SENSITIVITY ANALYSIS IN CONVEX QUADRATIC OPTIMIZATION: INVARIANT SUPPORT SET INTERVAL[†]

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In sensitivity analysis one wants to know how the problem and the optimal solutions change under the variation of the input data. We consider the case when variation happens in the right hand side of the constraints and/or in the linear term of the objective function. We are interested to find the range of the parameter variation in Convex Quadratic Optimization (CQO) problems where the support set of a given primal optimal solution remains invariant. This question has been first raised in Linear Optimization (LO) and known as Type II (so called *Support Set Invariancy*) sensitivity analysis. We present computable auxiliary problems to identify the range of parameter variation in support set invariancy sensitivity analysis for CQO. It should be mentioned that all given auxiliary problems are LO problems and can be solved by an interior point method in polynomial time. We also highlight the differences between characteristics of support set invariancy sensitivity analysis for LO and CQO.

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1 Introduction

The origins of parametric CQO trace back to Markowitz [19, 20]. His primary interest was using the results of parametric CQO in mean-variance analysis of investment portfolios. Later, Houthakker [14] and Wolfe [24] used a parametric algorithm to solve quadratic optimization problems. Their results were based on the simplex method. We refer to [5] for a survey of their method.

The seminal paper of Karmakar [17], that triggered extensive research on Interior Point Methods (IPMs), led also to reconsider sensitivity analysis for linear [1, 8, 12, 23], quadratic [3, 8, 9] and conic linear [25] optimization problems. The new wave of investigation in sensitivity analysis is concentrated on the behavior of the optimal value function and its characteristics. The concept of optimal partition, that was introduced originally in LO [11], was also generalized to CQO problems [4]. Berkelaar et al. [3, 8] studied sensitivity analysis of CQO problems when perturbation occurs in the Right Hand Side (RHS) or in the Linear Term of the Objective Function (LTOF) data. Recently, Ghaffari et al. [9] studied sensitivity analysis of CQO problems when simultaneous perturbation occurs both in the RHS and LTOF data. They also specialized their findings to LO problems.

Categorization of various sensitivity analysis questions in LO was introduced first by Koltai and Terlaky [18]. They considered three types of sensitivity analysis. It should be mentioned that this classification is also valid for CQO. Similar to the LO case, we prefer to have more descriptive names for them[1]. These three types of sensitivity analysis can be briefly described as follows:

<u>Type I</u> (*Basis Invariancy*) sensitivity analysis aims to find the range of parameter variation so that the given optimal basic remains optimal. This kind of sensitivity analysis is based on simplex methods. In the case when the problem has multiple optimal (and thus degenerate) solutions, different methods lead to different optimal basic solutions and consequently, different and confusing optimality ranges are obtained. One can find clarifying examples for LO in [16, 23].

The goal of <u>Type II</u> (*Support Set Invariancy*) sensitivity analysis is to identify the range of the parameter variation so that for all values

[1] The authors would like to thank H.J. Greenberg [13] for his suggestion in having descriptive titles for these three types of sensitivity analysis.

of the parameter in this range, the perturbed problem has an optimal solution with the same support set that the given optimal solution of the original problem has. Since the given optimal solution of the problem is not necessarily a basic solution, the optimality range in support set invariancy sensitivity analysis may be different from the optimality range that is obtained in basis invariancy sensitivity analysis. Observe that one may consider more general cases[2]:

1. $\sigma(x^*(\epsilon)) \subseteq \sigma(x^*)$,
2. $\sigma(x^*(\epsilon)) \supseteq \sigma(x^*)$,
3. $\sigma(x^*(\epsilon)) = \sigma(x^*)$.

An economic interpretation of cases 1 and 2 might be as follows. In case 1, a shareholder aims not to open a new portfolio but may decide to set off some of the active ones. Case 2 can be interpreted as the shareholder wants to maintain existing portfolios , meanwhile he may open some new portfolios. Identifying the range of parameter ϵ with the aforementioned goals are the objective of support set invariancy sensitivity analysis in cases 1 and 2. We assert that the methodology presented in this paper answers case 1 as well as case 3 but not case 2. The only difference between these cases 1 and 3 is that the obtained ranges in case 1 are always closed intervals, while they might be an open interval in case 3.

Type III (*Optimal Partition Invariancy*) sensitivity analysis is concerned with the behavior of the optimal value function under variation of the input data. In case of LO, it is proved that when variation occurs in either the RHS or in the objective function data, the optimal value function is piecewise linear and it has constant slopes on these subintervals [23]. Thus, Type III sensitivity analysis for LO problems is aimed to find the range of parameter variation for which the rate of change of the optimal value function is constant. Determining the rate of changes i.e., the left and right derivatives of the optimal value function is also aimed. It is also proved that when simultaneous variation occurs both in the RHS and the objective function data, the optimal value function is piecewise quadratic [10, 12]. In this case, Type III sensitivity analysis is equivalent to identifying those subintervals where the optimal value function has different quadratic representations. These intervals are referred to as *invariancy* intervals, because the optimal partition is invariant on these subintervals as well. It is proved that these subintervals can be characterized by the corresponding optimal partitions [10, 12].

For CQO, the optimal value function is always piecewise quadratic and the optimal partitions are different in these subintervals [8, 10], as they

[2]The authors want to acknowledge H.J. Greenberg [13] for mentioning this general classification of support set invariancy sensitivity analysis.

are different at the points that separate them. Consequently, Type III sensitivity analysis in CQO problems stands for finding those subintervals that the optimal value function has different quadratic representations.

Support set invariancy sensitivity analysis in LO problems was only considered in [18], but the authors did not give a method to calculate the range of parameter variation.

Recently, Ghaffari and Terlaky [10] investigated support set invariancy sensitivity analysis for LO problems. They referred to the optimality interval in support set invariancy sensitivity analysis as the *Invariant Support Set* (ISS) interval and introduced auxiliary problems that allow to identify the ISS intervals. They also introduced the concept of the *actual* invariancy interval, i.e., the invariancy interval around the current parameter value zero. The relationship between ISS intervals and actual invariancy intervals has been investigated as well.

Let us review briefly the results of support set invariancy sensitivity analysis for LO [10]. The optimal value function is linear on all ISS intervals if variation happens either in the RHS or objective function data. When variation occurs in the RHS data, the interior of the ISS interval is a subset of the actual invariancy interval. In this case, the ISS interval could be open or half-closed if it is not the singleton $\{0\}$. When variation happens in the objective function data and the current point 0 is not a break point of the optimal value function, then the ISS interval coincides with the actual invariancy interval. If zero is a break point of the optimal value function, then the ISS interval coincides with the closure of either the right or the left immediate invariancy intervals. The ISS interval is always closed when variation happens in the objective function data. In case of simultaneous perturbation of the RHS and objective function data, the ISS interval is the intersection of the ISS intervals obtained by the corresponding non-simultaneous perturbations of the RHS and objective function data. ISS intervals can not include more than one invariancy intervals except possibly the end points neighboring to the actual invariancy interval. Support set invariancy and basis invariancy sensitivity analysis coincide when the given optimal solution is a basic solution. Meanwhile, support set invariancy and optimal partition invariancy sensitivity analysis are not identical even if the given pair of primal-dual optimal solutions is strictly complementary.

CQO is a generalization of LO. From the practical perspective, probably the most important application area of parametric CQO is the portfolio selection problem where an optimal solution of CQO problem corresponds to an optimized portfolio of assets. Risk preferences and changes in budget or other constraints affect the optimality of the portfolio. Thus, support set invariancy sensitivity analysis might be interpreted as knowing the range of parameter variation for which the original portfolio configuration remains unchanged, but assets holdings are adjusted. This essentially means that trade should only occur with those assets that initially included in the portfolio.

In spite of some similarities, there are some differences in support set invariancy sensitivity analysis when applied to LO and CQO problems. In this paper, we highlight these differences and present computable algorithms to determine the ISS intervals for CQO. Special cases when support set invariancy sensitivity analysis of CQO problems is specialized as support set invariancy sensitivity analysis of LO problems is also presented. We prefer to maintain the terminology and notation introduced by Ghaffari and Terlaky [10].

The paper is organized as follows. Section 2 includes some basic concepts and notation. Section 3 is devoted to study support set invariancy sensitivity analysis of the perturbed CQO problem. Some fundamental properties of ISS intervals and the optimal value function on ISS intervals are studied. Computable methods that lead to identify the ISS intervals are also presented in this section. We consider three cases: When perturbation occurs in the RHS data and/or in the LTOF. We also investigate the relationship between actual invariancy intervals and ISS intervals. Illustrative examples are presented in Section 4 to display the results in $\mathbb{R}^2$. The closing section contains the summary of our findings, as well as suggestions for future work.

2 Preliminaries

Consider the primal CQO problem as

$$QP \qquad \begin{aligned} \min \ & c^T x + \tfrac{1}{2} x^T Q x \\ \text{s.t.} \ & Ax = b \\ & x \geq 0, \end{aligned}$$

and its Wolfe Dual

$$QD \qquad \begin{aligned} \max \ & b^T y - \tfrac{1}{2} u^T Q u \\ \text{s.t.} \ & A^T y + s - Qu = c \\ & s \geq 0, \end{aligned}$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are fixed data and $x, u, s \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are unknown vectors. Any vector $x \geq 0$ that satisfies $Ax = b$ is called a *primal feasible* *solution* of QP and any (u, y, s) with $u, s \geq 0$ that satisfies $A^T y + s - Qu = c$ is referred to as *dual feasible solution* of QD. Feasible solutions x and (u, y, s) of QP and QD are optimal if and only if $Qx = Qu$ and $x^T s = 0$ [6]. The equation $x^T s = 0$ is equivalent to $x_i s_i = 0$, $i = 1, 2, \ldots, n$ and is known as the *complementary slackness property*. Let $\mathcal{QP}$ and $\mathcal{QP}^*$ denote sets of primal feasible and primal optimal solutions of QP, respectively. Similar notation can be used for sets of feasible and optimal solutions of QD. It is well known [6] that there exist optimal solutions $x^* \in \mathcal{QP}^*$ and $(u^*, y^*, s^*) \in \mathcal{QD}^*$ for which $x^* = u^*$. Since we are concerned only with optimal solutions when $x = u$, we may denote dual feasible solutions by (x, y, s).

The *support* set of a nonnegative vector v is defined as

$$\sigma(v) = \{i : v_i > 0\}.$$

The index set $\{1, 2, \cdots, n\}$ can be partitioned as

$$\begin{aligned}
\mathcal{B} &= \{i : x_i > 0 \text{ for an } x \in \mathcal{QP}^*\}, \\
\mathcal{N} &= \{i : s_i > 0 \text{ for an } (x, y, s) \in \mathcal{QD}^*\}, \\
\mathcal{T} &= \{1, 2, \cdots, n\} \setminus (\mathcal{B} \cup \mathcal{N}) \\
&= \{i : x_i = s_i = 0 \text{ for all } x \in \mathcal{QP}^* \text{ and } (x, y, s) \in \mathcal{QD}^*\}.
\end{aligned}$$

We refer to this partition as the *optimal partition* and denote it by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. The optimal partition π is unique [3]. A *maximally complementary solution* (x, y, s) is a pair of primal-dual optimal solutions of QP and QD for which

$$\begin{aligned}
x_i > 0 &\text{ if and only if } i \in \mathcal{B}, \\
s_i > 0 &\text{ if and only if } i \in \mathcal{N}.
\end{aligned}$$

The existence of maximally complementary solutions is a consequence of the convexity of the optimal solution sets $\mathcal{QP}^*$ and $\mathcal{QD}^*$. IPMs are widely used to solve CQO problems in polynomial time [22] and sufficiently accurate solutions obtained by an IPM can be used to produce maximally complementary solutions [15]. By knowing a maximally complementary solution, one can easily determine the optimal partition as well. If in an optimal partition, the relation $\mathcal{T} = \emptyset$ holds, then any maximally complementary solution is *strictly complementary*. It is worth mentioning that for any pair of primal-dual optimal solutions (x^*, y^*, s^*), the relations $\sigma(x^*) \subseteq \mathcal{B}$ and $\sigma(s^*) \subseteq \mathcal{N}$ hold, and equality at both inclusions hold if and only if the given pair of primal-dual optimal solutions is maximally (strictly) complementary.

The parametric CQO problem can be defined as follows:

$$\begin{array}{rl} \min & (c+\epsilon_c \triangle c)^T x + \frac{1}{2} x^T Q x \\ \text{s.t.} & Ax = b + \epsilon_b \triangle b \\ & x \geq 0, \end{array} \tag{1}$$

where ϵ_b and ϵ_c are two real parameters, $\triangle b \in \mathbb{R}^m$ and $\triangle c \in \mathbb{R}^n$ are nonzero perturbation vectors. When ϵ_b and ϵ_c vary independently, the problem is known as a *multi-parametric quadratic optimization*. There are some studies in multi-parametric optimization in both LO (see e.g., [7]) and CQO (see e.g., [2]). Multi-parametric quadratic optimization is frequently used in control theory (see e.g., [21]). Though, multi-parametric quadratic optimization problems are the most studied one, there are practical problems [9], where the RHS and LTOF data vary simultaneously, and thus one needs to consider the special properties in the case when $\epsilon_b = \epsilon_c$. Therefore, we restrict our attention to a simpler case, when $\epsilon_b = \epsilon_c = \epsilon$. With this assumption, the domain of the optimal value function reduces to one dimension. In this case, the primal and dual perturbed CQO problems are

$$QP(\triangle b, \triangle c, \epsilon) \qquad \begin{array}{rl} \min & (c+\epsilon \triangle c)^T x + \frac{1}{2} x^T Q x \\ \text{s.t.} & Ax = b + \epsilon \triangle b \\ & x \geq 0, \end{array}$$

and

$$QD(\triangle b, \triangle c, \epsilon) \qquad \begin{array}{rl} \max & (b+\epsilon \triangle b)^T y - \frac{1}{2} x^T Q x \\ \text{s.t.} & A^T y + s - Qx = c + \epsilon \triangle c \\ & s \geq 0. \end{array}$$

Let $\mathcal{QP}(\triangle b, \triangle c, \epsilon)$ and $\mathcal{QD}(\triangle b, \triangle c, \epsilon)$ denote the feasible regions of problems $QP(\triangle b, \triangle c, \epsilon)$ and $QD(\triangle b, \triangle c, \epsilon)$, respectively. Moreover, let $\mathcal{QP}^*(\triangle b, \triangle c, \epsilon)$ and $\mathcal{QD}^*(\triangle b, \triangle c, \epsilon)$ be the optimal solution sets of problems $QP(\triangle b, \triangle c, \epsilon)$ and $QD(\triangle b, \triangle c, \epsilon)$, respectively.

It is proved that the optimal value function is piecewise quadratic function of the parameter ϵ and the optimal partition is invariant in these subintervals [3, 8, 9]. The maximal open subinterval where the optimal partition is invariant is called *invariancy interval* and the points that separate adjacent invariancy intervals are referred to as *transition points*. Observe that every transition point as a singleton is a special invariancy interval, because any change in ϵ at a transition point, changes the optimal partition. Due to the convexity of invariancy intervals (see Theorem 4.1 [9]) and since the number of optimal partition are finite (they are different partitions of the finite index set $\{1, 2, \ldots, n\}$), one can easily conclude that the number of transition points is finite, as well. We also need the concept of *actual invariancy interval* that is the invariancy interval which contains zero. We assert that the actual invariancy interval might be the singleton $\{0\}$ if zero is a transition point of the optimal value function.

Let (x^*, y^*, s^*) denote a pair of primal-dual optimal solutions of QP and QD with $\sigma(x^*) = P$. It should be mentioned that these optimal solutions are not necessarily maximally (strictly) complementary nor basic solutions. Further, let $Z = \{1, 2, \ldots, n\} \setminus P$. This way, the partition (P, Z) of the index set $\{1, 2, \ldots, n\}$ is defined. It is obvious that $P \subseteq \mathcal{B}$ and $\mathcal{N} \cup \mathcal{T} \subseteq Z$ when $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ is the optimal partition at $\epsilon = 0$.

We identify the range of the variation for parameter ϵ such that for any ϵ in this range, the perturbed problem $QP(\triangle b, \triangle c, \epsilon)$ still has an optimal solution $x^*(\epsilon)$ with $\sigma(x^*(\epsilon)) = \sigma(x^*) = P$. In other words, we want to have invariant support set for at least one primal optimal solution $x^*(\epsilon)$ for all ϵ values in this interval. We refer to the partition (P, Z) as *the ISS partition* w.r.t. the given optimal solution x^* and if the problem $QP(\triangle b, \triangle c, \epsilon)$ has such a solution, we say that it satisfies the *ISS property*. It will be proved that the set of all ϵ values where the problem $QP(\triangle b, \triangle c, \epsilon)$ satisfies the ISS property is an interval on the real line. This interval is referred to as an *ISS interval* (or the ISS interval w.r.t. the support set P).

Let $\Upsilon_Q(\triangle b, \triangle c)$ denote the set of all ϵ values such that problem $QP(\triangle b, \triangle c, \epsilon)$ satisfies the ISS property, i.e.,

$$\Upsilon_Q(\triangle b, \triangle c) = \{\epsilon | \exists\ x^*(\epsilon) \in \mathcal{QP}^*(\triangle b, \triangle c, \epsilon) \text{ with } \sigma(x^*(\epsilon)) = P\}.$$

Because $x^* \in \mathcal{QP}(\triangle b, \triangle c, 0)$ and $\sigma(x^*) = P$, it follows that $0 \in \Upsilon_Q(\triangle b, \triangle c)$, so the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$ is not empty. Analogous notation is used when $\triangle b$ or $\triangle c$ is zero.

3 Support Set Invariancy Sensitivity Analysis for Convex Quadratic Optimization

In this section we first investigate some general properties of the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$. We also study the optimal value function on them. Further, auxiliary LO problems are presented that allow us to identify the ISS intervals. We consider the simultaneous perturbation of the RHS and LTOF data and then, specialize the results for non-simultaneous perturbation cases.

Let (x^*, y^*, s^*) denote a pair of primal-dual optimal solutions of QP and QD, where $\sigma(x^*) = P$. Considering the ISS partition (P, Z) of the index set $\{1, 2, \ldots, n\}$ for matrices A, Q and vectors c, x and s we have

$$Q = \begin{pmatrix} Q_{PP} & Q_{PZ} \\ Q_{PZ}^T & Q_{ZZ} \end{pmatrix}, \quad A = \begin{pmatrix} A_P, & A_Z \end{pmatrix},$$
$$c = \begin{pmatrix} c_P \\ c_Z \end{pmatrix}, \quad x = \begin{pmatrix} x_P \\ x_Z \end{pmatrix}, \quad \text{and } s = \begin{pmatrix} s_P \\ s_Z \end{pmatrix}. \tag{2}$$

3.1 General Properties

The following lemma talks about a general property of the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$. It shows that $\Upsilon_Q(\triangle b, \triangle c)$ is indeed a convex set, when (P, Z) is the given ISS partition.

Lemma 3.1 *Let problem QP satisfy the ISS property with a pair of primal-dual optimal solutions $x^* \in \mathcal{QP}^*$ and $(x^*, y^*, s^*) \in \mathcal{QD}^*$, where $\sigma(x^*) = P$. Then, $\Upsilon_Q(\triangle b, \triangle c)$ is a convex set.*

Proof: Let a pair of primal-dual optimal solutions $(x^*, y^*, s^*) \in \mathcal{QP}^* \times \mathcal{QD}^*$, where $\sigma(x^*) = P$ be given. For any two $\epsilon_1, \epsilon_2 \in \Upsilon_Q(\triangle b, \triangle c)$, let $(x^*(\epsilon_1), y^*(\epsilon_1), s^*(\epsilon_1))$ and $(x^*(\epsilon_2), y^*(\epsilon_2), s^*(\epsilon_2))$ be given pairs of primal-dual optimal solutions of problems $QP(\triangle b, \triangle c, \epsilon)$ and $QD(\triangle b, \triangle c, \epsilon)$ at ϵ_1 and ϵ_2, respectively. By the assumption we have $\sigma(x^*(\epsilon_1)) = \sigma(x^*(\epsilon_2)) = P$. For any $\epsilon \in (\epsilon_1, \epsilon_2)$, with $\theta = \frac{\epsilon_2 - \epsilon}{\epsilon_2 - \epsilon_1} \in (0, 1)$ we have $\epsilon = \theta\epsilon_1 + (1-\theta)\epsilon_2$. Let us define

$$x^*(\epsilon) = \theta x^*(\epsilon_1) + (1-\theta) x^*(\epsilon_2); \tag{3}$$
$$y^*(\epsilon) = \theta y^*(\epsilon_1) + (1-\theta) y^*(\epsilon_2); \tag{4}$$
$$s^*(\epsilon) = \theta s^*(\epsilon_1) + (1-\theta) s^*(\epsilon_2). \tag{5}$$

It is easy to verify that $x^*(\epsilon) \in \mathcal{QP}(\triangle b, \triangle c, \epsilon)$ and $(x^*(\epsilon), y^*(\epsilon), s^*(\epsilon)) \in \mathcal{QD}(\triangle b, \triangle c, \epsilon)$. Moreover, one can easily verify the optimality property $x^*(\epsilon)^T s^*(\epsilon) = 0$. We also have $\sigma(x^*(\epsilon)) = P$, i.e., $\epsilon \in \Upsilon_Q(\triangle b, \triangle c)$ that completes the proof. $\square$

Lemma 3.1 proves that the ISS set $\Upsilon_Q(\triangle b, \triangle c)$ is an interval on the real line. We refer to $\Upsilon_Q(\triangle b, \triangle c)$ as the *ISS interval* of problem $QP(\triangle b,$

$\triangle c, \epsilon)$ at $\epsilon = 0$. We will see later that the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$ might be open, closed or half-closed.

The following theorem proves that the optimal value function is a quadratic function on ISS intervals. It is established in the case of simultaneous perturbation and is valid in case of non-simultaneous perturbations as well.

Theorem 3.2 *The optimal value function is quadratic on the ISS interval* $\Upsilon_Q(\triangle b, \triangle c)$.

Proof: Let $\phi(\epsilon)$ denote the optimal value function. Let $\overline{\Upsilon}_Q(\triangle b, \triangle c) = [\epsilon_\ell, \epsilon_u]$, where ϵ_ℓ and ϵ_u might be $-\infty$ and $+\infty$, respectively, and $\overline{\Upsilon}_Q(\triangle b, \triangle c)$ denotes the closure of the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$. If $\epsilon_\ell = \epsilon_u$ the statement is trivial, so we may assume that $\epsilon_\ell < \epsilon_u$. Let $\epsilon_\ell < \epsilon_1 \leq \epsilon \leq \epsilon_2 < \epsilon_u$ be given and let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be pairs of primal-dual optimal solutions corresponding to ϵ_1 and ϵ_2, respectively, where $\sigma(x^{(1)}) = \sigma(x^{(2)}) = P$. Due to Lemma 3.1, for any $\epsilon \in (\epsilon_1, \epsilon_2)$, there is a pair of primal-dual optimal solutions $(x(\epsilon), y(\epsilon), s(\epsilon)) \in \mathcal{QP}^*(\triangle b, \triangle c, \epsilon)$ such that $x(\epsilon) = (x_P(\epsilon)^T, x_Z(\epsilon)^T)^T$ with $\sigma(x(\epsilon)) = P$ and $x_Z(\epsilon) = 0$. Moreover, we have $s(\epsilon) = (s_P(\epsilon)^T, s_Z(\epsilon)^T)^T$ where $s_P(\epsilon) = 0$ and $s_Z(\epsilon) \geq 0$. Thus, for $\theta = \frac{\epsilon - \epsilon_1}{\epsilon_2 - \epsilon_1} \in (0, 1)$ we have

$$\begin{aligned} \epsilon &= \epsilon_1 + \theta \triangle \epsilon, \\ x_P(\epsilon) &= x_P^{(1)} + \theta \triangle x_P = x_P^{(1)} + \frac{\epsilon - \epsilon_1}{\epsilon_2 - \epsilon_1} \triangle x_P, \\ y(\epsilon) &= y^{(1)} + \theta \triangle y = y^{(1)} + \frac{\epsilon - \epsilon_1}{\epsilon_2 - \epsilon_1} \triangle y, \\ s_Z(\epsilon) &= s_Z^{(1)} + \theta \triangle s_Z = s_Z^{(1)} + \frac{\epsilon - \epsilon_1}{\epsilon_2 - \epsilon_1} \triangle s_Z, \end{aligned}$$

where $\triangle \epsilon = \epsilon_2 - \epsilon_1$, $\triangle x_P = x_P^{(2)} - x_P^{(1)}$, $\triangle y = y^{(2)} - y^{(1)}$ and $\triangle s_Z = s_Z^{(2)} - s_Z^{(1)}$. Further, we have $x_Z(\epsilon) = 0$ and $s_P(\epsilon) = 0$. We also have

$$A_P \triangle x_P = \triangle \epsilon \triangle b, \tag{6}$$
$$A_P^T \triangle y - Q_{PP} \triangle x_P = \triangle \epsilon \triangle c_P. \tag{7}$$

The optimal value function at ϵ is given by

$$\begin{aligned} \phi(\epsilon) &= (b + \epsilon \triangle b)^T y(\epsilon) - \tfrac{1}{2} x_P(\epsilon)^T Q_{PP} x_P(\epsilon) \\ &= (b + (\epsilon_1 + \theta \triangle \epsilon) \triangle b)^T (y^{(1)} + \theta \triangle y) \\ &\quad - \tfrac{1}{2} (x_P^{(1)} + \theta \triangle x_P)^T Q_{PP} (x_P^{(1)} + \theta \triangle x_P) \\ &= (b + \epsilon_1 \triangle b)^T y^{(1)} + \theta (\triangle \epsilon \triangle b^T y^{(1)} + (b + \epsilon_1 \triangle b)^T \triangle y) \\ &\quad + \theta^2 \triangle \epsilon \triangle b^T \triangle y - \tfrac{1}{2} x_P^{(1)^T} Q_{PP} x_P^{(1)} - \theta x_P^{(1)^T} Q_{PP} \triangle x_P \\ &\quad - \tfrac{1}{2} \theta^2 \triangle x_P^T Q_{PP} \triangle x_P. \end{aligned} \tag{8}$$

From equations (6) and (7), we have

$$\triangle x_P^T Q_{PP} \triangle x_P = \triangle\epsilon(\triangle b^T \triangle y - \triangle c_P^T \triangle x_P), \tag{9}$$
$$x_P^{(1)^T} Q_{PP} \triangle x_P = (b + \epsilon_1 \triangle b)^T \triangle y - \triangle\epsilon \triangle c_P^T x_P^{(1)}. \tag{10}$$

Substituting (9) and (10) into (8) we obtain

$$\begin{aligned}\phi(\epsilon) &= \phi(\epsilon_1 + \theta\triangle\epsilon)\\ &= \phi(\epsilon_1) + \theta\triangle\epsilon(\triangle b^T y^{(1)} + \triangle c_P^T x_P^{(1)}) + \tfrac{1}{2}\theta^2\triangle\epsilon(\triangle c_P^T \triangle x_P + \triangle b^T \triangle y),\end{aligned}$$

that by using the notation

$$\begin{aligned}\gamma_1 &= \triangle b^T y^{(1)} + \triangle c_P^T x_P^{(1)},\\ \gamma_2 &= \triangle b^T y^{(2)} + \triangle c_P^T x_P^{(2)},\\ \gamma &= \frac{\gamma_2 - \gamma_1}{\epsilon_2 - \epsilon_1} = \frac{\triangle c_P^T \triangle x_P + \triangle b^T \triangle y}{\epsilon_2 - \epsilon_1},\end{aligned}$$

can be rewritten as

$$\phi(\epsilon) = (\phi(\epsilon_1) - \epsilon_1\gamma_1 + \frac{1}{2}\epsilon_1^2\gamma) + (\gamma_1 - \epsilon_1\gamma)\epsilon + \frac{1}{2}\gamma\epsilon^2. \tag{11}$$

Because ϵ_1 and ϵ_2 are two arbitrary elements from the interval $(\epsilon_\ell, \epsilon_u)$, the claim of the theorem follows directly from (11) and the continuity of the optimal value function [8, 9]. The proof is complete. □

Remark 3.3 *It is known that the optimal value function is a continuous piecewise quadratic function and it has unique representations on each invariancy intervals* [3, 8, 9]. *By using Theorem 3.2 one can easily conclude that an ISS interval can not cover more than one (actual) invariancy intervals with the exception that an ISS interval might be a non-singleton invariancy interval augmented with its end point(s).*

3.2 Identifying the IACS Intervals

Let us assume that both vectors $\triangle b$ and $\triangle c$ are not zero and consider the general perturbed primal and dual CQO problems $QP(\triangle b, \triangle c, \epsilon)$ and $QD(\triangle b, \triangle c, \epsilon)$. Let (x^*, y^*, s^*) be a pair of primal-dual optimal solutions of QP and QD with $\sigma(x^*) = P$. We want to identify the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$ w.r.t. the support set $\sigma(x^*) = P$.

The following theorem presents a computational method to determine ϵ_ℓ and ϵ_u, i.e., the extreme points of $\overline{\Upsilon}_Q(\triangle b, \triangle c)$.

Theorem 3.4 *Let (x^*, y^*, s^*) be a pair of primal-dual optimal solutions of QP and QD with $\sigma(x^*) = P$. Then, ϵ_ℓ and ϵ_u can be obtained by solving the following two auxiliary LO problems:*

$$\epsilon_\ell = \min_{\epsilon, x_P, y, s_Z} \{\epsilon : A_P x_P - \epsilon \triangle b = b, A_P^T y - Q_{PP} x_P - \epsilon \triangle c_P = c_p,\ x_P \geq 0,$$
$$A_Z^T y + s_Z - Q_{PZ}^T x_P - \epsilon \triangle c_Z = c_Z,\ s_Z \geq 0\}, \tag{12}$$

$$\epsilon_u = \max_{\epsilon, x_P, y, s_Z} \{\epsilon : A_P x_P - \epsilon \triangle b = b, A_P^T y - Q_{PP} x_P - \epsilon \triangle c_P = c_p,\ x_P \geq 0,$$
$$A_Z^T y + s_Z - Q_{PZ}^T x_P - \epsilon \triangle c_Z = c_Z,\ s_Z \geq 0\}. \tag{13}$$

Proof: Using the ISS partition (P, Z), we have $x^* = ({x_P^*}^T, {x_Z^*}^T)^T$ and $s^* = ({s_P^*}^T, {s_Z^*}^T)^T$ with $x_P^* > 0$, $x_Z^* = 0$, $s_P^* = 0$ and $s_Z^* \geq 0$.

First we prove that $(\epsilon_\ell, \epsilon_u) \subseteq \text{int}(\Upsilon_Q(\triangle b, \triangle c))$. If $\epsilon_\ell = \epsilon_u = 0$, then $\Upsilon_Q(\triangle b, \triangle c) = \{0\}$ and inclusion holds trivially. Let us consider that at least one of ϵ_ℓ and ϵ_u is not zero and $\epsilon \in (\epsilon_\ell, \epsilon_u)$ be arbitrary. We need to prove that, there is a primal optimal solution $x^*(\epsilon) \in \mathcal{QP}^*(\triangle b, \triangle c, \epsilon)$ with $\sigma(x^*(\epsilon)) = P$. Without loss of generality, one can consider that $\epsilon_\ell < \epsilon < 0 \leq \epsilon_2$. In this case, there is an $\bar{\epsilon}$ such that $\epsilon_\ell \leq \bar{\epsilon} < \epsilon < 0$. Let $(\bar{\epsilon}, x_P^*(\bar{\epsilon}), y^*(\bar{\epsilon}), s_Z^*(\bar{\epsilon}))$ be a feasible solution of problems (12). Thus, the relation $\sigma(x_P^*(\bar{\epsilon})) \subseteq P$ holds. Let us define

$$x^*(\epsilon) = ((\theta x_P^* + (1-\theta) x_P^*(\bar{\epsilon}))^T, 0^T)^T; \tag{14}$$
$$y^*(\epsilon) = \theta y^* + (1-\theta) y^*(\bar{\epsilon}); \tag{15}$$
$$s^*(\epsilon) = (0^T, (\theta s_Z^* + (1-\theta) s_Z^*(\bar{\epsilon}))^T)^T, \tag{16}$$

where $\theta = 1 - \frac{\epsilon}{\bar{\epsilon}} \in (0, 1)$. One can easily verify that $Ax^*(\epsilon) = A_P x_P^*(\epsilon) = b + \epsilon \triangle b$ and $A^T y^*(\epsilon) + s^*(\epsilon) - Qx^*(\epsilon) = c + \epsilon \triangle c$. On the other hand, (14) and (16) show that $x^*(\epsilon)$ and $s^*(\epsilon)$ are complementary and thus the duality gap is zero. It is obvious that $\sigma(x^*(\epsilon)) = \sigma(x_P^*(\epsilon)) = \sigma(x_P^*(\bar{\epsilon})) \cup \sigma(x_P^*) = P$. Thus $(\epsilon_\ell, \epsilon_u) \subseteq \text{int}(\Upsilon_Q(\triangle b, \triangle c))$.

Now we prove that $\text{int}(\Upsilon_Q(\triangle b, \triangle c)) \subseteq (\epsilon_\ell, \epsilon_u)$. Let us assume to the contrary that there is a nonzero $\bar{\epsilon} \in \text{int}(\Upsilon_Q(\triangle b, \triangle c))$ with $\bar{\epsilon} < \epsilon_\ell$. Let $(x^*(\bar{\epsilon}), y^*(\bar{\epsilon}), s^*(\bar{\epsilon}))$ be a pair of primal-dual optimal solutions of problems $QP(\triangle b, \triangle c, \bar{\epsilon})$ and $QD(\triangle b, \triangle c, \bar{\epsilon})$ with $\sigma(x^*(\bar{\epsilon})) = P$. Thus, $x^*(\bar{\epsilon})$ and $s^*(\bar{\epsilon})$ can be partitioned as $x^*(\bar{\epsilon}) = (x_P^*(\bar{\epsilon})^T, x_Z^*(\bar{\epsilon})^T)^T$ and $s^*(\bar{\epsilon}) = (s_P^*(\bar{\epsilon})^T, s_Z^*(\bar{\epsilon})^T)^T$, respectively, where $x_Z^*(\bar{\epsilon}) = 0$, $s_P^*(\bar{\epsilon}) = 0$ and $s_Z^*(\bar{\epsilon}) \geq 0$. It is easy to verify that $(\bar{\epsilon}, x_P^*(\bar{\epsilon}), y^*(\bar{\epsilon}), s_Z^*(\bar{\epsilon}))$ satisfies the constraints of problem (12), while $\bar{\epsilon} < \epsilon_\ell$, that contradicts to the optimality of ϵ_ℓ. Thus, $\text{int}(\Upsilon_Q(\triangle b, \triangle c)) \subseteq (\epsilon_\ell, \epsilon_u)$ that completes the proof. □

In applying Theorem 3.4, if $\epsilon_\ell = \epsilon_u = 0$, then it is impossible to perturb the vector c in the given perturbing direction $\triangle c$ while keeping the ISS property of $QP(\triangle b, \triangle c, \epsilon)$ w.r.t. the support set $\sigma(x^*) = P$. On the other hand, if the auxiliary LO problem (12) is unbounded then the lower bound of the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$ is $-\infty$. Analogous conclusion is valid when the auxiliary LO problem (13) is unbounded.

When $\epsilon_\ell = 0$ (or $\epsilon_u = 0$) then the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$ is obviously half-closed. Let us consider the case when $\epsilon_\ell < 0$ and $(\epsilon_\ell, x_P^*(\epsilon_\ell), y^*(\epsilon_\ell), s_Z^*(\epsilon_\ell))$ is an optimal solution of problem (12). It is obvious that $\sigma(x_P^*(\epsilon_\ell)) \subseteq P$ and the equality holds if and only if $\epsilon_\ell \in \Upsilon_Q(\triangle b, \triangle c)$. Analogous discussion is valid for ϵ_u. The following corollary summarizes these discussions and presents the closedness conditions of the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$.

Corollary 3.5 *Let $(\epsilon_\ell, x_P^*(\epsilon_\ell), y^*(\epsilon_\ell), s_Z^*(\epsilon_\ell))$ and $(\epsilon_u, x_P^*(\epsilon_u), y^*(\epsilon_u), s_Z^*(\epsilon_u))$ be optimal solutions of problems (12) and (13), respectively. If $\sigma(x_P^*(\epsilon_\ell)) = P$ (or $\sigma(x_P^*(\epsilon_u)) = P$), then $\epsilon_\ell \in \Upsilon_Q(\triangle b, \triangle c)$ (or $\epsilon_u \in \Upsilon_Q(\triangle b, \triangle c)$).*

Remark 3.6 *Let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ be the optimal partition of problem QP. If the given pair of primal-dual optimal solutions (x^*, y^*, s^*) of problems QP and QD is a maximally complementary solution, then relations $P = \mathcal{B}$ and $Z = \mathcal{N} \cup \mathcal{T}$ hold. Without loss of generality, one can assume that (ϵ_1, ϵ_2) is the actual invariancy interval of problem $QP(\triangle b, \triangle c, \epsilon)$. Let $\pi_1 = (\mathcal{B}_1, \mathcal{N}_1, \mathcal{T}_1)$ be the optimal partition at ϵ_1. Since the optimal partition changes at break points, we have $\pi \neq \pi_1$. Meanwhile, we may have a situation that $\mathcal{B}$ stays unchanged but $\mathcal{N}$ and $\mathcal{T}$ change, i.e., $\mathcal{B} = \mathcal{B}_1$, $\mathcal{N} \neq \mathcal{N}_1$ and $\mathcal{T} \neq \mathcal{T}_1$. Even, it might be instances that the $\mathcal{B}$ part of the optimal partition changes while we still have a primal optimal solution with the desired support set. Thus, there might be an optimal solution $x^*(\epsilon_1) \in \mathcal{QP}^*(\triangle b, \triangle c, \epsilon_1)$ with $\sigma(x^*(\epsilon_1)) = P$. Analogous argument is valid for the transition point ϵ_2 (see Example 1). In this case the actual invariancy interval is a strict subset of the ISS interval. This observation implies that support set invariancy and optimal partition invariancy sensitivity analysis (see* [10, 16] *for details) are not necessarily identical when the given pair of primal-dual optimal solutions is maximally complementary. This situation might happen even when the given pair of primal-dual optimal solutions is strictly complementary (see Example 1).*

Remark 3.7 *It is obvious that when $Q = 0$ or when $Q \neq 0$ but $Q_{PP} = 0$ and $Q_{PZ} = 0$, the two auxiliary problems (12) and (13) reduce to the following problems:*

$$\epsilon_\ell = \min_{\epsilon, y, s_Z} \{\epsilon : A_P^T y - \epsilon \triangle c_P = c_P, \ A_Z^T y + s_Z - \epsilon \triangle c_Z = c_Z, \ s_Z \geq 0\},$$
$$\epsilon_u = \max_{\epsilon, y, s_Z} \{\epsilon : A_P^T y - \epsilon \triangle c_P = c_P, \ A_Z^T y + s_Z - \epsilon \triangle c_Z = c_Z, \ s_Z \geq 0\}.$$

These two problems are the same two auxiliary LO problems that were introduced as problems (8) and (9) in [10] *for the LO case. It means that in these cases, the results obtained in the CQO case, are specialized to the LO case.*

Remark 3.8 *When the given pair of primal-dual optimal solutions is maximally (strictly) complementary, then* $P = \mathcal{B}$ *and* $Z = \mathcal{N} \cup \mathcal{T}$. *In this case problems (12) and (13) have the following form:*

$$\epsilon_\ell = \min_{\epsilon, x_\mathcal{B}, y, s_Z} \{\epsilon : A_\mathcal{B} x_\mathcal{B} - \epsilon \triangle b = b, A_\mathcal{B}^T y - Q_{\mathcal{B}\mathcal{B}} x_\mathcal{B} - \epsilon \triangle c_\mathcal{B} = c_\mathcal{B},\ x_\mathcal{B} \geq 0, \\ A_Z^T y + s_Z - Q_{\mathcal{B}Z}^T x_\mathcal{B} - \epsilon \triangle c_Z = c_Z,\ s_Z \geq 0\}, \tag{17}$$

$$\epsilon_u = \max_{\epsilon, x_\mathcal{B}, y, s_Z} \{\epsilon : A_\mathcal{B} x_\mathcal{B} - \epsilon \triangle b = b, A_\mathcal{B}^T y - Q_{\mathcal{B}\mathcal{B}} x_\mathcal{B} - \epsilon \triangle c_\mathcal{B} = c_\mathcal{B},\ x_\mathcal{B} \geq 0, \\ A_Z^T y + s_Z - Q_{\mathcal{B}Z}^T x_\mathcal{B} - \epsilon \triangle c_Z = c_Z,\ s_Z \geq 0\}, \tag{18}$$

where $Q_{\mathcal{B}Z} = (\, Q_{\mathcal{B}\mathcal{N}} \quad Q_{\mathcal{B}\mathcal{T}}\,)$. *On the other hand, the two auxiliary problems to identify the actual invariancy interval (see Theorem 4.4 in* [9]*) are*

$$\epsilon_1 = \min_{\epsilon, x_\mathcal{B}, y, s_Z} \{\epsilon : A_\mathcal{B} x_\mathcal{B} - \epsilon \triangle b = b, A_\mathcal{B}^T y - Q_{\mathcal{B}\mathcal{B}} x_\mathcal{B} - \epsilon \triangle c_\mathcal{B} = c_\mathcal{B}, \\ A_\mathcal{N}^T y + s_\mathcal{N} - Q_{\mathcal{N}\mathcal{B}}^T x_\mathcal{B} - \epsilon \triangle c_\mathcal{N} = c_\mathcal{N}, \\ A_\mathcal{T}^T y - Q_{\mathcal{T}\mathcal{B}}^T x_\mathcal{B} - \epsilon \triangle c_\mathcal{T} = c_\mathcal{T},\ x_\mathcal{B} \geq 0,\ s_\mathcal{N} \geq 0\}, \tag{19}$$

and

$$\epsilon_2 = \max_{\epsilon, x_\mathcal{B}, y, s_Z} \{\epsilon : A_\mathcal{B} x_\mathcal{B} - \epsilon \triangle b = b, A_\mathcal{B}^T y - Q_{\mathcal{B}\mathcal{B}} x_\mathcal{B} - \epsilon \triangle c_\mathcal{B} = c_\mathcal{B}, \\ A_\mathcal{N}^T y + s_\mathcal{N} - Q_{\mathcal{N}\mathcal{B}}^T x_\mathcal{B} - \epsilon \triangle c_\mathcal{N} = c_\mathcal{N}, \\ A_\mathcal{T}^T y - Q_{\mathcal{T}\mathcal{B}}^T x_\mathcal{B} - \epsilon \triangle c_\mathcal{T} = c_\mathcal{T},\ x_\mathcal{B} \geq 0,\ s_\mathcal{N} \geq 0\}. \tag{20}$$

Observe, that the feasible region of problems (19) and (20) are subsets of the feasible regions of problems (17) and (18), respectively. Thus $\epsilon_\ell \leq \epsilon_1$ *and* $\epsilon_2 \leq \epsilon_u$. *It means that when the given pair of primal-dual optimal solutions is maximally (strictly) complementary solutions, then the actual invariancy interval is a subset of the ISS interval.*

Remark 3.9 *Comparing problems (12) and (13) with problems (19) and (20), respectively, shows that if the given pair of primal-dual optimal solutions is not a maximally (strictly) complementary solution, then the ISS interval might be included in the actual invariancy interval.*

Remark 3.10 *If the given pair of primal-dual optimal solutions* (x^*, y^*, s^*) *is a primal nondegenerate basic solution, then basis invariancy and support set invariancy sensitivity analysis are identical.*

Let us summarize the results so far. We have proved that ISS intervals are convex and the optimal value function is a continuous quadratic function on these intervals. Moreover, we have presented two auxiliary problems to identify the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$. The ISS interval $\Upsilon_Q(\triangle b, \triangle c)$ might be open, closed or half-closed, when it is not the singleton $\{0\}$, and closedness conditions for this interval were provided. Support set invariancy and optimal partition invariancy sensitivity analysis are not necessarily identical when the given pair of primal-dual optimal solutions (x^*, y^*, s^*) is maximally (strictly) complementary. We refer to Examples 1 and 2 as an illustration of the results in $\mathbb{R}^2$.

In the rest of this section, we specialize the results for non-simultaneous perturbation cases, when either $\triangle c$ or $\triangle b$ is a zero vector.

• **The case $\triangle \mathbf{c} = \mathbf{0}$:** In this case, We want to identify the ISS interval $\Upsilon_Q(\triangle b, 0)$ w.r.t. the given ISS partition (P, Z). The following theorem is the specialization of Theorem 3.4 offering two auxiliary LO problems that enable us to identify the ISS interval $\Upsilon_Q(\triangle b, 0)$. The proof is similar to the proof of Theorem 3.4 and it is omitted.

Theorem 3.11 *Let* (x^*, y^*, s^*) *be a pair of primal-dual optimal solutions of* QP *and* QD*, where* $\sigma(x^*) = P$*. Then* $\overline{\Upsilon}_Q(\triangle b, 0) = [\epsilon_\ell, \epsilon_u]$*, where*

$$\begin{aligned}\epsilon_\ell = \min_{\epsilon, x_P, y, s_Z} \{\epsilon : \; & A_P x_P - \epsilon \triangle b = b,\; x_P \geq 0,\; A_P^T y - Q_{PP} x_P = c_P \\ & A_Z^T y + s_Z - Q_{PZ}^T x_P = c_P,\; s_Z \geq 0\},\end{aligned} \tag{21}$$

$$\begin{aligned}\epsilon_u = \max_{\epsilon, x_P, y, s_Z} \{\epsilon : \; & A_P x_P - \epsilon \triangle b = b,\; x_P \geq 0,\; A_P^T y - Q_{PP} x_P = c_P \\ & A_Z^T y + s_Z - Q_{PZ}^T x_P = c_P,\; s_Z \geq 0\}.\end{aligned} \tag{22}$$

• **The case $\triangle \mathbf{b} = \mathbf{0}$:** In this case, We want to identify the ISS interval $\Upsilon_Q(0, \triangle c)$ w.r.t. the given ISS partition (P, Z). The following theorem is the specialization of Theorem 3.4 offering two auxiliary LO problems that enable us to identify the ISS interval $\Upsilon_Q(0, \triangle c)$. The proof is similar to the proof of Theorem 3.4 and it is omitted.

Theorem 3.12 *Let* (x^*, y^*, s^*) *be a pair of primal-dual optimal solutions* *of QP and QD, where* $\sigma(x^*) = P$. *Then* $\overline{\Upsilon}_Q(0, \triangle c) = [\epsilon_\ell, \epsilon_u]$, *where*

$$\epsilon_\ell = \min_{\epsilon, x_P, y, s_Z} \{\epsilon : A_P x_P = b,\ A_P^T y - Q_{PP} x_P - \epsilon \triangle c_P = c_P,\ x_P \geq 0,$$
$$A_Z^T y + s_Z - Q_{PZ}^T x_P - \epsilon \triangle c_Z = c_Z,\ s_Z \geq 0\}, \tag{23}$$

$$\epsilon_u = \max_{\epsilon, x_P, y, s_Z} \{\epsilon : A_P x_P = b,\ A_P^T y - Q_{PP} x_P - \epsilon \triangle c_P = c_P,\ x_P \geq 0,$$
$$A_Z^T y + s_Z - Q_{PZ}^T x_P - \epsilon \triangle c_Z = c_Z,\ s_Z \geq 0\}. \tag{24}$$

Remark 3.13 *It should be mentioned that, unlike the LO case when the ISS interval* $\Upsilon_L(0, \triangle c)$ *is always a closed interval (see Theorems 2.10 and 2.11 in* [10]*), due to Corollary 3.5,* $\Upsilon_Q(0, \triangle c)$ *might be an open, closed or half-closed interval.*

Remark 3.14 *It should be mentioned that the relation* $\Upsilon_L(\triangle b, \triangle c) = \Upsilon_L(\triangle b, 0) \cap \Upsilon_L(0, \triangle c)$ *holds in the LO case (see Theorem 2.12 in* [10]*). This relation does not necessarily hold in the case of CQO, because the constraints in problems (12) and (13) are not independent in their variables, and consequently these two problems could not be separated into two pairs of such problems as (21),(22) and (23),(24).*

In this subsection we have investigated support set invariancy sensitivity analysis for CQO problems when the RHS and LTOF data change simultaneously and he specialized the results for nonsimult- aneous cases. Auxiliary LO problems are presented that allow us to identify the ISS interval $\Upsilon_Q(\triangle b, \triangle c)$ in all cases. We also have emphasized the differences of the results in performing support set invariancy sensitivity analysis between LO and CQO. For an illustrative example of the results in $\mathbb{R}^2$, we refer to examples presented in the following section.

4 Illustrative Examples

In this section, we apply the methods derived in the previous section for some simple examples to illustrate support set invariancy sensitivity analysis and the behavior of the optimal value function. Example 1 shows that support set invariancy and optimal partition invariancy sensitivity analysis are not identical even if the given pair of primal-dual optimal solutions is maximally (strictly) complementary. Examples 2 and 3 are intentionally designed in $\mathbb{R}^2$ in order to have an illustrative picture of the feasible regions as well as the optimal solution sets.

Example 1: Consider the following CQO problem

$$\begin{array}{llllllllll} \min & -10x_1 & -16x_2 & +x_1^2 & +x_1x_2 & +x_2^2 & & & & \\ \text{s.t.} & 2x_1 & +x_2 & & +x_4 & +x_5 & & +x_7 & = 20 & \\ & 2x_1 & +2x_2 & +x_3 & & +x_5 & +x_6 & +x_7 & = 20 & \\ & x_1 & +5x_2 & +x_3 & +x_4 & & & +x_7 & = 20 & \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq 0, & \end{array} \tag{25}$$

with perturbation vectors $\triangle b = (1,0,0)^T$ and $\triangle c = (1,-1,-1,0,1,0,1)^T$. It is easy to determine that $\epsilon = 0$ is a transition point of the optimal value function and thus, the actual invariancy interval is the singleton $\{0\}$. The optimal partition at $\epsilon = 0$ is $\pi(0) = (\mathcal{B}(0), \mathcal{N}(0), \mathcal{T}(0))$, where

$$\mathcal{B}(0) = \{1,2,5\}\,, \quad \mathcal{N}(0) = \{3,6,7\} \quad \text{and } \mathcal{T}(0) = \{4\}\,.$$

Further, $x^*(0) = (\frac{10}{3}, \frac{10}{3}, 0, 0, 10, 0, 0)^T$, $y^*(0) = (1,-1,-1)^T$ and $s^*(0) = (0,0,2,0,0,1,1)^T$ is a maximally complementary optimal solution of problem (25). Thus, the support set of the given optimal solution $x^*(0)$ is $P = \mathcal{B}(0) = \{1,2,5\}$. The left immediate invariancy interval is $(-1.5, 0)$. Let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ denote the optimal partition on this invariancy interval. It is easy to see that

$$\mathcal{B} = \{1,2,5\}\,, \quad \mathcal{N} = \{3,4,6,7\} \text{ and } \mathcal{T} = \emptyset,$$

and the problem has a strictly complementary optimal solution for any $\epsilon \in (-1.5, 0)$. It means that for any $\epsilon \in (-1.5, 0)$ the perturbed problem has an optimal solution $x^*(\epsilon)$ with $\sigma(x^*(\epsilon)) = P$. For example, for $\epsilon = -1$, $(x^*(-1), y^*(-1), s^*(-1))$ is a strictly complementary optimal solution, where

$$\begin{aligned} x^*(-1) &= (2.5, 3.5, 0, 0, 10.5, 0, 0)^T, \\ y^*(-1) &= (-\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})^T, \\ s^*(-1) &= (0, 0, \frac{13}{6}, \frac{3}{2}, 0, \frac{1}{3}, \frac{5}{6})^T. \end{aligned}$$

On the other hand, the optimal partition at the transition point $\epsilon = -1.5$ is

$$\mathcal{B}(-1.5) = \{1,2,5\}\,, \quad \mathcal{N}(-1.5) = \{3,4,7\} \quad \text{and } \mathcal{T}(-1.5) = \{6\}\,,$$

and the perturbed problem has an optimal solution $x^*(-1.5) = (\frac{25}{12}, \frac{43}{12}, 0, 0, \frac{43}{4}, 0, 0)^T$ with $y^*(-1.5) = (-1.5, 0. - 0.75)^T$ and $s^*(-1.5) = (0, 0,$

$2.25, 2.25, 0, 0, .75)^T$ that is a maximally complementary optimal solution at the transition point $\epsilon = -1.5$. One can determine from Theorem 3.4 and Corollary 3.5 that $\Upsilon_Q(\triangle b, \triangle c) = [-1.5, 0]$.

This example shows that support set invariancy and optimal partition invariancy sensitivity analysis in CQO are not identical when the given pair of optimal-dual optimal solutions is maximally complementary (see Remark 3.6). Observe that the actual invariancy interval is the singleton$\{0\}$, because zero is a transition point of the optimal value function in this case. Therefore, the ISS interval is

$$\Upsilon_Q(\triangle b, 0) = [-1.5, 0] = \{-1.5\} \cup (-1.5, 0) \cup \{0\},$$

that is the union of three adjacent invariancy intervals.

The following example shows the case when perturbation occurs only in the RHS data and the CQO problem has multiple optimal solutions.

Example 2: Consider the following standard primal CQO problem

$$\begin{array}{llllllll} \min & x_1^2 & +2x_1x_2 & +x_2^2 & & & & \\ \text{s.t.} & x_1 & +x_2 & -x_3 & & & = 2 & \\ & & x_2 & & +x_4 & & = 2 & \\ & -x_1 & +x_2 & & & +x_5 & = 2 & \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0. & \end{array} \tag{26}$$

Let $\triangle b = (1, -1, -2)^T$ with $\triangle c = 0$ be the perturbing direction. The feasible region of problem (26) in the space of two variables x_1 and x_2 is depicted in Figure 1. It is easy to verify that problem (26) has multiple optimal solutions. In Figure 1, all points on the dashed line segment $x_1 + x_2 = 2,\ x_1 \geq 0,\ x_2 \geq 0$ are optimal. The actual invariancy interval of this problem is $(-2, 2)$. One can categorize the optimal solutions of this problem in three classes as follows:

- **Case 1. Primal optimal nondegenerate basic solution**, such as $x^{(1)} = (2, 0, 0, 2, 4)^T$. For this given optimal solution we have

  $$P = \{1, 4, 5\}, \qquad Z = \{2, 3\},$$

  and the following partition is defined for Matrix Q,

  $$Q_{PP} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ Q_{ZP}^T = Q_{PZ} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Q_{ZZ} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

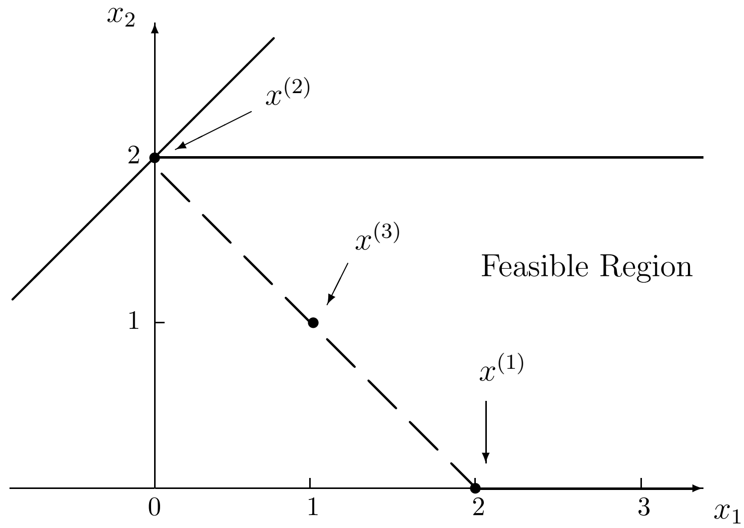

Figure 1: The feasible region and optimal solution set of Example 2.

By solving the corresponding two auxiliary LO problems (21) and (22), one can identify the ISS interval as $(-2, 2)$. For any given $\epsilon \in (-2, 2)$, say $\epsilon = 1$, the perturbed problem has an optimal solution, i.e., as $x^*(1) = (3, 0, 0, 1, 3)^T$ with $\sigma(x^*(1)) = P$ (see Figure 2). In this case, the ISS interval $\Upsilon_Q(\triangle b, 0)$ and the actual invariancy interval coincide.

- **Case 2. Primal optimal degenerate basis solution**, such as $x^{(2)} = (0, 2, 0, 0, 0)^T$. For this optimal solution we have

$$P = \{1\}, \qquad Z = \{2, 3, 4, 5\},$$

and therefore, we have the following partition for matrix Q,

$$Q_{PP} = \begin{bmatrix} 2 \end{bmatrix}, Q_{ZP}^T = Q_{PZ} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix} \text{ and } Q_{ZZ} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, the ISS Interval $\Upsilon_Q(\triangle b, 0)$ is the singleton $\{0\}$. Thus, for the given optimal solution, there is no room to perturb the RHS of the constraints in the specified perturbing direction $\triangle b$ while keeping the ISS property of the problem. Observe that the

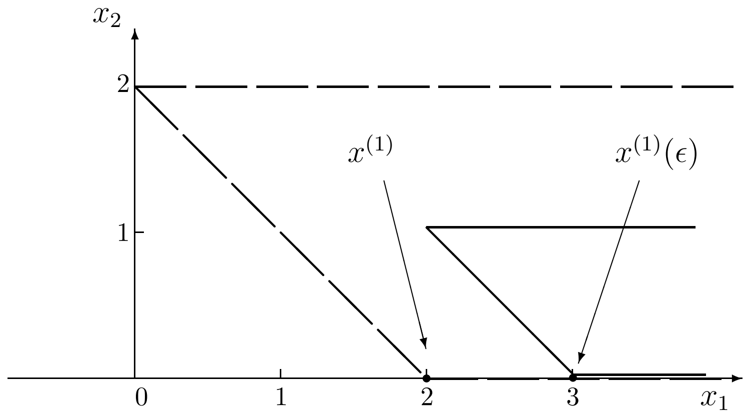

Figure 2: Case 1: Optimal solutions before and after perturbation with $\epsilon = 1$ in Example 2.

ISS interval $\Upsilon_Q(\triangle b, 0)$ is a subset of the actual invariancy interval in this case.

- **Case 3. Strictly complementary optimal solution**, such as $x^{(3)} = (1, 1, 0, 1, 2)^T$. For the given optimal solution

$$P = \{1, 2, 4, 5\}, \qquad Z = \{3\}.$$

This ISS partition defines the following partitioning for matrix Q:

$$Q_{PP} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_{ZP}^T = Q_{PZ} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } Q_{ZZ} = \begin{bmatrix} 0 \end{bmatrix}.$$

Observe that the ISS interval $\Upsilon_Q(\triangle b, 0)$ of the main perturbed problem is $(-2, 2)$. For any $\epsilon \in (-2, 2)$, say $\epsilon = -1$, $x^*(-1) = (\frac{2}{3}, \frac{1}{3}, 0, \frac{8}{3}, \frac{1}{3})^T$ with $\sigma(x^*(-1)) = P$ can be an optimal solution of the perturbed problem (see Figure 3).

The following example designed to show the results of support set invariancy sensitivity analysis when $\triangle b = 0$ and the problem has multiple optimal solutions.

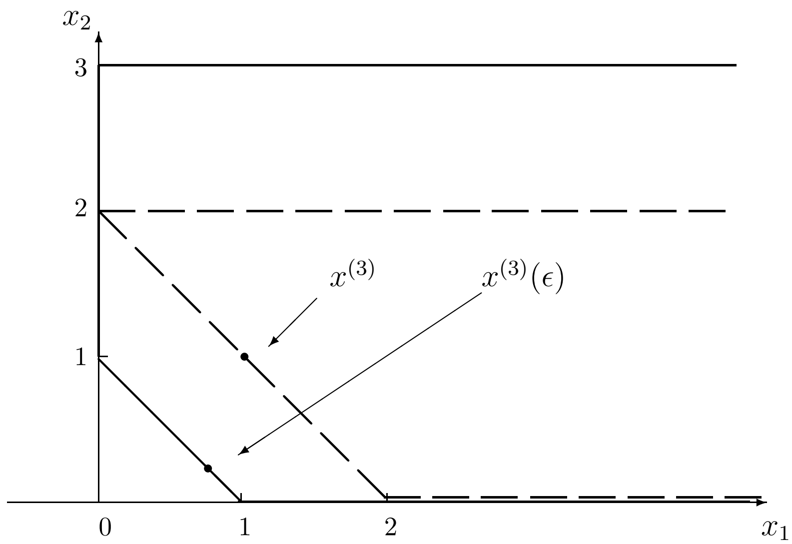

Figure 3: Case 3: Optimal solutions before and after perturbation with $\epsilon = -1$ in Example 2.

Example 3: Consider the following problem.

$$
\begin{array}{lllllll}
\min & -4x_1 & -4x_2 & +x_1^2 & +2x_1\, x_2 + & x_2^2 & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & & & = 2 \\
 & x_1 & & & +x_4 & & = 2 \\
 & & +x_2 & & & +x_5 & = 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
\tag{27}
$$

Let us consider the perturbation as $\triangle c = (-2, 1, 0, 0, 0)^T$ with $\triangle b = 0$. It is easy to verify that $\epsilon = 0$ is a transition point of the optimal value function and thus, the actual invariancy interval is the singleton $\{0\}$. The optimal partition at $\epsilon = 0$ is $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T}) = (\{1, 2, 4, 5\}, \emptyset, \{3\})$.

It is easy to see that the problem has multiple optimal solutions

$$\{(t, 2-t, 0, 2-t, t-1) |\ t \in [1, 2]\}.$$

The feasible region and the optimal solution set are shown in Figure 4. Optimal solutions of problem (27) can be categorized in three classes.

- **Case 1. Primal optimal nondegenerate basic solution**, such as $x^{(1)} = (1, 1, 0, 1, 0)^T$. For this given optimal solution we have

$$P = \{1, 2, 4\}, \qquad Z = \{3, 5\},$$

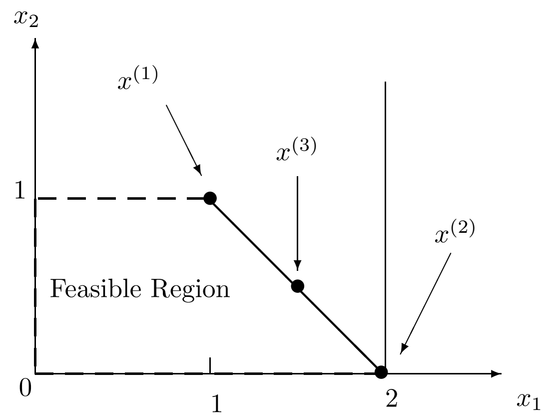

Figure 4: The feasible region and the optimal solution set in Example 3.

$$Q_{PP} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_{ZP}^T = Q_{PZ} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Q_{ZZ} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By solving the corresponding two auxiliary LO problems (23) and (24), one can identify the ISS interval $\Upsilon_Q(0, \triangle c)$ as the singleton $\{0\}$. It means that any variation in vector c, in the given direction $\triangle c$, changes the ISS partition (P, Z).

- **Case 2. Primal optimal degenerate basic solution**, such as $x^{(2)} = (2, 0, 0, 0, 1)^T$. For this kind of optimal solutions, we have

$$P = \{1, 5\}, \qquad Z = \{2, 3, 4\},$$

$$Q_{PP} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, Q_{ZP}^T = Q_{PZ} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } Q_{ZZ} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving the corresponding two auxiliary LO problems (23) and (24), one can identify that $\Upsilon_Q(0, \triangle c) = [0, \infty)$. Since for any $\epsilon \in [0, \infty)$, the support set of the given optimal solution is invariant and the optimal solution is basic, so it is again an optimal solution. In other words, for all $\epsilon \geq 0$, $x^*(\epsilon) = (2, 0, 0, 0, 1)^T$ is an optimal solution of the perturbed problem (see Figure 5). This example

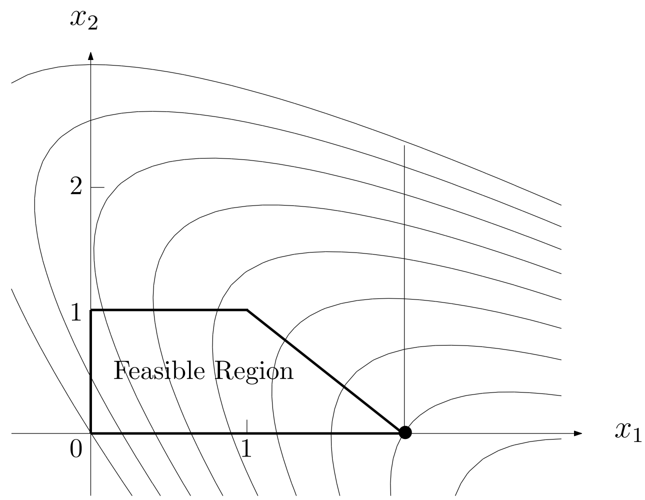

Figure 5: The feasible region and the optimal solution of problem (27: Case 3) before and after perturbation.

shows that the actual invariancy interval is a subset of the ISS interval $\Upsilon_Q(0, \triangle c)$ (see Remark 3.8).

- **Case 3. Maximally complementary optimal solution**, such as $x^{(3)} = (\frac{3}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})^T$. For the given optimal solution

$$P = \{1, 2, 4, 5\}, \qquad Z = \{3\},$$

$$Q_{PP} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_{ZP}^T = Q_{PZ} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } Q_{ZZ} = \begin{bmatrix} 0 \end{bmatrix}.$$

In this case, the ISS interval $\Upsilon_Q(0, \triangle c)$ is the singleton $\{0\}$.

5 Conclusions

We have developed procedures to calculate ISS intervals for CQO problems. We investigated the effect of simultaneous perturbation of both the vectors b and c and specialized the results for the non-simultaneous cases. It is proved that the ISS intervals in case of CQO are convex as they are in the case of LO. Moreover, the optimal value function is quadratic on ISS intervals. It is shown that ISS intervals can be identified by solving two auxiliary LO problems. Similar to the LO case, support set invariancy and optimal partition invariancy sensitivity analysis are not identical even if the given pair of primal-dual optimal solutions is maximally (strictly) complementary.

To identify the ISS intervals we need to solve two auxiliary LO problems that typically have smaller size than the original problem. It is worthwhile to mention that all the auxiliary LO problems can be solved in polynomial time by an IPM. We illustrated our results by some simple examples.

Since Convex Conic Optimization (CCO) is a generalization of LO and CQO problems, future work may be directed to interpret and perform support set invariancy sensitivity analysis for CCO and to find computable methods to determine support set invariancy sensitivity analysis optimality ranges for CCO, i.e., to find ISS intervals. support set invariancy sensitivity analysis could also be studied in such special cases of CCO as Second Order Conic and Semi-Definite Optimization.

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