A Piecewise Linearization Framework for Retail Shelf Space Management Models

Jens Irion
Universitaet Karlsruhe (TH)
Kaiserstr. 12
76131 Karlsruhe, Germany

Faiz Al-Khayyal and Jye-Chyi Lu
School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0205 U.S.A.

9 July 2004

Abstract

Managing shelf space is critical for retailers to attract customers and to optimize profit. This paper develops a shelf space allocation optimization model that explicitly incorporates essential in-store costs and considers space- and cross-elasticities. We propose a piecewise linearization technique for approximating the complicated nonlinear model that relaxes the nonconvex optimization problem into a linear Mixed Integer Program (MIP). This MIP not only generates near-optimal solutions, but also provides an *a posteriori* error bound to evaluate the quality of the solution. Consequently, our approach can solve single category shelf space management problems with as many products as are typically encountered in practice and with more complicated cost and profit structures than currently possible by existing methods. Numerical experiments on small test cases show the accuracy of the proposed method comparing the optimal solutions of our approximating linear MIP to the known global solutions of the exact nonlinear model. Several extensions of the main model are investigated to illustrate the flexibility of the proposed methodology.

1 Introduction

The decisions of which products to stock among the large number of competing products and how much shelf space to allocate to those products is a question central to retailing. Because shelf space is a scarce and fixed resource and the number of potentially available products continually increases, retailers have a high incentive to make these decisions correctly. If customers were completely brand-loyal, they would look for a specific item and buy it if it were available or delay their decisions if it were not. Thus, space allocated to a product would have no effect on its sales (Anderson [3]). However, marketing research shows that most customer decisions are made at the point of purchase (see, e.g., POPAI [20]). In addition, Ehrenberg [15] discovers that, "except in relatively short time periods ... buyers of any particular brand therefore buy other brands more often than the brand itself." This indicates that the product choice of customers may be influenced by in-store factors including shelf space allocated to a product. With a well-designed shelf space management system, retailers can attract customers, prevent stockouts and, more importantly, increase the financial performance of the store while reducing operating costs (Yang and Chen [26]). Further, close-to-optimal shelf space allocations provide the basis for distributing promotional resources among the different product categories (Chen *et al.* [10]). However, the optimization problem is very complex, because products usually have different profit margins and widely varying space- and cross-elasticities.

The objectives of this paper are to formulate realistic shelf space management optimization models and to provide a solution procedure that can handle realistic problem sizes and that is flexible enough to be applied to a wide range of shelf space management models. To achieve this, we extend the well-known model of Corstjens and Doyle [11] in three directions. First, our model requires the shelf space allocated to a product to be equal to an integer number of its facing. Second, it allows simultaneous shelf space and assortment decisions. Third, cost elements are modeled individually. More importantly, we approximate the resulting nonconvex optimization model, using piecewise linear functions, in such a way as to relax the model into an approximating linear Mixed Integer Program (MIP) that generates both a feasible solution and a bound on the global optimal objective value of the exact nonlinear model. This allows for the calculation of an *a posteriori* error bound[1] on the optimal MIP solution. We then extend the model formulation to incorporate the following additional effects that are mentioned in the literature: marketing variables other than space (Yang and Chen [26]), fixed procurement costs and the possibility of storing items in a warehouse (Urban [25]), and substitution effects due to temporary or permanent unavailability of products (Borin *et al.* [4]).

The remainder of this paper is organized as follows. We review the relevant literature in Section 2 and develop the main shelf space management model in Section 3. The piecewise linearization technique, which transforms the nonconvex optimization model into an approximating linear MIP, is illustrated for the main model in Section 4. Computational experiments and test results are presented in Section 5. These experiments include comparisons of the solutions of the approximating MIP to the known global optima for small test cases and indicate that excellent results and tight *a posteriori* bounds can be expected when using the MIP. Various extensions of the main model and their linearizations are discussed in Section 6, demonstrating that the proposed linearization technique can be applied to a wide range of shelf space management models in which the demand function is of signomial form. To clarify the presentation, we have standardized our notation as follows: decision variables and variables that depend on decision variables are written in lower-case Roman characters; constants and given quantities are defined as upper-case Roman characters; parameters and quantities that need to be estimated or user-provided are expressed in lower-case Greek characters; and functions, without arguments, are specified by upper-case Greek characters.

2 Literature Review

Relevant and previous work can be generally divided into two types: commercial models and optimization models . The discussion below focuses on literature that addresses models and procedures that relate to the application studied in this paper.

2.1 Commercial Models

Commercial software and hardware systems that apply modeling principles have gained many customers within the retailing industry due to their general simplicity and their easily implementable decisions (Zufryden [28]). Today there are various PC-based systems available to retailers including Apollo (IRI) and Spaceman (Nielsen). These software products can provide the retailer with a realistic view of the shelves and are capable of allocating shelf space according to simple heuristics such as turnover, gross profit or margin, using handling and inventory costs as constraints (Desmet and Renaudin [13]). The drawback of all these systems results from their failure to incorporate demand effects; all ignore the existing effects of shelf space on product sales. Thus, none of the available systems can be considered seriously as an optimization tool (Desmet and Renaudin [13]). Consequently, it is not surprising that most retailers "use them mainly for planogram accounting purposes so as to reduce the amount of time spent on manually manipulating the shelves" (Drèze *et al.* [14]).

[1]Some heuristics (none proposed in the literature for shelf space management models) guarantee *a priori* worst case bounds on solutions to any instance of a problem. *A posteriori* bounds are problem-specific and can only be evaluated after a problem is solved.

2.2 Optimization Models

One of the first optimization models was developed by Hansen and Heinsbroek [17]. They use a posynomial demand function, which incorporates individual space-elasticities but disregards cross-elasticities from similar products. Constraints of total available shelf space, minimum allocations, and integer solutions are taken into account. Binary variables for handling assortment decisions are also included. The proposed algorithm is based on a generalized Lagrange multiplier technique, which, in general, is only guaranteed to find local solutions of nonconvex programs.

The model of Corstjens and Doyle [11] incorporates both space- and cross-elasticities and takes into account constraints similar to those considered by Hansen and Heinsbroek [17]. However, it incorporates a more detailed cost structure including procurement costs, carrying costs and out-of-stock costs, which are jointly modeled as a signomial form with respect to allocated shelf space. A signomial geometric programming approach is used to optimize the shelf space allocation; however, Borin et al. [4] point out that the reported solutions of seven of ten problems violated the model's constraints.

The SHARP model developed by Bultez and Naert [6] and Bultez *et al.* [7] is similar to the one developed by Corstjens and Doyle [11]. However, [6] and [7] do not develop an explicit function that relates shelf space to product sales. Instead, space-elasticities are estimated using a symmetric attraction model for SHARP 1 and an asymmetric model for SHARP 2. A heuristic procedure is proposed to solve these models.

Borin *et al.* [4] extend the demand function of Corstjens and Doyle [11] to allow simultaneous decisions about assortment selections and shelf space allocations. In addition, they explicitly consider substitution effects due to temporary or permanent unavailability of products. The resulting model optimizes return on inventory and is solved using the simulated annealing heuristic procedure.

Yang and Chen [26] simplify the model of Corstjens and Doyle [11]. The authors disregard cross-elasticities and assume that a product's profit is linear within a small number of facings, which are constrained by the product's lower and upper bound. They allow the profit of each product to vary when allocated to different shelves by formulating the shelf space allocation problem in a way similar to a knapsack problem. Allowing profit to depend on shelf placement is consistent with the experimental study of Drèze *et al.* [14], who conclude that the location of products on the shelves is more important for determining sales of a product than the amount of space allocated to the product. Yang [27] proposes a heuristic to solve the model in [26]. His solution technique extends an approach often applied to solve simple knapsack problems. Lim *et al.* [18] combine a local search technique with metaheuristics to solve the model of Yang and Chen [26]. They also extend the model to account for nonlinear profit functions and product groupings.

Although the two research directions of Born *et al.* [4] and Yang and Chen [26] are fundamentally different, both have the following drawbacks. They focus on the revenue side and do not incorporate the cost side of the operation explicitly. Clearly, some of the relevant costs are actually not independent of the shelf space allocation. For example, the smaller the shelf space allocated to a product the greater the frequency of restocking and the higher the resulting restocking costs of this product. In addition, they use heuristics to solve the problem. Although these techniques provide good feasible solutions for the test cases considered, they cannot guarantee an optimal or close-to optimal solution. Furthermore, the solution techniques in the current literature do not provide a method to determine how close the computed solution actually is to true global optimality.

Because of the large number of products found in most retail stores, it is clearly not practical to solve one of the foregoing optimization models for all potentially available products within a store (which can exceed 60,000 in some case) since the size of the optimization problem would be prohibitively large. Consequently, current shelf space allocation models only solve *subproblems* of the overall store optimization problem. One such typical subproblem that has received much attention is the allocation of shelf space to products *within* a product category. Although not explicitly stated, it is easy to show that a number of the models and procedures in the literature can be modified to the storewide problem of allocating shelf space *to* product categories. This gives rise to a two-stage procedure whereby store shelf space is allocated to product categories in the first stage and then individual products within a category are allocated to the assigned category space in the second stage. Obviously, the solution obtained from such a procedure is suboptimal (Yang and Chen [26]) as the different allocation models are not integrated. Although the model and solution technique presented herein also can be used within such a two-stage solution procedure, we deal with retail shelf space for a single product category in this paper, and address the optimal allocation of store shelf space to product categories in a forthcoming companion paper.

3 Model Formulation

This section develops a model for allocating shelf space to individual products within a product category. The model maximizes product category profit and takes into account space- and cross-elasticities of the sales, essential cost elements, and various important constraints. We first define terminology and notation used in the model, followed by the main assumptions on the retail environment for which the model applies.

3.1 Terminology and Notation

N number of products in category.

n_i number of *facings*[2] allocated to product i.

z_i indicator variable is 1 if product i is selected for shelving, and 0 otherwise.

F_i shelf space of one facing for product i (inches).

S total amount of available shelf space within the product category (inches).

U_i upper bound on the number of facings allocated to product i.

L_i lower bound on the number of facings allocated to product i.

G_i number of units of product i that can be stored in one facing.

P_i selling price of product i ($).

C_i unit cost of product i ($).

CR_i cost each time product i is restocked ($).

CF_i fixed cost to include product i in the assortment ($).

CP_i unit replenishment cost for product i ($). This cost covers product insurance, deterioration, and the processing costs of sending units back to the supplier (e.g., if they are broken or no longer needed).

β_i *space-elasticity* of product i.

α_i *scale factor* for product i.

[2] A *facing* is a segment of shelf space with linear dimension width. The sizes (or widths) of facings can vary with products, so that each facing dedicated to product i would have width F_i. Moreover, for the purposes of our model, different products *cannot* share the same facing. If there is enough height and depth space available, products can be stacked up more than one row high and lined up many rows deep. The total number of products that fit on a facing is G_i, which allows for stacking multi-rows high and deep.

δ_{ij} *cross-elasticity* between products i and j.

I current investment/interest rate (%).

Ω product category profit ($).

Note that, with the exception of the decision variables (lower-case Roman letters) n_i and z_i, and the objective function (upper-case Greek letter) Ω, all other quantities are known, since they are either given constants (upper-case Roman letters) or parameters that need to be estimated (lower-case Greek letters).

3.2 Assumptions

(i) The retailer's objective is to maximize product category profit.

(ii) Consistent with prior research, the direct space-elasticity for product i satisfies $0 \leq \beta_i \leq 1$, the cross-elasticity between product i and product j satisfies $-1 \leq \delta_{ij} \leq 0$, and the scale factor α_i for product i is generally taken to be positive.[3]

(iii) Effects other than space (e.g., price discounts, special marketing efforts, etc.) are not present.

(iv) All shelved products are owned by the retailer.

(v) Products are restocked individually. As soon as the number of units on the shelves is zero, the product is fully restocked.

(vi) There is no backroom space to store additional inventory.

(vii) The unit cost of each product already contains all procurement costs.

Assumption (v) allows inventory holding costs to be calculated easily and also makes possible a disregard of substitution effects due to temporary stockouts. This assumption will be relaxed in our third extension (presented in Section 6). Assumption (vi) allows that only inventory holding costs of product-units stored on the shelves are considered. Assumption (vii) ignores the effects of fixed order costs. Assumptions (vi) and (vii) will be relaxed in our second extension (presented in Section 6), which incorporates inventory-related decisions into the shelf space allocation model and acknowledges that costs are actually not independent of the order size, inventory level, and frequency of ordering.

3.3 Model Development

Following the vast majority of prior research, our analysis will focus on the situation in which the demand function has *monomial*[4] form. The unit *demand* for an individual product i is modeled as

$$d_i = \alpha_i (F_i n_i)^{\beta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} (F_j n_j)^{\delta_{ij}} \tag{1}$$

[3] Please note that our model assigns shelf space to products within a product category. Since such products are usually very similar in nature, we would expect them to have substitution properties amongst each other ($\delta_{ij} \leq 0$). Nevertheless, it is straightforward to extend our model and the linearization technique to the more general case with substitute and complementary products ($-1 \leq \delta_{ij} \leq 1$).

[4] A *monomial* is a single term polynomial with real valued *exponents*; i.e., the power of each variable can be positive or negative real numbers, as opposed to positive integers for pure polynomials. When weighted monomials are added up, the resulting function is called a *posynomial* if all monomial *coefficients* are positive, and it is called a *signomial* if at least one monomial coefficient is negative. In our model, the demand (1) is a monomial posynomial.

where N is the total number of products to choose from, α_i is a scale factor for product i demand, F_i is the shelf-width of one *facing*, n_i is the decision variable for the number of facings to allocate to product i, parameter β_i is the product i *space-elasticity*, and δ_{ij} is the *cross-elasticity* between products i and j.

In practice, the parameters α_i, β_i and δ_{ij} can be determined via regression analysis using cross-sectional data. Please note that for given cross-sectional data, the magnitude of the scale factor α_i depends on the size of the time interval considered, while the elasticities β_i and δ_{ij} can be assumed to be independent of the time interval. The demand d_i defined by (1) is for an arbitrary interval, and all products must have the same sized interval.

Because our decision variable n_i is the number of facings allocated to a product (instead of space assigned) and we allow facings to be product-specific, our demand function is slightly different from those of Corstjens and Doyle [11], and Borin *et al.* [4]. With P_i as the product i selling price and C_i as the product's unit cost, the unit profit is $P_i - C_i$ and the *total gross margin* for product i is

$$a_i = (P_i - C_i)d_i \tag{2}$$

where the unit cost C_i includes all costs of bringing the product from the supply source to the store.

Turning to in-store costs for shelf space allocation, in addition to the *fixed cost* CF_i, we propose the following structure for the *variable costs*

$$v_i = CP_i d_i + \left(\frac{C_i I G_i}{2}\right) n_i + \left(\frac{CR_i}{G_i}\right)\frac{d_i}{n_i}. \tag{3}$$

With a unit replenishment cost of CP_i, the first term gives the total replenishment cost for product i. With G_i as the number of units of product i that can be stored in a single facing, the second term gives the inventory holding cost for product i. As demand is deterministic and product i is restocked (instantaneously) to its maximum level of $G_i n_i$ only when the shelves are depleted (by Assumption(v)), the *average* shelf-inventory level is $\left(\frac{G_i}{2}\right) n_i$, and this is multiplied by the unit cost C_i and the investment rate I to get the opportunity cost of capital tied up in inventory for product i. The last term is the restocking cost component, since the shelves for product i are restocked $\left(\frac{1}{G_i}\right)\frac{d_i}{n_i}$ times, each at a cost of CR_i.

There are a number of constraints in a retailing environment that have to be included in the model formulation. Similar to Corstjens and Doyle [11], our model includes capacity and control constraints. The capacity constraint ensures that any shelf space allocation must not exceed total available shelf space (Constraint (5)). Control constraints impose lower and upper bounds for the number of facings allocated to each product (Constraints (7)). However, in contrast to the model of Corstjens and Doyle [11], we impose integer restrictions, which guarantee that the amount of space assigned to a product is limited to blocks of its physical (footprint) size (Constraints (8)). Following the model of Yang and Chen [26], we do not need the availability constraint of Corstjens and Doyle [11] if we assume that it is automatically satisfied by an effective logistics system employed by the retailer.

Now the unit profit for product i is $a_i - v_i - CF_i$. However, in our universe of N products, it may be more profitable not to include all of them on the store shelves. To that end, we define the logical variable

$$z_i = \begin{cases} 1 & \text{if product } i \text{ is included in the assortment} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, our store profit function is

$$\begin{aligned} \Omega &= \sum_{i=1}^{N} z_i(a_i - v_i - CF_i) \\ &= \sum_{i=1}^{N} z_i \left[\left(P_i - C_i - CP_i - \left(\frac{CR_i}{G_i}\right)\frac{1}{n_i}\right) d_i - \left(\frac{C_i I G_i}{2}\right) n_i - CF_i\right] \end{aligned}$$

Using (1) to write the objective function only in terms of (n_i, z_i), incorporating the capacity and control constraints and the integer restriction, we obtain the *retail shelf space optimization model* for products within any given category:

Find (n_i, z_i), for $i = 1, 2, \ldots, N$, that maximize

$$\Omega = \sum_{i=1}^{N}\left[\alpha_i F_i^{\beta_i}\prod_{\substack{j=1\\ j\neq i}}^{N} F_j^{\delta_{ij}}\left((P_i - C_i - CP_i) z_i n_i^{\beta_i}\prod_{\substack{j=1\\ j\neq i}}^{N} n_j^{\delta_{ij}} - \left(\frac{CR_i}{G_i}\right) z_i n_i^{\beta_i - 1}\prod_{\substack{j=1\\ j\neq i}}^{N} n_j^{\delta_{ij}}\right)\right] - \sum_{i=1}^{N}\left[\left(\frac{C_i IG_i}{2}\right) z_i n_i - CF_i z_i\right] \tag{4}$$

subject to

$$\sum_{i=1}^{N} F_i n_i z_i \leq S \tag{5}$$

$$(F_i n_i - F_i)(F_i z_i - F_i) \geq 0 \qquad i = 1, \ldots, N \tag{6}$$

$$L_i \leq n_i \leq U_i \qquad i = 1, \ldots, N \tag{7}$$

$$n_i \in \aleph^{+} \qquad i = 1, \ldots, N \tag{8}$$

$$z_i \in \{0, 1\} \qquad i = 1, \ldots, N \tag{9}$$

where $\aleph^{+}$ is the set of *positive* integers and the ojective is a signomial function, which makes our model NP-Hard. Since $F_i > 0$, it is clear that constraint (6) can be replaced by $(n_i - 1)(z_i - 1) \geq 0$.

To understand why n_i is restricted to be a positive integer (instead of a nonnegative integer), observe that the demand function (1) yields zero sales for a given product if the number of facings allocated to *any* of the category's *other* products is set to zero. To overcome this modeling limitation, binary variables z_i and fixed costs CF_i are introduced into the assortment decision model. Furthermore, the following rule must be enforced: if $z_i = 0$ (product i is not in the assortment), then $n_i = 1$. This is achieved by nonlinear constraints (6). Borin *et al.* [4] satisfy this requirement in their simulated annealing heuristic approach by simply setting $n_i = 1$ whenever product i is not included in an assortment.

The above model falls into the class of Mixed Integer Nonlinear Programming (MINLP) problems, which has recently experienced a flourish of research activity (Bussieck and Pruessner [8]). MINLP problems are very hard to solve since they encompass both the combinatorial nature of Mixed Integer Programs (MIP) and the difficulties of solving Nonlinear Programs (NLP). Indeed, our model is further complicated by being a *nonconvex* NLP, which could have several local optima. Fortunately, there are various solvers currently available, including the Global Solver of LINGO, which provide global optimal solutions for MINLP problems in *relatively low dimensions.* These solvers use branch-and-bound techniques that solve linear MIP subproblems over subregions defined by a partition of the original feasible region. Our experience is with LINGO 8.0; however, due to the use of similar techniques, we expect that other global solvers such as BARON (see Tawarmalani and Sahinidis [24]) will have comparable behavior. For a more in-depth discussion of the LINGO solver, the interested reader is referred to Gau and Schrage [16].

Several attempts of applying the LINGO 8.0 Global Solver (see Lindo Systems, Inc. [19] for a manual) to our model showed that LINGO 8.0 had difficulties handling our problems with six or more products. During this analysis it was discovered that the less similar the products are, the easier and faster the solver can find the solution. However, products in the same product category usually have similar characteristics in their selling prices, unit costs, and space- and cross-elasticities. This challenges LINGO 8.0 greatly and increases computational time significantly. Thus, a solution procedure using existing global solvers is expected to become impractical for a realistic number N of potential products. To overcome this limitation, the next section develops a new optimization procedure for effectively solving the shelf space management problem for a large number of products.

4 Reformulation and Piecewise Linearization Technique

Focusing first on the constraints of the product category model, observe that only the bilinear term $z_i n_i$ in constraints (5) and (6) is nonlinear. In order to linearize these constraints we use the technique of Al-Khayyal and Falk ([2]) and Al-Khayyal ([1]), who propose a reformulation technique for finding global solutions of bilinear programming problems (see also Sherali and Adams [21] who extended this technique). The technique involves the use of the convex and concave envelope of a bilinear function over a rectangular region. Each bilinear term (in our case $z_i n_i$) is replaced by a new variable (in our case w_i) and four additional linear constraints are imposed on each of these variables. In case the reader is not familiar with this technique, Appendix B provides a summary of the main ideas (without proofs). It is crucial to note that the nonlinear constraints (5) and (6) are replaced, using the technique in Appendix B, by an *equivalent* system of *linear* inequalities.

Turning our attention to the objective function (4), we define the following two intermediate variables that facilitate the description of our linearization scheme

$$r_i = n_i^{\beta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} n_j^{\delta_{ij}} \tag{10}$$

$$u_i = n_i^{\beta_i - 1} \prod_{\substack{j=1 \\ j \neq i}}^{N} n_j^{\delta_{ij}}. \tag{11}$$

Here, r_i represents the nonlinear terms quantifying the difference between gross margin and replenishment costs, and similarly, u_i is related to restocking costs. Our objective function can now be written more concisely as

$$\Omega = \sum_{i=1}^{N} \left[\alpha_i F_i^{\beta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} F_j^{\delta_{ij}} \left((P_i - C_i - CP_i) z_i r_i - \left(\frac{CR_i}{G_i} \right) z_i u_i \right) - \left(\frac{C_i IG_i}{2} \right) z_i n_i - CF_i z_i \right] \tag{12}$$

which exhibits a linear component z_i and bilinear components $z_i n_i$, $z_i r_i$ and $z_i u_i$. The bilinear components can be reformulated into equivalent linear forms subject to additional side constraints using the reformulation technique cited above; however, that would still leave nonlinear monomial constraints (10) and (11). To circumvent this difficulty, we replace r_i with e^{m_i} and u_i with $e^{m_i'}$ in (12) to obtain the equivalent objective function

$$\Omega = \sum_{i=1}^{N} \left[\alpha_i F_i^{\beta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} F_j^{\delta_{ij}} \left((P_i - C_i - CP_i) z_i e^{m_i} - \left(\frac{CR_i}{G_i} \right) z_i e^{m_i'} \right) - \left(\frac{C_i IG_i}{2} \right) z_i n_i - CF_i z_i \right] \tag{13}$$

with side constraints, for all $i = 1, \ldots, N$, given by

$$m_i = \beta_i \ln(n_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \ln(n_j) \quad (14)$$

$$m'_i = (\beta_i - 1)\ln(n_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \ln(n_j). \quad (15)$$

These constraints are still nonlinear, but each individual function $\ln(n_i)$ can be approximated by a piecewise linear function over its interval $[L_i, U_i]$. Since the functions $\ln(n_i)$ are concave, their piecewise linear representations are greatly simplified by using separable programming theory (see Stefanov [23]). The essence of this approach is to subdivide the interval by introducing a fixed number of grid points. For each subinterval a line segment is constructed that agrees with the function at the end points of the subinterval; i.e., at each grid point. The convex combination of the grid points is then taken to represent any point in the interval $[L_i, U_i]$. In order to guarantee a *unique* representation, an *adjacent weights restriction* (AWR) is imposed which ensures to be nonzero only the weights associated with the adjacent grid points that define the subinterval where a point lies. Satisfying this adjacent weights restriction is achieved by a restricted basis entry rule in simplex-based solvers. However, for our model, since $n_i \in \aleph^+$, by taking integer grid points and requiring the weights to be binary, the piecewise linear underestimating function of $\ln(n_i)$ is *exact* for all *feasible* points and the adjacent weights restriction is redundant. Please note that we need to use $V_i = U_i - L_i + 1$ integer grid points in order to contain all integer n_i within their bounds.

To ascertain which of the two formulations is better, we conducted numerical experiments on our model that compare the solution times of the binary weights formulation to the continuous weights with AWR formulation. These tests indicate that the binary weights formulation dominates the continuous weights formulation with AWR. This is not surprising since the binary (respectively, continuous) weights formulation gives an exact (respectively, approximate) representation of constraints (14) and (15) for discrete values of n_i. Apparently, the expense of solving a problem with more binary variables is mitigated by the extra work needed to apply the AWR. Moreover, the solution obtained by the latter method satisfies (14) and (15) only approximately.

For reasons that will soon become clear, we need to derive lower and upper bounds, denoted as A_i and B_i, on m_i given by (14). Similarly, we compute lower bounds A'_i and upper bounds B'_i on m'_i given by (15). These bounds are relatively easy to find. For this discussion, please recall that $0 \leq \beta_i \leq 1$, whereas $\delta_{ij} \leq 0$. Therefore, m_i is at its upper bound B_i if n_i is at its maximum value U_i and n_j is at its minimum value L_j. A similar but opposite argument can be made for finding the lower bound A_i of m_i. Thus, we have

$$A_i = \beta_i \ln(L_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \ln(U_j) \quad (16)$$

$$B_i = \beta_i \ln(U_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \ln(L_j) \quad (17)$$

and

$$A_i \leq m_i \leq B_i. \quad (18)$$

Since $\beta_i \leq 1$, the bounds A'_i and B'_i for m'_i are

$$A'_i = (\beta_i - 1)\ln(U_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \ln(U_j) \tag{19}$$

$$B'_i = (\beta_i - 1)\ln(L_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \ln(L_j) \tag{20}$$

and

$$A'_i \leq m'_i \leq B'_i. \tag{21}$$

Although the two new sets of constraints (14) and (15) can be linearized using the foregoing separable programming arguments, the objective function (13) is still nonlinear. We deal with the exponential terms by judiciously approximating them over the bounds on their arguments. In particular, we want our approximating objective to *overestimate* (13) so that the optimal objective value of the estimating problem provides an upper bound on the true optimal object value. Specifically, we approximate e^{m_i} by a *convex* piecewise linear *over*estimating function, and $e^{m'_i}$ is approximated by a *convex* piecewise linear *under*estimating function. Notice that we want *both* lower and upper approximations of a convex function to be convex. Our choice of which approximation (lower or upper) to choose is based on the objective coefficient of the exponential term in (13). Since we must have $P_i - C_i - CP_i \geq 0$ (otherwise, $z_i = 0$ would always be optimal), the coefficient of e^{m_i} is nonnegative, so we overestimate it. On the other hand, the coefficient of $e^{m'_i}$ is nonpositive, so we need an underestimate of $e^{m'_i}$ in order to have an overestimate of its negation.

Figure 1 shows a convex piecewise linear overestimating function of e^{m_i} created by choosing one grid point $E_i \in (A_i, B_i)$. The two linear functions are given by

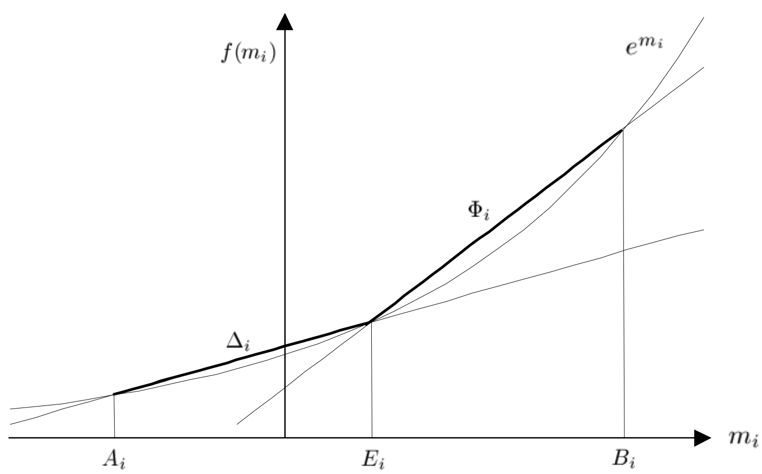

Figure 1: Piecewise-Linear Convex Overestimate of Exponential Function.

$$\Delta_i(m_i) = e^{A_i} + \left(\frac{e^{E_i} - e^{A_i}}{E_i - A_i} \right) (m_i - A_i)$$

$$\Phi_i(m_i) = e^{E_i} + \left(\frac{e^{B_i} - e^{E_i}}{B_i - E_i} \right) (m_i - E_i).$$

In general, E_i can be taken anywhere in the open interval (A_i, B_i), but we use

$$E_i = \ln\left(\frac{e^{B_i} - e^{A_i}}{B_i - A_i}\right)$$

which maximizes the absolute difference between e^{m_i} and the line segment connecting e^{A_i} and e^{B_i}; namely, $\left(\frac{e^{B_i}-e^{A_i}}{B_i-A_i}\right)(m_i - A_i) + e^{A_i}$. The piecewise linear function $\max\{\Delta_i(m_i), \Phi_i(m_i)\}$ overestimates e^{m_i} over the interval $[A_i, B_i]$ with $\Delta_i(m_i)$ defined on the subinterval $[A_i, E_i]$ and $\Phi_i(m_i)$ defined on the subinterval $[E_i, B_i]$. Hence, we have, for all $i = 1, \ldots, N$,

$$\begin{aligned} e^{m_i} &\leq \max\{\Delta_i(m_i), \Phi_i(m_i)\} \\ &\equiv y_i\Delta_i(m_i) + (1 - y_i)\Phi_i(m_i) \\ &= s_i \end{aligned}$$

for all y_i satisfying

$$y_i(E_i - m_i) \geq 0 \quad (22)$$

$$(1 - y_i)(E_i - m_i) \leq 0 \quad (23)$$

$$m_i \in [A_i, B_i] \quad (24)$$

$$y_i \in \{0, 1\}. \quad (25)$$

Therefore, replacing e^{m_i} by s_i in our objective (13) and incorporating the constraints (22)–(25) yields a linear mixed integer reformulation of the piecewise linear overestimating function of the exponential term.

In general, $K \geq 2$ grid points can be taken in the open interval (A_i, B_i), so that the whole interval is divided into $K + 1$ subintervals, each having overestimating line segment $\Delta_{ij}(m_i)$. It follows that, for every $i = 1, \ldots, N$, the piecewise linear overestimating function of e^{m_i} can be expressed as $\sum_{j=1}^{K+1} y_{ij}\Delta_{ij}(m_i)$ for $m_i \in [A_i, B_i]$ and $y_{ij} \in \{0, 1\}$ satisfying $\sum_{j=1}^{K+1} y_{ij} = 1$. If done for all products, this would introduce $N(K + 1)$ auxiliary binary variables. For the $K = 1$ case above, we used the constraints (22) and (23) to save one binary variable for each i, resulting in only N auxiliary binary variables. The remainder of the paper is restricted to the case $K = 1$, since the nominal improvement in the approximate solutions of several test problems did not justify the significant increase in the additional computation times for $K \geq 2$.

With bounds on s_i easily computed from (16) and (17), and after replacing all e^{m_i} by s_i in the objective function (13), we linearize all occurrences of the bilinear terms $z_i s_i$ in (13) and $y_i m_i$ in the constraints (22) and (23) using convex and concave envelopes (as detailed in Appendix B). This scheme produces equivalent linearly constrained linear reformulations of all bilinear terms.

Turning to the other exponential term in (13), recall that we need to construct an underestimating function of $e^{m'_i}$ over $[A'_i, B'_i]$ since its coefficient in (13) is nonpositive. We will take these underestimating linear segments to be defined by tangent lines of $e^{m'_i}$. In the spirit of the foregoing, we initially restrict our attention to the case of two segments defined by tangent lines at the interval end points; namely, lines tangent to the graph at $(A'_i, e^{A'_i})$ and $(B'_i, e^{B'_i})$. These are given by (see Figure 2)

$$\begin{aligned} \Theta_i(m'_i) &= e^{A'_i} + e^{A'_i}(m'_i - A'_i) \\ \Psi_i(m'_i) &= e^{B'_i} + e^{B'_i}(m'_i - B'_i). \end{aligned}$$

Hence, we have

$$\begin{aligned} -e^{m'_i} &\leq -\max\{\Theta_i(m'_i), \Psi_i(m'_i)\} \\ &= \min\{-\Theta_i(m'_i), -\Psi_i(m'_i)\} \\ &= g_i. \end{aligned}$$

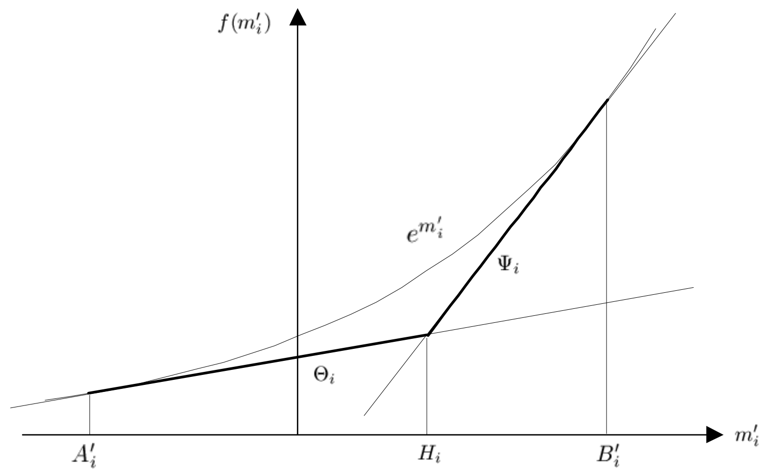

Figure 2: Piecewise-Linear Convex Underestimate of Exponential Function.

If we replace $-e^{m_i'}$ in (13) by g_i, we must add the constraint $g_i = \min\{-\Theta_i(m_i'), -\Psi_i(m_i')\}$. Since our objective is to maximize Ω and the coefficient of g_i is nonnegative (i.e., $\left(\frac{CR_i}{G_i}\right) z_i \geq 0$), then, by separability, the constraint $g_i = \min\{-\Theta_i(m_i'), -\Psi_i(m_i')\}$ is satisfied by maximizing $\left(\frac{CR_i}{G_i}\right) z_i g_i$ subject to the constraints

$$\begin{aligned} g_i &\leq -\Theta_i(m_i') \\ g_i &\leq -\Psi_i(m_i'). \end{aligned}$$

To complete the overestimating linearization of (13), the bilinear terms $z_i g_i$ are linearized via an equivalent linearly constrained reformulation using the convex and concave envelope technique of Appendix B, as the bounds for g_i are easily computed from (19) and (20). If more than two underestimating linear segments of $e^{m_i'}$ are desired, we only need to include one constraint for each new support. In general, if there are K tangent lines $\Theta_{ij}(m_i')$ that support the graph of $e^{m_i'}$ within the interval $[A_i', B_i']$, then $-e^{m_i'} \leq \min_{j=1,\ldots,K}\{-\Theta_{ij}(m_i')\}$. By the same argument as for the $K = 2$ case, we may overestimate $-e^{m_i'}$ by replacing it with g_i and introducing the K linear constraints $g_i \leq \Theta_{ij}(m_i')$ for $j = 1, \ldots, K$. A natural choice for a third support would be at $H_i = \frac{e^{B_i'}(B_i'-1)-e^{A_i'}(A_i'-1)}{e^{B_i'}-e^{A_i'}}$, which maximizes the difference between $e^{m_i'}$ and the piecewise linear underestimating function $\max\{\Theta_i(m_i'), \Psi_i(m_i')\}$. However, to be consistent with our use of two linear segments to overestimate e^{m_i}, we conducted all of our numerical experiments with two supports of Figure 2 to underestimate $e^{m_i'}$.

Thus, we have derived a linear MIP whose feasible region, when projected onto the decision space of the shelf space model, is identical to that of the nonconvex shelf space allocation model, and whose optimal objective value is an upper bound on the optimal objective value of the nonconvex model. The complete linearized model is presented in Appendix C.

4.1 Discussion

Careful consideration has been given to alternative methods of linearization. In particular, the foregoing approximations of e^{m_i} and $e^{m_i'}$ appear to be the best ways of linearizing these functions for our purposes. At first glance, separable programming approximations of e^{m_i} and $e^{m_i'}$ seem to be good alternatives and would be expected to generate better results because they estimate the exponential functions more closely. However, such approximations lead to a contradiction because of the following argument. If e^{m_i} and $e^{m'_i}$ were estimated with a separable programming technique, the model would have two equality constraints for each m_i and two others for each m'_i. Please note that for m_i, one of these constraints is the piecewise linear representation of (14) and the additional constraint would be the output of the overestimating piecewise linear approximation of e^{m_i}. This additional constraint imposes another equation that m_i must satisfy. Please recall that the separable programming approximations of (14) and (15) with binary weights define the exact same feasible region as (14) and (15), while the separable programming approximations of e^{m_i} and $e^{m'_i}$ would be inexact. Thus, we would have two contradictory constraints for each m_i and two for each m'_i, one of which gives exact values whereas the other gives approximate values. The linearized model becomes infeasible when this is done.

As noted earlier, our approximations of e^{m_i} and $e^{m'_i}$ can certainly be improved by utilizing more than two linear segments for each function. However, this requires the introduction of supplementary constraints and/or binary variables, which, in turn, increases computational time. We used only two linear segments in all of our computational experiments and observed excellent results for all test problems solved.

Please recall that the difference between gross margin and replenishment costs are overestimated as shown in Figure 1. Further, restocking costs are underestimated as illustrated in Figure 2. We did this because we made the reasonable assumption that the difference between gross margin and replenishment costs is always nonnegative (which holds if and only if $P_i - C_i - CP_i \geq 0$); else, $z_i = 0$. Under this assumption, the value of the objective function at the optimal solution of the linearized model provides an *a posteriori* upper error bound, denoted by P_u, on the overall problem. Since the optimal solution of the linearized model is feasible for the true model, we may use it to evaluate the *true* objective function (4) to obtain the actual profit, denoted by P_a . Although the true optimal solution, P_o , is unknown, the ratio $\frac{P_a}{P_u} \leq 1$ can be used to evaluate the quality of the returned solution. Please note that this ratio represents a "worst case" analysis as $P_u \geq P_o$ always holds.

5 Numerical Examples and Test Problems

To assess the tightness of our linear approximations, we applied our procedure to nine different product categories with up to six distinct products in each category. Rather than developing a hypothetical example, actual data was collected from a large retail store in order to make the results more practical. Space- and cross-elasticities were estimated, as cross-sectional data, needed to estimate them via regression, was not available. However, based on past research, (e.g., Curhan [12], Corstjens and Doyle [11]), space-elasticity typically ranges between 0.06 and 0.25, whereas cross-elasticities were assumed to take on values between -0.01 and -0.05. Similar to the classification of Brown and Tucker [5], the product categories were placed into three different classes. Space-elasticity was assumed to fall in the following closed intervals: $[0.06, 0.1]$, $[0.16, 0.20]$, and $[0.21, 0.25]$ for the first, second and third class, respectively. Space- and cross-elasticities were assigned randomly to the investigated products within these ranges. See Table 2 in Appendix A for actual values used, where δ_{ii} denotes space-elasticity and δ_{ij} is our standard notation for cross-elasticity between products i and j. With the product sales data as our demand and estimated elasticities in Table 2, the scaling factors α_i were easily calculated using equation (1).

According to information obtained from the store management, the following parameters were estimated and assumed to be equal for all product categories:

- Restocking cost $CR_i = \$5$ each time a product is restocked
- Unit replenishment cost $CP_i = 0.01C_i$
- Fixed cost $CF_i = \$25$ to include product i in the assortment

- Investment/interest rate $I = 1\%$ per month.

Individual product facings F_i were assumed to be equal to the width of each product. All other input parameters of the model were easily calculated using the data shown in Table 2 in Appendix A.

For each product category, the approximating linearized model was solved for three different levels of total available shelf space S; namely, the observed amount S_o, as well as $1.5S_o$ and $2S_o$. No restrictive lower and upper bounds on n_i were introduced. The optimal objective value P_a for the approximating model was found using CPLEX 8.1 and compared to the global optimal objective value P_o of the exact (retail shelf space optimization) model (4)–(9) obtained by the global solver LINGO 8.0. The results of our numerical experiments are displayed in Table 1. All tests were run on a personal computer with a single 0.93 GHz Pentium 3 processor and 384 KB RAM.

Product Category	Available Space S (in)	Approximate Profit P_a (\$)	$\frac{P_a}{P_u} \times 100$ (%)	$\frac{P_a}{P_o} \times 100$ (%)	ST_a (sec)	ST_o (sec)
Bulbs	166	1501.98	92.35	99.62	0.64	1914.13
	249	1593.23	91.13	99.41	0.64	2607.24
	332	1653.48	90.19	99.15	0.80	7115.56
Enamel	182	2364.97	98.90	100.00	0.31	7.75
	273	2403.95	98.72	100.00	0.34	10.19
	364	2426.31	98.57	100.00	0.28	12.45
Rust Stopper	87	381.89	88.59	96.47	0.81	428.95
	131	410.07	88.83	96.90	1.09	960.39
	174	427.31	88.24	96.49	1.55	1858.21
Plastic Sheets	422	2487.86	91.50	100.00	0.55	469.54
	633	2656.36	90.50	100.00	1.02	849.52
	844	2783.20	89.83	100.00	0.97	932 92
Flood Lights	249	3333.79	99.39	100.00	0.30	12.02
	374	3374.20	99.31	100.00	0.33	14.63
	498	3398.45	99.25	100.00	0.33	20.53
Tube Lights	116	240.66	95.76	99.88	0.72	20.46
	174	265.63	95.54	99.89	0.97	18.92
	232	282.23	95.31	99.85	0.81	30.54
Glass Cleaner	119	328.06	92.37	99.63	0.53	309.73
	179	338.35	92.08	98.96	1.13	585.64
	238	343.73	91.81	98.70	0.92	253.28
Degreaser	161	379.76	96.28	99.87	0.41	8.28
	242	411.00	96.51	99.97	0.59	9.06
	322	431.22	96.54	99.98	0.36	12.16
Trash Bags	190	2104.03	97.21	99.57	0.44	104.37
	285	2166.12	96.44	99.57	0.58	262.25
	380	2210.57	95.84	99.67	0.64	565.36

Table 1: Summary of Computational Experiments

The CPU times ST_o of the global solver of LINGO 8.0 ranged from 7.75 seconds to 7115.56 seconds with an average value of 718.30 seconds. In contrast, the CPU times of the approximating model ST_a, using CPLEX 8.1, ranged from 0.28 seconds to 1.55 seconds with an average value of 0.67 seconds. The approximating model found the exact global optimal solution in 9 out of 27 test cases. The worst result in terms of the value $\frac{P_a}{P_u}$ occurred for the product category "Rust Stopper" for total available shelf space of 174 inches. Since $P_u \geq P_o$, it is possible to claim for this test case that the calculated objective value P_a is no worse than 88.24% of the global optimal objetive value P_o. However, our comparison to the known global solution of the *true* nonconvex model shows that the calculated solution is significantly better, determining the value of P_a to be 96.49% of P_o. The average ratio $\frac{P_a}{P_u}$ was calculated to be 94.33% whereas the average ratio $\frac{P_a}{P_o}$ was 99.39%. Hence, on average our method found a feasible solution whose objective value is *guaranteed* to be within 6% of optimum, but is in fact within 1% of optimum. This indicates that near-optimal solutions and tight *a posteriori* error bounds can be expected when solving the proposed approximating MIP. Both measurements $\frac{P_a}{P_u}$ and $\frac{P_a}{P_o}$ can be improved by imposing lower and upper bounds on each variable n_i. Please note that such bounds tighten the intervals $[A_i, B_i]$ and $[A'_i, B'_i]$, and, therefore, improve the piecewise linear estimations of the exponential functions. For the test cases considered, we have allowed the total available shelf space to be allocated to a single product. If we make an assumption that a retailer wishes to limit the shelf space allocated to a product to be no more than 25 percent of the total available shelf space, then the average ratios $\frac{P_a}{P_u}$ and $\frac{P_a}{P_o}$ increase to 97.77% and 99.69% respectively. This indicates that the additional bounds not only improve the quality of the calculated solution but also allow the returned solution to be compared to a tighter *a posteriori* bound.

To further investigate the ability of the approximating model to handle problems with larger numbers of individual products, three additional tests were conducted for a problem size of twenty. For each test, we chose products that had similar characteristics to those of a specific category ("Bulbs" for the first test, "Enamel" for the second test, and "Rust Stopper" for the third test). We assumed that not all of the products within a product category were sufficiently similar to be regarded as substitute products. Thus, we classified the twenty products into four subcategories, each having five individual products, and further assumed that only cross-elasticities within a subcategory were negative. Cross-elasticities across subcategories were taken as zero. The amount of total available shelf space was increased to 2.5 times the amount observed in the store. The upper bound for each product was set in a way that allowed at most 20 percent of the total available shelf space to be allocated to a product.

The approximating model was solved with an average ST_a value of 1.15 seconds using CPLEX 8.1 with $\frac{P_a}{P_u}$ values of 97.44%, 99.64% and 96.95% for the three tests. In contrast, LINGO 8.0 was no longer able to compute the global solution in a reasonable time. After a processing time of three hours on the personal computer mentioned above, the first test case had not been solved and an optimal solution was expected to take several days. This indicates that realistic problem sizes are intractable for general global solvers, whereas the proposed approximating model is suitable for a practical number of products. Please note that a problem size of twenty individual products is already a realistic size for a product category. The average solution time of the investigated tests with twenty products was only slightly larger than the average solution time found for six products. This clearly indicates that our solution technique is capable of solving all realistic problem sizes in a given product category within reasonable computational time. In contrast to the work of Lim *et al.* [18], our shelf space optimization model is not a simplification but an extension of the original optimization model of Corstjens and Doyle [11]. In addition, our solution procedure not only finds excellent feasible solutions, but also provides tight *a posteriori* error bounds to evaluate the quality of the solution.

6 Extensions

The main model only considers space- and cross-elasticities. This section extends the model to incorporate additional factors that might be important to a retailer.

6.1 Incorporation of Other Marketing Variables

In the presentation above, we have assumed that effects other than space- and cross-elasticities are not present (Assumption (iii)). Clearly, this assumption neglects effects of price and other marketing variables. Therefore, it might be desirable to include price as well as other variables that could be used to influence the market. Instead of (1), we now formulate the demand function as (see Yang and Chen [26] for a similar formulation):

$$d_i = \alpha_i (F_i n_i)^{\beta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} (F_j n_j)^{\delta_{ij}} p_i^{-\varepsilon_i} \prod_v w_{iv}^{\gamma_{iv}}$$

where, for product i, the (unknown) selling price is given by p_i and $\varepsilon_i \geq 1$ denotes the price elasticity, while, for each v, the influence on the demand for product i (e.g., special in-store advertising) can be modeled in a multiplicative way via a new variable w_{iv} with *shape parameter* $0 \leq \gamma_{iv} \leq 1$). Each variable w_{iv} is likely to cause further expenditures (such as advertising expenditures) and, therefore, needs to be included in the cost terms of the objective function. In such a setting, the decision variables are no longer simply the number of facings allocated to each product, but also its price p_i and all other relevant demand influencing variables w_{iv}.

Although the demand function is now more complex, the model can still be solved using an extension to the approach followed in Section 4. Now, however, the functions r_i and u_i in (10) and (11), respectively, include all variables n_i, n_j, p_i, and w_{iv}.

6.2 Incorporation of Fixed Procurement Costs and Warehouse Space

So far we have not considered fixed procurement costs and the possibility of storing items in a warehouse. However, such a scenario appears to be very practical in retailing where products are usually placed in a "backroom" storage area before being brought to the display area. As stated before, demand within our model is assumed to be deterministic and products are fully restocked immediately when they stock out, allowing a simple calculation of the reorder point for each product. Thus, the only decision variables in this setting are the order quantity and the number of facings allocated to each product.

In order to model this new scenario, the following additional assumptions are made:

(viii) There are no quantity-related or transport-related discounts; and

(ix) If an order arrives, it is either partly or totally used to restock the shelves. (If the number of units ordered exceeds the available space for shelved units, then the rest of the order is placed in the warehouse).

Because of Assumption (v), order quantity can only be an integer multiple of the total number of product-units that fill the shelves, as any other variable would generate a suboptimal solution. To understand why this must be the case, assume that a retailer orders 1.5 times the number of product-units that can be placed on the shelves. This means that 33 percent of the ordered units have to be placed in the warehouse. However, the number of units sitting in the warehouse is not sufficient to restock this product completely, making it necessary (by Assumption (v)) to place another order and keep the excess inventory in storage until the new order arrives. Thus, such a scenario faces additional inventory holding costs, making this strategy apparently suboptimal.

Maintaining the demand function (1), the objective function of the extended model can be written as:

$$\Omega = \sum_{i=1}^{N} z_i \left[d_i \left(P_i - C_i - CP_i - \left(\frac{CR_i}{G_i} \right) \frac{1}{n_i} - \left(\frac{COF_i}{G_i} \right) \frac{1}{h_i n_i} \right) - \left(\frac{C_i IG_i}{2} \right) h_i n_i - CF_i \right]$$

where COF_i are the fixed procurement costs per order and h_i is a positive integer variable such that $h_i - 1$ is the number of restocks possible from one order quantity. Observe that $G_i h_i n_i$ represents the order quantity (say, q_i) of which $G_i n_i$ are shelved immediately, whereas $\left(\frac{C_i IG_i}{2}\right)(h_i - 1)n_i$

accounts for the additional inventory holding costs of product-units stored in the warehouse, and $(COF_i)\frac{d_i}{q_i} = \left(\frac{COF_i}{G_i}\right)\frac{d_i}{h_i n_i}$ gives the fixed procurement costs. As the available warehouse space is usually limited, an additional constraint has to be incorporated into the model formulation in order to ensure that the sum of all order quantities q_i can never exceed the sum of available shelf and warehouse space.

Please observe that the appearance of the new decision variable h_i in the denominator of the fixed procurement costs represents the only difficulty in linearizing this extended model in the same ways as demonstrated in Section 4. However, if lower and upper bounds on h_i are introduced, it is possible to approximate each function $\frac{1}{h_i}$ by a piecewise linear function. The term $z_i d_i \left(\frac{COF_i}{G_i}\right)\frac{1}{n_i}$ can then be approximated in the same way as $z_i d_i \left(\frac{CR_i}{G_i}\right)\frac{1}{n_i}$ using linear reformulations analogous to those derived in Section 4.

6.3 Incorporation of Substitution Effects

So far we have assumed that each product is restocked as soon as it stocks out. However, many retailers do not restock their products individually, but restock each product at specific times (e.g., every morning). In such a scenario, each product will face a stockout if its predicted demand during restocking cycle R exceeds its number of available units on the shelves. This excess demand (called a "stockout demand" by Borin *et al.* [4]) is then potentially available for other substitute products. In addition, we have ignored the effects of products that are not included in the assortment. In reality, each of the stocked products receives a portion of the available sales that other products would have obtained if they had not been excluded. This concept is introduced by Borin *et al.* [4] who define this additional demand as "acquired demand." The authors state that if a product is excluded from the assortment, its "modified demand" can be potentially transferred to the stocked products. Their definition of "modified demand" is identical to the demand function (1).

Similar to the work of Smith and Agrawal [22] we allow for one round of substitution. This means that a customer might either refuse to buy any other product or decide to purchase another item if his/her first choice product is either temporarily or permanently unavailable. If the customer's substitution choice is a product that is not available, the sale is lost. We propose the following structure for the substitution probabilities, μ_{ij}, between products i and j, for $i \neq j$:

$$\mu_{ij} = \eta_i \left(\frac{Z_j}{\sum_{\substack{k=1 \\ k\neq i}}^{N} Z_k} \right)$$

where η_i represents the fraction of the customers who are willing to compromise their initial choice of product i, and Z_j is the national market share of product j. In the denominator, Z_i is taken out of the total market share for product i to ensure that $\sum_{j=1}^{N} \mu_{ij} = \eta_i$. This formulation is similar to the one of Smith and Agrawal [22] and redistributes additional demand in proportion to the product's market share. It further allows the determination of the values μ_{ij} by solely estimating η_i. Please note that the substitution choice of a customer is not restricted to the products that are included in the assortment. If a customer wants to substitute his/her first choice with a product that is not shelved, the sale is consequently lost.

The objective function of the extended maximization model can now be formulated as:

$$\Omega = \sum_{i=1}^{N} z_i \left[\min\left\{ d_i, \left(\frac{T}{R}\right) G_i n_i \right\} (P_i - C_i - CP_i) - CR_i \left(\frac{T}{R}\right) - \left(\frac{C_i I G_i}{2}\right) n_i - CF_i \right] \tag{26}$$

where

$$d_i = f_i + \mu_{ir} \sum_{\substack{r=1 \\ r \neq i}}^{N} \left[(1 - z_r) f_r + z_r \max \left\{ 0, f_r - \left(\frac{T}{R} \right) G_i n_i \right\} \right] \tag{27}$$

$$\tag{28}$$

and

$$f_i = \alpha_i (F_i n_i)^{\beta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} (F_j n_j)^{\delta_{ij}}. \tag{29}$$

Here, T represents the time interval of each demand function (with $T = 1$ for demand per day) and R is the restocking cycle time.

The minimum operator in objective function (26) guarantees that product sales cannot exceed available units on the shelves, given by $\left(\frac{T}{R}\right) G_i n_i$, within time interval T. The first term within the brackets of demand function (27) represents "acquired demand" of products not included in the assortment. Here, f_r is the "modified demand" as defined by (29). The maximum operator in the second term within the brackets of demand function (27) represents excess demand for product r (similar to the "stockout demand" of Borin *et al.* [4]), which is transferred to product i according to the substitution probability μ_{ir}.

With this formulation, only "modified demand" is substituted. Note that demand for some products is partly reallocated to other products and might then exceed the number of shelved units of those products, implying additional substitutions. Because we assume that only one round of substitution can occur, these effects are ignored. Restocking costs of each product in objective function (26) are now independent of the number of facings allocated to the product. Inventory holding costs are only an approximation of the actual costs, which are higher if a product is restocked (even if the number of units sitting on the shelves is greater than zero) and lower if a product stocks out during restocking cycle R. However, this difference is small and, thus, neglected in the objective function shown above.

Although the demand function is now significantly more complex, the model can still be linearized using an analogous approach to that demonstrated in Section 4. By making the reasonable assumption that $P_i - C_i - CP_i$ is always nonnegative (else, $z_i = 0$), we can replace $\min \left\{ d_i, \left(\frac{T}{R}\right) G_i n_i \right\}$ by e_i, and add the following two linear constraints, since our objective is to maximize Ω:

$$\begin{aligned} e_i &\leq d_i \\ e_i &\leq \left(\frac{T}{R} \right) G_i n_i. \end{aligned}$$

Please note that we can only linearize the resulting bilinear term $z_i e_i$ in the objective function if we know the upper and lower bounds of e_i. Fortunately, these bounds are easy to find. The variable d_i is at its upper bound if n_i is at its maximum value U_i and n_j is at its minimum value L_j. A similar but opposite argument can be made for finding the lower bound of d_i . An analogous argument applies for finding the lower and upper bounds of $\left(\frac{T}{R}\right) G_i n_i$.

The maximum operator of (27) can be reformulated by introducing N supplementary binary variables and $2N$ additional constraints. The individual functions f_i and f_r can be approximated in the same way as the functions r_i are approximated in Section 4.

7 Conclusions and Future Work

This paper proposes a piecewise linearization technique, which is accurate, easy to implement, and flexible enough to be applied to a wide range of shelf space management models in which the demand function is of signomial form. It allows large-scale instances of such highly nonlinear models to be solved efficiently. More importantly, unlike heuristic methods employed in the literature, our procedure further provides an *a posteriori* error bound on the closeness of our computed solution to the true global optimal objective value of the nonconvex shelf space management model.

The optimization models developed in this paper, as well as most optimization models in the literature, are capable of allocating shelf space to products within a product category. Such models use a bottom-up approach to assign shelf space to individual products and implicitly assume that the amount of shelf space assigned to a product category is predetermined and unalterable. Because of the large number of products found in most retail stores, it is clearly not feasible to solve one of these existing models for all potentially available products within a store (sometimes more than 60,000). Please note that the assignment rules of most models, which allocate shelf space to individual products, are flexible enough to be applied to allocation problems at a higher level of aggregation. Thus, similar models can be used to allocate store shelf space to product categories using a top-down approach (see, e.g., Campo *et al.* [9] for an optimization model that allocates store shelf space to product categories using the assignment rule originally proposed by Bultez and Naert [6]). In theory, the output of the top-down approach (shelf space allocated to a product category) becomes a constraint for the bottom-up procedure (amount of available shelf space within a product category). However, these two different hierarchical approaches have not yet been connected. Consequently, a globally optimal allocation of store shelf space to products has not yet been achieved. In a forthcoming companion paper, the authors overcome this limitation by developing a bilevel hierarchical modeling and optimization method that finds accurate solutions for very large and complex shelf space management problems in a computationally efficient fashion.

Appendix A: Data for Computational Experiments

(Please see Table 2 at end of paper.)

Appendix B: Convex and Concave Envelopes of a Bilinear Form in $\Re^2$

For a bilinear form xy, where $(x,y) \in \Re^2$, Al-Khayyal and Falk [2] prove that the *convex envelope*[5] over the rectangular domain $[x^L, x^U] \times [y^L, y^U]$, where L (U) denotes a known lower (upper) bound on the variable, is given by

$$\max\{x^L y + y^L x - x^L y^L, x^U y + y^U x - x^U y^U\}.$$

Analogously, the *concave envelope*[6] of xy, where $(x,y) \in \Re^2$, over the rectangular domain $[x^L, x^U] \times [y^L, y^U]$ is given by (see Al-Khayyal [1])

$$\min\{x^U y + y^L x - x^U y^L, x^L y + y^U x - x^L y^U\}.$$

By definition we have,

$$\begin{aligned} \max\{x^L y + y^L x - x^L y^L, x^U y + y^U x - x^U y^U\} &\leq xy \qquad (30) \\ &\leq \min\{x^U y + y^L x - x^U y^L, x^L y + y^U x - x^L y^U\} \end{aligned}$$

for all $(x,y) \in [x^L, x^U] \times [y^L, y^U]$. It is easy to show that if either x or y is at one of its bounds, then equality in (30) holds throughout.

If we replace the term xy everywhere in a model by a new variable g, then from the convex envelope we have

$$g \geq x^L y + y^L x - x^L y^L \qquad (31)$$
$$g \geq x^U y + y^U x - x^U y^U \qquad (32)$$

and from the concave envelope we have

$$g \leq x^U y + y^L x - x^U y^L \qquad (33)$$
$$g \leq x^L y + y^U x - x^L y^U. \qquad (34)$$

Replacing xy everywhere in a model by g and augmenting the model's constraint set by the four constraints (31) through (34) linearizes the xy term in the model and ensures that the product xy satisfies (30). However, in general, there are infinitely many feasible points for which $g \neq xy$. Fortunately, as pointed out above, if either $x \in \{x^L, x^U\}$ or $y \in \{y^L, y^U\}$ we must have $g = xy$.

Appendix C: Summary of Linear MIP Approximating Model

Find $(z_i, n_i, w_i, l_i, t_i, o_i, s_i, y_i, g_i, m_i, m_i', \lambda_{iv})$ that

$$\text{maximize} \quad \widetilde{\Omega} = \sum_{i=1}^{N} \left[\alpha_i F_i^{\beta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} F_j^{\delta_{ij}} \left((P_i - C_i - CP_i) t_i + \left(\frac{CR_i}{G_i} \right) o_i \right) - \left(\frac{C_i IG_i}{2} \right) w_i - CF_i z_i \right]$$

subject to

[5] The convex envelope of a function over a convex domain is the highest convex underestimating function over the domain.

[6] The concave envelope of a function over a convex domain is the lowest concave overestimating function over the domain.

$$\sum_{i=1}^{N} F_i w_i \leq S \tag{35}$$

$$F_i^2 w_i - F_i^2 n_i - F_i^2 z_i + F_i^2 \geq 0 \qquad i = 1, \ldots, N \tag{36}$$

$$w_i \leq U_i z_i \qquad i = 1, \ldots, N \tag{37}$$

$$w_i \leq L_i z_i + n_i - L_i \qquad i = 1, \ldots, N \tag{38}$$

$$w_i \geq L_i z_i \qquad i = 1, \ldots, N \tag{39}$$

$$w_i \geq U_i z_i + n_i - U_i \qquad i = 1, \ldots, N \tag{40}$$

$$l_i \leq B_i y_i \qquad i = 1, \ldots, N \tag{41}$$

$$l_i \leq A_i y_i + m_i - A_i \qquad i = 1, \ldots, N \tag{42}$$

$$l_i \geq A_i y_i \qquad i = 1, \ldots, N \tag{43}$$

$$l_i \geq B_i y_i + m_i - B_i \qquad i = 1, \ldots, N \tag{44}$$

$$t_i \leq e^{B_i} z_i \qquad i = 1, \ldots, N \tag{45}$$

$$t_i \leq e^{A_i} z_i + s_i - e^{A_i} \qquad i = 1, \ldots, N \tag{46}$$

$$t_i \geq e^{A_i} z_i \qquad i = 1, \ldots, N \tag{47}$$

$$t_i \geq e^{B_i} z_i + s_i - e^{B_i} \qquad i = 1, \ldots, N \tag{48}$$

$$o_i \leq -e^{A'_i} z_i \qquad i = 1, \ldots, N \tag{49}$$

$$o_i \leq -e^{B'_i} z_i + g_i + e^{B'_i} \qquad i = 1, \ldots, N \tag{50}$$

$$o_i \geq -e^{B'_i} z_i \qquad i = 1, \ldots, N \tag{51}$$

$$o_i \geq -e^{A'_i} z_i + g_i + e^{A'_i} \qquad i = 1, \ldots, N \tag{52}$$

$$s_i = l_i(D_i - D'_i) + y_i(e^{A_i} - D_i A_i - e^{E_i} + D'_i E_i) + e^{E_i} + D'_i(m_i - E_i) \qquad i = 1, \ldots, N \tag{53}$$

$$y_i E_i - l_i \geq 0 \qquad i = 1, \ldots, N \tag{54}$$

$$(1 - y_i) E_i - m_i + l_i \leq 0 \qquad i = 1, \ldots, N \tag{55}$$

$$g_i \leq -(e^{A'_i} + e^{A'_i}(m'_i - A'_i)) \qquad i = 1, \ldots, N \tag{56}$$

$$g_i \leq -(e^{B'_i} + e^{B'_i}(m'_i - B'_i)) \qquad i = 1, \ldots, N \tag{57}$$

$$m_i = \beta_i \sum_{v=1}^{V_i} \lambda_{iv} \ln(X_{iv}) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \sum_{v=1}^{V_j} \lambda_{jv} \ln(X_{jv}) \qquad i = 1, \ldots, N \tag{58}$$

$$m'_i = (\beta_i - 1) \sum_{v=1}^{V_i} \lambda_{iv} \ln(X_{iv}) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{ij} \sum_{v=1}^{V_j} \lambda_{jv} \ln(X_{jv}) \qquad i = 1, \ldots, N \tag{59}$$

$$n_i = \sum_{v=1}^{V_i} \lambda_{iv} X_{iv} \qquad i = 1, \ldots, N \tag{60}$$

$$\sum_{v=1}^{V_i} \lambda_{iv} = 1 \qquad i = 1, \ldots, N \tag{61}$$

$$\lambda_{iv} \in \{0, 1\} \qquad i = 1, \ldots, N; \; v = 1, \ldots, V_i \tag{62}$$

$$L_i \leq n_i \leq U_i \qquad i = 1, \ldots, N \tag{63}$$

$$n_i \in \aleph^+ \qquad i = 1, \ldots, N \tag{64}$$

$$z_i \in \{0, 1\} \text{ and } y_i \in \{0, 1\} \qquad i = 1, \ldots, N \tag{65}$$

where $D_i = \frac{e^{E_i} - e^{A_i}}{E_i - A_i}$ and $D'_i = \frac{e^{B_i} - e^{E_i}}{B_i - E_i}$.

Constraints (37) through (52) are the output of the linearization technique of Al-Khayyal *et al.* [2] and Al-Khayyal [1] and guarantee that $w_i = n_i z_i$, $l_i = m_i y_i$, $t_i = s_i z_i$, and $o_i = g_i z_i$ hold at an optimal solution, since $z_i \in \{0, 1\}$ and $y_i \in \{0, 1\}$ (see Appendix B). Constraints (58) through (62) are the piecewise linear function representations of constraints (14) and (15), where V_i denotes the number of grid points used for the piecewise linear approximation of each $\ln(n_i)$ and X_{iv} specifies the value of grid point v.

Acknowledgments. The research of the first author was supported in part by the German Academic Exchange Service (DAAD), while author was visiting the School of Industrial and Systems Engineering at the Georgia Institute of Technology, Atlanta, GA, 30332-0205, USA. The research of the second author was supported in part by US National Science Foundation Grant DMI-990826.

References

[1] Al-Khayyal, F. A., Jointly Constrained Bilinear Programs and Related Problems: An Overview, *Computers & Mathematics with Applications*, **19**(8), pp. 53-62, 1990.

[2] Al-Khayyal, F. A. and J. E. Falk, Jointly Constrained Biconvex Programming, *Mathematics of Operations Research*, **8**(2), pp. 273-286, 1983.

[3] Anderson, E. E., An Analysis of Retail Display Space: Theory and Methods, *Journal of Business*, **52**(1), pp. 103-118, 1979.

[4] Borin, N., P. Farris and J. Freeland, A Model for Determining Retail Product Category Assortment and Shelf Space Allocation, *Decision Science*, **25**(3), pp. 359-384, 1994.

[5] Brown, W. M. and W.T. Tucker, Vanishing Shelf Space, *Atlanta Economic Review*, (October), pp. 9-13, 1961.

[6] Bultez, A. and P. Naert, SHARP: Shelf Space Allocation for Retailers Profit, *Marketing Science*, **7**(3). pp. 211-231, 1988.

[7] Bultez, A., P. Naert, E. Gijsbrechts and P. V. Abelle, Asymmetric Cannibalism in Retail Assortments, *Journal of Retailing*, **65**(2), pp. 153-192, 1989.

[8] Bussieck, M. R. and A. Pruessner, Mixed-Integer Nonlinear Programming, *SIAG/OPT Newsletter: Views & News*, **14**(1), 2003.

[9] Campo, K., E. Gijsbrechts, T. Goossens and A. Verhetsel, The Impact of Location Factors on the Attractiveness and Optimal Space Shares of Product Categories, *International Journal of Research in Marketing*, **17**, pp. 225-279, 2000.

[10] Chen, Y., J. D. Hess, R. T. Wilcox and Z. H. Zhang, Accounting Profits Versus Marketing Profits: A Relevant Metric for Category Management, *Marketing Science*, **18**(3), pp. 208-229, 1999.

[11] Corstjens, M. and P. Doyle, A Model for Optimizing Retail Shelf Space Allocations, *Management Science*, **27**(7), pp. 822-833, 1981.

[12] Curhan, R. C., The Relationship Between Shelf Space and Unit Sales in Supermarkets, *Journal of Marketing Research*, **9**, pp. 406-412, 1972.

[13] Desmet, P. and V. Renaudin, Estimation of Product Category Sales Responsiveness to Allocated Shelf Space, *International Journal of Research in Marketing*, **15**, pp. 443-457, 1998.

[14] Drèze, X., S. J. Hoch and M. E. Purk, Shelf Management and Space Elasticity, *Journal of Retailing*, **70**(4), pp. 301-326, 1994.

[15] Ehrenberg, A.S.C., *Repeat Buying: Theories and Applications*, North Holland, Amsterdam, 1972.

[16] Gau, C.-Y.and L. E. Schrage, Implementation and Testing of a Branch-and-Bound Based Method for Deterministic Global Optimization: Operations Research Applications, in *Frontiers in Global Optimization*, C.A. Floudas and P.M. Pardalos (Eds.), pp. 1-20, Kluwer Academic Publishers, Boston, 2003.

[17] Hansen, P. and H. Heinsbroek, Product Selection and Space Allocation in Supermarkets, *European Journal of Operational Research*, **3**, pp. 474-484, 1979.

[18] Lim, A., B. Rodrigues and X. Zhang, Metaheuristics with Local Search Technique for Retail Shelf-Space Optimization, *Management Science*, **50**(1), pp. 117-131, 2004.

[19] LINDO Systems, Inc., *LINGO user's Manual*, LINDO Systems, Chicago, IL. 2002.

[20] POPAI, *Consumer Buying Habits Study*, Washington DC, Point of Purchase Advertising Institute, 1997.

[21] Sherali, H. D. and W. P. Adams, *A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*, Kluwer Academic Publisher, Dordrecht, 1999.

[22] Smith, S. A. and N. Agrawal, Management of Multi-Item Retail Inventory Systems with Demand Substitution, *Operations Research*, **48**(1), pp. 50-64, 2000.

[23] Stefanov, S. M., *Separable Programming: Theory and Methods*, Kluwer Academic Publishers, Dordrecht, 2001.

[24] Tawarmalani, M. and N. V. Sahinidis, *Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Software, and Applications*, Vol. 65 in "Nonconvex Optimization and its Applications" series, Kluwer Academic Publishers, Dordrecht, 2002.

[25] Urban, T. L., An Inventory-Theoretic Approach to Product Assortment and Shelf-Space Allocation, *Journal of Retailing*, **74**(1), pp. 15-35, 1998.

[26] Yang, M.-H. and W.-C. Chen, A Study on Shelf Space Allocation and Management, *International Journal of Production Economics*, **60-61**, pp. 309-317, 1999.

[27] Yang, M.-H., An Efficient Algorithm to Allocate Shelf Space, *European Journal of Operational Research*, **131**, pp. 107-118, 2001.

[28] Zufryden, F. S., A Dynamic Programming Approach for Product Selection and Supermarkt Shelf-Space Allocation, *Journal of the Operational Research Society*, **37**(4), pp. 413-422, 1986.

Product Category	Product i	Product Dimensions (in)			Assigned Space(in)	Sales Price($)	Unit Cost($)	Units Sold	Shelf Space (in)		Cross Elasticity δ_{ij} for Product j						Scale α_i
		Height	Depth	Width					Height	Depth	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	
Bulbs	1	7.0	3.0	5.0	30	1.37	0.92	651	40	37	0.23	-0.04	-0.02	-0.01	-0.03	-0.02	442.22
	2	7.0	3.0	5.0	30	1.37	0.92	627			-0.05	0.24	-0.01	-0.02	-0.03	-0.05	474.64
	3	7.0	3.0	4.0	16	2.17	1.45	130			-0.03	-0.03	0.21	-0.05	-0.04	-0.04	138.59
	4	7.0	3.0	5.0	30	1.37	0.92	451			-0.05	-0.02	-0.03	0.23	-0.04	-0.01	337.15
	5	7.0	3.0	5.0	30	0.96	0.64	1268			-0.03	-0.03	-0.05	-0.03	0.25	-0.03	936.05
	6	7.0	3.0	5.0	30	1.37	0.92	845			-0.02	-0.04	-0.03	-0.04	-0.05	0.21	748.78
Enamel	1	8.0	7.0	7.0	42	22.97	14.59	100	26	37	0.09	-0.01	-0.03	-0.01	-0.01	-0.04	99.68
	2	8.0	7.0	7.0	28	23.97	15.50	123			-0.02	0.06	-0.02	-0.02	-0.02	-0.05	156.58
	3	8.0	7.0	7.0	28	20.97	14.59	22			-0.05	-0.03	0.06	-0.01	-0.01	-0.04	29.31
	4	8.0	7.0	7.0	28	18.97	13.59	17			-0.03	-0.03	-0.04	0.08	-0.04	-0.02	22.46
	5	8.0	7.0	7.0	28	17.98	12.57	10			-0.01	-0.03	-0.01	-0.03	0.06	-0.02	11.47
	6	8.0	7.0	7.0	28	15.98	12.53	25			-0.04	-0.02	-0.02	-0.05	-0.04	0.08	34.29
Rust Stopper	1	8.0	3.0	3.0	18	3.27	2.33	61	10	37	0.18	-0.03	-0.05	-0.03	-0.04	-0.03	58.04
	2	8.0	3.0	3.0	18	3.27	2.33	135			-0.03	0.16	-0.02	-0.03	-0.02	-0.01	113.61
	3	8.0	3.0	3.0	15	3.67	1.53	31			-0.02	-0.05	0.20	-0.02	-0.02	-0.03	26.28
	4	8.0	3.0	3.0	12	3.27	2.33	61			-0.01	-0.03	-0.02	0.18	-0.03	-0.01	51.05
	5	8.0	3.0	3.0	12	3.27	2.33	90			-0.05	-0.04	-0.03	-0.02	0.16	-0.04	98.76
	6	8.0	3.0	3.0	12	3.98	2.39	104			-0.03	-0.04	-0.03	-0.05	-0.01	0.18	102.49
Plastic Sheets	1	7.0	34.0	6.0	84	15.97	9.35	51	26	37	0.24	-0.03	-0.03	-0.02	-0.02	-0.04	31.48
	2	7.0	28.0	6.0	72	18.97	9.71	12			-0.01	0.23	-0.05	-0.04	-0.01	-0.03	7.94
	3	7.0	34.0	6.0	36	15.97	9.35	72			-0.05	-0.04	0.23	-0.05	-0.05	-0.01	75.62
	4	9.0	35.0	10.0	80	39.97	28.05	47			-0.05	-0.02	-0.03	0.21	-0.01	-0.03	33.64
	5	9.0	35.0	10.0	80	39.97	28.05	27			-0.03	-0.02	-0.04	-0.04	0.24	-0.02	17.57
	6	7.0	34.0	7.0	70	24.97	14.03	43			-0.05	-0.05	-0.03	-0.05	-0.05	0.25	39.65
Flood Lights	1	5.5	4.5	4.5	36	5.97	3.98	140	26	37	0.06	-0.04	-0.03	-0.03	-0.03	-0.02	192.56
	2	5.5	4.5	4.5	27	5.97	3.98	167			-0.02	0.07	-0.03	-0.01	-0.05	-0.01	207.01
	3	6.0	4.0	4.0	16	5.99	4.02	132			-0.03	-0.02	0.09	-0.04	-0.02	-0.04	181.15
	4	5.5	4.0	4.0	40	4.27	2.92	286			-0.02	-0.01	-0.01	0.08	-0.04	-0.02	312.34
	5	6.0	4.0	2.0	72	9.97	8.51	141			-0.02	-0.03	-0.02	-0.02	0.08	-0.03	152.33
	6	5.0	5.0	9.0	54	8.97	5.98	424			-0.02	-0.05	-0.03	-0.04	-0.05	0.07	633.67
Tube Lights	1	2.0	35.0	3.0	39	3.99	2.66	103	26	37	0.21	-0.05	-0.03	-0.01	-0.04	-0.04	75.11
	2	1.0	35.0	2.0	20	3.99	2.66	19			-0.04	0.24	-0.01	-0.05	-0.02	-0.05	15.03
	3	2.0	35.0	3.0	18	5.97	4.95	38			-0.01	-0.01	0.24	-0.02	-0.03	-0.01	23.63
	4	2.0	35.0	3.0	18	5.97	4.95	69			-0.01	-0.01	-0.01	0.23	-0.03	-0.03	44.97
	5	1.5	35.0	1.5	11	1.99	1.33	43			-0.02	-0.03	-0.04	-0.04	0.22	-0.05	42.77
	6	1.5	35.0	1.5	11	0.98	0.65	57			-0.04	-0.01	-0.04	-0.04	-0.01	0.23	51.08
Glass Cleaner	1	11.0	3.0	5.0	15	1.98	1.35	131	13	37	0.08	-0.03	-0.03	-0.01	-0.02	-0.02	142.59
	2	10.0	3.0	5.0	15	2.50	2.13	151			-0.02	0.08	-0.03	-0.01	-0.03	-0.05	177.91
	3	12.0	6.0	6.0	12	5.97	3.70	30			-0.03	-0.05	0.09	-0.03	-0.04	-0.01	38.28
	4	12.0	6.0	6.0	48	8.92	3.70	30			-0.04	-0.04	-0.03	0.07	-0.02	-0.02	34.07
	5	11.0	3.0	5.0	15	1.98	1.50	78			-0.04	-0.01	-0.03	-0.04	0.09	-0.02	92.81
	6	12.0	5.0	7.0	14	7.97	5.40	48			-0.02	-0.05	-0.02	-0.03	-0.05	0.10	60.23
Degreaser	1	10.0	3.0	5.0	15	2.50	2.15	78	13	37	0.18	-0.01	-0.03	-0.03	-0.01	-0.02	67.54
	2	11.0	3.0	5.0	15	2.99	2.10	32			-0.04	0.17	-0.03	-0.01	-0.05	-0.02	32.22
	3	12.0	6.0	6.0	48	8.97	4.55	60			-0.02	-0.04	0.17	-0.03	-0.02	-0.05	49.91
	4	12.0	6.0	6.0	48	8.93	7.32	50			-0.04	-0.02	-0.02	0.17	-0.03	-0.05	41.23
	5	10.0	3.0	5.0	20	2.50	2.15	51			-0.03	-0.02	-0.05	-0.02	0.18	-0.01	45.88
	6	11.0	3.0	5.0	15	3.49	1.80	10			-0.02	-0.01	-0.04	-0.05	-0.02	0.17	10.30
Trash Bags	1	9.0	5.5	11.0	33	13.77	7.40	65	40	37	0.18	-0.04	-0.02	-0.04	-0.01	-0.03	56.41
	2	9.0	5.5	11.0	33	14.93	8.11	58			-0.05	0.18	-0.05	-0.01	-0.05	-0.04	61.12
	3	9.0	4.5	9.0	36	9.77	4.53	110			-0.03	-0.05	0.18	-0.02	-0.05	-0.01	98.50
	4	10.0	4.5	8.0	32	9.88	5.44	85			-0.03	-0.04	-0.02	0.16	-0.02	-0.03	79.20
	5	11.0	5.0	10.0	20	9.96	7.00	71			-0.02	-0.03	-0.02	-0.01	0.18	-0.01	56.85
	6	9.0	4.5	12.0	36	9.77	6.07	67			-0.04	-0.05	-0.02	-0.05	-0.05	0.17	74.06

Table 2: Data and Parameters for Nine Product Categories