Perturbation Analysis of Second-Order Cone Programming Problems

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Abstract

We discuss first and second order optimality conditions for nonlinear second-order cone programming problems, and their relation with semidefinite programming problems. For doing this we extend in an abstract setting the notion of optimal partition. Then we state a characterization of strong regularity in terms of second order optimality conditions.

1 Introduction

$$\operatorname*{Min}_{x\in I\!R^n, s^j\in I\!R^{m_j+1}} f(x)\,;\; g^j(x)=s^j,\; (s^j)_0 \geq \|\bar{s}^j\|,\;\; j=1,\ldots,J, \tag{SOCP}$$

where f and g^j, $j=1,\ldots,J$ are C^1 mappings from $I\!R^n$ into $I\!R$ and $I\!R^{m_j+1}$, respectively. We use the standard convention of indexing components of vectors of $I\!R^{m_j+1}$ from 0 to m_j, while vectors in $I\!R^n$ are indexed from 1 to n. Given $s\in I\!R^{m_j+1}$, we also denote $\bar{s}:=(s_1,\ldots,s_{m_j})^\top$.

The second-order cone (or ice-cream cone, or Lorentz cone) of dimension $m+1$ is defined as

$$Q_{m+1} := \{s\in I\!R^{m+1}\,;\; s_0\geq \|\bar{s}\|\},$$

and the order relation $\succeq_{Q_{m+1}}$ induced by Q_{m+1} is given by

$$s\succeq_{Q_{m+1}} 0 \qquad \text{iff} \qquad s\in I\!R^{m+1},\; s_0\geq\|\bar{s}\|.$$

The interior of this cone is the set of $s\in I\!R^{m+1}$ such that $s_0>\|\bar{s}\|$. In that case we say that $s\succ_{Q_{m+1}} 0$. We also denote $\mathcal{Q}:=\Pi_{j=1}^J Q_{m_j+1}$. A second-order cone $Q=Q_{m+1}$ can be described as a linear matrix inequality by using the known equivalence (e.g. [1])

$$s\succeq_Q 0 \qquad \text{iff} \qquad \operatorname{Arw}(s) := \begin{pmatrix} s_0 & \bar{s}^\top \\ \bar{s} & s_0 I_m \end{pmatrix} \succeq 0, \tag{1}$$

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where I_m denotes the identity matrix in $I\!R^{m\times m}$, $\mathrm{Arw}(s)$ is the *arrow matrix* of the vector s, and $\succeq$ denotes the positive semidefinite order, that is, $A \succeq B$ iff A, B are symmetric matrices and $A - B$ is a positive semidefinite matrix. We also denote the set of $m+1$ by $m+1$ symmetric matrices by $\mathcal{S}^{m+1}$, indexed from 0 to m, equipped with the inner product $A \cdot B := \mathrm{Tr}(AB) = \sum_{i,j=0}^{m} A_{ij}B_{ij}$; the subset of symmetric positive semidefinite matrices is denoted $\mathcal{S}_+^{m+1}$. Finally, for two arbitrary vectors x and z of any dimension we set $x \cdot z := x^\top z = \sum_i x_i z_i$ the corresponding Euclidian inner product, and for an arbitrary optimization problem (P) we denote by $S(P)$, $F(P)$ and $\mathrm{val}(P)$ its solution set, feasible set and optimal value, respectively. The equivalence (1) implies that (SOCP) is *SDP-representable*, i.e., can be written as the nonlinear semidefinite problem

$$\operatorname*{Min}_{x\in I\!R^n} f(x)\,;\, G^j(x) := \mathrm{Arw}(g^j(x)) \succeq 0, \;\; j = 1, \dots, J. \qquad \text{(SDP)}$$

For a general view of semidefinite programming problems, see [18, 19]. A first objective in this paper is to compare the linear second-order programming problem (see (LSOCP) below) and its linear SDP-representation (see (LSDP) below) in terms of duality results. We show that their dual problems are no longer equivalent, and some important notions as the uniqueness of Lagrange multipliers (or equivalently, dual problems solutions) do not simultaneously hold for both problems (LSOCP) and (LSDP). We perform this analysis in an abstract framework. When specialized to second order cone problems, we recover some of the results of Sim and Zhao [17]. Still our main result is the characterization of the *strong regularity property* for SOCP problems in terms of second-order optimality conditions. This is a well studied subject in nonlinear programming and the reader can see two different approaches in the articles of Bonnans and Sulem [7], and Dontchev and Rockafellar [9]. Nevertheless, it is still an open problem in a general conic optimization framework, even in particular instances as semidefinite programming. Necessary and sufficient second-order conditions to obtain the strong regularity property in SDP are studied by the authors in [4].

The paper is organized as follows. Section 2 breaks into three subsections. In the first one, we review the main duality results concerning the linear second-order programming problem (LSOCP) and their comparison to linear SDP problems. Section 2.2 deals with an abstract framework involving two equivalent linear conic optimization problems with constraints in product form, that are related by a linear mapping (as in relation (1)). It introduces a notion of optimal partition of active constraints. It allows us to deduce several duality statements and related properties. Subsection 2.3 applies this abstract framework to linear problems (LSOCP) and (LSDP). Section 3 we discuss briefly the duality theory for nonlinear SOCP problems. Section 4 recalls some key notions as the *nondegeneracy condition* and the *reduction approach*, mainly for their use in section 5 where is stated our main result: the characterization of the strong regularity property for SOCP problems in terms of second-order optimality conditions. For this, we use the concepts given in section 4 as well some suitable known theorems and SOCP techniques.

2 Duality theory for linear SOCP problems

2.1 Dual linear SOCP problems

We assume in this section that $f(x) = c \cdot x$ and $g^j(x) = A^j x - b^j$, $j = 1, \dots, J$, where $c \in I\!R^n$ and A^j are $(m_j + 1) \times n$ matrices. In that case we speak of a linear SOCP problem:

$$\operatorname*{Min}_{x \in I\!R^n, s^j \in I\!R^{m_j+1}} c \cdot x \,;\; A^j x - b^j = s^j, \; (s^j)_0 \geq \|\bar{s}^j\|, \quad j = 1, \dots, J. \tag{LSOCP}$$

The dual problem of (LSOCP) is given by

$$\operatorname*{Max}_{y^j \in I\!R^{m_j+1}} \sum_{j=1}^{J} b^j \cdot y^j; \;\; \sum_{j=1}^{J} (A^j)^\top y^j = c, \;\; (y^j)_0 \geq \|\bar{y}^j\|, \;\; j = 1, \dots, J. \tag{LSOCP*}$$

Since both the primal and dual problems are convex, we have the following results of convex analysis Rockafellar [14]. The *weak duality* inequality val(LSOCP) $\geq$ val(LSOCP*) holds, with the convention that the optimal value (val) of problem (LSOCP) (resp. (LSOCP*)) is equal to $+\infty$ (resp. $-\infty$) if this problem is infeasible. If the value of (LSOCP) is finite, it is known that (LSOCP) is *strictly feasible*, i.e., there exists a point $\hat{x}$ such that $A^j \hat{x} - b^j \in \operatorname{int} Q_{m_j+1}$ for all $j = 1, \dots, J$, iff the set of solutions of the dual problem is nonempty and bounded. In that case we have the *strong duality* property, i.e., val(LSOCP) = val(LSOCP*). A symmetric statement holds by permuting the words "primal" and "dual" (we will see in lemma 2 a refinement of this statement). If the strong duality property holds, then a pair of primal-dual solution $(x^*, y^*) \in I\!R^n \times \Pi_{j=1}^J I\!R^{m_j+1}$ is characterized by the following optimality system

$$A^\top y^* = c, \quad Ax^* - b \in \mathcal{Q}, \quad y^* \in \mathcal{Q}, \quad (Ax^* - b) \circ y^* = 0, \tag{2}$$

where we have defined $A := (A^1; \cdots; A^J)$ as the matrix whose rows are those of A^1 to A^J and whose columns a_i are equal to $\operatorname{vec}(a_i^1, \dots, a_i^J)$, with a_i^j the i-th column of A^j, $b := \operatorname{vec}(b^1, \dots, b^J)$ and the operation $\circ$ (e.g. [1]) is given by

$$x \circ s := \operatorname{Arw}(x) s = \begin{pmatrix} x^\top s \\ x_0 \bar{s} + s_0 \bar{x} \end{pmatrix}, \;\; \text{for all } x, s \in I\!R^{m+1},$$

and for x, s in $\Pi_{j=1}^J I\!R^{m_j+1}$ we set

$$x \circ s := \operatorname{vec}(x^1 \circ s^1, \dots, x^J \circ s^J).$$

We may write $(Ax^* - b) \cdot y^* = 0$ instead of the last relation in (2), in view of the well known property (e.g. [1, Lemma 15])

$$\text{For all } x, \, s \in Q_{m+1}, \quad x \circ s = 0 \text{ iff } x \cdot s = 0. \tag{3}$$

In fact it is easily checked that relations in (3) are satisfied iff x and s belong to Q_{m+1} and

$$\text{Either } x = 0 \text{ or } s = 0, \;\; \text{or there exists } \; \alpha > 0 \; \text{ such that } \; s_0 = \alpha x_0 \; \text{ and } \; \bar{s} = -\alpha \bar{x}. \tag{4}$$

Similar duality results hold for the linear semidefinite problem, which can be written as

$$\operatorname*{Min}_{x\in I\!R^n} c\cdot x\,;\ \sum_{i=1}^{n} x_i G_i^j \succeq G_0^j,\ \ j=1,\dots,J, \tag{LSDP}$$

where we have set

$$G_0^j := \operatorname{Arw}(b^j) \qquad \text{and} \qquad G_i^j := \operatorname{Arw}(a_i^j),\quad i=1,\dots,n. \tag{5}$$

In this case, the dual problem of (LSDP) is

$$\operatorname*{Max}_{Y^j\in\mathcal{S}^{m_j+1}} \left\{ \sum_{j=1}^{J} G_0^j\cdot Y^j\,;\ \sum_{j=1}^{J} \mathcal{G}^j(Y^j)+c=0,\ \ Y^j\succeq 0,\ \ j=1,\dots,J, \right\}, \tag{LSDP*}$$

where the mappings $Y\in\mathcal{S}^{m_j+1}\to\mathcal{G}^j(Y) := (G_1^j\cdot Y,...,G_n^j\cdot Y)^\top$ are the adjoint operators of G^j, and a primal-dual solution $(x,Y)\in I\!R^n\times\Pi_{j=1}^J\mathcal{S}^{m_j+1}$ is characterized by

$$\sum_{j=1}^{J}\mathcal{G}^j(Y^*)^j+c=0,\quad G^j(x)\succeq 0,\, Y^j\succeq 0,\quad G^j(x)Y^j=0,\ \ j=1,\dots,J. \tag{6}$$

In the sequel we denote $\mathcal{G}(Y) := \sum_{j=1}^J \mathcal{G}^j(Y^*)^j$.

Note that a linear second-order cone programming problem as (LSOCP) satisfies the strong duality property if both problems (LSOCP) and its dual (LSOCP*) are feasible, see Shapiro and Nemirovski [16], whereas this is no longer true for a linear semidefinite programming problem, see [18, page 65].

2.2 An abstract framework

The aim of this section is to clarify some properties of optimization problems with constraints in product form, as well as relations between the dual solutions of (LSOCP) and (LSDP). For this, we consider a general linear conic optimization problem with constraints in product form, i.e.,

$$\operatorname*{Min}_{x\in I\!R^n} c\cdot x\,;\ A^j x-b^j\in K_j,\ \ j=1,\dots,J, \tag{COP}$$

where K_j are closed convex cones in $I\!R^{q_j}$. We set $K := K_1\times\cdots\times K_J$, and define $A=(A^1;\cdots;A^J)$ as the matrix whose rows are those of A^1 to A^J, and $b := \operatorname{vec}(b^1,\dots,b^J)$ so that (COP) is equivalent to $\operatorname{Min}_{x\in I\!R^n}\{c\cdot x;\ Ax-b\in K\}$. The dual problem is

$$\operatorname*{Max}_{y^1,\dots,y^J}\sum_{j=1}^{J} b^j\cdot y^j;\ \ \sum_{j=1}^{J}(A^j)^\top y^j=c,\ \ y^j\in K_j^+,\ \ j=1,\dots,J, \tag{COP*}$$

where the (positive) polar of a set $C\subset I\!R^m$ is defined as $C^+ := \{y\in I\!R^m;\ y\cdot z\geq 0,\ \ \text{for all } z\in C\}$. If the primal and dual values are equal, a pair (x,y) of the primal and dual problems is characterized by the optimality system

$$A^j x-b^j\in K_j,\quad y^j\in K_j^+,\quad y^j\cdot(A^j x-b^j)=0,\ j=1,\dots,J;\quad A^\top y=c. \tag{COPOS}$$

We denote by $S(\text{COPOS})$ the set of solutions of relations (COPOS). In the sequel we introduce notions of componentwise strict feasibility and strict complementarity.

Definition 1. *We say that strict primal (resp. dual) feasibility holds for $j \in \{1, \dots, J\}$ if there exists $x \in F(COP)$ such that $A^j x - b^j \in \operatorname{int} K_j$ (resp. $y \in F(COP^*)$ such that $y^j \in \operatorname{int} K_j^+$).*

Lemma 2. *Let j be strictly primal (resp. dual) feasible. Then the set $\{y^j; y \in S(COP^*)\}$ (resp. $\{A^j x - b^j; x \in S(COP)\}$) is bounded.*

Proof. If j is strictly primal feasible, there exists $\varepsilon > 0$ such that $s = Ax - b$ satisfies $s^j + \varepsilon B \subset K_j$, or equivalently $\varepsilon B \subset s^j - K_j$. Let $y \in S(COP^*)$. Since $y^j \in K_j^+$, it follows that $\varepsilon \|y^j\| \leq y^j \cdot s^j$. Using also $y^{j'} \cdot s^{j'} \geq 0$, for all j', we get

$$0 = x \cdot (c - A^\top y) = c \cdot x - y \cdot Ax = c \cdot x - b \cdot y - y \cdot s \leq c \cdot x - b \cdot y - \varepsilon \|y^j\|.$$

In other words, $\varepsilon \|y^j\| \leq c \cdot x - b \cdot y = c \cdot x - \operatorname{val}(COP^*)$, which gives the desired estimate. The proof for the dual statement is similar. ■

One says (e.g., [6, Def. 4.74]) that the *strict complementarity hypothesis* holds for problem (COP) if there exists a pair (x, y) solution of the optimality system, such that $-y \in \operatorname{ri} N_K(Ax - b)$, where N_K is the normal cone of convex analysis. Since K is a closed convex cone, we have for $s \in K$ that

$$N_K(s) = (-K^+) \cap s^\perp, \tag{7}$$

(where $s^\perp$ denotes the set of all orthogonal vectors to s) and $N_K(s) = \emptyset$ if $s \notin K$.

For problems with constraints in product form, it is worthwhile to introduce the concept of *componentwise strict complementarity hypothesis*, which for each component j means that there exists a pair $(x, y) \in S(\text{COPOS})$, such that $-y^j \in \operatorname{ri} N_{K_j}(A^j x - b^j)$.

We can extend and refine for this framework the notion of optimal partition, well known for linear programming and monotone linear complementarity problems, see e.g. [3, Section 18.2.4].

Lemma 3. *If $S(\text{COPOS})$ is not empty, there exists a partition (B, N, R, T) of $\{1, \dots, J\}$ such that,* (i) *The set B is the union of j such that there exists $(x(j), y(j)) \in S(\text{COPOS})$ satisfying $A^j x(j) - b^j \in \operatorname{int} K_j$,* (ii) *The set N is the union of j such that there exists $(x(j), y(j)) \in S(\text{COPOS})$ satisfying $y^j(j) \in \operatorname{int} K_j^+$,* (iii) *The set R is the union of j, not belonging to B or N, such that there exists $(x(j), y(j)) \in S(\text{COPOS})$ with $-y^j(j) \in \operatorname{ri} N_{K_j}(A^j x(j) - b^j)$, and* (iv) *for all $j \in T$, every $(x, y) \in S(\text{COPOS})$ does not satisfy strict complementarity for component j.*

Proof. Let (B, N, R, T) be defined as in the lemma; we have to check that this is a partition. The definition of T implies that their union equals $\{1, \dots, J\}$, and by definition of R and T, we have that $(B \cup N, R, T)$ is a partition of $\{1, \dots, J\}$. It remains to prove that $B \cap N = \emptyset$. Since $S(\text{COPOS})$ is not empty, we know that $S(\text{COPOS}) = S(COP) \times S(COP^*)$. Therefore $\hat{x} := |B|^{-1} \sum_{j \in B} x(j)$ satisfy $\hat{x} \in S(COP)$. We see that $A^j \hat{x} - b^j \in \operatorname{int} K_j$, for all $j \in B$. Therefore any $y \in S(COP^*)$ is such that $y^j = 0$, for all $j \in B$. This proves that $B \cap N = \emptyset$. ■

Remark 4. *Note that, for monotone linear complementarity problems the optimal partition is of the form (B, N, T), since in that case a strictly complementary component belongs either to B or N. Therefore the main novelty consists in introducing the set R.*

Definition 5. *Any pair $(x, y) \in S(\mathrm{COPOS})$ satisfying the relations below is said to be of maximal complementarity:*

$$\left\{ \begin{array}{l} \text{(i) } A^i x - b^i \in \operatorname{int} K_i, \;\; \forall i \in B, \quad \text{(ii) } y^i \in \operatorname{int} K_i^+, \;\; \forall i \in N, \\ \text{(iii) } -y^i \in \operatorname{ri} N_{K_i}(A^i x - b^i), \;\; \forall i \in R. \end{array} \right. \tag{8}$$

Let $x(j)$ and $y(j)$ be as in lemma 3. We define

$$\hat{x} := (|B| + |R|)^{-1} \sum_{j \in B \cup R} x(j); \quad \hat{y} := (|N| + |R|))^{-1} \sum_{j \in N \cup R} y(j).$$

Let us state some properties of the set of maximal complementarity solutions. We need a preliminary lemma.

Lemma 6. *Let K be a closed convex cone. Let $s^i \in K$, for $i = 1, 2$, $-y^1 \in N_K(s^1)$, and $-y^2 \in \operatorname{ri} N_K(s^2)$. Given $\alpha \in]0, 1[$, set $(s, y) := \alpha(s^1, y^1) + (1-\alpha)(s^2, y^2)$. If $-y \in N_K(s)$, then $-y \in \operatorname{ri} N_K(s)$.*

Proof. Since $-N_K(s) = K^+ \cap s^\perp$, we have that $-y \in \operatorname{ri} N_K(s)$ iff, for all $z \in N_K(s)$, $y \pm \varepsilon z \in K^+$ for small enough $\varepsilon > 0$. As K^+ is a cone, $y + \varepsilon z \in K^+$ always holds. Therefore we have to prove that for $z \in N_K(s)$, $y - \varepsilon z \in K^+$ for small enough $\varepsilon > 0$. Using $N_K(s) = N_K(s^1) \cap N_K(s^2)$, obtain $z \in N_K(s^2)$, and hence, $y^2 - \varepsilon' z \in K^+$ for some $\varepsilon' > 0$. Let $\varepsilon := (1-\alpha)\varepsilon'$. Then $y - \varepsilon z = \alpha y^1 + (1-\alpha)(y^2 - \varepsilon' z)$ belongs to K^+. The conclusion follows. ■

Lemma 7. (i) *The pair $(\hat{x}, \hat{y})$ is of maximal complementarity.* (ii) *Any pair $(\hat{x}, \hat{y}) \in$* $\operatorname{ri} S(\mathrm{COPOS})$ *(set equal to* $\operatorname{ri} S(COP) \times \operatorname{ri} S(COP^*)$*) is of maximal complementarity.*

Proof. (i) That $A^j \hat{x} - b^j \in \operatorname{int} K_j$, for all $j \in B$, is a classical property. Similarly, $\hat{y}^j \in \operatorname{int} K_j^+$, for all $j \in N$. Finally, that $-\hat{y}^j \in \operatorname{ri} N_{K_j}(A^j \hat{x} - b^j)$, for all $j \in R$, is consequence of lemma 6.
(ii) Let $(\hat{x}, \hat{y}) \in \operatorname{ri} S(\mathrm{COPOS})$, and $(\tilde{x}, \tilde{y}) \in S(\mathrm{COPOS})$ be of maximal complementarity. Then there exists $\varepsilon > 0$ such that $(\hat{x}, \hat{y}) - \varepsilon(\tilde{x}, \tilde{y}) \in S(\mathrm{COPOS})$. Set $\alpha = 1/(1+\varepsilon) \in (0, 1)$. We may write

$$\alpha(\hat{x}, \hat{y}) = \alpha[(\hat{x}, \hat{y}) - \varepsilon(\tilde{x}, \tilde{y})] + (1-\alpha)(\tilde{x}, \tilde{y}).$$

Similarly, setting $\hat{s} := A\hat{x} - b$ and $\tilde{s} := A\tilde{x} - b$, we have that

$$\alpha(\hat{s}^j, \hat{y}^j) = \alpha[(\hat{s}^j, \hat{y}^j) - \varepsilon(\tilde{s}^j, \tilde{y}^j)] + (1-\alpha)(\tilde{s}^j, \tilde{y}^j).$$

We conclude by applying lemma 6 to the above relation. ■

We now introduce another problem related to (COP), having in mind the relations between SOCP and SDP problems. Let $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_J$ be another finite family of closed convex cones in $I\!R^{r_j}$, $j = 1$ to J, and M^j be $r_j \times q_j$ matrices such that

$$s^j \in K_j \quad \text{iff} \quad M^j s^j \in \mathcal{K}_j, \; j = 1, \ldots, J. \tag{9}$$

Let $M = (M^1; \cdots; M^J)$ be the matrix whose rows are those of M^j. Then (COP) is equivalent to the linear conic problem

$$\operatorname*{Min}_{x \in I\!R^n} c \cdot x\,;\ M^j(A^j x - b^j) \in \mathcal{K}_j,\ j = 1, \ldots, J, \tag{MCOP}$$

whose dual is

$$\operatorname*{Max}_{z \in \mathcal{K}^+} \sum_{j=1}^{J} b^j \cdot M^\top z^j;\quad \sum_{j=1}^{J} (A^j)^\top (M^j)^\top z = c;\quad z^j \in \mathcal{K}_j^+,\ j = 1, \ldots, J. \tag{MCOP*}$$

If the primal and dual values are equal, a pair (x, y) of the primal and dual problems is characterized by the optimality system

$$\begin{cases} M^j(A^j x - b^j) \in \mathcal{K}_j,\ \ z^j \in \mathcal{K}_j^+,\ \ z^j \cdot M^j(A^j x - b^j) = 0,\ j = 1, \ldots, J; \\ \sum_{j=1}^{J} (A^j)^\top (M^j)^\top z^j = c. \end{cases} \tag{10}$$

We first state two lemmas that deal with properties that do not involve explicitly the product form.

Lemma 8. *The following relations hold:* (i) $MS(COP) = S(MCOP)$, $M^\top \mathcal{K}^+ \subset K^+$, *and* $M^\top S(MCOP^*) \subset S(COP^*)$. (ii) *If* $M^\top \mathcal{K}^+$ *is closed, then* $M^\top \mathcal{K}^+ = K^+$ *and* $M^\top S(MCOP^*) = S(COP^*)$. (iii) *Closeness of* $M^\top \mathcal{K}^+$ *holds if* $M^\top$ *is coercive on* $\mathcal{K}^+$, *i.e., if* $\|M^\top z\| \geq c\|z\|$ *for all* $z \in \mathcal{K}^+$. *In that case,* $S(MCOP^*)$ *is bounded iff* $S(COP^*)$ *is bounded.*

Proof. (i) That $MS(COP) = S(MCOP)$ is a consequence of (9). Since $MK \subset \mathcal{K}$, any $z \in \mathcal{K}^+$ is such that $M^\top z \in K^+$. It follows from the expression of dual problems that $M^\top S(MCOP^*) \subset S(COP^*)$.
(ii) Assume now that $\hat{K} := M^\top \mathcal{K}^+$ is closed. Since we know that $M^\top \mathcal{K}^+ \subset K^+$, we have to prove the converse inclusion. If this is not true, then there exists $y \in K^+$, $y \notin \hat{K}$. By the separation theorem there exists $h \in \hat{K}^+$ such that $h^\top y < 0$. That $h \in \hat{K}^+$ is equivalent to $Mh \in \mathcal{K}$, hence to $h \in K$; but since $y \in K^+$, this contradicts $h^\top y < 0$. This proves $M^\top \mathcal{K}^+ \subset K^+$, from which $M^\top S(MCOP^*) = S(COP^*)$ follows easily.
(iii) Finally, that the closeness of $M^\top \mathcal{K}^+$ is a consequence of coercivity of $M^\top$ is easy and left to the reader, as well as the equivalence of boundedness of $S(MCOP^*)$ and $S(COP^*)$. ■

Lemma 9. *Assume that* M *is one to one. Then the following holds.* (i) *The mapping* $M^\top$ *is onto, and* $M^\top \operatorname{int} \mathcal{K}^+ \subset \operatorname{int} K^+$. (ii) *If in addition* $M^\top \mathcal{K}^+$ *is closed, then* $M^\top \operatorname{int} \mathcal{K}^+ = \operatorname{int} K^+$ *and* $M^\top \operatorname{int} S(MCOP^*) = \operatorname{int} S(COP^*)$. (iii) *Under the same assumptions as in* (ii) *we also have that, for all* $s \in K$, $M^\top \operatorname{ri}(\mathcal{K}^+ \cap (Ms)^\perp) \subset \operatorname{ri}(K^+ \cap s^\perp)$.

Proof. (i) That the transposition of an injective mapping is surjective is well-known. If $z \in \operatorname{int} \mathcal{K}^+$, then there exists $\varepsilon > 0$ such that $z + \varepsilon B \subset \mathcal{K}^+$ (where B denotes the Euclidean ball). Since $M^\top$ is onto, $M^\top B \supset \alpha B$ for some $\alpha > 0$, and hence, $K^+ \supset M^\top(z + \varepsilon B) \supset M^\top z + \varepsilon \alpha B$, which proves that $M^\top z \in \operatorname{int} K^+$.
(ii) Since $M^\top$ is onto, $M^\top \operatorname{int} \mathcal{K}^+$ is an open set. As $M^\top \mathcal{K}^+$ is closed, the closure of

$M^\top \operatorname{int}\mathcal{K}^+$ is $M^\top\mathcal{K}^+$, and the latter is equal to K^+ by lemma 8. This means that $M^\top \operatorname{int}\mathcal{K}^+ = \operatorname{int} K^+$. The equality between $M^\top \operatorname{int} S(MCOP^*)$ and $\operatorname{int} S(COP^*)$ is proved in a similar manner.
(iii) We know that $M^\top\mathcal{K}^+ = K^+$, and that for all $x \in K^+$, $z \cdot Mx = 0$ iff $(M^\top z)\cdot x = 0$. It follows that $M^\top(\mathcal{K}^+ \cap (Ms)^\perp) = (K^+ \cap s^\perp)$.

Let $z \in \operatorname{ri}(\mathcal{K}^+ \cap (Ms)^\perp)$, and set $y = M^\top z$. Let $y' \in K^+ \cap s^\perp$. We know that there exists $z' \in \mathcal{K}^+ \cap (Ms)^\perp$ such that $y' = M^\top z'$. Since $z \in \operatorname{ri}(\mathcal{K}^+ \cap (Ms)^\perp)$, there exists $\varepsilon' > 0$ such that $z \pm \varepsilon z' \in \mathcal{K}^+ \cap (Ms)^\perp$. It follows that $y \pm \varepsilon y' \in K^+ \cap s^\perp$. The conclusion follows. ■

We denote by $(B_{COP}, N_{COP}, R_{COP}, T_{COP})$ and $(B_{MCOP}, N_{MCOP}, R_{MCOP}, T_{MCOP})$ the optimal partitions of (COP) and $(MCOP)$, respectively.

Lemma 10. *Assume that $M^\top\mathcal{K}^+$ is closed, that M is one to one, and that*

$$\text{For all } s^j \in K_j, M^j s^j \in \partial\mathcal{K}_j \text{ iff } s^j \in \partial K_j. \tag{11}$$

Then the following relations hold between the optimal partitions of problems (COP) and $(MCOP)$:

$$B_{COP} = B_{MCOP}, \quad N_{COP} = N_{MCOP}, \quad R_{COP} \supset R_{MCOP}, \quad T_{COP} \subset T_{MCOP}. \tag{12}$$

In particular, the strict complementarity hypothesis holds for (COP) if it holds for $(MCOP)$.

Proof. That $B_{COP} = B_{MCOP}$ is an immediate consequence of (9) and (11). Applying the first part of lemma 9(ii) to $(K_i, \mathcal{K}_i, M^i)$ we deduce that $N_{COP} = N_{MCOP}$. Finally that $R_{COP} \supset R_{MCOP}$ follows from lemma 9(iii) applied to $(K_i, \mathcal{K}_i, M^i)$. The relation $T_{COP} \subset T_{MCOP}$ follows from the three others.

As a consequence, if T_{MCOP} is empty then T_{COP} is also empty, which means that the strict complementarity hypothesis holds for (COP) if it holds for $(MCOP)$. ■

2.3 Application of the abstract framework

We apply the results of the above section. Here $K_j = Q_{m_j+1}$, $\mathcal{K}_j := \mathcal{S}_+^{m_j+1}$, and $M^j s^j = \operatorname{Arw} s^j$. Note that we can write

$$\operatorname{Arw}(s) = (s_0 - \|\bar{s}\|)I_{m+1} + \begin{pmatrix} \|\bar{s}\| & \bar{s}^\top \\ \bar{s} & \|\bar{s}\| I_m \end{pmatrix}. \tag{13}$$

This shows that for $s \in Q_{m+1} \setminus \{0\}$, $\operatorname{Arw}(s)$ is of rank m iff $s \in \partial Q_{m+1}$, and of rank $m+1$ otherwise. In particular, $\operatorname{Arw} \partial Q_{m+1} \subset \partial \mathcal{S}_+^{m+1}$, and $\operatorname{Arw} \operatorname{int} Q_{m+1} \subset \operatorname{int} \mathcal{S}_+^{m+1}$. Therefore (11) holds. Let us decompose any matrix $Y \in \mathcal{S}^{m+1}$ as follows

$$Y = \begin{pmatrix} Y_{00} & \bar{Y}_0^\top \\ \bar{Y}_0 & \bar{Y} \end{pmatrix}, \tag{14}$$

where $Y_{00} \in I\!R$, $\bar{Y}_0 \in I\!R^m$ and $\bar{Y} \in \mathcal{S}^m$. We note that for any $s \in I\!R^{m+1}$ we get

$$\operatorname{Arw}(s) \cdot Y = s_0 \operatorname{Tr}(Y) + 2\bar{s} \cdot \bar{Y}_0. \tag{15}$$

It follows that $\mathrm{Arw}^\top : \mathcal{S}^{m+1} \to I\!R^{m+1}$ is nothing but

$$\mathrm{Arw}^\top Y := \begin{pmatrix} \mathrm{Tr}(Y) \\ 2\bar{Y}_0 \end{pmatrix}. \tag{16}$$

Consequently

$$M^\top(Y^1, \ldots, Y^J) = \mathrm{vec}\left(\begin{pmatrix} \mathrm{Tr}(Y^1) \\ 2\bar{Y}_0^1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathrm{Tr}(Y^J) \\ 2\bar{Y}_0^J \end{pmatrix} \right). \tag{17}$$

Proposition 11. (i) *We have that y is solution of $(LSOCP^*)$ iff there exists z solution of $(LSDP^*)$ such that $y = M^\top z$.* (ii) *One of these dual problems has a bounded set of solutions iff the other one has the same property.* (iii) *One of these dual problems has an interior feasible point iff the other one has the same property.* (iv) *Problems $(LSOCP)$ and $(LSDP)$ have the same optimal partition.*

Proof. Since $\mathrm{Arw}^\top$ is coercive on $\mathcal{S}_+^{m+1}$, $M^\top$ is also coercive. By lemma 8, we have that $S(LSOCP^*) = M^\top S(LSDP^*)$ and $S(LSDP^*)$ is bounded iff $S(LSOCP^*)$ is bounded. This proves points i) and (ii). Point (iii) is consequence of lemma 9(ii). We now prove (iv). By lemma 10, $B_{LSOCP} = B_{LSDP}$, $N_{LSOCP} = N_{LSDP}$, and $R_{LSOCP} \supset R_{LSDP}$; it remains to prove that $R_{LSOCP} \subset R_{LSDP}$ since (B, N, R, T) is a partition. Let $j \in R_{LSOCP}$. Then there is a pair (x, y) solution of (2) such that $s^j \neq 0 \neq y^j$, and both s^j and y^j belong to the boundary of Q_{m^j+1}. As observed after (13), this implies that $\mathrm{Arw}\, s^j$ is of rank m^j, and hence, the corresponding set of normals is a half line (of rank one semidefinite positive matrices, orthogonal to $\mathrm{Arw}\, s^j$). Since the corresponding multiplier Y for problem $(LSDP)$ is such that $0 \neq y^j = \mathrm{Arw}^\top Y^j$, we have that $Y^j \neq 0$, proving that $-Y^j$ belongs to the relative interior of the normal cone (to the set of semidefinite positive matrices) at $\mathrm{Arw}\, s^j$. ■

The above analysis shows that strong duality holds for problem (LSOCP) iff it holds for problem (LSDP). The next proposition states an interesting relation between the solutions of (LSOCP*) and (LSDP*).

Proposition 12. *Let the strong duality property hold for problem* (LSOCP)*. Let I be the set of indexes in $1, \ldots, J$ such that there exists $x^* \in S$*(LSOCP) *satisfying $A^j x^* \neq b^j$. Then every $Y \in S$*(LSDP*) *is such that, for some $y \in S$*(LSOCP*)*, the following relation holds:*

$$Y^j = 0, \ \ \text{if } y^j = 0; \qquad Y^j = \tfrac{1}{2} \begin{pmatrix} \|\bar{y}^j\| & (\bar{y}^j)^\top \\ \bar{y}^j & \bar{y}^j (\bar{y}^j)^\top / \|\bar{y}^j\| \end{pmatrix}, \ \ \text{otherwise.} \tag{18}$$

Proof. Let $j \in I$, x^* be the associated solution of (LSOCP), and let $Y \in S$(LSDP*). We claim that

$$Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top = 0, \tag{19}$$

where Y_{00}^j, $\bar{Y}^j$ and $\bar{Y}_0^j$ are given by (14). Since $Y^j \in \mathcal{S}_+^{m_j+1}$, by Schur complement the matrix $Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top$ is positive semidefinite, and hence, it is enough to show that

$$\mathrm{Tr}\left(Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top \right) \leq 0. \tag{20}$$

By strong duality, any primal-dual solution (x^*, y^*) of (LSOCP) is solution of (2). Since $A^j x^* \neq b^j$, the complementarity condition implies that any $y \in S(\text{LSOCP}^*)$ satisfies $y_0^j = \|\bar{y}^j\|$. Taking $y^j = \text{Arw}^\top Y^j$, we deduce $\text{Tr}(Y^j) = y_0^j = \|\bar{y}^j\| = 2\|\bar{Y}_0^j\|$, which implies

$$\text{Tr}\left(Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top\right) = Y_{00}^j \,\text{Tr}\left(Y^j\right) - (Y_{00}^j)^2 - \|\bar{Y}_0^j\|^2 = -\left(Y_{00}^j - \|\bar{Y}_0^j\|\right)^2 \leq 0, \quad (21)$$

proving (20) and therefore also (19). Combining (16) and (21), obtain

$$Y_{00}^j = \|\bar{Y}_0^j\| = \tfrac{1}{2}\|\bar{y}^j\|. \quad (22)$$

Now, we distinguish two cases: a) If $Y_{00}^j = 0$, we obtain from (22) that $\bar{Y}_0^j = \bar{y}^j = 0$ and then $\text{Tr}(Y^j) = y_0^j = 0$. Hence, since Y^j is positive semidefinite this implies $Y^j = 0$. b) Else if $Y_{00}^j \neq 0$, we get directly from (19) and (22) that

$$\bar{Y}^j = (Y_{00}^j)^{-1}(\bar{Y}_0^j)(\bar{Y}_0^j)^\top = \frac{2}{\|\bar{y}^j\|}(\bar{y}^j/2)(\bar{y}^j/2)^\top = \tfrac{1}{2}(\bar{y}^j)(\bar{y}^j)^\top/\|\bar{y}^j\|,$$

which, combined with (22), allows to conclude the proof. ■

3 Duality theory for nonlinear SOCP problems

The Lagrangian function associated with problem (SOCP) (stated in the introduction) is $L(x, y) := f(x) - \sum_{j=1}^m y^j \cdot g^j(x)$, and the dual problem is

$$\operatorname*{Max}_{y \in \mathcal{Q}} \inf_x L(x, y), \quad \text{(DSOCP)}$$

where we have set $\mathcal{Q} := \Pi_j Q_{m_j+1}$. If problems (SOCP) and (DSOCP) have the same finite value, then a pair (x, y) of primal and dual solution is characterized by the optimality system

$$L(x, y) = \min_{x'} L(x', y); \quad g^j(x) \in Q_{m_j+1}; \quad y^j \in Q_{m_j+1}; \quad y^j \circ g^j(x) = 0, \; j = 1, \ldots, J. \quad (23)$$

The above statement is of special interest when problem (SOCP) is convex, i.e. (see e.g. [6, Def 2.163]) if $f(x)$ is convex, and the mapping $g(x)$ is convex with respect to the set $\mathcal{Q}' := -\mathcal{Q}$. The latter means [6, Section 2.3.5] that

$$g(tx + (1-t)x') \succeq_{\mathcal{Q}} tg(x) + (1-t)g(x'), \quad \text{for all } x, x' \in I\!R^n \text{ and } t \in [0, 1]. \quad (24)$$

Since $\mathcal{Q}$ is in product form, this is equivalent to say that $g^j(x)$ is convex w.r.t. Q_{m_j+1} for all j, that is, $x \to \|\bar{g}^j(x)\| - g_0^j(x)$ is convex for all j. This holds, for instance, if $\bar{g}^j(x)$ is affine and $g_0^j(x)$ is concave for all j.

The results of the previous sections have a natural extension to nonlinear second order cone problems. Since, for smooth problems, Lagrange multipliers are solutions of the dualization of the linearized problems we have that, for a nonconvex problem, there is a natural notion of optimal partition of constraints (B, N, R, T). For convex nonlinear second order cone problems, we can in the same way define the optimal partition of constraints (B, N, R, T), defined as follows. The set B is the union of j such that there exists $x \in S(SOCP)$ satisfying $g_0^j(x) > \bar{g}^j(x)$, the set N is the union of j such that there exists $y \in S(DSOCP)$ satisfying $y_0^j > \|\bar{y}^j\|$, the set R is the union of j, such that there exists $x \in S(SOCP)$ and $y \in S(DSOCP)$ satisfying $g^j(x) \neq 0 \neq y^j$, and for all $j \in T$, $x \in S(SOCP)$ and $y \in S(DSOCP)$, either $g^j(x)$ or y^j is equal to 0, or both are zero, and neither $g^j(x)$ or y^j belong to the interior of Q_{m_j+}.

Remark 13. *For second order cone problems we can even partition T as T_0, T_P and T_D, with T_0 the set of j for which, if $x \in S(SOCP)$ and $y \in S(DSOCP)$, then $g^j(x) = 0 = y^j$, T_P is the set of $j \in T$ such that there exists $x \in S(SOCP)$ with $\|g^j(x)\| > 0$, and T_D is the set of $j \in T$ such that there exists $y \in S(DSOCP)$ with $\|y^j\| > 0$. It is easy to see that such a refined partition is invariant under the reformulation as a semidefinite programming problem.*

4 Nondegeneracy Condition and Reduction Approach

We recall the basic concepts of the *reduction approach*, see [6, Sec. 3.4.4].

Definition 14. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two finite dimensional spaces. Let $K \subseteq \mathbb{X}$ and $\hat{K} \subseteq \mathbb{Y}$ be closed, convex sets. We say that the set K is* reducible *to $\hat{K}$ at $s^* \in K$ if there exist a neighborhood V of s^* and a smooth mapping $\phi : V \to \mathbb{Y}$ such that: i) for all $s \in V$, $s \in K$ iff $\phi(s) \in \hat{K}$, and ii) $D\phi(s^*) : \mathbb{X} \to \mathbb{Y}$ is onto. If the set K is* reducible *to $\hat{K}$ at all $s^* \in K$, we just say that the set K is reducible to $\hat{K}$. If in addition $\phi(s^*) = 0$, and $\hat{K}$ is a pointed cone, we say that K is cone reducible.*

For our purposes, a smooth mapping will be a twice continuously differentiable (C^2) mapping. For problems with constraints in product form, i.e. $K = K_1 \times \cdots \times K_J$, the reduction approach has the following obvious decomposition property: cone reducibility holds whenever it holds for each set K_j, $j = 1$ to J.

Lemma 15. *The second-order cone Q_{m+1} is cone reducible at every point $\hat{s} \in Q_{m+1}$, in the following way:* (i) *If $\hat{s} = 0$, take $\hat{K} = Q_{m+1}$ and $\phi(s) = s$,* (ii) *If $\hat{s}_0 > \|\bar{\hat{s}}\|$, take $\hat{K} = \{0\}$ and $\phi(s) = 0$,* (iii) *If $0 \neq \bar{s}_0 = \|\bar{s}\|$, take $\hat{K} = \mathbb{R}_-$ and $\phi(s) = \|\bar{s}\| - s_0$.*

Definition 16. *Consider an arbitrary problem* (P) $\text{Min}_{x \in \mathbb{X}}\{f(x)\,;\, g(x) \in K\}$*, where f, g are smooth functions, $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ are finite dimensional spaces and $K \subseteq \mathbb{Y}$ is a closed convex cone, reducible to a closed convex cone $\hat{K} \subseteq \mathbb{Z}$ at $g(x^*) \in K$ by a mapping ϕ. We say that x^* is* nondegenerate *(with respect to the reduction given by ϕ) if the derivative $D\mathcal{A}(x^*)$ of the function $\mathcal{A} := \phi \circ g$ is onto.*

This notion, introduced in [5], generalizes to problems with general constraints the corresponding concept used in linear or nonlinear programming. Note that there are other definitions of nondegeneracy, e.g. [1, Def. 18] and references therein. In the case of second order cones all these definitions are essentially equivalent.

One of the main implication of nondegeneracy is stated in the next proposition, proved in [6, Prop. 4.75].

Proposition 17. *Consider the problem* (P) *given in definition 16. Let x^* be a solution of* (P) *and suppose that the set K is reducible to a pointed closed convex cone $\hat{K}$ at the point $g(x^*)$. If x^* is nondegenerate then there exists a unique Lagrange multiplier y^* associated. Conversely, if the pair (x^*, y^*) is strictly complementarity, and y^* is the unique Lagrange multiplier associated with x^*, then x^* is nondegenerate.*

Proposition 18. *Let x^* be a solution of the second-order problem (LSOCP) with $J = 1$. Set $s^* = Ax^* - b$ and $m = m_1$. Then, x^* is* nondegenerate *if and only if one of the following conditions holds: a) $s^* \in \operatorname{int} Q_{m+1}$, b) $s^* = 0$ and the matrix A is onto, c) $A^\top R_m(Ax^* - b) \neq 0$, where $R_m := \begin{pmatrix} 1 & 0^\top \\ 0 & -I_m \end{pmatrix}$.*

Proof. The result is a direct consequence of lemma 15. ■

We extend the above result to the case $J > 1$.

Proposition 19. *Let x^* be a solution of the second-order problem (LSOCP), and set $s^j = A^j x^* - b^j$. Set $I^* = \{1 \le j \le J; s^j \in \operatorname{int} Q_{m_j+1}\}$, $Z^* = \{1 \le j \le J; s^j = 0\}$, and $B^* = \{1 \le j \le J; s^j \in \partial Q_{m_j+1} \setminus \{0\}\}$, where ∂Q_{m_j+1} is the boundary of Q_{m_j+1}. Then, x^* is* nondegenerate *if and only if the following conditions holds: The matrix $\mathbb{A}$ whose rows are the union of those of A^j, for $j \in Z^*$, and the vectors rows $(A^j x^* - b^j)^\top R_{m_j} A^j$, for $j \in B^*$, is onto.*

Proof. This is once again a consequence of lemma 15. Indeed, for $\mathcal{A}^j(x) := \phi(g^j(x)) = \phi(A^j x - b^j)$, where ϕ is the reduction map of lemma 15, its derivative at x^* is given by

$$D\mathcal{A}^j(x^*) = \Big\{0, \ \text{ if } j \in I^*; \ A^j, \ \text{ if } j \in Z^*; \ -(s_0^j)^{-1}(A^j x^* - b^j)^\top R_{m_j} A^j, \ \text{ if } j \in B^* \Big\}.$$

So, the derivative $D\mathcal{A}(x^*)$ of function $\mathcal{A} := (\mathcal{A}^1; \dots; \mathcal{A}^J)$ is onto iff the matrix $\mathbb{A}$ is onto. ■

Remark 20. *We recover the result of [1, Thm 20]. Obviously, if $(A^1; \cdots; A^J)$ is onto, then any feasible point is nondegenerate.*

For problem (LSOCP), the Lagrange multipliers y^* are the solutions of (LSOCP*), so, if x^* is nondegenerate then proposition 17 says us that the dual problem (LSOCP*) has a unique solution y^*. On the other hand, we know from proposition 11 that any $Y^* \in S(\text{LSDP}^*)$ is such that $y^* = Arw^\top Y^* \in S(\text{LSOCP}^*)$. By proposition 12, if $A^j x^* \neq b^j$ for all j, uniqueness of solution of (LSOCP*) implies uniqueness of the solution of (LSDP*). Yet it may happen that $S(\text{LSDP}^*)$ is not a singleton, even when x^* is nondegenerate for problem (LSOCP), as the next example shows.

Example 21. *Consider just one block $J = 1$. Let $A = I_3 \in I\!R^{3\times 3}$ the identity matrix, $m = 2$, $b = 0$ and $c = (1, 0, 0)^\top$. It follows that $x^* = 0$ is the unique solution of* (LSOCP) *(and then of* (LSDP)*), which is actually nondegenerate, and $y^* = (1, 0, 0)^\top$ is the unique solution of* (LSOCP*)*. Using proposition 11(i) and* (16)*, and since $Ax^* - b = 0$, we see that $Y \in S(\text{LSDP}^*)$ iff $Y \succeq 0$, $\operatorname{Tr}(Y) = 1$ and $\bar{Y}_0 = 0$. For instance, $y^*(y^*)^\top$ and $Y^* = \frac{1}{3} I_3$ belong to $S(\text{LSDP}^*)$.*

5 Strongly Regular Solutions of SOCP

In this section we consider the problem (SOCP) defined in the introduction as follows:

$$\operatorname*{Min}_{x\in \mathbb{R}^n, s^j\in\mathbb{R}^{m_j+1}} f(x);\;\; g^j(x) = s^j \succeq_{Q_{m_j+1}} 0,\; j=1,\dots,J, \tag{SOCP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g^j : \mathbb{R}^n \to \mathbb{R}^{m_j+1}$ are smooth functions (at least C^2). The first-order optimality system is

$$D_x L(x^*, y) = Df(x^*) - \sum_{j=1}^{J} Dg^j(x^*)^\top y^j = 0, \tag{25a}$$

$$g^j(x) = s^j \succeq_{Q_{m_j+1}} 0,\;\; y^j \succeq_{Q_{m_j+1}} 0,\;\; s^j \circ y^j = 0,\;\; j = 1,\dots,J, \tag{25b}$$

where $L : \mathbb{R}^n \times \mathbb{R}^{m+1} \to \mathbb{R}$ is the Lagrangian function of problem (SOCP)

$$L(x, y) := f(x) - \sum_{j=1}^{J} y^j \cdot g^j(x). \tag{26}$$

If (x^*, y^*) satisfies (25), then x^* will be called a critical or stationary point of (SOCP). Let us recall the definition of *strongly regular* solutions [13]:

Definition 22. *We say that* (x^*, y^*) *is a* strongly regular *solution of KKT-conditions* (25) *if there exists a neighborhood* V *of* (x^*, y^*) *such that for every* $\delta := (\delta_1, \delta_2) \in \mathbb{R}^n \times \Pi_{j=1}^{J} \mathbb{R}^{m_j+1}$ *close enough to 0, the "linearized" system:*

$$D^2_{xx} L(x^*, y^*)(x - x^*) - Dg(x^*)^\top (y - y^*) = \delta_1, \tag{27a}$$

$$g(x^*) \circ y + Dg(x^*)(x - x^*) \circ y = \delta_2 \circ y, \tag{27b}$$

$$g(x^*) + Dg(x^*)(x - x^*) - \delta_2 \succeq_{\mathcal{Q}} 0,\; y \succeq_{\mathcal{Q}} 0, \tag{27c}$$

has a unique solution $(x, y) = (x^*(\delta), y^*(\delta))$ *in* V*, which is a Lipschitz continuous map of* δ*.*

It can be shown that the strong regularity condition implies *Robinson's constraint qualification* condition:

$$\text{There exists } h^* \in \mathbb{R}^n \text{ such that } g(x^*) + Dg(x^*)h^* \in \operatorname{int} \mathcal{Q}, \tag{28}$$

which coincides with the *Slater* (or *primal strict feasibility*) condition for linear problem (LSOCP). This condition is discussed in [6, Section 2.3.4].

In this section we characterize the strong regularity in the context of problem (SOCP) by using second order optimality conditions. This characterization is a consequence of a well developed theory in a general conic optimization framework given by problem (P) stated in definition 16. Note that the strong regularity condition (definition 22) can be written in this general framework as

$$D^2_{xx} L(x^*, y^*)(x - x^*) - Dg(x^*)^\top (y - y^*) = \delta_1, \tag{29a}$$

$$(g(x^*) + Dg(x^*)(x - x^*) - \delta_2) \cdot y = 0, \tag{29b}$$

$$g(x^*) + Dg(x^*)(x - x^*) - \delta_2 \in K,\; y \in K^-. \tag{29c}$$

In order to establish our main result we will recall some key notions and theorems. For instance, a useful definition involved in this section is the following *uniform second order growth* condition [12]. For this, we define a family of perturbation of (P), denoted (P_u), as follows

$$\operatorname*{Min}_{x\in\mathbb{X}}\{f(x,u)\,;\, g(x,u)\in K\}, \tag{30}$$

where $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{U}$ are finite dimensional spaces, $u \in \mathbb{U}$ (perturbation space) is the perturbation parameter and the functions $f(x,u) : \mathbb{X}\times\mathbb{U} \to I\!R$ and $g(x,u) : \mathbb{X}\times\mathbb{U} \to \mathbb{Y}$ are at least twice continuously differentiable and satisfy $f(\cdot,0) := f(\cdot)$ as well as $g(\cdot,0) := g(\cdot)$.

Definition 23. *Let x^* be a stationary (or critical) point of problem* (P). *It is said that the* uniform second order growth *condition holds at x^* if there exist $\alpha > 0$ and a neighborhood $\mathcal{N}$ of x^* such that for any $u \in \mathbb{U}$ (perturbation space) close enough to 0 and any stationary (or critical) point $x^*(u) \in \mathcal{N}$ of the perturbed problem* (P_u)*, we have that*

$$f(x,u) \geq f(x^*(u),u) + \alpha\|x - x^*(u)\|^2, \quad \forall x \in \mathcal{N},\ g(x,u) \in K. \tag{31}$$

We say that the second order growth *condition holds at x^* if* (31) *holds for problem* (P)*, that is, there exist $\alpha > 0$ and a neighborhood $\mathcal{N}$ of x^* such that condition* (31) *is satisfied at $u = 0$ and $x^*(0) = x^*$.*

We need the next result, obtained in [6, Th. 5.24], that states a first characterization which is valid in a general context.

Theorem 24. *Let x^* be a local solution of problem (P) and y^* its corresponding Lagrange multiplier. Suppose that K is reducible to a pointed closed convex cone $\hat{K} \subseteq \mathbb{Z}$ at the point $g(x^*)$. Then (x^*, y^*) is a strongly regular solution of the Karush-Kuhn-Tucker conditions if and only if x^* is nondegenerate (definition 16) and the uniform second order growth condition holds at x^*.*

Theorem 24 means that we can completely characterize the strong regularity condition by giving sufficient and necessary conditions to obtain the uniform second order growth condition, under a nondegeneracy hypothesis. Unfortunately, such a characterization (in terms of derivatives of data at the nominal point) is known only in very specific examples as for nonlinear programming problems with C^2 data, see e.g. Bonnans and Sulem [7] and Dontchev and Rockafellar [9] and their references. For conic optimization problems, such a characterization is not known. In fact the (non uniform) second order growth condition itself can be characterized essentially in two situations. The first is when the cone is second order regular, see Bonnans, Cominetti and Shapiro [2], and the second is when reduction to a pointed cone holds. We will apply this second approach later in this section. Let us denote by $\Lambda(x^*)$ the set of Lagrange multiplier associated with x^* for problem (P), i.e., $y^* \in \Lambda(x^*)$ iff $D_x L(x^*, y^*) = 0$ and $-y^* \in N_K(g(x^*))$ (the *normal cone* to K at $g(x^*)$), where $L(x,y) := f(x) - y \cdot g(x)$ is the Lagrangian function of problem (P). We define the *tangent cone* to the set $K \subseteq \mathbb{Y}$ at the point $y \in K$ as

$$T_K(s) := \{d \in \mathbb{Y} \,:\, s + td + o(t) \in K,\ \forall t > 0\}, \tag{32}$$

and the *critical directions cone* at x^* for problem (P) as follows

$$C(x^*) := Df(x^*)^{\perp} \cap Dg(x^*)^{-1} T_K(g(x^*)) \tag{33}$$

or equivalently, if $\Lambda(x^*)$ is not empty, say contains some y^*:

$$C(x^*) := \{h \in \mathbb{X} \, : \, Dg(x^*)h \in T_K(g(x^*)) \cap (y^*)^\perp\}.$$

Lemma 25. *Consider the second order cone $Q := Q_{m+1}$ and let $s \in Q$. Then,*

$$T_Q(s) = \left\{ \begin{array}{ll} I\!\!R^{m+1}, & s \in \operatorname{int} Q, \\ Q, & s = 0, \\ d \in I\!\!R^{m+1} \, : \, \bar{d} \cdot \bar{s} - s_0 d_0 \le 0, & s \in \partial Q \setminus \{0\}. \end{array} \right\} \tag{34}$$

Proof. The cases when $s \in \operatorname{int} Q$ and $s = 0$ follow directly from the definition of $T_Q(s)$ and the fact that Q is a cone. Suppose then that $s \in \partial Q \setminus \{0\}$, that is, $s_0 = \|\bar{s}\| \ne 0$.

Since the set Q can be written in the form $Q = \{s \in I\!\!R^{m+1} \, : \, \phi(s) \le 0\}$, where $\phi(s) := \|\bar{s}\| - s_0$ is a convex differentiable function at all s such that $\bar{s} \ne 0$, by [6, Prop. 2.61] the tangent cone $T_Q(s)$ is given by

$$T_Q(s) = \{d \in I\!\!R^{m+1} \, : \, \phi'(s; d) \le 0\}.$$

Therefore, we conclude by noting that the directional derivative $\phi'(s; d)$ when $\bar{s} \ne 0$ is equal to $\phi'(s; d) = D\phi(s) \cdot d = \bar{s} \cdot \bar{d} / \|\bar{s}\| - d_0$, and using $0 \ne s_0 = \|\bar{s}\|$. ■

Corollary 26. *Let x^* be a stationary (or critical) point of problem (SOCP) and $y \in \Lambda(x^*)$. Given $h \in I\!\!R^n$, denote $d^j(h) := Dg^j(x^*)h$, as well as $s^j = g^j(x^*)$. Then, the critical directions cone $C(x^*)$ is given by*

$$C(x^*) = \left\{ \begin{array}{ll} h \in I\!\!R^n \, : \, \text{for all } j = 1, \dots, J, & \\ d^j(h) \in T_{Q_{m_j+1}}(s^j), & y^j = 0, \\ d^j(h) = 0, & y^j \in \operatorname{int} Q_{m_j+1}, \\ d^j(h) \in I\!\!R_+(y_0^j, -\bar{y}^j), & y^j \in \partial Q_{m_j+1} \setminus \{0\}, \; s^j = 0, \\ d^j(h) \cdot y^j = 0, & y^j, \, s^j \in \partial Q_{m_j+1} \setminus \{0\}. \end{array} \right. \tag{35}$$

Proof. Since the constraints are in product form, the critical cone has the following decomposition property:

$$C(x^*) = \Big\{ h \in I\!\!R^n; \; d^j(h) \in T_{Q_{m_j+1}}(s^j), \; d^j(h) \cdot y^j = 0, \; j = 1, \dots, J \Big\}. \tag{36}$$

It suffices to establish the equivalence between the relations in (35) and (36) concerning a given j. The case when $y^j = 0$ is obvious. If $y^j \in \operatorname{int} Q_{m_j+1}$, then $s^j = 0$ (by (25b)), and hence, $T_{Q_{m_j+1}}(s^j) = Q_{m_j+1}$, concluding that $T_{Q_{m_j+1}}(s^j) \cap (y^j)^\perp = Q_{m_j+1} \cap (y^j)^\perp = \{0\}$ and the result follows.

Suppose now that $y^j \in \partial Q_{m_j+1} \setminus \{0\}$. If $s^j = 0$ then, $T_{Q_{m_j+1}}(s^j) = Q_{m_j+1}$ again. Using (3), we obtain after elementary computations that $Q_{m_j+1} \cap (y^j)^\perp$ is the set of d^j satisfying $d_0^j(h) = \|\bar{d}^j(h)\|$ as well as $\bar{d}^j(h) \in I\!\!R_- \bar{y}^j$. If $s^j \ne 0$, we obtain by similar computations that $T_{Q_{m_j+1}}(s^j) \cap (y^j)^\perp$ is the set of d^j satisfying $\bar{d}^j(h) \cdot \bar{s}^j - s_0^j d_0^j(h) = 0$. The conclusion follows. ■

For the second-order analysis we need the notion of *(outer) second order tangent set* at $s \in K$ in the direction $d \in T_K(s)$, defined as follows

$$T_K^2(s,d) := \{w \in \mathbb{Y}\,;\, \exists t_n \downarrow 0 \text{ s.t. } s + t_n d + \tfrac{1}{2}t_n^2 w + o(t_n^2) \in K\}. \tag{37}$$

Let us characterize this set when $K = Q$.

Lemma 27. *Let $s \in Q = Q_{m+1}$, and $d \in T_Q(s)$. Then,*

$$T_Q^2(s,d) = \begin{cases} I\!R^{m+1}, & d \in \operatorname{int} T_Q(s), \\ T_Q(d), & s = 0, \\ \{w \in I\!R^{m+1} \,:\, \bar{w}\cdot\bar{s} - w_0 s_0 \le d_0^2 - \|\bar{d}\|^2\}, & \textit{otherwise.} \end{cases} \tag{38}$$

Note that the last case in (38) is when $s \in \partial Q \setminus \{0\}$ and $d \in \partial T_Q(s)$, the latter being, by lemma 25, equivalent to $\bar{d}\cdot\bar{s} - s_0 d_0 = 0$.

Proof. The first two cases follow directly from the definitions of second order tangent set, and the fact that Q is a cone. Suppose now that $s \in \partial Q \setminus \{0\}$ and $d \in \partial T_Q(s)$. As in lemma 25, since Q has the form $Q = \{s \in I\!R^{m+1} \,:\, \phi(s) \le 0\}$, where $\phi(s) := \|\bar{s}\| - s_0$, by [6, Prop. 3.30], the set $T_Q^2(s,d)$ is given by

$$T_Q^2(s,d) = \{d \in I\!R^{m+1} \,:\, \phi''(s;d,w) \le 0\},$$

where

$$\phi''(s;d,w) := \lim_{t\downarrow 0} \frac{\phi(s + td + \frac{1}{2}t^2 w) - \phi(s) - t\phi'(s;d)}{\frac{1}{2}t^2}$$

is the *(parabolic) second order directional derivative* of ϕ. But ϕ is twice differentiable at all s such that $\bar{s} \neq 0$ which implies that (e.g. [6, Eq. 2.81])

$$\phi''(s;d,w) = D\phi(s)w + D^2\phi(s)(d,d) = \frac{\bar{s}\cdot\bar{w}}{\|\bar{s}\|} - w_0 + \frac{\|\bar{d}\|^2}{\|\bar{s}\|} - \frac{(\bar{d}\cdot\bar{s})^2}{\|\bar{s}\|^3}, \tag{39}$$

and the desired result follows using $s_0 = \|\bar{s}\|$ and $d_0 s_0 = \bar{d}\cdot\bar{s}$ (the latter being consequence of lemma 25 and the fact that $d \in \partial T_Q(s)$). ■

Roughly speaking, the characterization of the second order growth condition (definition 23), established in [2, Th. 3.2], assumes a notion of set regularity on K, called *second order regularity*, that holds under the hypothesis that the set K is reducible to a cone $\hat{K}$ (e.g. [6, Prop.3.136]). The result presented below is a simplified version of this characterization. (cf. [6, Th. 3.137].)

Theorem 28. *Let x^* be a feasible point of problem (P) satisfying Robinson's constraint qualification condition*

$$0 \in \operatorname{int}\{\, g(x^*) + Dg(x^*)\mathbb{X} - K\} \tag{40}$$

Suppose that the set K is reducible to a closed convex cone $\hat{K}$ at the point $g(x^*)$. Then, the second order growth condition holds at x^* iff the next second order condition holds:

$$\sup_{y^*\in\Lambda(x^*)} D_{xx}^2 L(x^*,y^*)(h,h) - \sigma(-y^*;\mathcal{T}^2) > 0, \quad \forall h \in C(x^*)\setminus\{0\}, \tag{41}$$

where $\sigma(\cdot\,;\mathcal{T}^2)$ denotes the support function of the set $\mathcal{T}^2 := T_K^2(g(x^*), Dg(x^*)h)$.

In the case of problem (SOCP) (i.e., $K = \mathcal{Q}$), the set $\mathcal{T}^2 := T^2_{\mathcal{Q}}(g(x^*), Dg(x^*)h)$ can be written in the product form $\mathcal{T}^2 = \mathcal{T}^2_1 \times \ldots \times \mathcal{T}^2_J$ such that each $\mathcal{T}^2_j$ is given by formula (38) where $Q = Q_{m_j+1}$, $s = s^{*j}$ and $d = d^j(h)$. We have set $s^* := g(x^*)$ and $d(h) := Dg(x^*)h$. Since $-y^* \in N_{\mathcal{Q}}(s^*) \cap d^\perp$, we always have that $y^* \cdot w \geq 0$, for all $w \in \mathcal{T}^2$. So, formula (38) implies that $0 \in \mathcal{T}^2$ and hence $\sigma(-y^*; \mathcal{T}^2) = 0$, except in the case when $s^{*j} \in \partial Q_{m_j+1} \setminus \{0\}$ and $d^j(h) \in \partial T_{Q_{m_j+1}}(s^{*j}) \setminus \{0\}$, for some index $j \in \{1, \ldots, J\}$. Dealing with the latter case means, thanks to (38), to maximize $-(y_0 w_0 + \bar{y} \cdot \bar{w})$ over the set of w satisfying $\bar{w} \cdot \bar{s} - w_0 s_0 \leq d_0^2 - \|\bar{d}\|^2$, where we have considered the notation $y = y^{*j}$, and s and d given above, with j given by the case. Since $\bar{y} = -(y_0/s_0)\bar{s}$, we have that $-(y_0 w_0 + \bar{y} \cdot \bar{w}) = (y_0/s_0)(\bar{w} \cdot \bar{s} - w_0 s_0)$. It follows that

$$\sigma(-y^*; \mathcal{T}^2) = \sum_{j \in \mathcal{J}} (y_0^{*j}/s_0^{*j})(d^j(h)_0^2 - \|\bar{d}^j(h)\|^2), \tag{42}$$

where $\mathcal{J}$ is the set of index j s.t. $s^{*j} \in \partial Q_{m_j+1} \setminus \{0\}$ and $d^j(h) \in \partial T_{Q_{m_j+1}}(s^{*j}) \setminus \{0\}$. On the other hand, we know that $\mathcal{Q}$ is reducible, (cf. lemma 15), so we can apply theorem 28 to problem (SOCP) and state the following theorem.

Theorem 29. *Let x^* be a feasible point of the problem (SOCP) satisfying Robinson's constraint qualification condition* (28)*. Then, the second order growth condition holds at x^* iff the following second order condition holds:*

$$\sup_{y \in \Lambda(x^*)} D^2_{xx} L(x^*, y)(h, h) + h^\top \mathcal{H}(x^*, y) h > 0, \quad \forall h \in C(x^*) \setminus \{0\}, \tag{43}$$

where the critical directions cone $C(x^*)$ is established in (35)*, and the $n \times n$ matrix $\mathcal{H}(x^*, y)$ is defined by $\mathcal{H}(x^*, y) = \sum_{j=1}^J \mathcal{H}^j(x^*, y^j)$, where for $s^j = g^j(x^*)$, $j = 1$ to J,*

$$\mathcal{H}^j(x^*, y^j) := -\frac{y_0^j}{s_0^j} Dg^j(x^*)^\top R_{m_j} Dg^j(x^*) = -\frac{y_0^j}{s_0^j} Dg^j(x^*)^\top \begin{pmatrix} 1 & 0^\top \\ 0 & -I_{m_j} \end{pmatrix} Dg^j(x^*), \tag{44}$$

if $s^j \in \partial Q_{m_j+1} \setminus \{0\}$, and $\mathcal{H}^j(x^*, y^j) := 0$ otherwise.

In the next theorem we give a characterization of the strong regularity condition.

Theorem 30. *Let x^* be a local solution of problem (SOCP) and y^* its corresponding Lagrange multiplier. Then, (x^*, y^*) is a strongly regular solution of optimality conditions* (25) *iff x^* is nondegenerate (definition 16) and the next second order condition holds at x^*:*

$$Q_0(h) := D^2_{xx} L(x^*, y^*)(h, h) + h^\top \mathcal{H}(x^*, y^*) h > 0, \quad \forall h \in \mathrm{Sp}(C(x^*)) \setminus \{0\}. \tag{45}$$

Proof. a) We establish some preliminary results. By theorem 24 we know that (x^*, y^*) is a strongly regular solution of (25) iff x^* is nondegenerate and the uniform growth condition holds at x^* for problem (SOCP). So, under the nondegeneracy hypothesis, we just need to prove that second order condition (45) is equivalent to the uniform growth condition. It is not difficult to check that, under this hypothesis, the linear space generated by the critical cone has the following expression:

$$\operatorname{Sp}(C(x^*)) = \begin{cases} h \in I\!R^n : & \text{for all } j = 1, \dots, J, \\ d^j(h) = 0, & y^j \in \operatorname{int} Q_{m_j+1}, \\ d^j(h) \in I\!R(y_0^j, -\bar{y}^j), & y^j \in \partial Q_{m_j+1} \setminus \{0\},\ s^j = 0, \\ d^j(h) \cdot y^j = 0, & y^j \in \partial Q_{m_j+1} \setminus \{0\},\ s^j \in \partial Q_{m_j+1} \setminus \{0\}, \end{cases} \tag{46}$$

where throughout this proof we will denote by y^j the j-th vector block of y^*. (In particular, there is no condition on $d^j(h)$ if $y^j = 0$.)
b) Let us prove that the uniform growth condition implies (45). Consider the vector space E defined by

$$E := \begin{cases} h \in I\!R^n : & \text{for all } j = 1, \dots, J, \\ d^j(h) = 0, & y^j \in \operatorname{int} Q_{m_j+1}, \\ d^j(h) \cdot y^j = 0, & y^j \in \partial Q_{m_j+1} \setminus \{0\}. \end{cases} \tag{47}$$

(Again, there is no restriction of $d^j(h)$ if $y^j = 0$.) We have that $\operatorname{Sp}(C(x^*)) \subset E$. The key idea is to consider a perturbed version of problem (SOCP) in such a way that x^* is still a local solution with the same Lagrange multiplier y^*, but with a bigger critical cone, equal to E. This perturbed problem is of the form

$$\operatorname*{Min}_{x \in I\!R^n} f(x)\,;\ g^j(x,u) := g^j(x) + u\delta^j \succeq_{Q_{m_j+1}} 0,\ \ j = 1, \dots, J, \tag{SOCP$_u$}$$

where for all j, e_1^j denotes the first element of the natural basis of $I\!R^{m_j+1}$, $u > 0$ is the perturbation parameter, and

$$\delta^j = \begin{cases} e_1^j & \text{if } y^j = 0, \\ (y_0^j, -\bar{y}^j) & \text{if } \ s^j := g^j(x^*) = 0,\ y^j \in \partial Q_{m_j+1} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{48}$$

This means that, if $y^j = 0$, the constraint $g^j(x) \succeq_{Q_{m_j+1}} 0$ is made inactive (in a neighborhood of x^*), and if $s^j = 0$ and $y^j \in \partial Q_{m_j+1} \setminus \{0\}$, then the constraint $g^j(x) \succeq_{Q_{m_j+1}} 0$ is still active, but at a point different from 0 where the set of tangent directions to Q_{m_j+1} is a half space. The point (x^*, y^*) is still solution of the optimality system of (SOCP$_u$). It is easily seen that the expression of the critical cone for problem (SOCP$_u$) at the point (x^*, y^*) is given by (47).

Define

$$I := \{1 \le j \le J;\ g^j(x^*) = 0,\ y^j \in \partial Q_{m_j+1} \setminus \{0\}\}.$$

Let $\mathcal{H}(x^*, y^j, u)$ denote the matrices in the expression of second order conditions, for the perturbed problem. We have that $\mathcal{H}(x^*, y^j, u) = \mathcal{H}(x^*, y^j)$ for all $j \notin I$, whereas for $j \in I$ we obtain

$$\mathcal{H}^j(x^*, y^j, u) = \frac{1}{u}\hat{\mathcal{H}}^j(x^*, y^j), \ \ \text{where } \ \hat{\mathcal{H}}^j(x^*, y^j) := -Dg^j(x^*)^\top \begin{pmatrix} 1 & 0^\top \\ 0 & -I_{m_j} \end{pmatrix} Dg^j(x^*). \tag{49}$$

Set

$$Q_1(h) := \sum_{j\in I} h^\top \hat{\mathcal{H}}^j(x^*, y^j)h = \sum_{j\in I} \left(\|\bar{d}^j(h)\|^2 - (d^j(h)_0)^2\right). \tag{50}$$

Note that, if $h \in E$, then since $d^j(h)\cdot y^j = 0$ and $y_0^j = \|\bar{y}^j\|$:

$$|d^j(h)_0| = |\bar{d}^j(h)\cdot \bar{y}^j|/y_0^j \le \|\bar{d}^j(h)\|, \tag{51}$$

with equality iff $d^j(h) \in I\!R(y_0^j, -\bar{y}^j)$. Combining with (50), we obtain that, for all $h \in E$, $Q_1(h) \ge 0$, and that $Q_1(h) = 0$ iff $h \in \mathrm{Sp}(C(x^*))$.

We see that the uniform second-order growth for the perturbed problem implies

$$Q_0(h) + \frac{1}{u}Q_1(h) > 0 \quad \text{for all } \ h \in E\setminus\{0\}, \tag{52}$$

for u small enough. This implies that $Q_0(h) > 0$, for all $h \in E$ such that $Q_1(h) = 0$. Therefore, the uniform second-order growth condition implies (45).
c) Conversely, assume that the second order condition (45) holds. If the uniform second order growth condition at x^* is not satisfied, then there exists a family of perturbed functions $f(x,u)$ and $g(x,u)$ such that, for some sequences $u_n \to 0$, there exist (x_n, y_n) solution of the optimality system (25) of the perturbed problem satisfying $x_n \to x^*$, $h_n \to 0$ in $I\!R^n$, with $h_n \ne 0$, such that $x_n + h_n$ is a feasible point of (P_{u_n}) (cf. (30)) (that is, $g(x_n + h_n, u_n) \in \mathcal{Q}$) and they also satisfy that

$$f(x_n + h_n, u_n) \le f(x_n, u_n) + o(\|h_n\|^2). \tag{53}$$

The nondegeneracy condition being stable under small perturbations, for large enough n, there exists a unique Lagrange multiplier y_n associated with each stationary (or critical primal) point x_n of (P_{u_n}), and since $x_n \to x^*$, we have that $y_n \to y^*$.

Extracting if necessary a subsequence, we may assume that $h_n/\|h_n\|$ converges to some $h^* \ne 0$. Let us check that $h^* \in \mathrm{Sp}(C(\bar{x}))$. Since $g^j(x_n + h_n, u_n) \in Q_{m_j+1}$ we have that

$$g^j(x_n + h_n, u_n) = g^j(x_n, u_n) + D_x g^j(x_n, u_n)h_n + o(\|h_n\|) \succeq_{Q_{m_j+1}} 0. \tag{54}$$

Since $g^j(x_n, u_n)$ and $(y_n)^j$ are orthogonal this implies

$$(y_n)^j \cdot D_x g^j(x_n, u_n)h_n + o(\|h_n\|) \ge 0. \tag{55}$$

Dividing by $\|h_n\|$, setting $d^j(h^*) := Dg^j(x^*)h^*$, and passing to the limit, obtain $y^j \cdot Dg^j(x^*)h^* \ge 0$ for all j. Passing to the limit in (53) and combining with (25a), we obtain $0 \ge \nabla f(x^*)\cdot h^* = y \cdot Dg(x^*)h^* = \sum_{j=1}^J y^j \cdot Dg^j(x^*)h^*$. We have proved that

$$d^j(h^*)\cdot y^j = 0, \quad j = 1,\ldots,J. \tag{56}$$

Consider the case when $y^j \in \mathrm{int}\, Q_{m_j+1}$. Since $y_n^j \to y^j$, we have that $g^j(x_n, u_n) = 0$ for large enough n. Let $\varepsilon > 0$ be such that $y^j + 2\varepsilon B \subset Q_{m_j+1}$. Then for all unit vector z, $y_n^j + \varepsilon z \in Q_{m_j+1}$ for large enough n. Computing the scalar product of (54) by $y_n^j + \varepsilon z$, and passing to the limit as was done before, obtain $(y^j + \varepsilon z) \cdot Dg^j(x^*)h^* \geq 0$. Using (56), since this is true for any unit norm z, we get

$$d^j(h^*) = 0, \qquad \text{for all } j;\ y^j \in \operatorname{int} Q_{m_j+1}. \tag{57}$$

Now in the case when $y^j \in \partial Q_{m_j+1} \setminus \{0\}$ and $g^j(x^*) = 0$, we have that $g^j(x_n, u_n) \in \partial Q_{m_j+1}$ for all n large enough (otherwise we obtain from complementarity condition that $y_n^j = 0$ for some sequence $y_n^j \to y^j \neq 0$). Let us set $g_n^j := g^j(x_n, u_n)$ and $d_n^j := D_x g^j(x_n, u_n)h_n$. Of course $d_n^j \to d^j(h^*) := Dg^j(x^*)h^*$. By the very definition of Q_{m_j+1}, condition (54) can be equivalently written as follows

$$(g_n^j)_0 + (d_n^j)_0 \geq \|\bar{g}_n^j + \bar{d}_n^j\| + o(\|h_n\|).$$

Since $g_n \in \partial Q_{m_j+1}$, that is $(g_n^j)_0 = \|\bar{g}_n^j\|$, we obtain that

$$(d_n^j)_0 \geq \|\bar{g}_n^j + \bar{d}_n^j\| - \|\bar{g}_n^j\| + o(\|h_n\|) \geq \|\bar{d}_n^j\| + o(\|h_n\|).$$

Hence, by dividing by $\|h_n\|$ and tending $n \to +\infty$, we deduce that $d^j(h^*) \in Q_{m_j+1} = T_{Q_{m_j+1}}(g^j(x^*))$. This together with relations (56)-(57) proves that $h^* \in \mathrm{Sp}(C(\bar{x}))$.

We now use the same reduction argument as in lemma 15. It suffices for indexes in

$$I := \{1 \leq j \leq J : \ \ g^j(x^*) \neq 0 \neq y^j\}. \tag{58}$$

to change the formulation of corresponding constraint of the perturbed problem, that is, $g^j(x,u) \succeq_{Q_{m_j+1}} 0$, into $\phi(g^j(x,u)) \leq 0$, where $\phi(s) := \|\bar{s}\| - s_0$. The corresponding component of Lagrange multiplier is y_0^j (see the discussion of relation between Lagrange multipliers before and after reduction in [6, Section 3.4.4], especially equation (3.267)). We have that, for each feasible point of the perturbed problem (P_{u_n}), and denoting by y_n the Lagrange multiplier associated with x_n,

$$\sum_{j \notin I} (y_n)^j \cdot g^j(x,u) + \sum_{j \in I} (y_n)_0^j \phi(g^j(x,u)) \geq 0. \tag{59}$$

Writing this inequality at point $(x_n + h_n, u_n)$ and noticing that equality holds at (x_n, u_n) in view of the complementarity conditions, obtain

$$\begin{aligned} &\sum_{j \notin I} (y_n)^j \cdot (g^j(x_n + h_n, u_n) - g^j(x_n, u_n)) \\ &\qquad + \sum_{j \in I} (y_n)_0^j (\phi(g^j(x_n + h_n, u_n) - \phi(g^j(x_n, u_n)) \geq 0. \end{aligned} \tag{60}$$

Adding it to (53), in order to get a difference of Lagrangian functions, and after a second-order expansion (using the fact that the derivative of Lagrangian function w.r.t. x, at (x_n, u_n), is zero), it follows that

$$\begin{aligned} &D_{xx}^2 f(x_n, u_n)(h_n, h_n) - \sum_{j \notin I} (y_n)^j \cdot D_{xx}^2 g(x_n, u_n)(h_n, h_n) \\ &\quad - \sum_{j \in I} (y_n)_0^j D_{xx}^2 \phi(g^j(x_n, u_n))(d_n^j(h_n), d_n^j(h_n)) \leq o(\|h_n\|^2), \end{aligned} \tag{61}$$

where $d_n^j(h_n) := D_x g^j(x_n, u_n)h_n$. Using the expression of the expansion of ϕ, computed in (39), and passing to the limit in n, obtain $Q_0(h^*) \leq 0$. Since $h^* \in \mathrm{Sp}(C(x^*)) \setminus \{0\}$, this contradicts (45). The conclusion follows. ■

Remark 31. *A related result is [6, Thm 5.25], where it is proved that a necessary condition for uniform quadratic growth, assuming uniqueness of the Lagrange multiplier, is that the Hessian of Lagrangian function is positive definite over the space spanned by radial critical directions. By contrast, our result is a characterization involving additional terms in the quadratic form, and space spanned by all critical directions. There is also a second part in [6, Thm 5.25] that involves the space spanned by all critical directions, but under a certain "strong extended polyhedricity condition" that is not satisfied here.*

References

[1] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Mathematical Programming*, 95:Ser. B, pp. 3–51, 2003.

[2] J. F. Bonnans, R. Cominetti, and A. Shapiro. Second-order optimality conditions based on parabolic second-order tangent sets. *SIAM J. on Optimization*, 9(2):pp. 466–492, 1999.

[3] J. F. Bonnans, J. Ch. Gilbert, C. Lemaréchal, and C. Sagastizábal. *Numerical Optimization: theoretical and numerical aspects.* Universitext. Springer-Verlag, Berlin, 2004.

[4] J. F. Bonnans and H. Ramírez C. Strong regularity and sensitivity in semidefinite programming. Working paper.

[5] J. F. Bonnans and A. Shapiro. Nondegeneracy and quantitative stability of parameterized optimization problems with multiple solutions. *SIAM Journal on Optimization*, 8:940–946, 1998.

[6] J. F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems.* Springer-Verlag, New York, 2000.

[7] J. F. Bonnans and A. Sulem. Pseudopower expansion of solutions of generalized equations and constrained optimization problems. *Mathematical Programming*, 70:123–148, 1995.

[8] R. Correa and H. Ramírez C. A global algorithm for nonlinear semidefinite programming. Research Report 4672 (2002), INRIA, Rocquencourt, France. To appear in SIAM J. on Optimization.

[9] A. L. Dontchev and R. T. Rockafellar. Characterization of strong regularity for variational inequalities over polyhedral convex sets. *SIAM Journal on Control and Optimization*, 6:1087–1105, 1996.

[10] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret. Applications of second-order cone programming. *Linear Algebra and Its Applications*, 284:pp. 193–228, 1998.

[11] Y. Nesterov and A. Nemirovsky. *Interior-point polynomial methods in convex programming.* vol. 13 of Studies in Applied Mathematics, SIAM, Philadelphia, PA, 1994.

[12] S. M. Robinson. Generalized equations and their solutions, part II: Applications to nonlinear programming. *Math Programming Stud.*, 19:pp. 200–221, 1982.

[13] S.M. Robinson. Strongly regular generalized equations. *Mathematics of Operations Research*, 5:43–62, 1980.

[14] R. T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, New Jersey, 1970.

[15] A. Shapiro. First and second-order analysis of nonlinear semidefinite programs. *Mathematical Programming*, 77(2):pp. 301–320, 1997.

[16] A. Shapiro and A. Nemirovski. Duality of linear conic problems. Submitted, 2004.

[17] C. Sim and G. Zhao. A note on treating second-order cone problems as a special case of semidefinite problems. Presented at ISMP 2003, Copenhagen, Denmark.

[18] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):pp. 49–95, 1996.

[19] H. Wolkowitz, R. Saigal, and L. Vandenberghe, editors. *Handbook of Semidefinite Programming: Theory, Algorithms and Applications.* Kluwer's International Series in Operations Research and Management Science, 2000.