Note: A Graph-Theoretical Approach to Level of Repair Analysis

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This paper is dedicated to the memory of Lillian Barros

Abstract

Level of Repair Analysis (LORA) is a prescribed procedure for defence logistics support planning. For a complex engineering system containing perhaps thousands of assemblies, sub-assemblies, components, etc. organized into several levels of indenture and with a number of possible repair decisions, LORA seeks to determine an optimal provision of repair and maintenance facilities to minimize overall life-cycle costs. We consider the LORA optimization problem (LORAP) introduced by Barros (1998) and Barros and Riley (2001) who solved LORAP using branch-and-bound heuristics. The surprising result of this paper is that LORAP is, in fact, polynomial-time solvable. We prove it by reducing LORAP to the maximum weight independent set problem on a bipartite graph.

(*Level of Repair Analysis; Independent Sets; Bipartite Graphs*)

1 Introduction

Level of Repair Analysis (LORA) is a prescribed procedure for defence logistics support planning (see, e.g., Crabtree and Sandel (1989) and the website of the UK MoD Acquisition Management System at www.ams.mod.uk/ams). For a complex engineering system containing perhaps thousands of assemblies, sub-assemblies, components etc. organized into $\ell \geq 2$ levels of *indenture* and with $r \geq 2$ possible repair decisions, LORA seeks to determine an optimal provision of repair and maintenance facilities to minimize overall life-cycle costs.

Barros (1998) and Barros and Riley (2001) provide a generic integer programming formulation of the LORA optimization problem (LORAP) for systems with ℓ levels of indenture and r possible repair decisions (including the non-repair option). The case of $\ell = 2$ and $r = 3$ is of importance because it corresponds to recommendations in certain UK and US military standard handbooks, see Barros and Riley (2001). In French military standards, $\ell = 2$ and $r = 5$.

While Barros (1998) solves her LORA integer programming (IP) model using a general purpose IP solver, Barros and Riley (2001) outline a specialized branch-and-bound heuristic, which appears to be more efficient in computational experiments. Their heuristic is based on a relaxation of LORA into a pair of uncapacitated facility location (UFLP) problems. A branch-and-bound procedure then employs local search heuristics to satisfy additional side constraints ensuring consistency between repair decisions for pairs of items nested on adjacent indenture levels. Since UFLP is *NP*–hard, it could be expected that LORAP would also be intractable. However, the surprising result of this paper is that LORAP is polynomially solvable and this is achieved by reducing it to the maximum weight independent set problem on a bipartite graph. For standard graph-theoretical terminology and notation, see, e.g., Asratian, Denley and Haggkvist (1998) or West (1996).

In Section 2, we provide a graph-theoretical formulation of LORAP. In Section 3, we show how to solve LORAP in polynomial time. Finally, in Section 4 we apply our approach to solve the bipartite case of the critical independent set problem in polynomial time.

2 Graph-Theoretical Formulation of LORAP

Consider first LORAP with $\ell = 2$ and $r = 3$ following Barros (1998) and Barros and Riley (2001) (we will call this special case of LORAP, *LORAI*). We refer to the first level of indenture in LORAI as *subsystems* $s \in S$ and the second level of indenture as *modules* $m \in M$. The distribution of modules in subsystems is given by a $(|S| \times |M|)$ binary matrix $T = (t_{sm})$ with $t_{sm} = 1$ if and only if module m is located in subsystem s.

There are $r = 3$ available repair decisions for each level of indenture: "discard", "local repair" and "central repair", labelled respectively D, L, C (subsystems) and d, l, c (modules). To be able to use a decision $k \in \{D, L, C, d, l, c\}$, we have to pay a fixed cost c_k. Assume also known additive costs (over a system life-cycle) $c_{1,j}(s)$ $c_{2,j}(m)$ of prescribing repair decision j for subsystem s, module m respectively.

We wish to minimize the total cost of choosing a subset of the six repair decisions and assigning available repair options to the subsystems and modules subject to the following constraints:

If a module m occurs in subsystem s so that $t_{sm} = 1$, we impose the following logical restrictions on the repair decisions for the pair (s, m) motivated through practical considerations:

R1: $D_s \Rightarrow d_m$, R2: $l_m \Rightarrow L_s$,

where D_s, d_m denote the decisions to discard subsystem s, module m respectively etc. Notice that even though module m may be common to several subsystems we are required to prescribe a unique repair decision for that module.

R1 has the interpretation that a decision to discard subsystem s necessarily entails discarding all enclosed modules. R2 is a consequence of R1 and a policy of "no backshipment" which rules out the local repair option for any module enclosed in a subsystem which is sent for central repair.

LORAI can be interpreted as the following graph-theoretical problem. Consider a bipartite graph $G = (V_1, V_2; E)$ with partite sets $V_1 = S$ and $V_2 = M$. For arbitrary $s \in V_1$ and $m \in V_2$, $sm \in E$ if and only if module m is in subsystem s. Instead of the repair options, we assign colors 1,2,3 to vertices of G. Define the color correspondence $D \to 1$, $C \to 2$, $L \to 3$; $d \to 3$, $c \to 2$, $l \to 1$. R1 (R2) means that if $u \in V_1$ (V_2) is assigned color 1, all its neighbors must be assigned color 3. An assignment of colors to vertices of G satisfying R1 and R2 is called an *acceptable coloring*. If we assign a color j to a vertex $u \in V_1 \cup V_2$, we have to pay cost $c_j(u)$.

For each $i = 1, 2$, we wish to choose a subset L_i of $\{1, 2, 3\}$ and find an acceptable coloring of the vertices of G such that a vertex $u \in V_i$ gets color $k(u) \in L_i$ in order to minimize

$$\sum_{u\in V_1\cup V_2} c_{k(u)}(u) + \sum_{j\in L_1} c_{1j} + \sum_{j\in L_2} c_{2j},$$

where c_{ij} is the cost of using color j for vertices of V_i.

We may replace R1 and R2 by a bipartite graph T with partite sets $\{1',2',3'\}$ and $\{1'',2'',3''\}$ and with edges $\{1'3'',2'3'',2'2'',3'3'',3'2'',3'1''\}$. Indeed, in an acceptable coloring, we may assign color j to a vertex $u\in V_1$ and color k to a vertex $v\in V_2$ if and only if $j'k''\in E(T)$. We call T a *color-acceptability graph.*

We call a bipartite graph B with partite sets $\{1',\dots,r'\}$ and $\{1'',\dots,r''\}$ *monotone* if $p'q''\in E(B)$ implies that $s't''\in E(B)$ for each $s\geq p$ and $t\geq q$. Clearly, T is a monotone bipartite graph. Interestingly, monotone bipartite graphs form a family of so-called convex bipartite graphs; several families of convex bipartite graphs have been found useful in various applications, see Asratian, Denley and Haggkvist (1998).

The above graph-theoretical formulation of LORAI can be easily extended to (a generalization of) LORAP with $\ell=2$ and arbitrary $r\geq 2$ by using r colors rather than just 3 and allowing T to be an arbitrary monotone bipartite graph with r vertices in each partite set. (The monotonicity significantly extends a simple generalization of R1 and R2 given in Barros (1998).) We will go further than that and provide a graph-theoretical formulation of (a generalization of) LORAP for arbitrary fixed $\ell,r\geq 2$.

Let $G=(V,E)$ a graph with an ℓ-partition $V_1\cup\ldots\cup V_\ell$ of V such that if $u_iu_j\in E$, $u_i\in V_i$ and $u_j\in V_j$, then $|i-j|=1$. Clearly, G is a bipartite graph with partite sets $\cup\{V_i:\ 1\leq i\leq \ell,\ i\equiv 1 \pmod 2\}$ and $\cup\{V_i:\ 1\leq i\leq \ell,\ i\equiv 0 \pmod 2\}$. Let F be a monotone bipartite (color-acceptability) graph with partite sets $\{1',\dots,r'\}$ and $\{1'',\dots,r''\}$. An assignment of colors from $\{1,\dots,r\}$ to V that assigns a vertex u a color $k(u)$ is called an *acceptable coloring* if for each edge $uv\in G$, where $u\in V_p$, $v\in V_q$ and $q=p+1$, we have $k'(u)k''(v)\in E(F)$.

For each $i=1,2,\dots,\ell$, we wish to choose a subset L_i of $\{1,\dots,r\}$ and find an acceptable coloring of the vertices of G such that a vertex $u\in V_i$ gets color $k(u)\in L_i$ in order to minimize

$$\sum_{i=1}^{\ell}\Big(\sum_{u\in V_i} c_{k(u)}(u) + \sum_{j\in L_i} c_{ij}\Big), \tag{1}$$

where $c_{k(u)}(u)$ is the cost of assigning color $k(u)$ to u and c_{ij} is the cost of using color j for vertices of V_i.

3 Solving LORAP

In this section we reduce LORAP to the maximum weight independent set problem on bipartite graphs with weights on the vertices. Recall that a vertex set I of a graph is *independent* if there is no edge between vertices of I.

In the next theorem, we will consider a bipartite graph B with partite sets U_1, U_2 and nonnegative vertex weights $p(u)$, $u \in V(B)$, and the following (s,t)-*network* $N(B)$: add new vertices s and t to B, append all arcs su of capacity $p(u)$, vt of capacity $p(v)$ for all $u \in U_1$ and $v \in U_2$, and orient every edge xy of B, where $x \in U_1$, from x to y (these arcs are of capacity ∞). For many results on flows and cuts in networks see Ahuja, Magnanti and Orlin (1993).

Theorem 3.1 *If (S,T) is a minimum cut in $N(B)$, $s \in S$, then $(S \cap U_1) \cup (T \cap U_2)$ is a maximum weight independent set in B. One can find a maximum weight independent set in B in time $O(n_1^2\sqrt{m} + n_1 m)$, where $n_1 = |U_1|$ and $m = |E(B)|$.*

The structural part of Theorem 3.1 is well-known, cf. Frahling and Faigle (2004). Since the selection problem introduced by Rhys (1970) and Balinski (1970) is equivalent, in a sense, to the maximum weight independent set problem on bipartite graphs, a similar result can be found in Hochbaum (2004). The complexity claim follows from the fact that one can find a minimum cut in $N(B)$ in time $O(n_1^2\sqrt{m} + n_1 m)$ by first finding a maximum flow by the bipartite preflow-push algorithm of Ahuja et al. (1994) and then finding a minimum cut (e.g., by finding vertices reachable from s in the residual network using depth-first search).

Let us return to LORAP and formulate it as a maximization problem. If $c_1(u) > c_2(u)$ for some $u \in V(G)$, then by the monotonicity of color-acceptability graph F there is an optimal solution in which u is not colored 1. Thus, we may set $c_1(u) := c_2(u)$ and keep a record, say $(u, 1, 2)$, that indicates that if, in an optimal acceptable coloring that we found u is colored 1, we recolor it 2. Similar arguments allow us to assume that $c_1(u) \leq c_2(u) \leq \cdots \leq c_k(u)$ for each $u \in V(G)$.

Let M be the maximum of all costs in LORAP (i.e., $c_j(u)$'s and c_{ij}'s). For each color j and vertex u let $w_j(u) = M - c_j(u)$ and for each $i = 1, \ldots, \ell$ and color j let $w_{ij} = M - c_{ij}$. Then minimization of (1) can be replaced by maximization of

$$\sum_{i=1}^{\ell}\left(\sum_{u \in V_i} w_j(u) + \sum_{j \in L_i} w_{ij}\right) \tag{2}$$

Notice that all weights $w_j(u)$ and w_{ij} are nonnegative and $w_1(u) \geq w_2(u) \geq \cdots \geq w_k(u)$ for each $u \in V(G)$.

Let U_1, U_2 be partite sets of G, $n_1 = \min\{|U_1|, |U_2|\}$ and $m = |E(G)|$. We first prove the main mathematical result of the paper and then illustrate it using an instance of LORAI.

Theorem 3.2 *For fixed subsets L_i, $i = 1, 2, \ldots, \ell$, LORAP can be solved in time $O(n_1^2\sqrt{m} + n_1 m)$.*

Proof: Since L_i, $i = 1, 2, \ldots, \ell$, are fixed, for simplicity, we will assume that all weights $w_{ij} = 0$ in (2). Let M be a constant larger than $\max\{w_1(u) : u \in V(G)\}$. Let $r(i)$ be the largest element of L_i, $i = 1, 2, \ldots, \ell$.

Construct a new graph H with $\sum_{i=1}^{\ell} |L_i| \times |V_i|$ vertices:

$$V(H) = \cup_{i=1}^{\ell}\{u_j : u \in V_i, j \in L_i\}.$$

Let an edge $u_j v_k$ be in H if $uv \in E(G)$, $u \in V_p$, $v \in V_{p+1}$ for some p, and $j'k'' \notin E(F)$. For every $i = 1, \ldots, \ell$, $u \in V_i$ and $j \in L_i$, let the weight $w(u_j)$ be equal to $w_{r(i)}(u) + M$, if $j = r(i)$, and be equal to $w_j(u) - w_k(u)$, where k is the smallest number in L_i larger than j, otherwise. Observe that the weights of the vertices of H are nonnegative.

Clearly, if we replace, in G, a vertex $u \in V_i$ by $|L_i|$ independent copies such that there is an edge between a copy of x and a copy of y if and only if $xy \in E(G)$, then we obtain a supergraph G^* of H. Since G is bipartite, so is G^* and, thus, H.

Observe that, by monotonicity of F, if $u_j, u_{j'}, v_k, v_{k'}$ are vertices of H, $j' \geq j$, $k' \geq k$ and $u_j v_k \notin E(H)$, then $u_{j'} v_{k'} \notin E(H)$ as well. We call this property of H *index-antimonotonicity* of H.

Let G have an acceptable coloring, where u gets color $k(u)$. Then the set $\{u_{k(u)} : u \in V(G)\}$ is independent in H. Moreover, by index-antimonotonicity of H,

$$S = \cup_{i=1}^{\ell} \cup_{u \in V_i} \{u_j : j \in L_i, j \geq k(u)\} \qquad (3)$$

is an independent set in H. Observe that S contains $S' = \cup_{i=1}^{\ell}\{u_{r(i)} : u \in V_i\}$ and the weight of S is equal to that of the coloring plus $M \times |V(G)|$ (we use telescopic sums).

Assume that a maximum weight independent set S in H contains S'. Then assign each $u \in V(G)$ color $k(u) = \min\{j : u_j \in S\}$. By maximality, S is of the form (3) or, due to index-antimonotonicity of H, S may be extended to (3) by adding some vertices of zero weight. Observe that the weight of S is equal to that of the coloring plus $M \times |V(G)|$. If a maximum weight independent set S in H does not contain

S', then S' is not an independent set in H (since the weight of S' is larger than the weight of S) and, thus, G has no acceptable coloring.

Thus, an optimal acceptable coloring corresponds to a maximum weight independent set S in H and S contains S', and vise versa. It remains to observe that we may apply Theorem 3.1 to find a maximum weight independent set of H. $\square$

The construction of the graph H in the proof above is illustrated by the following example.

Example 3.3 *Consider LORAI with graph G whose partite sets are $V_1 = \{u^i :\ 1 \leq i \leq p\}$ and $V_2 = \{v^i :\ 1 \leq i \leq p\}$, and whose edge set is $E(G) = \{u^i v^k :\ 1 \leq i \neq k \leq p\}$. Fix $L_1 = \{1, 3\}$ and $L_2 = \{1, 2, 3\}$. Let $w_j(u^i) = 10 \times (4 - j)$ and $w_j(v^i) = 4 - j$ for each $j = 1, 2, 3$ and $i = 1, \ldots, p$. According to the color-acceptability graph T, one may color two adjacent vertices of G in colors j, k as long as $j + k > 3$. It is easy to see that the optimal acceptable coloring consists of assigning color 1 to all vertices of V_1 and color 3 to all vertices of V_2. The total weight (without fixed weights for using L_1 and L_2) is $30 \times p + 1 \times p = 31p$.*

Now we solve LORAI with L_1, L_2 fixed as above by constructing H. Let $M = 40$. We have

$$V(H) = \{u_1^i, u_3^i, v_1^i, v_2^i, v_3^i :\ 1 \leq i \leq p\}$$

and $E(H) = \{u_1^i v_b^k :\ 1 \leq i \neq k \leq p, b = 1, 2\}$. Also, $w(u_1^i) = 20$, $w(u_3^i) = 50$, $w(v_1^i) = w(v_2^i) = 1$ and $w(v_3^i) = 41$ for each $i = 1, \ldots, p$. Observe that $S = V(H) - \{v_1^i, v_2^i :\ 1 \leq i \leq p\}$ is a maximum weight independent set in H, and S contains $S' = \{u_3^i, v_3^i :\ 1 \leq i \leq p\}$. Clearly, S implies that we assign color 1 to all vertices of V_1 and color 3 to all vertices of V_2. The weight of S is $20p + 50p + 41p = 31p + 40 \times 2p$.

Since r and ℓ are constants, there are less than $2^{r\ell}$ choices of nonempty $L_1, \ldots, L_\ell$. Thus, we obtain the following:

Theorem 3.4 *LORAP can be solved in polynomial time.*

4 LORAI and Critical Independent Set Problem

Let Q be an arbitrary graph. For a set $X \subseteq V(Q)$, let $N(X) = \cup_{x \in X}\{y \in V(Q) : xy \in E(Q)\}$. Let p, q be a pair of functions from $V(Q)$ to the set of nonnegative reals. In the *critical independent set problem (CISP)* we seek

$$\text{argmax}\{\sum_{a\in A} p(a) - \sum_{c\in N(A)} q(c): \ A \text{ is an independent vertex set in } Q\}.$$

Clearly, CISP is *NP*–hard as the maximum weight independent set problem on arbitrary graphs is reducible to CISP with $q(u) = 0$ for each $u \in V(Q)$. Ageev (1994) proved that CISP is polynomial time solvable if $p(u) = q(u)$ for each $u \in V(Q)$. This generalized the corresponding result of Zhang (1990) for $p(u) = q(u) = 1$ for each $u \in V(Q)$. We will show that CISP can be solved in polynomial time on bipartite graphs for arbitrary functions p and q.

Theorem 4.1 *CISP on a bipartite graph $G = (V_1, V_2; E)$ can be solved in time $O(n_1^2\sqrt{m} + n_1 m)$, where $n_1 = |V_1|$ and $m = |E|$.*

Proof: Consider LORAI. Recall that vertices of color 1 in T can be adjacent only to vertices of color 3. Consider a optimal acceptable coloring of G for $L_1 \cup L_2 = V_1 \cup V_2$, in which A is the set of vertices assigned color 1. Then A is independent, all vertices of $N(A)$ must have color 3 and all vertices of $B = V(G) - A - N(A)$ may have color 2. The total weight of the coloring is

$$\sum_{a\in A} w_1(a) + \sum_{c\in N(A)} w_3(c) + \sum_{b\in B} w_2(b) = \sum_{d\in V(G)} w_2(d) - \sum_{c\in N(A)} w_{2,3}(c) + \sum_{a\in A} w_{1,2}(a),$$

where $w_{2,3}(c) = w_2(c) - w_3(c)$, $w_{1,2}(a) = w_1(a) - w_2(a)$.

Choose weight functions w_1, w_2, w_3 as follows: $w_1(u) = p(u)+q(u)$, $w_2(u) = q(u)$, $w_3(u) = 0$ for each $u \in V(G)$. Since $\sum_{d\in V(G)} w_2(d)$ is a constant, we observe that CISP on G (and functions p and q) can be reduced to LORAI with fixed $L_1 = V_1$, $L_2 = V_2$. It remains to apply Theorem 3.2. □

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