The Core of Network Problems with Quotas

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Abstract

This paper proves the existence of non-empty cores for directed network and two-sided network problems with quotas.

Introduction

Networks among a group of agents, arise very often in society as well as in economic analysis. In a network, pairs of agents are linked to each other in a symmetric relationship. Slikker and van den Nouweland [2001], Dutta and Jackson [2003] and Jackson [2004], study the problem of network formation. In a recent work, Jackson and van den Nouweland [2004] study the existence of networks that are stable against changes in links by any coalition of individuals. However, not all interactions among individuals are of necessity symmetrical. Thus, for instance, when an agent decides to buy an object from another individual, it is not necessary that the resulting transaction, materializes in a direct exchange of objects. This problem was analyzed rigorously by Shapley and Scarf [1974], where each of a set of individuals was initially endowed with exactly one object and an allocation of the objects which could not be improved upon by any coalition of individuals by redistributing their initial endowments, was sought. An allocation such as this was called core stable and Shapley and Scarf [1974], used Gale's Top Trading Cycle Algorithm to show that a core stable allocation for such a situation would always exist.

In the situation that Shapley and Scarf[1974] analyzed, an allocation did not necessarily correspond to a network. If one agent received the object owned by a second, it did not follow that the second received the object owned by the first in return. An allocation would however correspond to what is known as a directed network. If a link was established from one agent to a second, all that it would imply is that the first agent received the object owned by the second.

The situation studied by Shapley and Scarf[1974] was characterized by two features: (a) each agent consumed exactly one object: (b) excludability in consumption. While, the first feature was perhaps only a simplifying assumption, meant largely to facilitate exposition as subsequent research as well as Lahiri [2004] reveals, the same cannot be said of the second feature. Excludability in consumption implies that at most one agent can consume a particular item, a characteristic associated with "private goods". There are many situations and objects which do not qualify this property. For instance, an internet server can be linked simultaneously to several other internet servers, not just one. A particular

object or facility can be simultaneously used by several users, whose number does not exceed a pre-assigned quota. Such facilities or goods are akin to public goods. It is precisely such goods that we have in mind in the present context.
As in Shapley and Scarf[1974], we consider a finite population of agents, each of whom is initially endowed with a single item. Each item has a capacity denoting the number of agents it can simultaneously cater too. The quota of each agent which is assumed equal to the capacity of the item she is initially endowed with, imposes an upper bound on the number of items she can consume. The requirement that the quota of an agent is equal to the capacity of her initial endowment, implies that in any directed network that would be of interest in the present context, the number of links that terminate at any agent is equal to the number of links that emanate from her. In particular a link can be a loop i.e. begin and terminate at the same agent. The problem we are concerned with here, is with the existence of a directed network which satisfies individual quotas and is core stable in the following sense: there does not exist any coalition of agents who can link up among themselves and do better than at the existing network. We show that a slight modification of the Gale's Top Trading Cycles Algorithm that was used by Shapley and Scarf[1974], proves the existence of a core stable network for every directed network problem with quotas.
A salient feature of many markets is to match one agent with another, which gives rise to a network. This is particularly true, in the case of assigning workers to firms, where the set of workers and the set of firms are disjoint. Such markets are usually studied with the help of "two sided matching models" introduced by Gale and Shapley [1962].
The solution concept proposed by Gale and Shapley [1962], called a stable matching, requires that there should not exist two agents each on a different side of the market, who prefer each other to the individual they have been paired with. It was shown in Gale and Shapley [1962], in a framework where every agent has preferences defined by a linear order over the entire set of agents, that a two-sided matching problem always admits a stable matching. Indeed, given a two-sided matching problem, there is always a stable matching which no firm considers inferior to any other stable matching, and there is always a stable matching that no worker considers inferior to any other stable matching. The first is called an F – optimal stable matching (i.e. stable matching optimal for firms) and the second one a W – optimal stable matching (i.e. stable matching optimal for workers). An overview of the considerable literature on two-sided matching problems that has evolved out of the work of Gale and Shapley [1962], is available in Roth and Sotomayor [1990].
While the existing literature on two-sided matching model, does allow the possibility of a firm employing more than one worker, the situation where a worker may be employed by more than one firm, or for that matter be employed by a firm part time and remain self-employed otherwise is considered less frequently. However, part time appointments and individuals working for more than one firm, particularly as consultants, is a common feature of modern job markets.
Roth [1984] considers the concept of stability in a many-to-many matching problem with money, whereas the model we study here does not use such a medium of exchange. An alternative linear programming formulation studied by Roth, Rothblum and Vande Vate [1993], introduces the concept of fractional matchings, which is also a type of many-to-many matching.

The model and the pair-wise stability concept we analyze here is the one proposed in Sotomayor [1999] for separable preferences. Sotomayor [1999] is concerned with establishing the inter-relationships between three different concepts of stability that exist in the literature on many-to-many matching, of which pair-wise stability is one. More recently, Echenique and Oveido [2003], provide conditions under which the set-wise stable set of Roth [1984] and Sotomayor [1999] is non-empty. However, Example 1 of Sotomayor [1999] establishes that a pair-wise stable network may not belong to the core and Example 2 of the same paper shows that a pair-wise stable network may not be set-wise stable. In that paper an existence result for "pair-wise stable matchings" for substitutable preferences is established, which when restricted to the domain of separable preference implies the existence of a pair-wise stable network as defined here.

In this paper, we go further. We show that by modifying the algorithm that Gale and Shapley [1962] used, it is possible to generalize the results pertaining to the existence and Weak Pareto Optimality for firms of a F-optimal stable matching, to the context where both firms and workers have quotas restricting the number of links that each agent on one side of the market can establish with the other side. This leads to the desired existence result of a F-optimal pair-wise stable network and its Weak Pareto Optimality for firms in the context on many-to-many matching matching model with separable preferences.

Through out the analysis reported here, we focus on stability as a solution concept. This clearly rules out the possibility of ex-post re-contracting among the agents.

In the mechanism design literature, strategy-proofness or dominant strategy implementation has often been invoked as the criterion that a mechanism is required to satisfy. Without going into the merits or validity of strategy-proofness for the networks we propose here, it is worth pointing out that whether a resource allocation mechanism is otherwise acceptable or not, if it fails to prevent ex-post re-contracting among agents, then the purpose of decentralization is obviously defeated. Thus, stability in the sense we use, is a minimal requirement that a mechanism ought to satisfy, in order to be credible.

The requirement of stability may sometimes make the additional requirement of strategy-proofness superfluous. The resource allocation obtained via strategic misrepresentation of one or more agent's preference, may lead to the possibility of ex-post re-contracting among the agents. The likelihood of the adverse consequences on oneself arising out instability, should by itself deter an agent from acting strategically.

The Directed Network Problem

Given a non-empty finite set I of agents, a preference relation for agent $i \in I$ is summarized by a linear order $R(i)$ over I.

A directed network is a function $A: I \rightarrow 2^I \backslash \{\phi\}$, where I is a non-empty finite set of agents. A directed network is said to be a network if for all $A \in \Lambda$ and $i,j \in I$: $[j \in A(i)$ implies $i \in A(j)]$.

We assume that each agent has a quota which is a natural number less than or equal to the cardinality of I. Hence, a quota function is a function $q: I \rightarrow \{0, \ldots, |I|\}$.

A directed network problem with quotas is the ordered pair $E = (\Lambda(q), <R(i) / i \in I>)$, where $\Lambda(q) = \{A /$ A is a directed network satisfying [for all $i \in I$: $|A(i)| \leq q(i)]$, $|A(i)| < q(i)$ implies $i \in A(i)$, and $|A(i)| = q(i)$ for some $i \in I\}$.

A directed network A for $E = (\Lambda(q), \langle R(i) / i \in I \rangle)$ is said to be a feasible network (or simply "feasible") if $A \in \Lambda(q)$.

The reason why there may be no directed network at which all agents exhaust their quota is illustrated by the following lemma.

Lemma 1: Let $I = \{1,\ldots,n\}$ for some positive integer $n \geq 3$ and let q be a quota function such that $q(1) = 1$, $q(i) = n$ for $I > 1$. Let $A \in \Lambda(q)$. Then, there exists a subset S of I, containing at least n-2 agents, such that $|A(i)| < q(i)$ for all $i \in S$.

Proof: Let $A \in \Lambda(q)$ be such that for at least one $i \in \{2,\ldots,n\}$: $|A(i)| = n$. Without loss of generality suppose $|A(n)| = n$. Thus, $A(n) = I$ and $A(1) = \{n\}$. Since $q(1) = 1$, $1 \notin \bigcup_{i=2}^{n-1} A(i)$. Thus, $|A(i)| < n$ for all $i \in \{2,\ldots,n-1\}$. Thus, at least n-2 agents in I must have unexhausted quotas at any $A \in \Lambda(q)$. Q.E.D.

A feasible network A is said to be blocked by a coalition (: a non-empty subset of agents) M, if there exists a permutation $p: M \rightarrow M$ and a function $y: M \rightarrow \bigcup_{i \in M} A(i)$ such that for all $i \in M$: (i)$p(i)\ R(i)\ y(i)$; (ii) $p(i) \in I \backslash A(i)$.
An alternative way of defining the concept of blocking by a coalition would be by using the concept of a unilateral hyper-relation due to Aizerman and Aleskerov [1995]. A unilateral hyper-relation on I is a subset of $2^I \times I$.
For $i \in I$ and $(S, j) \in 2^I \times I$, we write $S \geq_i j$ if and only if either $j \in S$ or $kR(i)j$ for all $k \in S$.
Clearly $\geq_i$ is a unilateral hyper-relation for all $i \in I$.
A feasible network A is said to be blocked by a coalition (: a non-empty subset of agents) M, if there exists a permutation $p: M \rightarrow M$ such that for no $i \in M$ is it the case that $A(i) \geq_i p(i)$.
A feasible network A is said to belong to the core of the directed network problem with quotas E, if it is not blocked by any coalition.
The core of E, denoted Core(E) is the set of feasible networks belonging to the core of E.

Given a list of distinct agents $i_1,\ldots,i_k$ we say that a transaction is completed along the cycle $(i_1,\ldots,i_k)$ if each $i_j \in \{i_2,\ldots,i_k\}$ receives i_{j-1} and i_1 receives i_k. Thus, if $k = 1$, then after completion of transaction along the cycle, agent i_1 receives i_1.

The proof of the following theorem, which is a generalization of the one in Shapley and Scarf[1974], relies on a minor variation of the Gale's Top Trading Cycles Algorithm. Our proof itself is a modification of the one in Shapley and Scarf[1974].

Theorem 1: If E is a directed network problem with quotas, then Core(E) is non-empty.

Proof: Stage 1: Each agent i points to the agent who owns her most preferred object according to the linear order R_i. Since, the number of agents is finite, there exists at least one subset of agents who form a cycle, i.e. there exists a set $i_1,\ldots,i_k$

of agents, such that i_j is the most preferred item of agent i_{j-1} for $i_j \in \{i_2,\ldots,i_k\}$ and i_1 is the most preferred item of agent i_k. Since each agent points to exactly one agent, no two distinct cycles can share an agent. Otherwise, there would exist an agent who points to two different agents, contrary to hypothesis. Complete the transaction along each such cycle.

Each agent who does not get an object she had pointed to, was not part of a cycle. Each agent who received an object at this stage, strikes that particular object off from her linear order.

Each agent who received an object up until this stage, reduces her quota by one, to obtain revised quotas. Any agent whose quota has been reduced to zero, withdraws from the procedure. If in the process all agents withdraw from the procedure, the procedure terminates. Otherwise the procedure moves to Stage 2, with participating agents being only those agents who either did not receive an item at Stage 1 or whose revised quota after stage 1 is positive. No agent whose quota is incomplete is removed from the linear order of any participating agent. Each agent who participate in the subsequent stage removes from her linear order all agents who have exhausted their quota. Each agent who received an object at Stage 1 and proceeds to participate in the subsequent stage, removes from her linear order the (owner of) the item she received at Stage 1.

Stage 2: Repeat Stage 1, among the participating agents. (This may involve an agent pointing to an agent she had at stage 1). Each agent who received an object up until this stage, reduces her quota by one, to obtain revised quotas.

Repeating the process at most a finite number of times, we arrive at a stage where either all agents have filled their quota, or the agents who have not filled there quota, have by now struck all agents off their list.

The procedure terminates now with the directed network A being defined such that for all $i \in I$, A(i) is the set of items currently in the possession of agent i.

We claim that A belongs to the core of E. A is clearly feasible. If for all $i \in I$ it is the case that $|A(i)| < q(i) \leq |I|$, then no agent would have struck all agents off from her list of preferences, and hence the procedure could not have terminated. Thus, there exists $i \in I$, such that $|A(i)| = q(i)$.

Suppose there is a coalition M which blocks A. Thus there exists a permutation p: $M \rightarrow M$ and a function y: $M \rightarrow \bigcup_{i \in M} A(i)$ such that for all $i \in M$: (i)p(i) R(i) y(i); (ii) $p(i) \in I \backslash A(i)$.

Without loss of generality let an agent in M whose quota was exhausted first among all agents in M, be denoted 1. If agent 1's quota was exhausted at the first stage of the procedure, she clearly got her best item and therefore could not be part of a blocking coalition. Hence no agent whose quota was exhausted at the first stage would be part of a blocking coalition. If agent 1's quota was exhausted at stage 2, then the only agents that she could form a blocking coalition with, must have exhausted their quota in stage 1. Since agents who exhausted their quota in stage 1 cannot belong to M, it is not possible for agent 1 to belong to M either. Thus, no agent whose quota was exhausted at stage 2 can belong to M. Proceeding thus, we see that if agent 1's quota was exhausted at stage k, then the owners of the items she could form a blocking coalition with must have exhausted their quota at a previous stage. Since agent 1 is assumed to be among the first to exhaust her quota among the agents in M, M cannot be a blocking coalition. This contradiction establishes the non-emptiness of the Core(E). Q.E.D.

The purpose of requiring the termination rule in the above procedure to permit agents whose quota may have remained unexhausted may once again be illustrated by the following example.

Example 1: Let I = {1,2} and suppose q(1) = 1 where as q(2) = 2. Let E be a directed network problem where agent 1 prefers 1 to 2. If A belongs to Core(E), then A(1) = {1}. Thus, whatever be the preference of agent 2, A(2) = {2}. In fact this would be the unique feasible network in Core(E). Clearly, the quota of agent 2 remains unexhausted at A.
However, thee feasible network where agent 1 gets 2 and agent 2 gets both 1 and 2, exhausts the quota of all agents. This network is blocked by agent 1 and hence does not belong to the core.

Note: Suppose the directed network A obtained in the proof of theorem 1 above, was the outcome of a procedure that terminated at stage $K \geq 1$. For all agents $i \in I$, who received item j at stage K, let $p_K = p_i(j) = 1$.
If K > 1, then having defined p_k for stages K, K-1,…, L < 1, define $p_{L-1} = \sum_{k=L}^{K} p_k + 1$. For all $i \in I$, who receive an item j at stage $k \in \{L-1,\ldots,K\}$, let $p_i(j) = p_k$.
For $i \in I$ and $j \in A(i)$, $p_i(j)$ may be interpreted as a personalized price of item j to agent i.
For all $j \in I$, let $p(j) = \min \{p_i(j) / j \in A(i), i \in I\}$. For $i \in I$ and $j \in I \setminus A(i)$, let $A^{-j}(i) = \{h \in A(i) / jR(i)h\}$.
The pair (A, <$p_i(j) / j \in A(i), i \in I$> satisfies the following property: (i) for all $i \in I$ and $j \in I \setminus A(i)$ with $A^{-j}(i) \neq \phi$: $p(j) > \sum_{h \in A^{-j}(i)} p_i(h)$; (ii) for all $i \in I$, $j \in A(i)$ and $h \in I \setminus A(i)$, [hR(i) j] implies [$p(h) > p_i(j)$]; (iii) for all $i \in I$: $\sum_{\{j / j \in A(i)\}} p_i(j) = \sum_{\{j / i \in A(j)\}} p_j(i)$.
(i) says that for any agent i and any item j not belonging to A(i), the total payment that agent i makes for items she does not prefer to item j, is less than the least personalized price paid for item j. Now, (ii) follows from (i) since all personalized prices computed above, are strictly positive. (iii) says that, given any agent i, the sum of payments made by i is equal to the sum of payments received by i .

The Core of a Two-Sided Network Problem with Quotas

A directed network problem with quotas E = (Λ(q), <R(i) / i∈I>), is said to be a network problem with quotas if Λ(q) = {A/ A is a network satisfying [for all i∈I: |A(i)| ≤ q(i)], |A(i)| < q(i) implies i∈A(i), and |A(i)| = q(i) for some i∈I}.
A network A∈Λ(q) is said to be blocked by a pair (i,j)∈I×I if : (a) j∉A(i) (and hence i∉A(j)); (b) there exists h∈A(i) and k∈A(j) such that jR(i)h and iR(j)k.
A network A ∈Λ(q) is said to be pair-wise stable if it is not blocked by any pair in I×I.
Gale and Shapley [1962] showed that there exists a network problem with quotas, where all agent has a quota of one, which does not admit any pair-wise stable network.

A network problem with quotas $E = (\Lambda^*(q), <R(i) / i\in I>)$ is said to be a two-sided network problem with quotas if there exists two non-empty, disjoint subsets F and W of I satisfying $I = F\cup W$ such that:

(a) For all $f\in F$, $q(f) \leq |W|$.

(b) For all $w\in W$, $q(w) \leq |F|$.

(b) $\Lambda^*(q) = \{A\in\Lambda(q)/$ (i) for all $f\in F$: $A(f) \subset W\cup\{f\}$; (ii) for all $w\in W$: $A(w) \subset F\cup\{w\}\}$.

If $E = <R(i(\Lambda^*(q),) / i\in I>)$ is a two-sided network problem with quotas as above, then we will represent it as $E = (\Lambda^*(q), <R^*(i) / i\in F\cup W>)$, where for all $f\in F$ and $w\in W$, $R^*(f)$ is the restriction of $R(f)$ to $W\cup\{f\}$ and $R^*(w)$ is the restriction of $R(w)$ to $F\cup\{w\}$.

If $A \in(\Lambda^*(q)$, then A is said to be feasible for E.

Following Sotomayor (1999) we introduce the following definitions:

A feasible network A for $E = (\Lambda^*(q), <R^*(i) / i\in F\cup W>)$, is said to be blocked by a pair $(f,w) \in F\times W$ if: (a) $w\notin A(f)$; (b) there exists $k\in A(f)$ and $h\in A(w)$ such that $wR^*(f)k$ and $fR^*(w)h$.

A feasible network A for $E = (\Lambda^*(q), <R^*(i) / i\in F\cup W>)$, is said to be pair-wise stable if it is not blocked by any pair (f,w) in $F\times W$.

Gale and Shapley [1962] showed that if for all $i\in I$: $q(i) = 1$, then a two-sided network problem with quotas has a pair-wise stable network.

For the sake of simplicity we assume that:

For all $f\in F$ and $w\in W$: $wR^*(f)f$ and $fR^*(w)w$.

Lemma 2: Let A be any pair-wise stable network for $E = (\Lambda^*(q), <R^*(i) / i\in F\cup W>)$. Then:

Either (i) $|A(w)| = q(w)$ for all $w\in W$;

Or (ii) $|A(f)| = q(f)$ for all $f\in F$.

Proof: Suppose towards a contradiction $|A(i)| < q(i)$ for all $i\in I$. Let $f\in F$. Thus, $f\in A(f)$ and $|A(f)| < q(f) \leq |W|$. Thus, there exists $w\in W$ such that $w\notin A(f)$. Since $|A(w)| < q(w)$, $w\in A(w)$ and $f\notin A(w)$, the pair (f,w) blocks A, leading to a contradiction. Q.E.D.

A matching for $E = (\Lambda^*(q), <R^*(i) / i\in F\cup W>)$ is a function $m: F\cup W \rightarrow F\cup W$ such that:

(a) for all $f\in F$ and $w\in W$: $m(f)\in W\cup\{f\}$ and $m(w)\in F\cup\{w\}$;

(b) for all $f\in F$ and $w\in W$: $m(m(f)) = f$ and $m(m(w)) = w$.

Given a network A and a matching m for $E = (\Lambda^*(q), <R^*(i) / i\in F\cup W>)$, agent $i\in I$ is said to prefer m to A if $m(i) \notin A(i)$ and there exists $k\in A(i)$ such that $m(i)R(i)k$.

A network $A\in\Lambda^*(q)$ is said to be Weakly Pareto Optimal for F(Weakly Pareto Optimal for W) if there is no matching m which all firms (respectively, workers) prefer to A.

A matching m is said to belong to a network B if for all $i\in I$: $m(i) \in B(i)$.

The ordered pair (B,m) where m is a matching belonging to a network B is called a compatible pair.

A compatible pair (B,m) is said to be a stable pair for the two-sided network problem E, if B is a pair-wise stable network.
Given a network A an agent i is said to prefer the compatible pair (B,m) to A if agent i prefers m to A.
A pair-wise stable network A for E = ($\Lambda^*(q)$, <R(i) / $i \in I$>) is said to be F- optimal (W-optimal) or optimal for F (optimal for W, respectively) if there does not exist a stable pair (B,m), which some firm (worker, respectively) prefers to A.

We now modify the Deferred Acceptance Procedure used by Gale and Shapley to show that every two-sided network problem with quotas has a F-optimal and a W-optimal pair-wise stable network.

Theorem 2: Every two-sided network problem with quotas has a unique F-optimal pair-wise stable network and a unique W-optimal pair-wise stable network.

Proof: We will only prove the existence of a unique F-optimal pair-wise stable network, the existence of a unique W-optimal one being analogous.
For the purpose of the algorithm that we use to prove the theorem, let F stand for a set of firms and W for a set of workers.
To start each firm f makes an offer to its best worker in W. Each worker w, receiving a set of offers $S^1(w)$ at this stage, chooses her best $\min\{q(w), |S^1(w)|\}$ offers and rejects the rest. If $\min\{q(w), |S^1(w)|\} = q(w)$, then w's quota is exhausted. If $\min\{q(w), |S^1(w)|\} < q(w)$, then w's quota is not exhausted. Any firm whose offer is not rejected at this point is kept "pending" by the worker who did not reject it. Each firm who is kept pending by a worker, temporarily reduces its quota by one, to obtain a revised quota. Each worker reduces her quota by the number of firms she has kept pending.
At this point there are two types of firms and two types of workers. There is a type of worker whose quota is exhausted and a type whose quota is not. There is a type of firm whose quota is temporarily exhausted and a type whose quota is not.
Each firm f who is kept pending by a worker, strikes that worker off its linear order and thus obtains a revised linear order over a subset of workers. Each firm f who has been rejected by a worker, strikes that worker off its linear order and thus obtains a revised linear order over a subset of workers.
Each firm f whose quota was not temporarily exhausted at the earlier stage, makes an offer to its best worker on its revised linear order. This could include a worker whose quota was exhausted at the previous round. Each worker w receiving an offer at this stage, adds these new offers to the set of firms she has kept pending from the previous stage, which now forms her current list of offers $S^2(w)$. Each worker w who received an offer at this stage, chooses her best $\min\{q(w), |S^2(w)|\}$ offers and rejects the rest. As in the earlier stage, if $\min\{q(w), |S^1(w)|\} = q(w)$, then w's quota is exhausted, otherwise not.
At this stage a firm revises its quota in the following manner: (temporary quota obtained from the previous stage – number of workers who had kept it pending but reject it at this stage) + number of workers who accept its offer at this stage = temporary quota of the firm at the end of this stage.
Note: There can be at most one worker who accepts the offer of a particular firm at any stage, since at any stage of the algorithm a firm makes at most one offer.

At this point again there are two types of firms and two types of workers. There is a type of worker whose quota is exhausted and a type whose quota is not. There is a type of firm whose quota is exhausted and a type whose quota is not.
Each firm f who is kept pending by a worker at this stage, strikes that worker off its earlier revised linear order, to obtain a new revised linear order over a subset of workers. Each firm f who has been rejected by one or more workers, strikes that (those) worker(s) off its earlier revised linear order, to obtain a new revised linear order over a subset of workers.
The algorithm stops after any step in which every firm whose quota is not exhausted has struck all workers off from its linear order. Otherwise, the algorithm proceeds to the next stage, with participating agents, their quotas and revised linear orders being determined as in earlier steps.
The network that is defined once the algorithm stops, associates to each firm the workers who have kept it pending, if there be any, and associates to each worker the firms she has kept pending. All agents whose quotas have not been exhausted as yet, include themselves in their own assignment.
In the above procedure, each firm, proceeds down its list of workers, and each worker proceeds up her list of firms.
Let A be the network thus defined. Suppose that for some f, $|A(f)| < q(f)$. Thus, f, must have been rejected by some worker w, in spite of having made an offer to her. This is possible, only if w could have chosen q(w) firms from the offers she had received, who she preferred to f. Since once the quota of a worker gets exhausted, she remains thus, in spite of making possible replacements, clearly $|A(w)| = q(w)$. Thus, A is feasible.
Suppose there is a firm f and a worker w, such that $f \notin A(w)$, $w \notin A(f)$, $h \in A(w)$, $k \in A(f)$ and fR(w)h and wR(f)k. Thus, f must have made an offer to w before arriving at k, and was rejected by w, in favor of a set of firms all of whom she preferred to f. Thus, w prefers h to f, leading to a contradiction. Thus, A is pair-wise stable.
Now suppose there is a stable pair (B,m) which some firm f prefers to A. Let m(f) = w. Thus, f must have made an offer to w at some stage of the algorithm and was rejected by w. If w had rejected f at the first step of the algorithm, then she did so in favor of a set of q(w) firms $F^1(w)$ each of whom she prefers to f. Since $f \in B(w)$ and $|B(w)| \leq q(w)$, there exist $f' \in F^1(w)$ such that $f' \notin B(w)$. Since w is f' first choice and $w \notin B(f')$, clearly the pair (f',w) blocks B contradicting its pair-wise stability. Thus, at the first stage of the algorithm no firm f that prefers B to A could have been rejected by m(f).
Suppose that up to a stage L in the algorithm no firm f that prefers B to A was rejected by m(f). Suppose at the next stage a firm f that prefers B to A is rejected and towards a contradiction suppose that this firm f was rejected by m(f) = w. Thus, w must have rejected f in favor of a set of q(w) q(w) firms $F^{L+1}(w)$ each of whom she prefers to f. Since none of the firms in F^{L+1} (w) were rejected at this stage, none of them could have been rejected by their assignment under m. Thus, for all $f' \in F^{L+1}(w)$ it must be the case that w is at least as good as m(f'). Since $f \in B(w)$ and $|B(w)| \leq q(w)$, there exist $f' \in F^{L+1}(w)$ such that $f' \notin B(w)$. Thus, w must be preferred by f' to m(f'). Since w prefers f' to f = m(w), the pair-wise stability of B stands contradicted. Thus, even at stage L+1, no firm f that prefers B to A could have been rejected by m(f). Hence at no stage of the algorithm is a firm f who prefers B to A rejected by m(f), leading to a contradiction.

Thus, there does not exist any pair-wise stable network derived from A via a matching which any firm would prefer to A.
Thus, A is F-optimal.
To prove uniqueness, suppose that A and B are two F-optimal pair-wise stable networks, such that for some $f \in F$: $A(f) \neq B(f)$. If both $A(f) \setminus B(f) \neq \phi$ and $B(f) \setminus A(f) \neq \phi$, then by choosing $i \in A(f) \setminus B(f)$ and $j \in B(f) \setminus A(f)$, we get the following: Either f prefers i to j or j to i. Without loss of generality suppose f prefers i to j. Let (A,m) be any compatible pair with m(f) = i. Thus, (A,m) is a stable pair which f prefers to B, contradicting the F-optimality of B. Thus, either $A(f) \setminus B(f) = \phi$ or $B(f) \setminus A(f) = \phi$.
Suppose, $A(f) \setminus B(f) \neq \phi$. Thus, $B(f) \setminus A(f) = \phi$. Thus, B(f) is a proper subset of A(f). Clearly, $f \in B(f)$. Let $w \in A(f) \setminus B(f)$. Let (A,m) be any compatible pair with m(f) = w. Thus, (A,m) is a stable pair, which f prefers to B, contradicting the F-optimality of B.
Thus, an F-optimal pair-wise stable network is unique. Q.E.D.

Let A_F be the unique F-optimal network that we obtain as a consequence of Theorem 2.

In the following theorem we show that A_F is Weakly Pareto Optimal for F(firms). This result generalizes a similar result due to Roth and Sotomayor [1990].

Theorem 3: Suppose $A_F(f) \subset W$ for all $f \in F$. Then A_F is Weakly Pareto Optimal for F.

Proof: Towards a contradiction suppose there is a matching m which all firms prefer to A_F. Clearly, $m(f) \in W$ for all $f \in F$.
Let f* be a firm whose offer was accepted at the last by a worker w at the last step of the algorithm leading to A_F. If at the time of accepting the offer made by f*, w's quota was not exhausted, then at no stage of the algorithm could her quota have been exhausted. Since $q(w) \leq |F|$, no $f \in F \setminus A_F(w)$ ever made an offer to w and hence all such firms prefer every worker in $A_F(f)$ to w. Since $m(w) \in F \setminus A_F(w)$, there is some firm $f \in F \setminus A_F(w)$ such that m(f) = w and prefers every worker in $A_F(f)$ to m(f), leading to a contradiction.
If at the time of accepting the offer made by f*, w's quota was exhausted, then w rejects f in favor of f*. This implies $|A(f)| < q(f)$ and hence $f \in A(f)$, leading to a contradiction.
Thus, A_F is Weakly Pareto Optimal for F. Q.E.D.

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