\boldsymbol{w}}$,

$$g_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2) > \boldsymbol{c}^\top \bar{\boldsymbol{w}}. \tag{17}$$

The above inequalities (16) and (17) show that

$$\max\{g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_1), g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_2)\} < g_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2)$$

and we see that the condition of quasi-convex function:

$$g_{\boldsymbol{u}}(\alpha\boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) \le \max\{g_{\boldsymbol{u}}(\boldsymbol{p}_1), g_{\boldsymbol{u}}(\boldsymbol{p}_2)\} \quad \text{for } \forall\boldsymbol{p}_1, \boldsymbol{p}_2 \in conv(P), \forall\alpha \in (0,1)$$

is violated at $\bar{\boldsymbol{p}}_1, \bar{\boldsymbol{p}}_2 \in conv(P)$ and $\bar{\alpha}$. Thus, if $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$, $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfies the condition of quasi $Q$-convex set-valued map (15). ∎

## 3.2 Functional Description of $V_{\boldsymbol{u}}(\boldsymbol{p})$

In this subsection, we focus on the case where the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ are described explicitly using constraints and obtain sufficient conditions for quasi $Q$-convexity of the mapping $V_{\boldsymbol{u}}$ in Propositions 4 and 5.

For an arbitrary closed convex cone $K$, we consider a vector-valued function $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ that satisfies

$$V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \le_K \boldsymbol{0}\}.$$

We now investigate conditions on functions $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ that guarantee $V_{\boldsymbol{u}}(\boldsymbol{p})$ to be a quasi $Q$-convex set-valued map. Not surprisingly, we observe that quasi $K$-convexity of $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ is sufficient for this purpose. We first define this property [8]:

**Definition 2.** *$F : D \to W$ is a quasi $K$-convex vector-valued function in $\boldsymbol{d}$ if for $\forall\boldsymbol{d}_1, \boldsymbol{d}_2 \in D$ and $\forall\alpha \in [0,1]$,*

$$F(\alpha\boldsymbol{d}_1 + (1-\alpha)\boldsymbol{d}_2) \le_K \boldsymbol{z} \text{ holds}$$
$$\text{for } \forall\boldsymbol{z} \text{ satisfying } F(\boldsymbol{d}_1) \le_K \boldsymbol{z},\;\; F(\boldsymbol{d}_2) \le_K \boldsymbol{z}.$$

**Proposition 4.** *Consider vector-valued functions $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) : conv(P) \times V \to W$ such that $V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \le_K \boldsymbol{0}\}$ for a given cone $K$. If $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ is quasi $K$-convex in $(\boldsymbol{v}, \boldsymbol{p})$ for all $\boldsymbol{u} \in U$, then $V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \le_K \boldsymbol{0}\}$ is a quasi $Q$-convex set-valued map for any closed convex cone $Q$.*

*Proof:* By definition, for $\forall\boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1)$ and $\forall\boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2)$,

$$F_{\boldsymbol{u}}(\boldsymbol{v}_1, \boldsymbol{p}_1) \le_K \boldsymbol{0}, \text{ and } F_{\boldsymbol{u}}(\boldsymbol{v}_2, \boldsymbol{p}_2) \le_K \boldsymbol{0}$$

holds, and under the assumption that $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ is quasi $K$-convex, we obtain

$$F_{\boldsymbol{u}}(\alpha\boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2, \alpha\boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) \le_K \boldsymbol{0}, \quad \forall\alpha \in [0,1],$$

which implies $\alpha\boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\alpha\boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$ for $\forall\alpha \in [0,1]$.

Now, if $\boldsymbol{w} \in V$ satisfies $\boldsymbol{v}_1 \le_Q \boldsymbol{w}$ and $\boldsymbol{v}_2 \le_Q \boldsymbol{w}$, $\alpha\boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2 \le_Q \boldsymbol{w}$ holds for $\forall\alpha \in [0,1]$, since $\boldsymbol{w} - \boldsymbol{v}_1 \in Q$, $\boldsymbol{w} - \boldsymbol{v}_2 \in Q$, and the convexity of $Q$ indicate $\alpha(\boldsymbol{w} - \boldsymbol{v}_1) +$

$(1-\alpha)(\boldsymbol{w}-\boldsymbol{v}_2) = \boldsymbol{w} - \{\alpha \boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2\} \in Q$. Therefore, $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfies the condition (15) of quasi $Q$-convexity. ∎

We stress that the cones $K$ and $Q$ in the proposition above need not coincide. Next, we consider an even more specific form for $V_{\boldsymbol{u}}(\boldsymbol{p})$ by defining $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}) = (f^1_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}),\ldots,f^m_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}))^\top$ where each $f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is a real-valued function and $K = R^m_+$. If $f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$, $i=1,\ldots,m$, are quasi-convex functions in $(\boldsymbol{v},\boldsymbol{p})$, then $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is a quasi $R^m_+$-convex vector-valued function, and therefore, $V_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi $Q$-convex set-valued map by the proposition.

In fact, when $V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}) \le 0, i=1,\ldots,m\}$ with quasi-convex functions $f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$, $i=1,\ldots,m$, the mapping $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfies the stronger $Q$-*convexity* property. The $Q$-convex set valued map is defined in [2, 7]:

$$\begin{array}{l}\text{for } \forall \boldsymbol{p}_1, \forall \boldsymbol{p}_2 \in conv(P), \forall \boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1), \forall \boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2),\ \text{and } \alpha \in (0,1),\\ \exists \boldsymbol{w} \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) \text{ s.t. } \boldsymbol{w} \le_Q \alpha \boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2.\end{array} \tag{18}$$

We end this section by presenting the following result which easily follows from Propositions 3 and 4.

**Proposition 5.** *Consider the problem:*

$$g_{\boldsymbol{u}}(\boldsymbol{p}) = \inf_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p})} f_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}),$$

*If the objective function $f_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is quasi-convex in $(\boldsymbol{v},\boldsymbol{p})$ and $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is a quasi $K$-convex vector-valued function for some convex cone $K$, then $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$.*

# 4 Examples with Quasi-convex Mappings

We investigate the condition (15) for quasi $Q$-convexity of set-valued mappings and two stronger (more restrictive) variants of this condition by studying three examples. The first one is an example of a $Q$-convex mapping, the second one is that of a naturally quasi $Q$-convex mapping, and the last one is that of a quasi $Q$-convex mapping. The $Q$-convexity condition was described above in (18). Before we present the examples, we define the naturally quasi $Q$-convexity condition [2, 7]:

**Definition 3.** *A set valued mapping $M : D \rightsquigarrow W$ is said to be naturally quasi $Q$-convex if for $\forall \boldsymbol{d}_1, \boldsymbol{d}_2 \in D$,*

$$\begin{array}{l}\forall \alpha \in (0,1), \forall \boldsymbol{m}_1 \in M(\boldsymbol{d}_1), \forall \boldsymbol{m}_2 \in M(\boldsymbol{d}_2)\\ \exists \boldsymbol{w} \in M(\alpha \boldsymbol{d}_1 + (1-\alpha)\boldsymbol{d}_2) \text{ and } \exists \beta \in [0,1] \text{ s.t.}\\ \boldsymbol{w} \le_Q \beta \boldsymbol{m}_1 + (1-\beta)\boldsymbol{m}_2.\end{array} \tag{19}$$

It is known that every convex set-valued map is also naturally quasi-convex, and every naturally quasi-convex set-valued map is also quasi-convex.

In the examples we describe below, the optimal solution of $\sup_{\boldsymbol{p}\in conv(P)} \inf_{\boldsymbol{v}\in V_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \boldsymbol{v}$ is obtained for $\boldsymbol{p}_i \in P$ for some $i$ and we can ignore the constraints induced from the interior points of $conv(P)$. However, Examples 2 and 3 do not satisfy the sufficient conditions of Propositions 4 and 5 for quasi $Q$-convex set valued maps $V_{\boldsymbol{u}}(\boldsymbol{p})$. Thus, these examples indicate that conditions given in Propositions 4 and 5 are not necessary for quasi $Q$-convexity of the mapping $V_{\boldsymbol{u}}$ and more general problems can be reduced to the single-level optimization problem (8).

**Example 1 ($Q$-convex $V_u(\boldsymbol{p})$) :** *Consider the ARC problem described below:*

$$\min_{u\in U} \max_{\boldsymbol{p}\in conv(P)} \min_{\boldsymbol{v}=(v_1,v_2)\in V(\boldsymbol{p})} (-v_1 - uv_2) \tag{20}$$

*with* $P = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ *and*

$$V(\boldsymbol{p}) = \{(v_1, v_2)| \ \ (v_1 - p_1)^2 + (v_2 - p_2)^2 \le 1, \ \ \boldsymbol{v} \ge \boldsymbol{0}\}.$$

*Note that* $conv(P) = \{\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \mid \boldsymbol{p} \ge \boldsymbol{0}, p_1 + p_2 = 1\}$. *We dropped the subscript* $u$ *from* $V_u(\boldsymbol{p})$ *since this set does not depend on* $u$. *The inner max-min problem in (20) is equivalent to:*

$$\min_{u\in U, v_0} \left\{ v_0 | \ \forall \boldsymbol{p} \in conv(P) \ \exists \boldsymbol{v} : \begin{array}{l} v_0 \ge -v_1 - uv_2, \\ (v_1 - p_1)^2 + (v_2 - p_2)^2 \le 1 \\ \boldsymbol{v} \ge \boldsymbol{0}. \end{array} \right\}$$

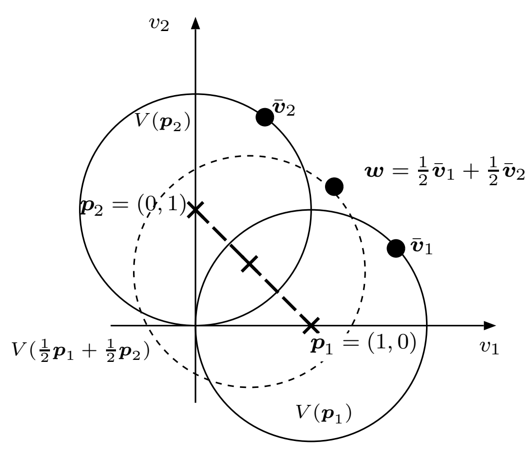


Figure 1: Feasible sets and optimal solutions (with $u = 1$) in Example 1

*We observe that the mapping $V(\boldsymbol{p})$ is $Q$-convex for every cone $Q$ containing zero. Indeed, for $\forall \boldsymbol{p}_1, \forall \boldsymbol{p}_2 \in conv(P)$, $\forall \alpha \in (0,1)$, $\forall \boldsymbol{v}_1 \in V(\boldsymbol{p}_1)$ and $\forall \boldsymbol{v}_2 \in V(\boldsymbol{p}_2)$, we can construct the inner point between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$: $\alpha\boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2$ which lies in the set $V(\alpha\boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$, since $\alpha V(\boldsymbol{p}_1) + (1-\alpha)V(\boldsymbol{p}_2) = V(\alpha\boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$. Now, the set-valued map $V$ satisfies the condition of $Q$-convexity (18) with $\boldsymbol{w}$ chosen as $\boldsymbol{w} = \alpha\boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2$, since $\alpha\boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2 - \boldsymbol{w} = \boldsymbol{0} \in Q$.*

*Since every convex set-valued map is also quasi-convex [7], the set-valued map $V$ is quasi $Q$-convex and therefore, $g_u(\boldsymbol{p})$ becomes a quasi-convex function in $\boldsymbol{p}$, $conv(P)$ in (20) can be replaced by $P$ and this problem reduces to the single-level optimization problem with finitely many constraints.*

*Indeed, Figure 1 shows that it is sufficient to focus on the extreme cases $V(\boldsymbol{p}_1)$ and $V(\boldsymbol{p}_2)$, since the objective function is linear and an optimal solution is attained in some scenario $V(\boldsymbol{p}_1)$ or $V(\boldsymbol{p}_2)$.*

**Remark 2.** *Note that this example contains a constraint in which the coefficients of adjustable vector variable $\boldsymbol{v}$ are affected by uncertainty:*

$$(v_1 - p_1)^2 + (v_2 - p_2)^2 \leq 1 \quad \Leftrightarrow \quad v_1^2 + v_2^2 - 2p_1v_1 - 2p_2v_2 + p_1^2 + p_2^2 \leq 1.$$

*But $g_u(\boldsymbol{p})$ is quasi-convex and $conv(P)$ can be replaced by the finite set $P$ in the ARC formulation. It is noted in [5] that when the constraint coefficients of the adjustable variables $\boldsymbol{v}$ are affected by uncertainty, the resulting ARC can be computationally intractable. For example, this case is excluded in Theorem 2.2 of [5]. The example above shows a special case where the resulting ARC problem is still tractable.*

**Example 2 (Naturally quasi $Q$-convex $V_u(p)$) :** *We focus on the inner problem $g_u(p) = \min_{\boldsymbol{v} \in V_u(p)} \boldsymbol{c}^\top \boldsymbol{v}$ with a fixed $u \in U$, where*

$$\begin{aligned} &V_u(p) = \{(v_1, v_2) |\ \ u \leq pv_1 \leq 2u,\ u \leq pv_2 \leq 2u\}, \\ &P = \{\tfrac{1}{2}, 1\}, \quad conv(P) = [\tfrac{1}{2}, 1]. \end{aligned}$$

*We'll show that the set-valued map $V_u(p)$ satisfies the condition of naturally quasi $Q$-convexity defined above. Indeed, for $\forall p_1, \forall p_2 \in conv(P)$, $\forall \alpha \in (0,1)$, $\forall \bar{\boldsymbol{v}}_1 \in V_u(p_1)$ and $\forall \bar{\boldsymbol{v}}_2 \in V_u(p_2)$, we can construct $\boldsymbol{w} = \beta\bar{\boldsymbol{v}}_1 + (1-\beta)\bar{\boldsymbol{v}}_2 \in V(\alpha p_1 + (1-\alpha)p_2)$ by computing $\beta \in [0,1]$ from*

$$\frac{1}{\alpha p_1 + (1-\alpha)p_2} = \frac{\beta}{p_1} + \frac{1-\beta}{p_2}.$$

*Therefore, the set-valued map $V_u$ of this example satisfies the condition (19) of naturally quasi $Q$-convexity*[1] *whenever we take any $Q$ which includes $\boldsymbol{0}$. Assuming $u = 1$, Figure 2 shows, in the case of $\alpha = 1/2$, the inner point of $\bar{\boldsymbol{v}}_1$ and $\bar{\boldsymbol{v}}_2$: $\boldsymbol{w} = \beta\bar{\boldsymbol{v}}_1 + (1-\beta)\bar{\boldsymbol{v}}_2 \in V_u(\alpha p_1 + (1-\alpha)p_2)$ with $\beta = 1/3$.*

[1] If an appropriate objective function $\bar{\boldsymbol{c}}^\top \boldsymbol{v}$ is given (for example, $\bar{\boldsymbol{c}} = \bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2$ and therefore, $Q' = \{\boldsymbol{q} : (\bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2)^\top \boldsymbol{q} \geq 0\}$), the set-valued map $V_u$ might be $Q'$-convex.

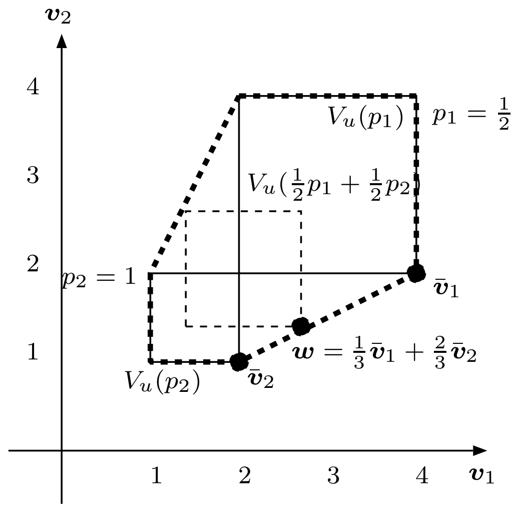


Figure 2: Feasible sets and optimal solutions (with $u = 1$) in Example 2

*It is shown in [7] that every naturally quasi-convex set-valued map is also quasi-convex, and we find that the set-valued map $V$ of this example is quasi $Q$-convex. However, we note that the constraint functions defining $V_u(p)$ do not satisfy the sufficient conditions given in Propositions 4 and 5 for the quasi $Q$-convexity of set-valued map $V_u$.*

**Example 3 (Quasi $Q$-convex $V_{\boldsymbol{u}}(\boldsymbol{p})$) :** *We now give a geometric example of a mapping $V_{\boldsymbol{u}}(\boldsymbol{p})$ that is quasi $Q$-convex for some convex cones $Q$ but not for others. We define the convex cone $Q$ depending on the linear objective function $\boldsymbol{c}^\top \boldsymbol{v}$ as follows: $Q = \{\boldsymbol{q} : \boldsymbol{c}^\top \boldsymbol{q} \geq 0\}$. In our previous examples and discussion, the cone $Q$ in the definition of quasi $Q$-convex functions and mappings was largely irrelevant. Note however that while the mapping on the left in Figure 3 does not satisfy the condition (15), the mapping on the right satisfies this condition. For the example on the right, $g_{\boldsymbol{u}}(\boldsymbol{p}) := \min_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \boldsymbol{v}$ is quasi-convex in $\boldsymbol{p}$. Therefore, we can focus on the extreme scenario-cases $V_{\boldsymbol{u}}(\boldsymbol{p}_1)$ and $V_{\boldsymbol{u}}(\boldsymbol{p}_2)$, and conv$(P)$ in (ARC) can be replaced as the set of finite points $P$.*

*Although in this example $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ of $V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \leq_K \boldsymbol{0}\}$ is not a quasi $K$-convex vector-valued function (if quasi $K$-convex, $\alpha \boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$ holds for $\forall \alpha \in [0, 1]$, which is clearly not the case), the set-valued map $V_{\boldsymbol{u}}(\boldsymbol{p})$ of the right figure satisfies the condition of quasi $Q$-convexity and $V_{\boldsymbol{u}}(\boldsymbol{p})$ is shown as a quasi $Q$-convex set-valued map.*

# 5 Multi-period Model

In this section, we extend our two-period formulation from the previous sections to a multi-period setting with 3 or more periods. In addition to our desire to solve these more general

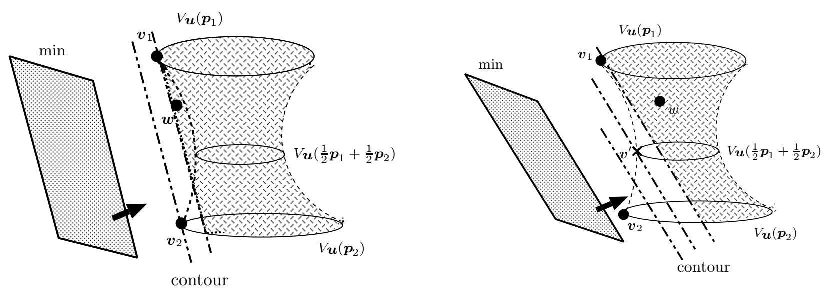


Figure 3: $V_{\boldsymbol{u}}(\boldsymbol{p})$ of the right figure is quasi $Q$-convex, but that of left figure is not.

classes of problems, this extension is motivated by the expectation that ARC models in this multi-period setting can be reduced to single-level optimization problems with similar assumptions to what we have already used above. As in the previous sections, we focus on polytopic uncertainty sets. We observe below that it takes little more than notational changes to address the more general multi-period models.

To simplify the notation in the succeeding discussion, we use the following convention: Given vectors $\boldsymbol{x}^1$, $\boldsymbol{x}^2, \dots \boldsymbol{x}^k$, we use $\bar{\boldsymbol{x}}^k$ to denote the collection $(\boldsymbol{x}^1, \boldsymbol{x}^2, \dots \boldsymbol{x}^k)$. The collection $\bar{\boldsymbol{p}}^k$ is defined similarly.

## 5.1 3-period Model

Let $\boldsymbol{g}^1(\bar{\boldsymbol{x}}^1)$, $\boldsymbol{g}^2(\bar{\boldsymbol{x}}^2, \bar{\boldsymbol{p}}^1)$ and $\boldsymbol{g}^3(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2)$ be vector valued functions, and we consider the following problem:

$$\inf_{\boldsymbol{x}^1 \in \mathcal{X}^1} \sup_{\boldsymbol{p}^1 \in conv(P^1)} \inf_{\boldsymbol{x}^2 \in \mathcal{X}^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1)} \sup_{\boldsymbol{p}^2 \in conv(P^2)} \inf_{\boldsymbol{x}^3 \in \mathcal{X}^3(\bar{\boldsymbol{x}}^2, \bar{\boldsymbol{p}}^2)} f(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2) \tag{21}$$

where

$$\begin{aligned}
\mathcal{X}^1 &= \left\{\boldsymbol{x}^1 \mid \boldsymbol{g}^1(\bar{\boldsymbol{x}}^1) \leq \boldsymbol{0}\right\}, \\
\mathcal{X}^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1) &= \left\{\boldsymbol{x}^2 \mid \boldsymbol{g}^2(\bar{\boldsymbol{x}}^2, \bar{\boldsymbol{p}}^1) \leq \boldsymbol{0}\right\}, \\
\mathcal{X}^3(\bar{\boldsymbol{x}}^2, \bar{\boldsymbol{p}}^2) &= \left\{\boldsymbol{x}^3 \mid \boldsymbol{g}^3(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2) \leq \boldsymbol{0}\right\}, \\
P^1 &= \left\{\boldsymbol{p}^1_1, \boldsymbol{p}^1_2, \dots, \boldsymbol{p}^1_{k_1}\right\}, \\
P^2 &= \left\{\boldsymbol{p}^2_1, \boldsymbol{p}^2_2, \dots, \boldsymbol{p}^2_{k_2}\right\}.
\end{aligned}$$

We define

$$V_{\bar{\boldsymbol{x}}^2\bar{\boldsymbol{p}}^1}(\boldsymbol{p}^2) = \left\{(\boldsymbol{x}^3, s) : f(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2) \leq s, \boldsymbol{x}^3 \in \mathcal{X}^3(\bar{\boldsymbol{x}}^2, \bar{\boldsymbol{p}}^2)\right\},$$

and

$$V_{\bar{\boldsymbol{x}}^1}(\boldsymbol{p}^1) = \left\{ (\boldsymbol{x}^2, \boldsymbol{x}^3_1, \dots, \boldsymbol{x}^3_{k_2}, s) : \begin{array}{l} f(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3_i, \boldsymbol{p}^1, \boldsymbol{p}^2_i) \leq s, \quad i = 1, 2, \dots, k_2 \\ \boldsymbol{x}^2 \in \mathcal{X}^2(\boldsymbol{x}^1, \boldsymbol{p}^1), \\ \boldsymbol{x}^3_i \in \mathcal{X}^3(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{p}^1, \boldsymbol{p}^2_i), \quad i = 1, 2, \dots, k_2 \end{array} \right\}.$$

We assume that the mapping $V_{\bar{\boldsymbol{x}}^1}(\cdot)$ is quasi $Q^1$-convex with

$$Q^1 = \left\{(\boldsymbol{x}^2, \boldsymbol{x}^3_1, \ldots, \boldsymbol{x}^3_{k_2}, s) : s \geq 0\right\},$$

and the mapping $V_{\bar{\boldsymbol{x}}^2\bar{\boldsymbol{p}}^1}(\cdot)$ is quasi $Q^2$-convex with

$$Q^2 = \left\{(\boldsymbol{x}^3, s) : s \geq 0\right\}.$$

Propositions 4 and 5 show that if we take quasi-convex functions

$$\begin{array}{ll} f(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2), & \text{in } (\boldsymbol{x}^3, \boldsymbol{p}^2), (\boldsymbol{x}^2, \boldsymbol{x}^3, \boldsymbol{p}^1) \text{ and } \boldsymbol{x}^1 \\ \boldsymbol{g}^1(\bar{\boldsymbol{x}}^1), & \text{in } \boldsymbol{x}^1 \\ \boldsymbol{g}^2(\bar{\boldsymbol{x}}^2, \bar{\boldsymbol{p}}^1), & \text{in } (\boldsymbol{x}^2, \boldsymbol{p}^1) \text{ and } \boldsymbol{x}^1 \\ \boldsymbol{g}^3(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2), & \text{in } (\boldsymbol{x}^3, \boldsymbol{p}^2), (\boldsymbol{x}^2, \boldsymbol{x}^3, \boldsymbol{p}^1) \text{ and } \boldsymbol{x}^1 \end{array} \tag{22}$$

the quasi $Q$-convexity of $V_{\bar{\boldsymbol{x}}^2\bar{\boldsymbol{p}}^1}(\boldsymbol{p}^2)$ and $V_{\bar{\boldsymbol{x}}^1}(\boldsymbol{p}^1)$ is ensured. If the above functions (22) are convex in all $\boldsymbol{x}^i$ variables and uncertainties $\boldsymbol{p}^i$, then the induced set valued maps $V_{\bar{\boldsymbol{x}}^2\bar{\boldsymbol{p}}^1}(\boldsymbol{p}^2)$ and $V_{\bar{\boldsymbol{x}}^1}(\boldsymbol{p}^1)$ are quasi $Q$-convex. A natural, separable structure for functions (22) arises when $t$-th period decision $\boldsymbol{x}^t$ depends only on the uncertainties $\boldsymbol{p}^{t-1}$ of the previous period as follows:

$$\begin{array}{lcl} f(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2) & = & d_1(\boldsymbol{x}^1) + d_2(\boldsymbol{x}^2, \boldsymbol{p}^1) + d_3(\boldsymbol{x}^3, \boldsymbol{p}^2), \\ \boldsymbol{g}^1(\bar{\boldsymbol{x}}^1) & = & \boldsymbol{m}(\boldsymbol{x}^1), \\ \boldsymbol{g}^2(\bar{\boldsymbol{x}}^2, \bar{\boldsymbol{p}}^1) & = & \boldsymbol{h}_1(\boldsymbol{x}^1) + \boldsymbol{h}_2(\boldsymbol{x}^2, \boldsymbol{p}^1), \\ \boldsymbol{g}^3(\bar{\boldsymbol{x}}^3, \bar{\boldsymbol{p}}^2) & = & \boldsymbol{k}_1(\boldsymbol{x}^1) + \boldsymbol{k}_2(\boldsymbol{x}^2, \boldsymbol{p}^1) + \boldsymbol{k}_3(\boldsymbol{x}^3, \boldsymbol{p}^2), \end{array}$$

where all of the components $d_i(\boldsymbol{x}^i, \boldsymbol{p}^{i-1})$, $\boldsymbol{m}(\boldsymbol{x}^1)$, $\boldsymbol{h}_i(\boldsymbol{x}^i, \boldsymbol{p}^{i-1})$, and $\boldsymbol{k}_i(\boldsymbol{x}^i, \boldsymbol{p}^{i-1})$ are convex vector functions in $(\boldsymbol{x}^i, \boldsymbol{p}^{i-1})$.

In this setting, the problem (21) can be written as:

$$\inf_{\boldsymbol{x}^1 \in \mathcal{X}^1} \sup_{\boldsymbol{p}^1 \in conv(P^1)} \left\{ \inf_{\boldsymbol{x}^2 \in \mathcal{X}^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1)} \sup_{\boldsymbol{p}^2 \in conv(P^2)} \inf_{\boldsymbol{x}^3, s) \in V_{\bar{\boldsymbol{x}}^2\bar{\boldsymbol{p}}^1}(\boldsymbol{p}^2)} s \right\}.$$

The inner min-max-min problem:

$$\inf_{\boldsymbol{x}^2 \in \mathcal{X}^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1)} \sup_{\boldsymbol{p}^2 \in conv(P^2)} \inf_{(\boldsymbol{x}^3, s) \in V_{\bar{\boldsymbol{x}}^2\bar{\boldsymbol{p}}^1}(\boldsymbol{p}^2)} s \tag{23}$$

is equivalent to

$$\begin{array}{lll} \inf\limits_{\boldsymbol{x}^2, \{\boldsymbol{x}^3(\boldsymbol{p})\}, \{t(\boldsymbol{p})\}, s_2} & s_2 & \\ \text{s.t.} & s(\boldsymbol{p}) \leq s_2, & \forall \boldsymbol{p} \in conv(P^2) \\ & f(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3(\boldsymbol{p}), \boldsymbol{p}^1, \boldsymbol{p}) \leq s(\boldsymbol{p}) & \forall \boldsymbol{p} \in conv(P^2) \\ & \boldsymbol{x}^2 \in \mathcal{X}^2(\boldsymbol{x}^1, \boldsymbol{p}^1), & \\ & \boldsymbol{x}^3(\boldsymbol{p}) \in \mathcal{X}^3(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{p}^1, \boldsymbol{p}) & \forall \boldsymbol{p} \in conv(P^2), \end{array}$$

where $(\boldsymbol{x}^3(\boldsymbol{p}), s(\boldsymbol{p}))$, $\boldsymbol{p} \in conv(P^2)$ corresponds to $(\boldsymbol{x}^3, s)$ depending on $\boldsymbol{p} \in conv(P^2)$ in (23). Since $V_{\bar{\boldsymbol{x}}^2 \bar{\boldsymbol{p}}^1}(\boldsymbol{p}^2)$ is quasi $Q^2$-convex, using variables $\boldsymbol{x}_i^3$, $i = 1, \ldots, k_2$ instead of $\boldsymbol{x}^3(\boldsymbol{p})$, $\boldsymbol{p} \in conv(P^2)$, we obtain:

$$
\begin{array}{rl}
& \inf_{\boldsymbol{x}^1 \in \mathcal{X}^1} \sup_{\boldsymbol{p}^1 \in conv(P^1)} \inf_{\boldsymbol{x}^2, \boldsymbol{x}_1^3, \ldots, \boldsymbol{x}_{k_2}^3, s_2} \quad s_2 \\
& \qquad\qquad \text{s.t.} \quad f(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}_i^3, \boldsymbol{p}^1, \boldsymbol{p}_i^2) \le s_2, \quad i = 1, 2, \ldots, k_2 \\
& \qquad\qquad\qquad \boldsymbol{x}^2 \in \mathcal{X}^2(\boldsymbol{x}^1, \boldsymbol{p}^1), \\
& \qquad\qquad\qquad \boldsymbol{x}_i^3 \in \mathcal{X}^3(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{p}^1, \boldsymbol{p}_i^2), \quad i = 1, 2, \ldots, k_2 \\
\Longleftrightarrow & \inf_{\boldsymbol{x}^1 \in \mathcal{X}^1} \sup_{\boldsymbol{p}^1 \in conv(P^1)} \inf_{(\boldsymbol{x}^2, \boldsymbol{x}_1^3, \ldots, \boldsymbol{x}_{k_2}^3, s_2) \in V_{\boldsymbol{x}^1}(\boldsymbol{p}^1)} s_2 \\
& \inf_{\boldsymbol{x}^1, \{\boldsymbol{x}_j^2\}, \{\boldsymbol{x}_{i,j}^3\}, s_1} \quad s_1 \\
\Longleftrightarrow & \qquad \text{s.t.} \quad f(\boldsymbol{x}^1, \boldsymbol{x}_j^2, \boldsymbol{x}_{i,j}^3, \boldsymbol{p}_j^1, \boldsymbol{p}_i^2) \le s_1, \quad i = 1, 2, \ldots, k_2, \; j = 1, 2, \ldots, k_1 \\
& \qquad\qquad \boldsymbol{x}^1 \in \mathcal{X}^1 \\
& \qquad\qquad \boldsymbol{x}_j^2 \in \mathcal{X}^2(\boldsymbol{x}^1, \boldsymbol{p}_j^1), \quad j = 1, 2, \ldots, k_1 \\
& \qquad\qquad \boldsymbol{x}_{i,j}^3 \in \mathcal{X}^3(\boldsymbol{x}^1, \boldsymbol{x}_j^2, \boldsymbol{p}_j^1, \boldsymbol{p}_i^2), \quad i = 1, 2, \ldots, k_2, \; j = 1, 2, \ldots, k_1
\end{array} \tag{24}
$$

When functions $f$, $\boldsymbol{g}^2$, and $\boldsymbol{g}^3$ are linear in $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, and $\boldsymbol{x}^3$ respectively, and $\mathcal{X}^1$ is polyhedral, this problem is an LP, since $\boldsymbol{p}_j^1$ $(j = 1, 2, \ldots, k_1)$ and $\boldsymbol{p}_i^2$ $(i = 1, 2, \ldots, k_2)$ are given, and all coefficients depending on uncertainty are determined uniquely.

## 5.2 Recurrence Formula for Multi-period Model

We propose the general formulation of the multi-period model using a recurrence formula. As above, $\bar{\boldsymbol{p}}^t$ denotes the set of $t$-period-uncertainty $(\boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^t)$, and $\bar{\boldsymbol{x}}^t$ denotes the set of $t$-period-decision variables $(\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^t)$. and we rewrite the objective function $f(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^T, \boldsymbol{p}^1, \ldots, \boldsymbol{p}^{T-1})$ as $f(\bar{\boldsymbol{x}}^T, \bar{\boldsymbol{p}}^{T-1})$. Also, the feasible set $\mathcal{X}^t(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{t-1}, \boldsymbol{p}^1, \ldots, \boldsymbol{p}^{t-1})$ for the $t$-th variable $\boldsymbol{x}^t$ is rewritten as

$$
\mathcal{X}^t(\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-1}) = \left\{ \boldsymbol{x}^t \mid \boldsymbol{g}^t(\bar{\boldsymbol{x}}^t, \bar{\boldsymbol{p}}^{t-1}) \le \boldsymbol{0} \right\}.
$$

Then, the $T$-period model we consider is

$$
\begin{array}{rl}
& \inf_{\boldsymbol{x}^1 \in \mathcal{X}^1} \sup_{\boldsymbol{p}^1 \in conv(P^1)} \inf_{\boldsymbol{x}^2 \in \mathcal{X}^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1)} \sup_{\boldsymbol{p}^2 \in conv(P^2)} \cdots \inf_{\boldsymbol{x}^T \in \mathcal{X}^T(\bar{\boldsymbol{x}}^{T-1}, \bar{\boldsymbol{p}}^{T-1})} f(\bar{\boldsymbol{x}}^T, \bar{\boldsymbol{p}}^{T-1}) \\
= & \inf_{\boldsymbol{x}^1 \in \mathcal{X}^1} \sup_{\boldsymbol{p}^1 \in conv(P^1)} \inf_{\boldsymbol{x}^2 \in \mathcal{X}^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1)} \sup_{\boldsymbol{p}^2 \in conv(P^2)} \cdots \inf_{(\boldsymbol{x}^T, s) \in V_{\bar{\boldsymbol{x}}^{T-1}, \bar{\boldsymbol{p}}^{T-2}}(\boldsymbol{p}^{T-1})} s,
\end{array} \tag{25}
$$

where

$$
V_{\bar{\boldsymbol{x}}^{T-1}, \bar{\boldsymbol{p}}^{T-2}}(\boldsymbol{p}^{T-1}) = \left\{ (\boldsymbol{x}^T, s) : \begin{array}{l} f(\bar{\boldsymbol{x}}^T, \bar{\boldsymbol{p}}^{T-1}) \le s \\ \boldsymbol{x}^T \in \mathcal{X}^T(\bar{\boldsymbol{x}}^{T-1}, \bar{\boldsymbol{p}}^{T-1}) \end{array} \right\}.
$$

Assuming $\boldsymbol{p}^i \in P^i$ at each period $(i = 1, \ldots, t)$, we describe constraints for the set of $t$-period-uncertainty as $\boldsymbol{P}_t^1 \in P^1 \times \ldots \times P^t$. We think of $F^t(\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-1})$ as a robust optimal value obtained from the $t$-th term to the $T$-th term under the earlier decisions $\bar{\boldsymbol{x}}^{t-1}$ and uncertainty $\bar{\boldsymbol{p}}^{t-1}$, and formulate it as

$$F^t(\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-1}) = \begin{cases} \inf\limits_{(\boldsymbol{x}^T, s) \in V_{\bar{\boldsymbol{x}}^{T-1}, \bar{\boldsymbol{p}}^{T-2}}(\boldsymbol{p}^{T-1})} s, & t = T \\ \inf\limits_{\boldsymbol{x}^t \in \mathcal{X}^t(\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-1})} \sup\limits_{\boldsymbol{p}^t \in conv(P^t)} F^{t+1}(\bar{\boldsymbol{x}}^t, \bar{\boldsymbol{p}}^t), & t = 1, \ldots, T-1. \end{cases}$$

The problem $F^{t-1}(\bar{\boldsymbol{x}}^{t-2}, \bar{\boldsymbol{p}}^{t-2})$:

$$\inf_{\boldsymbol{x}^{t-1} \in \mathcal{X}^{t-1}(\bar{\boldsymbol{x}}^{t-2}, \bar{\boldsymbol{p}}^{t-2})} \sup_{\boldsymbol{p}^{t-1} \in conv(P^{t-1})} F^t(\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-1})$$

is equivalent to

$$\begin{array}{ll} \inf\limits_{\boldsymbol{x}^{t-1}, s} & s \\ \text{s.t.} & F^t(\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-1}) \le s, \quad \forall \boldsymbol{p}^{t-1} \in P^{t-1} \\ & \boldsymbol{x}^{t-1} \in \mathcal{X}^{t-1}(\bar{\boldsymbol{x}}^{t-2}, \bar{\boldsymbol{p}}^{t-2}), \end{array}$$

if $F^t(\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-1})$ is a quasi-convex function, or

$$\begin{array}{l} V_{\bar{\boldsymbol{x}}^{t-1}, \bar{\boldsymbol{p}}^{t-2}}(\boldsymbol{p}^{t-1}) = \\ \left\{ \begin{array}{ll} \left(\boldsymbol{x}^t(\bar{\boldsymbol{p}}^{t-1}),\ \ldots, \{\boldsymbol{x}^T(\bar{\boldsymbol{p}}^{T-1}) : \boldsymbol{p}^t \in P^t, \ldots, \boldsymbol{p}^{T-1} \in P^{T-1}\},\ s\right) : & \\ & \\ f(\bar{\boldsymbol{x}}^T(\bar{\boldsymbol{p}}^{T-1}), \bar{\boldsymbol{p}}^{T-1}) \le s, & \boldsymbol{p}^t \in P^t, \ldots, \boldsymbol{p}^{T-1} \in P^{T-1} \\ \boldsymbol{x}^t(\bar{\boldsymbol{p}}^{t-1}) \in \mathcal{X}^t(\bar{\boldsymbol{x}}^{t-1}(\bar{\boldsymbol{p}}^{t-2}), \bar{\boldsymbol{p}}^{t-1}) & \\ \boldsymbol{x}^{t+1}(\bar{\boldsymbol{p}}^t) \in \mathcal{X}^{t+1}(\bar{\boldsymbol{x}}^t(\bar{\boldsymbol{p}}^{t-1}), \bar{\boldsymbol{p}}^t), & \boldsymbol{p}^t \in P^t \\ \ldots & \ldots \\ \boldsymbol{x}^T(\bar{\boldsymbol{p}}^{T-1}) \in \mathcal{X}^T(\bar{\boldsymbol{x}}^{T-1}(\bar{\boldsymbol{p}}^{T-2}), \bar{\boldsymbol{p}}^{T-1}), & \boldsymbol{p}^t \in P^t, \ldots, \boldsymbol{p}^{T-1} \in P^{T-1} \end{array} \right\} \end{array}$$

is a quasi $Q$-convex set-valued map.

Under one of these two quasi $Q$-convexity assumptions, the $T$-period ARC problem

$$\inf_{\boldsymbol{x}^1 \in \mathcal{X}^1} \sup_{\boldsymbol{p}^1 \in conv(P^1)} F^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1)$$

can be written as follows:

$$\begin{array}{lll} \inf\limits_{\boldsymbol{x}^1, \{\boldsymbol{x}^2(\bar{\boldsymbol{p}}^1)\}, \ldots, \{\boldsymbol{x}_T(\bar{\boldsymbol{p}}^{T-1})\}} s & & \\ \text{s.t} & f(\bar{\boldsymbol{x}}^T(\bar{\boldsymbol{p}}^{T-1}), \bar{\boldsymbol{p}}^{T-1}) \le s, \ \forall \bar{\boldsymbol{p}}^{T-1} & \\ & \boldsymbol{x}^1 \in \mathcal{X}^1 & \\ & \boldsymbol{x}^2(\bar{\boldsymbol{p}}^1) \in \mathcal{X}^2(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{p}}^1), & \forall \bar{\boldsymbol{p}}^1 = (\boldsymbol{p}^1) \in P^1 \\ & \boldsymbol{x}^3(\bar{\boldsymbol{p}}^2) \in \mathcal{X}^3(\bar{\boldsymbol{x}}^2(\bar{\boldsymbol{p}}^1), \bar{\boldsymbol{p}}^2), & \forall \bar{\boldsymbol{p}}^2 = (\boldsymbol{p}^1, \boldsymbol{p}^2) \in P^1 \times P^2 \\ & \ldots & \\ & \boldsymbol{x}^T(\bar{\boldsymbol{p}}^{T-1}) \in \mathcal{X}^T(\bar{\boldsymbol{x}}^{T-1}(\bar{\boldsymbol{p}}^{T-2}), \bar{\boldsymbol{p}}^{T-1}) & \\ & \qquad \forall \bar{\boldsymbol{p}}^{T-1} = (\boldsymbol{p}^1, \ldots, \boldsymbol{p}^{T-1}) \in P^1 \times \ldots \times P^{T-1}. & \end{array}$$

This problem is equivalent to the problem (24) when $T = 3$. Here, we use the notation $\boldsymbol{x}^t(\bar{\boldsymbol{p}}^{t-1})$ for adjustable variables at the $t$-th term, instead of $\boldsymbol{x}^t_{i_1,i_2,\ldots,i_{t-1}}$ in (24).

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