Domination between Traffic Matrices

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Abstract

A traffic matrix D^1 dominates a traffic matrix D^2 if D^2 can be routed on every (capacitated) network where D^1 can be routed. We prove that D^1 dominates D^2 if and only if D^1, considered as a capacity vector, supports D^2. We show several generalizations of this result.

1 Introduction

A common class of network design problems asks for the installation of capacity (bandwidth) over the edges of a network as to support a given set of pairwise traffic demands - a *traffic matrix* - with some additional constraints (integrality, unsplittable flows, resilience etc.). A crucial assumption in this model is that the traffic matrix is *single*.

This is not the case with several applications, where communication patterns change over time, and therefore we have to support a set $\mathcal{D}$ of *non-simultaneous* traffic matrices: this set can be given either explicitely (e.g. day/night traffic matrix) or implicitely (e.g. by a set of constraints defining a traffic polytope [1, 2]).

Unfortunately, moving from a single to a set of traffic matrices can easily increase the complexity of a network design problem. Consider, in fact, the following problem:

Given: an undirected graph $G(V, E)$ with per-unit capacity installation cost c_{uv} for each (potential) arc $uv \in E$, an integer p, a subset $X = \{v^0, v^1, \ldots, v^p\}$ of nodes and a set of non-simultaneous traffic matrices:

$$\mathcal{D} = \{D^1, \ldots, D^p\}, \text{ where } D^h: \quad d^h_{ij} = \begin{cases} 1 & \{i,j\} = \{v^0, v^h\} \\ 0 & else \end{cases}$$

Find: a minimum cost capacity installation $U : E \to Z_+$, such that *every* traffic matrix of the set $\mathcal{D}$ can be routed by integral flows on the network G equipped with capacity u_{ij} for every edge $ij \in E$.

It is easy to see that while with $p = 1$, i.e. a single traffic matrix, the problem above reduces to the shortest path problem, when $p > 1$ it reduces to the Steiner tree problem.

In the paper we consider therefore the following basic question: suppose that $\mathcal{D} = \{D^1, D^2\}$, when is it possible to discard one matrix and reduce therefore $\mathcal{D}$ to a single matrix?

In particular, we say that a traffic matrix D^1 *dominates* a traffic matrix D^2 if D^2 can be routed on every (capacitated) network supporting D^1, i.e. on every network where D^1 can be routed. This property may be easily expressed in term of containment between two suitable polytopes. Theorem 2.1 gives a good characterization for domination: D^1 dominates D^2 if and only if D^1, considered as a capacity installation, supports D^2.

We discuss several generalization of domination and Theorem 2.1: when demands are to be routed by either unsplittable or integral flows, when the routing for D^1 and D^2 has to be the same and when $|\mathcal{D}| = 3$.

1.1 Preliminaries

Let N be a set of nodes and $K_N(N, A_N)$ the complete directed network with node set N, i.e. $A(K_N) = \{(i,j) : i,j \in N\}$. Let $|N| = n$.

A *traffic matrix* over K_N is a non-negative, matrix of rationals D of size $n \times n$, where d_{ij} is the amount of *demand* from node i to node j; w.l.o.g. we assume $d_{ii} = 0$ for any i.

A *capacity matrix* - often simply a capacity - over K_N is a non-negative, matrix of rationals U of size $n \times n$, where u_{ij} is the amount of capacity we install over the arc (i,j) of K_N; w.l.o.g. we assume $u_{ii} = 0$ for any i.

A *routing* over K_N is a non-negative, matrix of rationals F of size $n \times n \times n \times n$, where f_{ijhk} is the fraction of demand d_{ij} that is routed on the arc (h,k). Therefore, F is such that[1]:

(a) $f_{ijhk} = 0$ if $i = j$ or $h = k$;

(b) for any pair $(i,j) \in N \times N, i \neq j$:

- (b1) $\sum_{k \in N} f_{ijik} - \sum_{k \in N} f_{ijki} = 1$;
- (b2) $\sum_{k \in N} f_{ijjk} - \sum_{k \in N} f_{ijkj} = -1$;
- (b3) $\sum_{k \in N} f_{ijvk} - \sum_{k \in N} f_{ijkv} = 0$ for any $v \in N \setminus \{i,j\}$;

We say that a capacity matrix U and a routing F *support* a traffic matrix D if D may be routed over K_N equipped with capacity U, via F. That is, U and F support D if $\sum_{(i,j) \in N \times N} f_{ijhk} d_{ij} \leq u_{hk}$, for each arc $(h,k) \in A(K_N)$.

We simply say that a capacity U *supports* a traffic D if there exists a routing F such that U and F support D.

A nice characterization of the latter property is a well-known result from the literature. The *metric polytope*, a normalization of the metric cone, is defined as follows:

$$P_\mu = \quad \{\mu \in \mathcal{R}_+^{N \times N} : \quad \mu_{ij} + \mu_{jk} \geq \mu_{ik} \ \forall \ (i,j,k) \in N \times N \times N; \quad \sum_{(i,j) \in N \times N} \mu_{ij} = 1; \quad \mu_{ii} = 0 \ \forall \ i \in N\}$$

[1]We do not assume that $0 \leq f_{ijhk} \leq 1$ for each (i,j,h,k). This is indeed the case if, for each pair (r,s), we deal with acyclic (r,s)-flows, that is, there are no directed cycles in the support of the (r,s)-flow.

Theorem 1.1. **[5, 7]** *A capacity matrix U supports a traffic matrix D if and only* $\sum_{(i,j)\in N\times N} \mu_{ij}u_{ij} \geq \sum_{(i,j):\in N\times N} \mu_{ij}d_{ij}$ *for any* $\mu \in P_\mu$.

Given a capacity U and a routing F, $\mathcal{D}(U,F)$ denotes the set of traffic matrices that are supported by U and F. That is, $\mathcal{D}(U,F) = \{D : \sum_{(i,j)\in N\times N} f_{ijhk}d_{ij} \leq u_{hk},\ (h,k) \in A(K_N)\}$. Analogously, given a traffic matrix D, $\mathcal{UF}(D)$ denotes the set of pairs (U,F) supporting D. That is $\mathcal{UF}(D) = \{U,F : \sum_{(i,j)\in N\times N} f_{ijhk}d_{ij} \leq u_{hk},\ (h,k) \in A(K_N)\}$.

Moreover, $\mathcal{D}(U)$ denotes the set of traffic matrices supported by a given capacity U. By Theorem 1.1, $\mathcal{D}(U) = \{D : \sum_{(i,j)\in N\times N} \mu_{ij}d_{ij} \leq \sum_{(i,j)\in N\times N} \mu_{ij}u_{ij},\ \mu \in P_\mu\}$. Observe that $\mathcal{D}(U)$ is a down-monotone polytope. Analogously, $\mathcal{U}(D)$ is the set of capacities supporting a given traffic matrix D. By Theorem 1.1, $\mathcal{U}(D) = \{U : \sum_{(i,j)\in N\times N} \mu_{ij}d_{ij} \leq \sum_{(i,j)\in N\times N} \mu_{ij}u_{ij},\ \mu \in P_\mu\}$. Observe that $\mathcal{U}(D)$ is an up-monotone polyhedron, and that $\mathcal{U}(D)$ is the projection over the U-space of $\mathcal{UF}(D)$.

2 Domination

Let D^1 and D^2 be two traffic matrices. We say that D^1 *dominates* D^2 if $\mathcal{U}(D^1) \subseteq \mathcal{U}(D^2)$. In other words, D^1 dominates D^2 if any capacity supporting D^1 supports also D^2. Trivially, D^1 dominates D^2 if $d^1_{ij} \geq d^2_{ij}$ for any pair (i,j), but this condition is only sufficient.

Theorem 2.1. *D^1 dominates D^2 if and only if D^1 (considered as a capacity matrix) supports D^2.*

Proof. Necessity. Trivially, D^1 as a capacity matrix supports D^1 as a traffic matrix. Hence D^1 must support D^2 too.

Sufficiency. Since D^1 supports D^2, by Theorem 1.1:

$$\sum_{(i,j)\in N\times N} \mu_{ij}d^1_{ij} \geq \sum_{(i,j)\in N\times N} \mu_{ij}d^2_{ij} \qquad \text{for any } \mu \in P_\mu \tag{1}$$

Let U be any capacity matrix U supporting D^1. Again, by Theorem 1.1:

$$\sum_{(i,j)\in N\times N} \mu_{ij}u_{ij} \geq \sum_{(i,j)\in N\times N} \mu_{ij}d^1_{ij} \qquad \text{for any } \mu \in P_\mu \tag{2}$$

and therefore, combining Equation (1) and Equation (2):

$$\sum_{(i,j)\in N\times N} \mu_{ij}u_{ij} \geq \sum_{(i,j)\in N\times N} \mu_{ij}d^2_{ij} \qquad \text{for any } \mu \in P_\mu$$

that is, U supports D^2 too. □

Corollary 2.2. *D^1 dominates D^2 if and only if* $\sum_{(i,j)\in N\times N} \mu_{ij}d^1_{ij} \geq \sum_{(i,j):\in N\times N} \mu_{ij}d^2_{ij}$ *for any* $\mu \in P_\mu$.

The previous theorem implies that recognizing if a traffic matrix D^1 dominates a traffic matrix D^2 is easy, since it is equivalent to solve a fractional multi-commodity problem. We also recall that recognizing if a polytope contains another polytope is easy if both polytopes are given by systems of inequalities [3], but, unfortunately, in our case the size of these systems is not polynomially bounded in the size of the original input.

2.0.1 Unsplittable flows

A routing is *unsplittable* if $f_{ijik} \in \{0,1\}$ for each (i,j,h,k). If U and F support D and F is unsplittable, then it is possible to route D over K_N equipped with capacity U by using a *single* path for each demand d_{ij}.

As usual, we say that U *supports* D by *unsplittable flows* if there exists an unsplittable routing F such that U and F support D.

Moreover, if D^1 and D^2 are two traffic matrices, we say that D^1 *dominates* D^2 *with respect to unsplittable flows* if any capacity U supporting D^1 by unsplittable flows also supports D^2 by unsplittable flows.

Theorem 2.3. *D^1 dominates D^2 with respect to unsplittable flows if and only if D^1 (considered as a capacity) supports D^2 by unsplittable flows.*

Proof. Necessity. Trivially, D^1 as a capacity supports D^1 as a traffic matrix by unsplittable flows. Hence D^1 must support D^2 by unsplittable flows too.

Sufficiency. Since D^1 supports D^2 by unsplittable flows, there exists an unsplittable routing F such that $\sum_{(r,s)\in N\times N} f_{rshk} d^2_{rs} \leq d^1_{hk}$, for each $(h,k) \in A(K_N)$. Also let U be any capacity supporting D^1 by unsplittable flows. Again, there exists an unsplittable routing Q such that $\sum_{(i,j)\in N\times N} q_{ijuv} d^1_{ij} \leq u_{uv}$, for each $(u,v) \in A(K_N)$.

We must show that there exists an unsplittable routing W such that U and W support D^2. We define W as follows:

$$w_{rsuv} = \sum_{(i,j)\in N\times N} q_{ijuv} f_{rsij} \qquad \text{for any } (r,s,u,v) \in N\times N\times N\times N$$

It is easy to see that W defines an unsplittable routing. Moreover, for any $(u,v) \in A(K_n)$:

$$\sum_{(r,s)\in N\times N} d^2_{rs} w_{rsuv} = \sum_{(r,s)\in N\times N} d^2_{rs} \sum_{(i,j)\in N\times N} q_{ijuv} f_{rsij} =$$
$$= \sum_{(i,j)\in N\times N} q_{ijuv} \sum_{(r,s)\in N\times N} d^2_{rs} f_{rsij} \leq \sum_{(i,j)\in N\times N} q_{ijuv} d^1_{ij} \leq u_{uv}$$

□

We point out that, since recognizing if a capacity matrix U supports a traffic matrix D as unsplittable [6] is NP-complete, then such is the complexity of checking if a traffic matrix D^1 dominates a traffic matrix D^2 with respect to unsplittable flows.

2.0.2 Integral flows

We say that U *supports* D by *integral flows* if there exists a routing F such that U and F support D and $f_{ijhk}d_{ij}$ is integral for any (i,j,h,k).

Moreover, if D^1 and D^2 are two traffic matrices, we say that D^1 *dominates* D^2 *with respect to integral flows* if any capacity U supporting D^1 by integral flows also supports D^2 by integral flows. The proof of the following theorem is similar to that of Theorem 2.3, so we omit it.

Theorem 2.4. *D^1 dominates D^2 with respect to integral flows if and only if D^1 (considered as a capacity) supports D^2 by integral flows.*

Again, since recognizing if a capacity matrix U supports a traffic matrix D with integral flows (the integer multi-commodity flow problem [4]) is NP-complete then such is the complexity of checking if a traffic matrix D^1 dominates a traffic matrix D^2 with respect to integral flows.

2.0.3 Strong domination

In many network applications "migrating" from a routing to another one is costly [1], so it is convenient to keep the *same* routing even if the traffic matrices change over time. We here characterize a stronger type of domination that, in a sense, allows to deal with this constraint.

Let D^1 and D^2 be two traffic matrices. We say that D^1 *strongly dominates* D^2 if, for any capacity U supporting D^1, there exists a routing $F(U)$ such that U and $F(U)$ support both D^1 and D^2. Trivially, if D^1 strongly dominates D^2, then D^1 dominates D^2.

As we show in the following, recognizing if a traffic matrix D^1 strongly dominates a traffic matrix D^2 is easy, since it is again equivalent to solve a fractional multi-commodity problem. First, we need a few definitions. For $h = 1, 2$, let $I(D^h) = \{(i,j) : d^h_{ij} > 0\}$. Also let:

$$\overline{D}^1 : \quad \overline{d}^1_{ij} = \begin{cases} d^1_{ij} - d^2_{ij} & (i,j) \in I(D^1) \cap I(D^2) \\ d^1_{ij} & else \end{cases}$$

$$\overline{D}^2 : \quad \overline{d}^2_{ij} = \begin{cases} 0 & (i,j) \in I(D^1) \cap I(D^2) \\ d^2_{ij} & else \end{cases}$$

Theorem 2.5. *D^1 strongly dominates D^2 if and only if:*

$d^1_{ij} \geq d^2_{ij}$ for any $(i,j) \in I(D^1) \cap I(D^2)$;

$\overline{D}^1$ (considered as a capacity matrix) supports $\overline{D}^2$.

Proof. We will prove the following statement: D^1 strongly dominates D^2 if and only if D^1 supports D^2 via a routing F such that $f_{ijij} = 1$, for any $(i,j) \in I(D^1) \cap I(D^2)$ - which, of course, implies the theorem.

Necessity. Trivially, $D^1 \in \mathcal{U}(D^1)$. Therefore there exists a routing $F(D^1)$ such that D^1 and $F(D^1)$ support both D^1 and D^2. On the other hand, any routing F such that D^1 and F support D^1 is such that $f_{ijij} = 1$, for any $(i,j) \in I(D^1)$. Since $I(D^1) \cap I(D^2) \subseteq I(D^1)$ we are done.

Sufficiency. Let $U \in \mathcal{U}(D^1)$ and let G be a routing such that U and G support D^1. Also by hypothesis there exists a routing F such that D^1 and F support D^2, and $f_{ijij} = 1$, for any $(i,j) \in I(D^1) \cap I(D^2)$.

In order, to prove our statement we have to show that there exists a routing $W = W(U)$ such that U and W support both D^1 and D^2. We define W as follows:

$$w_{ijhk} = \begin{cases} g_{ijhk} & (i,j) \in I(D^1) \setminus I(D^2), (h,k) \in N \times N \\ \sum_{(u,v)\in N\times N} f_{ijuv} g_{uvhk} & (i,j) \in I(D^2), (h,k) \in N \times N \\ 1 & (i,j) \notin I(D^1) \cup I(D^2), (h,k) = (i,j) \\ 0 & else \end{cases}$$

It is easy to see that W defines a routing. Moreover, since $w_{ijhk} = g_{ijhk}$ for any $(i,j) \in I(D^1)$ and $(h,k) \in N \times N$, it follows that U and W support D^1. Moreover, for any $(i,j) \in I(D^2)$ and $(h,k) \in N \times N$:

$$\sum_{(i,j)\in N\times N} d^2_{ij} w_{ijhk} = \sum_{(i,j)\in N\times N} d^2_{ij} \sum_{(u,v)\in N\times N} f_{ijuv} g_{uvhk} =$$
$$= \sum_{(u,v)\in N\times N} g_{uvhk} \sum_{(i,j)\in N\times N} d^2_{ij} f_{ijuv} \leq \sum_{(u,v)\in N\times N} g_{uvhk} d^1_{uv} \leq u_{hk}$$

and, therefore, U and W support D^2 too. □

Corollary 2.6. *If D^1 dominates D^2 and $I(D^1) \cap I(D^2) = \emptyset$, then D^1 strongly dominates D^2.*

2.0.4 Total domination

We say that D^1 *totally dominates* D^2 if $\mathcal{UF}(D^1) \subseteq \mathcal{UF}(D^2)$, i.e. any pair (U, F) supporting D^1 also supports D^2. Trivially, if D^1 totally dominates D^2, then D^1 strongly dominates D^2.

Theorem 2.7. *D^1 totally dominates D^2 if and only if $d^1_{ij} \geq d^2_{ij}$ for any (i,j).*

Proof. Necessity. Let F be the routing:

$$F: \quad f_{ijhk} = \begin{cases} 1 & (h,k) = (i,j) \\ 0 & (h,k) \neq (i,j) \end{cases}$$

Trivially, D^1 (as a capacity) and F support D^1. Therefore they must support D^2 too, and then, for any $(h,k) \in N \times N$:

$$d^1_{hk} \geq \sum_{(i,j)\in N\times N} d^2_{ij} f_{ijhk} = d^2_{hk}$$

Sufficiency. Let U and F support D^1. We claim that they support D^2 too. In fact:

$$u_{hk} \geq \sum_{(i,j)\in N\times N} d^1_{ij} f_{ijhk} \geq \sum_{(i,j)\in N\times N} d^2_{ij} f_{ijhk}$$

□

2.0.5 Domination among more matrices

The most interesting open question concerns the extension of domination to more traffic matrices. For instance, say that D^1 *and* D^2 *dominate* D^3 if $\mathcal{U}(D^1) \cap \mathcal{U}(D^2) \subseteq \mathcal{U}(D^3)$, that is, any capacity supporting *both* D^1 *and* D^2 supports D^3 too.

Unfortunately, we have not been able to generalize Theorem 2.1 to this case, and we do not know whether recognizing this kind of domination is easy (again, the problem may be formulated in terms of containment between two polytopes). Indeed, D^1 and D^2 dominate D^3 if:

$$\max\Big(\sum_{(i,j)\in N\times N} \mu_{ij} d^1_{ij}, \sum_{(i,j)\in N\times N} \mu_{ij} d^2_{ij}\Big) \geq \sum_{(i,j):\in N\times N} \mu_{ij} d^3_{ij}, \forall \mu \in P_\mu$$

but there are cases where this condition is not necessary. Vive versa, D^1 and D^2 dominate D^3 only if:

$$\sum_{(i,j)\in N\times N} \mu_{ij} \max(d^1_{ij}, d^2_{ij}) \geq \sum_{(i,j):\in N\times N} \mu_{ij} d^3_{ij}, \forall \mu \in P_\mu$$

but there are cases where this condition is not sufficient.

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