McMaster University

Advanced Optimization Laboratory

Title:

The Q Method for Symmetric Cone Programming

Authors:

Farid Alizadeh and Yu Xia

AdvOl-Report No. 2004/18

October 2004, Hamilton, Ontario, Canada

The Q Method for Symmetric Cone Programming

Farid Alizadeh* Yu Xia†

October 9, 2004

Abstract

We extend the Q method to the symmetric cone programming. An infeasible interior point algorithm and a Newton-type algorithm are given. We give convergence results of the interior point algorithm and prove that the Newton-type algorithm is good for "warm starting".

Key words. Symmetric cone programming, infeasible interior point method, polar decomposition, Jordan algebras, complementarity, Newton's method.

1 Introduction

The Q method [1] is a unique procedure originally for Semidefinite Programming (SDP). Basic idea of the Q method for SDP is the following. Let real symmetric matrices X and Z denote the primal and dual variables. When $X \bullet Z = \mu I$, they share a same complete system of eigenvectors. Hence, the eigenvalue decompositions can be written as $X = [Q^T \Lambda Q]$ and $Z = [Q^T \Omega Q]$, where Λ and Ω are diagonal matrices with eigenvalues of X and Z as the diagonal elements respectively. The Q method employs Newton's method to the primal-dual system on the central path by updating Q, Λ, Ω and $\mathbf{y}$ at each iteration separately. Instead of considering symmetric semidefinite matrices, the Q method works on the vectors in nonnegative orthant.

The Q method has many attractive properties. For other interior point methods of SDP, to maintain each iterate positive definite, i.e. given iterate X and search direction ΔX, except $\Delta X \succcurlyeq 0$, to find step size α so that $X + \alpha \Delta X \succ 0$, one needs to calculate $\alpha^{-1} = \lambda_{\max}\left(-L^{-1}\Delta X L^{-T}\right)$, where $\lambda_{\max}$ represents the largest eigenvalue, L is the Cholesky factor of $X : X = LL^T$; while for the Q method, one only needs to find $\alpha^{-1} \geq \max\{-\Delta\lambda_i/\lambda_i \colon \Delta\lambda_i < 0\}$. The Schur complement in the Q method is symmetric positive semidefinite; so it can be calculated by Cholesky factorization; while Cholesky factorization can not be used for some other directions for SDP because they are not symmetric. Disparate from other SDP search directions, the Q method doesn't require symmetrization; so its a pure Newton's method. Thus, one can expect fast local convergence. The Jacobian of the solution of the Q method is nonsingular at optimum under mild conditions; while no nonsingularity properties of the Jacobian at solution are known for SDP search directions except AHO direction. The condition numbers of the Jacobians on the central path are bounded for LP, but that for SDP is unbounded. Therefore, the Q method is able to compute solutions more accurately than any other interior point method is, as is shown by preliminary computation. The Q method for second-order cone programming (SOCP) also has the above good properties.

The symmetric cone programming includes linear programming, second-order cone programming, and semidefinite programming, which are current active research areas. In this paper, we carry on

*RUTCOR and Business School, Rutgers, the State University of New Jersey, U.S.A. alizadeh@rutcor.rutgers.edu research supported in part by the U.S. National Science Foundation.

†Computing and Software, McMaster University, 1280 Main Street West, Hamilton, ON L8S 4K1, Canada. yuxia@cas.mcmaster.ca

the Q method to symmetric cone programming. This gives a unified treatment of the Q method for SDP and SOCP. We show that the Q method for symmetric cone programming also has the above good properties. We give two algorithms for the Q method, i.e., an infeasible interior point algorithm and a pure Newton's method. We prove that the infeasible interior point algorithm converges to an optimum in finite steps while the Newton's algorithm is good for "warm starting" a perturbed problem.

The rest of the paper is organized as follows. In § refsc:scp, we show on the central path, primal and dual variables share a same orthogonal transformation and show the iteration system for the Q method is well-defined, one-to-one at optimum. We give an infeasible interior point algorithm and its convergence proof in § 3. A Newton-type algoritm and analysis for its "warm starting" properties are given in § 4. Concrete examples of the Q method for SDP and SOCP are given in § 5. We conclude the paper in § 6. In Appendix, we give some basics of Jordan algebras.

Notations

Throughout this paper, superscripts are used to represent iteration numbers while subscripts are used for block numbers of primal and dual vectors. We use capital letters for matrices, bold lower case letters for column vectors, lower case letters for entries of a vector. In this way, jth entry of vector $\mathbf{x}_i$ is written as $(x_i)_j$.

We assume V is a finite dimensional Euclidean Jordan algebra over $\mathbb{R}$. That means there exists a positive definite symmetric bilinear form on V which is associative (see [5, p. 42, chapter III.1]).

By [5, p. 54, proposition III 4.4], V is, in a unique way, a direct sum of simple ideals V_i, i.e. V_i doesn't contain any nontrivial ideal. We write the simple ideal decomposition of V as $V = V_1 \oplus \cdots \oplus V_n$. Correspondingly, write $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_n$ with $\mathbf{x}_i \in V_i$. Suppose the rank of V_i is r_i. Furthermore, assume the second version of spectral decomposition (Theorem A.2) of $\mathbf{x}_i$ is

$$\mathbf{x}_i = (\lambda_i)_1 (\mathbf{f}_i)_i + \cdots + (\lambda_i)_{r_i} (\mathbf{f}_i)_{r_i},$$

where $(\lambda_i)_j$ are eigenvalues of $\mathbf{x}$, and $(\mathbf{f}_i)_j$ is a Jordan frame of V.

The trace and determinant of $\mathbf{x}$ is denoted as:

$$\operatorname{tr}(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i)_j, \quad \det(\mathbf{x}) \stackrel{\text{def}}{=} \Pi_{i=1}^{n} \Pi_{j=1}^{r_i} (\lambda_i)_j.$$

By [5, p. 51, proposition III.4.1], in a simple Euclidean Jordan Algebra, every associative symmetric bilinear form is a scalar multiple of $\operatorname{tr}(\mathbf{xy})$. The symmetric bilinear form $\operatorname{tr}(\mathbf{xy})$ is positive definite by [5, p.46, proposition III.1.5]. So, there is no loss of generality to define the inner product on V as $\langle \mathbf{x}, \mathbf{y} \rangle \stackrel{\text{def}}{=} \operatorname{tr}(\mathbf{xy})$.

Since V is finite dimensional, it is complete. Hence V equipped with that inner product is a Hilbert space. Throughout the paper, the norm on V we use is that induced by the inner product.

Let $\mathbf{e}$ be the identity of V.

Let Sq be the set of all squares:

$$Sq \stackrel{\text{def}}{=} \left\{ \mathbf{x}^2 \colon \mathbf{x} \in V \right\}.$$

Then by [5, p. 46, theorem III.2.1], the interior of Sq is a symmetric cone. Further more, $\operatorname{Int} Sq$ is the connected component of $\mathbf{e}$ in the set of invertible elements. By [5, p. 55, proposition III 4.5], Sq is, in a unique way, a direct sum of irreducible symmetric cones Sq_i. It is obvious that $Sq_i \in V_i$. Let $L(\mathbf{x})$ denote the linear operator on V as $L(\mathbf{x})\mathbf{y} = \mathbf{xy}$. Then $\operatorname{Int} Sq$ is also the set of elements $\mathbf{x}$ in V for which $L(\mathbf{x})$ is positive definite.

In this paper, we fix a Jordan frame $\{(\mathbf{f}_i)_j \colon j = 1, \ldots, r_i;\ i = 1, \ldots, n\}$.

Denote

$$G(Sy) \stackrel{\text{def}}{=} \{ g \in GL(V) \colon g(Sy) = Sy \}.$$

Write G_i for the connected component of the identity in $G(Sq_i)$, G for the connected component of the identity in $G(Sq)$. It is shown in [5, p. 5, I.1.1] that G_i is transitive on Sq_i. By [5, p. 54, proposition III 4.5], G is the direct sum of G_i. Write $O(V_i)$ for the orthogonal group of V_i. Denote K_i to be

$$K_i \stackrel{\text{def}}{=} G_i \cap O(V_i),$$

and

$$K \stackrel{\text{def}}{=} G \cap O(V).$$

Use $\mathfrak{g}_i$ for the Lie algebra of G_i and $\mathfrak{k}_i$ for the Lie algebra of K_i; then

$$\mathfrak{k}_i = \{S \in \mathfrak{g}_i : S^T = -S\}.$$

We also use $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{k}$ for the Lie algebra of K.

Let $Aut(V)$ represent the automorphism group of V, i.e. for all $A \in Aut(V)$ and $\mathbf{x}$, $\mathbf{y}$ in V,

$$A(\mathbf{xy}) = A\mathbf{x} \cdot A\mathbf{y}.$$

By [5, p. 57], if the inner product in V is given by $\operatorname{tr}(xy)$, then

$$Aut(V) = G(Sq) \cap O(V).$$

Let $Der(V)$ represent the set of all derivations of V, i.e., $D \in Der(V)$ if D is a linear transformation of V such that

$$D(\mathbf{xy}) = D\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot D\mathbf{y},$$

for all $\mathbf{x}$, $\mathbf{y}$ in V.

The set $Der(V)$ is the Lie algebra of the Lie group $Aut(V)$.

Let the set W_i be

$$W_i \stackrel{\text{def}}{=} \{\mathbf{a} = \lambda_1(\mathbf{f}_i)_1 + \cdots + \lambda_{r_i}(\mathbf{f}_i)_{r_i} : \lambda_j \in \mathbb{R}\ (j = 1, \ldots, r_i)\},$$

and M_i be the subgroup of K_i fixing each point $\mathbf{a}$ in W_i :

$$M_i \stackrel{\text{def}}{=} \{k \in K_i : \forall\, \mathbf{a} \in W_i,\ k\mathbf{a} = \mathbf{a}\}.$$

Denote the Lie algebra of M_i as

$$\mathfrak{m}_i \stackrel{\text{def}}{=} \{S \in \mathfrak{k}_i : \forall\, \mathbf{a} \in W_i,\ S\mathbf{a} = \mathbf{0}\}.$$

For $j \neq k$, define

$$\mathfrak{l}_{i(jk)} \stackrel{\text{def}}{=} \left\{[L\left((\mathbf{f}_i)_j\right), L(\xi)] : \xi \in V_{i(jk)}\right\} \tag{1}$$

$$\mathfrak{l}_i \stackrel{\text{def}}{=} \sum_{j<k} \mathfrak{l}_{i(jk)}. \tag{2}$$

By [5, p. 103, proposition VI.2.2], $\mathfrak{l}_i$ is the direct sum of the $\mathfrak{l}_{i(jk)}$ $(j < k)$:

$$\mathfrak{l}_i = \bigoplus_{j<k} \mathfrak{l}_{i(jk)}.$$

2 The Symmetric Cone Programming

In this part, we give some propositions that are essential for developing the Q method. We show that the primal and dual variables share a same Jordan frame on the central path and how to update them in §§ 2.1. We present the Newton system on the central path and its well definedness in §§ 2.2. We prove that the linear system determined by the Q method is one-to-one at optimum in §§ 2.3.

2.1 Basic Properties

The symmetric cone programming is to minimize a linear functional over the intersection of an affine space and Sq. Denote the ordering relation $\mathbf{p} \geq_{Sq} \mathbf{q} \Leftrightarrow \mathbf{p} - \mathbf{q} \in Sq$. Then, the mathematical programming problem is generally written as

$$(3) \qquad \begin{array}{llll} \underline{\textbf{Primal}} & & \underline{\textbf{Dual}} & \\ \min_{\mathbf{x}} & \langle \mathbf{c}, \mathbf{x} \rangle & \max_{\mathbf{y},\mathbf{z}} & \langle \mathbf{b}, \mathbf{y} \rangle \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} & \text{s.t.} & A^*\mathbf{y} + \mathbf{z} = \mathbf{c} \\ & \mathbf{x} \geq_{Sq} \mathbf{0}, & & \mathbf{z} \geq_{Sq} \mathbf{0}. \end{array}$$

Here A is a linear transformation from V onto Y – another finite dimensional Hilbert space over $\mathbb{R}$; A^* is the adjoint of A; $\mathbf{y}$ is in Y. We use Y to denote both the space Y and its dual space.

We assume the Slater conditions for both the Primal and Dual. Then the strong duality holds: $\langle \mathbf{c}, \mathbf{x} \rangle = \langle \mathbf{b}, \mathbf{y} \rangle$, which is equivalent to $\langle \mathbf{z}, \mathbf{x} \rangle = 0$.

Employing logarithmic barrier to (3), the system on the central path is

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^*\mathbf{y} + \mathbf{z} &= \mathbf{c} \\ \mathbf{z} &= \mu \mathbf{x}^{-1} \end{aligned}$$

First we show that when $\langle \mathbf{x}, \mathbf{z} \rangle = 0$, $\mathbf{x}$ and $\mathbf{z}$ share a same Jordan frame in their spectral decomposition second version.

Note that for any primitive idempotent $\mathbf{f}$, by the second version of spectral theorem, $\operatorname{tr}(\mathbf{f}) = 1$. So

$$\|\mathbf{f}\| = \sqrt{\operatorname{tr}(\mathbf{f}^2)} = \sqrt{\operatorname{tr}(\mathbf{f})} = 1.$$

We first prove an exercise from [5, p. 79, exercise IV.7], which will be used later.

Proposition 2.1 *Let*

$$\mathbf{x} = \sum_{i=1}^{r} x_i \mathbf{f}_i + \sum_{i<j} \mathbf{x}_{ij}$$

be the Peirce decomposition (Definition A.6) with respect to the Jordan frame $\mathbf{f}_1, \ldots, \mathbf{f}_r$ *of an element* $\mathbf{x} \in Sq$. *Then*

$$x_i \geq 0, \quad \|\mathbf{x}_{ij}\|^2 \leq 2 x_i x_j.$$

Proof: For any $\mathbf{u} \in V\left(\mathbf{f}_i, \frac{1}{2}\right) \cap V\left(\mathbf{f}_j, \frac{1}{2}\right)$, $\|\mathbf{u}\|^2 = 2$, $\lambda^2 + \mu^2 = 1$, set

$$\mathbf{w} = \lambda^2 \mathbf{f}_i + \mu^2 \mathbf{f}_j + \lambda\mu \mathbf{u}.$$

By Proposition A.1,

$$\mathbf{u}^2 = \frac{1}{2}\|\mathbf{u}\|^2 (\mathbf{f}_i + \mathbf{f}_j) = \mathbf{f}_i + \mathbf{f}_j.$$

Then

$$\begin{aligned} \mathbf{w}^2 &= \lambda^4 \mathbf{f}_i + \mu^4 \mathbf{f}_j + 2\lambda^3\mu \mathbf{f}_i \mathbf{u} + 2\lambda\mu^3 \mathbf{f}_j \mathbf{u} + \lambda^2\mu^2 \mathbf{u}^2 \\ &= \lambda^4 \mathbf{f}_i + \mu^4 \mathbf{f}_j + \lambda^3\mu \mathbf{u} + \lambda\mu^3 \mathbf{u} + \lambda^2\mu^2 (\mathbf{f}_i + \mathbf{f}_j) \\ &= \lambda^2 \mathbf{f}_i + \mu^2 \mathbf{f}_j + \lambda\mu \mathbf{u}. \end{aligned}$$

So $\mathbf{w}$ is an idempotent. Because $L(\mathbf{x})$ is positive semidefinite,

$$\langle \mathbf{x}, \mathbf{w} \rangle = \langle \mathbf{w}, L(\mathbf{x})\mathbf{w} \rangle \geq 0.$$

The Peirce decomposition is orthogonal with respect to the inner product; so

$$\langle \mathbf{x}, \mathbf{w} \rangle = \lambda^2 x_i + \mu^2 x_j + \lambda\mu \langle \mathbf{u}, \mathbf{x}_{ij} \rangle.$$

Therefore,

$$\lambda^2 x_i + \mu^2 x_j + \lambda\mu \langle \mathbf{u}, \mathbf{x}_{ij} \rangle \geq 0. \tag{4}$$

Set $\lambda = 1$ and $\mu = 0$ in (4) get

$$x_i \geq 0, \quad (i = 1, \ldots, r).$$

If $\mathbf{x}_{ij} = \mathbf{0}$, $\|\mathbf{x}_{ij}\|^2 \leq 2x_i x_j$ is automatically satisfied. So in the rest of the proof, assume $\mathbf{x}_{ij} \neq \mathbf{0}$.

In (4), set

$$\mathbf{u} = \frac{\sqrt{2}}{\|\mathbf{x}_{ij}\|} \mathbf{x}_{ij}.$$

Then

$$\langle \mathbf{u}, \mathbf{x}_{ij} \rangle = \sqrt{2} \|\mathbf{x}_{ij}\|.$$

If neither x_i nor x_j is zero, $\|\mathbf{x}_{ij}\|^2 \leq 2x_i x_j$ is proved by setting

$$\lambda = \sqrt{\frac{x_j}{x_i + x_j}}, \quad \mu = -\sqrt{\frac{x_i}{x_i + x_j}}$$

in (4).

When $x_i = 0$, let $\{(\lambda_k, \mu_k)\}$ be a sequence such that $\forall\, k > 0$:

$$\lambda_k < 0, \quad \mu_k > 0, \quad \lambda_k^2 + \mu_k^2 = 1, \quad \lambda_k \to -1, \quad \mu_k \to 0.$$

Hence $\|\mathbf{x}_{ij}\| = 0$ by (4) with λ and μ being the above sequence.
Analogously, when $x_j = 0$, $x_{ij} = \mathbf{0}$ is proved by exchanging λ_k and μ_k in the above sequence. ■

Corollary 2.1 *Let* $\mathbf{x} \in \operatorname{Int} Sq$*. Then in the above decomposition*

$$x_i > 0, \qquad \|\mathbf{x}_{ij}\|^2 < 2x_i x_j.$$

Proof: For $\mathbf{x} \in \operatorname{Int} Sq$, $L(\mathbf{x})$ is positive definite. In the above proof, λ and μ can not both be zero, $V_{ii} + V_{jj} + V_{ij}$ is a direct sum. So $\mathbf{w} \neq \mathbf{0}$. Hence

$$\langle \mathbf{x}, \mathbf{w} \rangle > 0.$$

Therefore, "$\geq$" in (4); hence in the subsequence proof can be replaced by "$>$". ■

With the help of the above proposition, we can prove $\mathbf{x}$ and $\mathbf{z}$ share a same Jordan frame on the central path. We first consider $\mu = 0$.

Lemma 2.1 *Suppose* $\mathbf{x}$ *and* $\mathbf{z}$ *belong to* Sq*. Then the following statements are equivalent.*

1. $\langle \mathbf{x}, \mathbf{z} \rangle = 0$.

2. $\mathbf{x}\mathbf{z} = \mathbf{0}$.

3. *There is a Jordan frame* $\mathbf{f}_1, \ldots, \mathbf{f}_r$ *that simultaneously diagonalize* $\mathbf{x}$ *and* $\mathbf{z}$*:*

$$\mathbf{x} = \sum_{i=1}^{r} \lambda_i \mathbf{f}_i, \quad \mathbf{z} = \sum_{i=1}^{r} \omega_i \mathbf{f}_i.$$

Furthermore,

$$\lambda_i \omega_i = 0 \quad (i = 1, \ldots, r).$$

Proof: (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is obvious.

Next we will show (1) $\Rightarrow$ (3).

Let

$$\mathbf{z} = \sum_{i=1}^{r} \omega_i \mathbf{g}_i$$

be the spectral decomposition second version of $\mathbf{z}$. Because $\mathbf{z} \in Sq$, its eigenvalues are nonnegative. Without loss of generality, assume $\omega_i > 0\,(i \le l)$ and $\omega_i = 0\,(i > l)$.

Write the Peirce decomposition of $\mathbf{x}$ with respect to $\mathbf{g}_1, \ldots, \mathbf{g}_r$ as:

$$\mathbf{x} = \sum_{i=1}^{r} x_i \mathbf{g}_i + \sum_{i<j} \mathbf{x}_{ij}.$$

The Peirce decomposition is orthogonal with respect to the inner product; so

$$\langle \mathbf{x}, \mathbf{z} \rangle = \sum_{i=1}^{r} x_i \omega_i = \sum_{i=1}^{l} x_i \omega_i.$$

By Proposition 2.1, for all $i \in \{1, \ldots, r\} : x_i \ge 0$. So

$$x_i = 0 \quad (i \le l).$$

Hence

$$\mathbf{x}_{ij} = 0 \quad (i < j, i \le l).$$

Consider the Peirce decomposition of the space V with respect to $\mathbf{g}_1, \ldots, \mathbf{g}_r$, write

$$V_x \stackrel{\text{def}}{=} \bigoplus_{\substack{i \le j \\ i > l}} V_{ij}.$$

Then $\mathbf{x} \in V_x$ and $V_x = \left(\sum_{i=1}^{l} \mathbf{g}_i, 0\right)$. Besides, $\mathbf{g}_{l+1}, \ldots, \mathbf{g}_r$ is a Jordan frame in V_x.

A primitive idempotent in V is necessarily a primitive idempotent in a simple ideal of V. So Theorem A.5 is applicable to non-simple Euclidean Jordan algebras if the automorphism is the direct sum of automorphisms in simple Jordan algebras. Hence there exists a mapping $k \in G(V_x)$ such that $k\mathbf{g}_{l+1}, \ldots, k\mathbf{g}_r$ is a Jordan frame in V_x and

$$\mathbf{x} = \sum_{i=l+1}^{r} \lambda_i k \mathbf{g}_i.$$

Obviously

$$\sum_{i=l+1}^{r} k\mathbf{g}_i = \sum_{i=l+1}^{r} \mathbf{g}_i,$$

because both of them are Jordan frames in V_x.

By Theorem A.3, $V_x \cdot V_{ii} = \mathbf{0}\,(i \le l)$. So

$$(k\mathbf{g}_i)\mathbf{g}_j = \mathbf{0} \quad (i > l, j \le l).$$

Therefore $\mathbf{g}_1, \ldots, \mathbf{g}_l, k\mathbf{g}_{l+1}, \ldots, k\mathbf{g}_r$ is a Jordan frame in V that simultaneously diagonalize $\mathbf{x}$ and $\mathbf{z}$; and the product of the corresponding eigenvalues of $\mathbf{x}$ and $\mathbf{z}$ are zero. ∎

Next, we consider $\mu \neq 0$.

Proposition 2.2 *Assume $\mathbf{x}_i \in V_i$ is invertible. Write the polar decomposition (Theorem A.6) of $\mathbf{x}_i = Q_i \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j$. Then the polar decomposition of $\mathbf{x}_i^{-1} = Q_i \sum_{j=1}^{r_i} (\lambda_i)_j^{-1} (\mathbf{f}_i)_j$.*

Proof: Suppose the second version of spectral decomposition (Theorem A.2) of $\mathbf{x}_i$ is

$$\mathbf{x}_i = (\lambda_i)_1 (\mathbf{f}_{xi})_1 + \cdots + (\lambda_i)_{r_i} (\mathbf{f}_{xi})_{r_i}.$$

Here $(\mathbf{f}_{xi})_1, \ldots, (\mathbf{f}_{xi})_{r_i}$ is a Jordan frame.
From the uniqueness, the first version of spectral decomposition of $\mathbf{x}_i$ can be obtained by adding all the $(\mathbf{f}_{xi})_j$ corresponding to a same eigenvalue together:

$$\mathbf{x}_i = (\hat{\lambda}_i)_1 (\hat{\mathbf{f}}_{xi})_1 + \cdots + (\hat{\lambda}_i)_{t_i} (\hat{\mathbf{f}}_{xi})_{t_i}.$$

Here $(\hat{\lambda}_i)_j$ are distinct. Let

$$\mathbf{u} \stackrel{\text{def}}{=} (\hat{\lambda}_i)_1^{-1} (\hat{\mathbf{f}}_{xi})_1 + \cdots + (\hat{\lambda}_i)_{t_i}^{-1} (\hat{\mathbf{f}}_{xi})_{t_i}.$$

Let $\mathbf{e}_i$ represent the identity of V_i. Then $\mathbf{u}\mathbf{x}_i = \mathbf{e}_i$. And $\mathbf{u} \in \mathbb{R}[\mathbf{x}_i]$ since $(\hat{\mathbf{f}}_{xi})_j \in \mathbb{R}[\mathbf{x}_i]$ for $i = 1, \ldots, t_i$ by the first version of spectral theorem. The inverse is unique, so $\mathbf{x}_i^{-1} = \mathbf{u}$.
Decomposing $(\hat{\mathbf{f}}_{xi})_j$ back to $(\mathbf{f}_{xi})_j$, get

$$\mathbf{x}_i^{-1} = (\lambda_i)_1^{-1} (\mathbf{f}_{xi})_1 + \cdots + (\lambda_i)_{k_i}^{-1} (\mathbf{f}_{xi})_{r_i}.$$

The proposition is proved by transforming $(\mathbf{f}_{xi})_j$ to $(\mathbf{f}_i)_j$ through $Q_i \in K_i$. ∎

Corollary 2.2 *Let $\mathbf{x} \in \operatorname{Int} Sq$, $\mu \in \mathbb{R}$, and $\mathbf{x}\mathbf{z} = \mu\mathbf{e}$. Then $\mathbf{x}$ and $\mathbf{z}$ share a same Jordan frame in their spectral decomposition second version.*

Proof: When $\mu = 0$, the corollary is proved in Lemma 2.1.

Otherwise, since every element in $\operatorname{Int} Sq$ is invertible, $\mathbf{x}^{-1}$ exists and is unique. So $\mathbf{z} = \mu\mathbf{x}^{-1}$. The conditions of the proposition are satisfied. ∎

Next we give a procedure of approximating Q_i.

By Theorem A.5, given any two Jordan frames $\mathbf{a}_1, \ldots, \mathbf{a}_k$ and $\mathbf{b}_1, \ldots, \mathbf{b}_k$, there exists $A \in K$ such that $K\mathbf{a}_i = \mathbf{b}_i$ for $(i = 1, \ldots, k)$. One the other hand, given a Jordan frame $\mathbf{c}_1, \ldots, \mathbf{c}_k$, and $A \in K$, since A is an automorphism by Theorem A.4, we have

$$\begin{aligned} A\mathbf{c}_i \cdot A\mathbf{c}_i &= A(\mathbf{c}_i \cdot \mathbf{c}_i) = A\mathbf{c}_i, \\ A\mathbf{c}_i \cdot A\mathbf{c}_j &= A(\mathbf{c}_i \cdot \mathbf{c}_j) = \mathbf{0}, \\ A\mathbf{c}_i + \cdots + A\mathbf{c}_k &= A(\mathbf{c}_i + \cdots + \mathbf{c}_k) = A\mathbf{e} = \mathbf{e}. \end{aligned}$$

The last equality is due to $A \in GL(V)$. Therefore, $A\mathbf{c}_1, \ldots, A\mathbf{c}_k$ is also a Jordan frame.

For any $\mathbf{x} \in Sq$, assume its polar decomposition is $Q \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j$. Then, for any vector $\Delta\boldsymbol{\lambda}$ satisfying $\boldsymbol{\lambda} + \Delta\boldsymbol{\lambda} \geq \mathbf{0}$ and any automorphism $\Delta Q \in K$, $Q \Delta Q \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\lambda_i)_j + \Delta(\lambda_i)_j\right] (\mathbf{f}_i)_j \in Sq$.

To linearly update the Q, from the properties of exponential map (it maps some neighborhood of zero in a Lie algebra diffeomorphically onto a neighborhood of the identity in its Lie group), one can uptdate $Q_i I$ by $Q_i \exp(S_i)$ with $S_i \in \mathfrak{l}_i$, since $\mathfrak{k}_i = \mathfrak{l}_i \oplus \mathfrak{m}_i$ [5, p. 103, corollary VI.2.2]. Approximate $\exp(S_i)$ by its linear term $I + S_i$ at each iteration. Then, in the Newton's system, Q_i is replaced by $Q_i(I + S_i)$.

2.2 The System on the Central Path

In this part, we derive the linear system on the central path and show the system is well defined.

As convention, stack the eigenvalues of $\mathbf{x}$ and $\mathbf{z}$ respectively to generate vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\omega}$; let Λ and Ω be the diagonal matrices whose diagonal elements being the eigenvalues of $\mathbf{x}$ and $\mathbf{z}$ respectively.

Let Q^k be the direct sum of the automorphisms Q_i^k in the polar decomposition of $\mathbf{x}_i^k$. Given an iterate

$$(\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k) = \left(Q^k \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i^k)_j (\mathbf{f}_i)_j, \mathbf{y}^k, Q^k \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\omega_i^k)_j (\mathbf{f}_i)_j \right),$$

define

$$\mathbf{r}_p^k \overset{\text{def}}{=} \mathbf{b} - A\mathbf{x}^k, \quad \mathbf{r}_d^k \overset{\text{def}}{=} (Q^k)^* \left(\mathbf{c} - \mathbf{z}^k - A^* \mathbf{y}^k \right), \quad (r_{ci}^k)_j \overset{\text{def}}{=} \mu^k - (\lambda_i^k)_j (\omega_i^k)_j.$$

Denote $B^k \overset{\text{def}}{=} AQ^k$. Below is the resulting system of equations on the central path:

$$B^k S \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i^k)_j (\mathbf{f}_i)_j + B^k \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j = \mathbf{r}_p^k, \tag{5a}$$

$$(B^k)^* \Delta \mathbf{y} + S \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\omega_i^k)_j (\mathbf{f}_i)_j + \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j = \mathbf{r}_d^k, \tag{5b}$$

$$\Lambda^k \Delta \boldsymbol{\omega} + \Omega^k \Delta \boldsymbol{\lambda} = \mathbf{r}_c^k. \tag{5c}$$

The following lemma shows that the iterates are well defined.

Lemma 2.2 *System* (5) *has a unique solution on the analytic center if the iterate* $(\mathbf{x}^k, \mathbf{z}^k)$ *is regular.*

Proof: Since $S_i \in \mathfrak{l}_i$, it can be represented as

$$S_i = \sum_{j<l} \left[L\left((\mathbf{f}_i)_j \right), L(\xi_{i(jl)}^k) \right], \quad \xi_{i(jl)}^k \in V_{i(jl)}.$$

Then

$$S_i \sum_{j=1}^{r_i} (\omega_i^k)_j (\mathbf{f}_i)_j = \frac{1}{4} \sum_{j<l} \left[(\omega_i^k)_l - (\omega_i^k)_j \right] \xi_{i(jl)}^k,$$

$$S_i \sum_{j=1}^{r_i} (\lambda_i^k)_j (\mathbf{f}_i)_j = \frac{1}{4} \sum_{j<l} \left[(\lambda_i^k)_l - (\lambda_i^k)_j \right] \xi_{i(jl)}^k.$$

From (5c),

$$\Delta \boldsymbol{\lambda} = {\Omega^k}^{-1} \left(\mathbf{r}_c^k - \Lambda^k \Delta \boldsymbol{\omega} \right). \tag{6}$$

Replace $\Delta(\lambda_i)_j$ in (5a) by (6).
Recall $\mathbf{x}^k$ and $\mathbf{z}^k$ regular means

$$(\lambda_i^k)_l - (\lambda_i^k)_j \neq 0, \quad (\omega_i^k)_l - (\omega_i^k)_j \neq 0.$$

Because $\mathbf{x}^k$ and $\mathbf{z}^k$ are interior points, $(\lambda_i^k)_j / (\omega_i^k)_j > 0$. Also note that by Theorem A.3, the Peirce decomposition is a direct sum. Hence, there is a one-to-one linear operator P that maps

$$S \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\omega_i^k)_j (\mathbf{f}_i)_j + \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j$$

to

$$S \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i^k)_j (\mathbf{f}_i)_j - \sum_{i=1}^{n} \sum_{j=1}^{r_i} \frac{(\lambda_i^k)_j}{(\omega_i^k)_j} \Delta(\omega_i)_j (\mathbf{f}_i)_j. \tag{7}$$

Next, transform both sides of (5b) by $B^k P$ and subtract (7) from it. Since A is surjective, $B^k P (B^k)^*$ is a bijection. Therefore, $\Delta \mathbf{y}$ is uniquely determined by system (5). Then, by Proposition A.2 and the Peirce decomposition being a direct sum, $\Delta \boldsymbol{\omega}$ and S can be obtained from (5b). Finally, $\Delta \boldsymbol{\lambda}$ is solved for through (5c). ■

Remark 2.1 *We can always ensure that $\boldsymbol{\lambda}^{k+1}$ and $\boldsymbol{\omega}^{k+1}$ are regular by careful choice of step sizes. For example, assume $\boldsymbol{\omega}^k$ is regular. Only when $\Delta(\omega_i)_1 \neq \Delta(\omega_i)_2$ and $\beta = \frac{(\omega_i^k)_2 - (\omega_i^k)_1}{\Delta(\omega_i)_1 - \Delta(\omega_i)_2}$, it is possible that $(\omega_i^k)_1 + \beta \Delta(\omega_i)_1 = (\omega_i^k)_2 + \beta \Delta(\omega_i)_2$. Under this case, we can use a smaller step size β_i'. It is obvious that β' can be at least as large as $\frac{\beta}{2}$. And β_i' are not necessarily the same for all i.*

2.3 The Linear Transformation is One-to-One at Optimum

Assume the dimension of V_i is n_i. Denote $\nu \stackrel{\text{def}}{=} \sum_{i=1}^n r_i$, $N \stackrel{\text{def}}{=} \sum_{i=1}^n n_i$. Let F^k represent the linear transformation of $\mathfrak{l} \times \mathbb{R}^m \times \mathbb{R}^\nu \times \mathbb{R}^\nu$ into $\mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^\nu$ defined by the left hand side of (5) at kth iteration. Analogously, let F denote that linear transformation evaluated at optimum. By (28), Theorem A.3, and Proposition A.2, it is easy to see that the dimension of $\mathfrak{l}_i$ is $n_i - r_i$. Therefore, F^k maps a linear space into a same dimensional linear space. In this section, we give conditions under which F is one-to-one.

2.3.1 Definitions

To prove that F is one-to-one, we use related definitions and results from [6].

Let $(\mathbf{x}, \mathbf{z})$ be the primal-dual solutions of the symmetric cone program.

Definition 2.1 *[6, definition 3.4] The pair $(\mathbf{x}, \mathbf{z})$ is said to be strictly complementary if $\mathbf{x} + \mathbf{z} \in \operatorname{Int} Sq$.*

Through rearrangement if necessary, we can assume for $i = 1, \ldots, n$:

$$\begin{aligned} (\lambda_i)_j &\neq 0 \quad (j = 1, \ldots, t_i), \\ (\lambda_i)_j &= 0 \quad (j = t_i + 1, \ldots, r_i). \end{aligned}$$

Since $\langle \mathbf{x}, \mathbf{z} \rangle = 0$, by Lemma 2.1, $\mathbf{x}$ and $\mathbf{z}$ are strictly complementary if and only if for $i = 1, \ldots, n$:

$$\begin{aligned} (\omega_i)_j &= 0 \quad (j = 1, \ldots, t_i), \\ (\omega_i)_j &\neq 0 \quad (j = t_i + 1, \ldots, r_i). \end{aligned}$$

For $i = 1, \ldots, n$, define

$$e(\mathbf{x}_i) \stackrel{\text{def}}{=} \sum_{j=1}^{t_i} (\mathbf{f}_i)_j.$$

Note that $e(\mathbf{x}_i)$ is an idempotent in V_i. Analogously, $e(\mathbf{z}_i)$ can be defined as well.
Write the Peirce decomposition of V_i with respect to $e(\mathbf{x}_i)$ as:

$$V_i = V_i \left(e(\mathbf{x}_i), 1 \right) + V_i \left(e(\mathbf{x}_i), \frac{1}{2} \right) + V_i \left(e(\mathbf{x}_i), 0 \right).$$

Use the notations of [6], decompose the indices into three sets:

$$\Gamma_0 \stackrel{\text{def}}{=} \{(j,l) \colon t_i + 1 \le j \le l \le r_i\}$$
$$\Gamma_{\frac{1}{2}} \stackrel{\text{def}}{=} \{(j,l) \colon 1 \le j \le t_i,\, t_i + 1 \le l \le r_i\}$$
$$\Gamma_1 \stackrel{\text{def}}{=} \{(j,l) \colon 1 \le j \le l \le t_i\}.$$

Then for $k = 0, \frac{1}{2}, 1$,

$$V_i\left(e(\mathbf{x}_i), k\right) = \bigoplus_{j,l \in \Gamma_k} V_{i(jl)}\left(e(\mathbf{x}_i)\right).$$

Recall $B = AQ$.

Definition 2.2 *The solution* $\mathbf{x}$ *is primal nondegenerate if*

$$\left(\sum_{i=1}^{n} V_i\left(e(x_i), 0\right)\right) \cap range(B^*) = \mathbf{0}.$$

Definition 2.3 *The solution* $\mathbf{z}$ *is dual nondegenerate if*

$$\left(\sum_{i=1}^{n} V_i\left(e(z_i), 0\right)\right) \cap null(B) = \mathbf{0}.$$

Note that Q is one-to-one, the range of A is closed, the space V and Y are complete. Hence the range of A^* is just the annihilators of the null space of A, and the annihilators of the null space of A^* is the range of A. The spectral decomposition first version (Theorem A.1) can be obtained from the spectral decomposition second version by merging the idempotents corresponding to a same eigenvalues together. So, our definitions are the same as those in [6, definition 3.1].

By [6, theorem 3.7], primal nondegeneracy is satisfied iff the Dual has a unique solution; dual nondegeneracy is satisfied iff the Primal has a unique solution.

2.3.2 One-to-One

In this section, we prove that the linear transformation F is one-to-one at optimum.

We assume the optimal primal and dual solutions $\mathbf{x}$ and $\mathbf{z}$ satisfy the following conditions:

$$\begin{aligned} &(\lambda_i)_j \neq (\lambda_i)_k \quad (\text{for } (\lambda_i)_j \neq 0,\, (\lambda_i)_k \neq 0), \\ &(\omega_i)_j \neq (\omega_i)_k \quad (\text{for } (\omega_i)_j \neq 0,\, (\omega_i)_k \neq 0). \end{aligned} \tag{8}$$

Lemma 2.3 *Let*

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left(Q \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j, \mathbf{y}, Q \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\omega_i)_j (\mathbf{f}_i)_j\right)$$

be an optimal solution of the symmetric cone program satisfying strict complementarity, primal-dual nondegeneracy. Also assume (8) *are satisfied. Then* F *is one-to-one at* $(\boldsymbol{\lambda}, \boldsymbol{\omega}, Q, \mathbf{y})$.

Proof: We need to show that (5) has only zero solution at $(\boldsymbol{\lambda}, \boldsymbol{\omega}, Q, \mathbf{y})$ when its right hand side is zero.

For $i = 1, \ldots, n$, assume

$$\begin{aligned} &(\lambda_i)_j \neq 0 \quad (\omega_i)_j = 0 \quad (j = 1, \ldots, t_i), \\ &(\lambda_i)_j = 0 \quad (\omega_i)_j \neq 0 \quad (j = t_i + 1, \ldots, r_i). \end{aligned}$$

Then by (5c),

$$\Delta(\omega_i)_j = 0 \quad (j = 1, \dots, t_i),$$
$$\Delta(\lambda_i)_j = 0 \quad (j = t_i + 1, \dots, r_i).$$

For $i = 1, \dots, n$, write $S_i \in \mathfrak{l}_i$ as

$$S_i = \sum_{j<l} \left[L\left((\mathbf{f}_i)_j\right), L(\xi^k_{i(jl)}) \right], \quad \xi^k_{i(jl)} \in V_{i(jl)}.$$

Then

$$S_i \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j = \frac{1}{4} \sum_{j,l \in \Gamma_1} \left[(\lambda_i)_l - (\lambda_i)_j\right] \xi_{i(jl)} - \frac{1}{4} \sum_{j,l \in \Gamma_{\frac{1}{2}}} (\lambda_i)_j \xi_{i(jl)},$$
$$S_i \sum_{j=1}^{r_i} (\omega_i)_j (\mathbf{f}_i)_j = \frac{1}{4} \sum_{j,l \in \Gamma_{\frac{1}{2}}} (\omega_i)_l \xi_{i(jl)} + \frac{1}{4} \sum_{j,l \in \Gamma_0} \left[(\omega_i)_l - (\omega_i)_j\right] \xi_{i(jl)}.$$

Now calculate the inner product of (5b) and $\sum_{i=1}^n \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j$. Since the Peirce decomposition is orthogonal with respect to the inner product,

$$\langle S_i \sum_{j=1}^{r_i} (\omega_i)_j (\mathbf{f}_i)_j, \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \rangle = 0,$$
$$\langle \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j, \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \rangle = 0.$$

Therefore,

$$(9) \quad 0 = \langle B^* \Delta\mathbf{y}, \sum_{i=1}^n \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \rangle = \langle \Delta\mathbf{y}, B \sum_{i=1}^n \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \rangle$$
$$= \langle \Delta\mathbf{y}, BS \sum_{i=1}^n \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j \rangle.$$

The last equality is from (5a).

Next evaluate the inner product of (5b) and $S \sum_{i=1}^n \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j$. By orthogonality with respect to the inner product of the decomposition in Theorem A.3,

$$\langle \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j, S_i \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j \rangle = 0.$$

Along with (9):

$$0 = \langle S \sum_{i=1}^n \sum_{j=1}^{r_i} (\omega_i)_j (\mathbf{f}_i)_j, S \sum_{i=1}^n \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j \rangle$$
$$= -\frac{1}{16} \sum_{i=1}^n \sum_{j,l \in \Gamma_{\frac{1}{2}}} \langle (\omega_i)_l \xi_{i(jl)}, (\lambda_i)_j \xi_{i(jl)} \rangle.$$

For $i = 1, \dots, n$ and $j, l \in \Gamma_{\frac{1}{2}}$: $(\lambda_i)_j(\omega_i)_l > 0$. Therefore,

$$\xi_{i(jl)} = 0, \quad j, l \in \Gamma_{\frac{1}{2}}. \tag{10}$$

Hence

$$S_i \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j = \frac{1}{4} \sum_{j,l \in \Gamma_1} \left[(\lambda_i)_l - (\lambda_i)_j\right] \xi_{i(jl)}, \tag{11}$$

$$S_i \sum_{j=1}^{r_i} (\omega_i)_j (\mathbf{f}_i)_j = \frac{1}{4} \sum_{j,l \in \Gamma_0} \left[(\omega_i)_l - (\omega_i)_j\right] \xi_{i(jl)}. \tag{12}$$

Dual nondegeneracy, (11), and (5a) imply

$$\sum_{i=1}^{n} \sum_{j=1}^{r_1} \left[S(\lambda_i)_j (\mathbf{f}_i)_j + \Delta(\lambda_i)_j (\mathbf{f}_i)_j\right] = 0.$$

The conditions (8) imply $(\lambda_i)_l - (\lambda_i)_j \neq 0$. Note (11). From Theorem A.3, the decomposition is a direct sum. Therefore,

$$\xi_{i(jl)} = 0 \quad (j, l \in \Gamma_1), \tag{13}$$

$$\Delta \boldsymbol{\lambda} = 0. \tag{14}$$

Similarly, primal nondegeneracy, (12), and (5b) mean

$$B^* \Delta \mathbf{y} = 0, \tag{15}$$

$$S_i \sum_{j=1}^{r_i} (\omega_i)_j (\mathbf{f}_i)_j + \sum_{i=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j = 0. \tag{16}$$

By the surjectiveness of A, one-to-one of Q, and (15), we get $\Delta \mathbf{y} = 0$.
By (12), $(\omega_i)_l \neq (\omega_i)_j$, the decomposition in Theorem A.3 is a direct sum, and (16), we get

$$\xi_{i(jl)} = 0 \quad (j, l \in \Gamma_0), \tag{17}$$

$$\Delta \boldsymbol{\omega} = 0. \tag{18}$$

Combine (10), (13), (17):

$$\xi_{i(jl)} = 0, \quad (j < l).$$

Since

$$S = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[L\left((\mathbf{f}_i)_j\right), L\left(\xi_{i(jl)}\right)\right],$$

$$S = 0.$$

We have proved that (5) has only zero solution. ■

Hence we can expect the solution is numerically stable and accurate at optimum.

3 An Infeasible Interior Point Algorithm

In this section, we give an infeasible interior point algorithm based on [9, 7], which are originally for linear programming (LP). We describe the basic algorithm in §§ 3.1, convergence proof is presented in §§ 3.2. To ensure global convergence, the algorithm is modified in §§ 3.3.

As convention, the norm of a linear transformation L from a vector space E into another vector space F is defined as

$$\|L\| \overset{\text{def}}{=} \sup_{\substack{\mathbf{x} \neq \mathbf{0}, \\ \mathbf{x} \in E}} \frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|}.$$

3.1 The KMM Algorithm

Our interior point algorithm for the Q method is based on [9]. This algorithm is originally for infeasible LP with exact search directions, while the Newton system for the Q method is nonlinear on the central path and the search direction is not exact. This algorithm can start from an arbitrary infeasible interior point. So it doesn't employ big M method; consequently it doesn't have the drawback of the big M method – numerically instable and computationally inefficient, see [10]. accuracy measures for primal, dual infeasibility and complementarity can be chosen separately; primal and dual step sizes can be different.

Given $\epsilon_p > 0$, $\epsilon_d > 0$, $\epsilon_c > 0$, we want to find an approximate solution of the symmetric cone program such that

$$\|A\mathbf{x} - b\| \le \epsilon_p, \quad \|A^*\mathbf{y} + \mathbf{z} - \mathbf{c}\| \le \epsilon_d, \quad \langle \mathbf{x}, \mathbf{z} \rangle \le \epsilon_c.$$

Note that $\langle \mathbf{x}, \mathbf{z} \rangle = \boldsymbol{\lambda}^T \boldsymbol{\omega}$.

The neighborhood is defined as

$$\begin{aligned} \mathcal{N}(\gamma_c, \gamma_p, \gamma_d) \overset{\text{def}}{=} \Big\{ & (\boldsymbol{\lambda}, \boldsymbol{\omega}, \mathbf{y}, Q) \colon \boldsymbol{\lambda} \in \mathbb{R}^\nu, \boldsymbol{\omega} \in \mathbb{R}^\nu, \mathbf{y} \in Y, Q \in K, \, \boldsymbol{\lambda} > 0, \, \boldsymbol{\omega} > 0, \\ & (\lambda_i)_j (\omega_i)_j \ge \gamma_c \frac{\boldsymbol{\lambda}^T \boldsymbol{\omega}}{\nu} \; (j = 1, \dots, r_i; i = 1, \dots, n), \\ & \boldsymbol{\lambda}^T \boldsymbol{\omega} \ge \gamma_p \|A\mathbf{x} - \mathbf{b}\| \; \text{ or } \; \|A\mathbf{x} - \mathbf{b}\| \le \epsilon_p, \\ & \boldsymbol{\lambda}^T \boldsymbol{\omega} \ge \gamma_d \|A^*\mathbf{y} + \mathbf{z} - \mathbf{c}\| \; \text{ or } \; \|A^*\mathbf{y} + \mathbf{z} - \mathbf{c}\| \le \epsilon_d. \Big\} \end{aligned}$$

The first inequality is the centrality condition. It prevents the iterates from hitting the boundary before reaching the optimum. The second and third inequalities guarantee that the complementarity will not be achieved before the primal or the dual feasibility. Obviously, when $(\gamma_c', \gamma_p', \gamma_d') \le (\gamma_c, \gamma_p, \gamma_d)$,

$$\mathcal{N}(\gamma_c, \gamma_p, \gamma_d) \subseteq \mathcal{N}(\gamma_c', \gamma_p', \gamma_d').$$

And

$$\bigcup_{(\gamma_c, \gamma_p, \gamma_d) > 0} \mathcal{N}(\gamma_c, \gamma_p, \gamma_d) = \{(\boldsymbol{\lambda}, \boldsymbol{\omega}, \mathbf{y}, Q) \colon \boldsymbol{\lambda} > 0, \boldsymbol{\omega} > 0\}.$$

Clearly, when $\boldsymbol{\lambda}^T \boldsymbol{\omega}$ approaches 0, $\mathcal{N}(\gamma_c, \gamma_p, \gamma_d)$ tends to the optimal solution set of the symmetric cone program (3). The algorithm is the following.

KMM Algorithm
Choose $0 < \sigma_1 < \sigma_2 < \sigma_3 < 1$ and $\Upsilon > 0$. To start from an arbitrary point $(\boldsymbol{\lambda}^0, \boldsymbol{\omega}^0, \mathbf{y}^0, Q^0)$, one may select $0 < \gamma_c < 1, \gamma_p > 0, \gamma_d > 0$, so that $(\boldsymbol{\lambda}^0, \boldsymbol{\omega}^0, \mathbf{y}^0, Q^0) \in \mathcal{N}(\gamma_c, \gamma_p, \gamma_d)$.
Do until (1) $\|\mathbf{r}_p^k\| < \epsilon_p$, $\|\mathbf{r}_d^k\| < \epsilon_d$, and ${\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k < \epsilon_c$; or (2) $\left\| (\boldsymbol{\lambda}^k, \boldsymbol{\omega}^k) \right\|_1 > \Upsilon$.

1. Set $\mu = \sigma_1 \frac{{\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k}{\nu}$.
2. Compute the search direction $(\Delta\boldsymbol{\lambda}, \Delta\boldsymbol{\omega}, \Delta\mathbf{y}, S)$ from (5).
3. Choose step sizes α, β, γ, set

$$\begin{aligned} \Lambda^{k+1} &= \Lambda^k + \alpha \Delta\Lambda, \\ \mathbf{y}^{k+1} &= \mathbf{y}^k + \theta \Delta\mathbf{y}, \\ \Omega^{k+1} &= \Omega^k + \beta \Delta\Omega, \\ Q^{k+1} &= Q^k \exp(\gamma S). \end{aligned}$$

4. $k \leftarrow k+1$.

End

The stepsizes are chosen as the following. Let $\hat{\alpha}^k$ be the maximum of $\tilde{\alpha} \in [0,1]$, so that for any $\alpha \in [0, \tilde{\alpha}]$:

$$\left(\boldsymbol{\lambda}^k + \alpha\Delta\boldsymbol{\lambda}, \boldsymbol{\omega}^k + \alpha\Delta\boldsymbol{\omega}, \mathbf{y}^k + \alpha\Delta\mathbf{y}, Q^k \exp(\alpha S)\right) \in \mathcal{N},$$
$$(\boldsymbol{\lambda}^k + \alpha\Delta\boldsymbol{\lambda})^T(\boldsymbol{\omega}^k + \alpha\Delta\boldsymbol{\omega}) \leq [1 - \alpha(1-\sigma_2)]\, {\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k.$$

The step sizes $\alpha \in (0,1]$, $\theta \in (0,1]$, $\beta \in (0,1]$, $\gamma \in (0,1]$ are chosen so that

$$(\boldsymbol{\lambda}^{k+1}, \boldsymbol{\omega}^{k+1}, \mathbf{y}^{k+1}, Q^{k+1}) \in \mathcal{N}(\gamma_c, \gamma_p, \gamma_d),$$
$${\boldsymbol{\lambda}^{k+1}}^T \boldsymbol{\omega}^{k+1} \leq \left[1 - \hat{\alpha}^k(1-\sigma_3)\right] {\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k.$$

Because $\sigma_1 < \sigma_2 < \sigma_3$, the primal and dual step sizes are not necessarily the same.

3.2 Convergence Results

The global convergence of the preceding algorithm can be proved similarly to that in [9] by showing the boundedness of the step sizes.

For an operator T on V,

$$\|\exp(T) - I - T\| \leq \sum_{j=2}^{\infty} \frac{\|T\|^j}{j!} \leq \sum_{j=1}^{\infty} \frac{\|T\|^j}{j!} \|T\| \leq \exp(\|T\|)\|T\|^2. \tag{19}$$

Following the notations of [9], for each k, define

$$\begin{aligned}
f_{ij} &\stackrel{\text{def}}{=} \left[(\lambda_i^k)_j + \alpha\Delta(\lambda_i)_j\right]\left[(\omega_i^k)_j + \alpha\Delta(\omega_i)_j\right] - \frac{\gamma_c}{\nu}(\boldsymbol{\lambda}^k + \alpha\Delta\boldsymbol{\lambda})^T(\boldsymbol{\omega}^k + \alpha\Delta\boldsymbol{\omega}),\\
g_p(\alpha) &\stackrel{\text{def}}{=} (\boldsymbol{\lambda}^k + \alpha\Delta\boldsymbol{\lambda})^T(\boldsymbol{\omega}^k + \alpha\Delta\boldsymbol{\omega})\\
&\quad -\gamma_p \left\| B^k \exp(\alpha S) \sum_{i=1}^{n}\sum_{j=1}^{r_i} \left[(\lambda_i^k)_j + \alpha\Delta(\lambda_i)_j\right](\mathbf{f}_i)_j - \mathbf{b} \right\|,\\
g_d(\alpha) &\stackrel{\text{def}}{=} (\boldsymbol{\lambda}^k + \alpha\Delta\boldsymbol{\lambda})^T(\boldsymbol{\omega}^k + \alpha\Delta\boldsymbol{\omega})\\
&\quad -\gamma_d \left\| (B^k)^* \left(\mathbf{y}^k + \alpha\Delta\mathbf{y}\right) + \exp(\alpha S) \sum_{i=1}^{n}\sum_{j=1}^{r_i} \left[(\omega_i^k)_j + \alpha\Delta(\omega_i)_j\right](\mathbf{f}_i)_j - (Q^k)^*\mathbf{c} \right\|,\\
h(\alpha) &\stackrel{\text{def}}{=} [1 - \alpha(1-\sigma_2)]\, {\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k - (\boldsymbol{\lambda}^k + \alpha\Delta\boldsymbol{\lambda})^T(\boldsymbol{\omega}^k + \alpha\Delta\boldsymbol{\omega}).
\end{aligned}$$

Therefore, $\hat{\alpha}^k$ is determined by the following inequalities:

$$\begin{aligned}
&f_{ij}(\alpha) \geq 0 \quad (j = 1, \ldots, r_i;\ i = 1, \ldots, n),\\
&g_p(\alpha) \geq 0 \text{ or } \|\mathbf{r}_p^k\| \leq \epsilon_p,\\
&g_d(\alpha) \geq 0 \text{ or } \|\mathbf{r}_d^k\| \leq \epsilon_p,\\
&h(\alpha) \geq 0.
\end{aligned}$$

Let $\epsilon^* \stackrel{\text{def}}{=} \min(\epsilon_c, \gamma_p\epsilon_p, \gamma_d\epsilon_d)$. Then for each intermediate iterate:

$${\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k \geq \epsilon^*, \quad \left\|(\boldsymbol{\lambda}^k, \boldsymbol{\omega}^k)\right\|_1 \leq \Upsilon.$$

Suppose the solutions to (5) are uniformly upper bounded for all the iterations. Then there exists $\eta \geq 0$ such that

$$\left|\Delta(\lambda_i)_j \Delta(\omega_i)_j - \frac{\gamma_c}{\nu} \Delta\boldsymbol{\lambda}^T \Delta\boldsymbol{\omega}\right| \leq \eta, \quad |\Delta\boldsymbol{\lambda}^T \Delta\boldsymbol{\omega}| \leq \eta,$$

$$\|A\| \left\| S \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \right\| \leq \eta,$$

$$\|A\| \exp(\|\alpha S\|) \|S\|^2 \left\| \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\lambda_i^k)_j + \alpha \Delta(\lambda_i)_j\right] (\mathbf{f}_i)_j \right\| \leq \eta \ (0 \leq \alpha \leq 1),$$

$$\left\| S \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j \right\| \leq \eta,$$

$$\exp(\|\alpha S\|) \|S\|^2 \left\| \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\omega_i^k)_j + \alpha \Delta(\omega_i)_j\right] (\mathbf{f}_i)_j \right\| \leq \eta \ (0 \leq \alpha \leq 1).$$

First determine the lower bound of α for $g_p(\alpha)$.
When $\|\mathbf{r}_p^k\| > \epsilon_p$:

$$\begin{aligned}
g_p(\alpha) \geq\ & (1-\alpha) {\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k + \alpha \sigma_1 {\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k + \alpha^2 \Delta\boldsymbol{\lambda}^T \Delta\boldsymbol{\omega} \\
& - \gamma_p (1-\alpha) \left\| \mathbf{b} - B^k \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i^k)_j (\mathbf{f}_i)_j \right\| - \alpha^2 \gamma_p \left\| B^k S \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \right\| \\
& - \alpha^2 \gamma_p \|B^k\| \exp(\alpha \|S\|) \|S\|^2 \left\| \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\lambda_i^k)_j + \alpha \Delta(\lambda_i)_j\right] (\mathbf{f}_i)_j \right\| \\
& \geq \alpha \sigma_1 \epsilon^* - \alpha^2 \eta - 2\alpha^2 \gamma_p \eta.
\end{aligned}$$

The first inequality is from (19), (5c), and (5a); the second inequality is because the kth iterate is in $\mathcal{N}$, $Q \in O(V)$ and orthogonal transformations don't change the norm of a vector, and from the definitions of ϵ^* and η.
When $\|\mathbf{r}_p^k\| \leq \epsilon_p$:

$$\begin{aligned}
& \left\| AQ^k \exp(\alpha S) \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\lambda_i^k)_j + \alpha \Delta(\lambda_i)_j\right] (\mathbf{f}_i)_j - \mathbf{b} \right\| \\
& \leq (1-\alpha) \left\| AQ^k \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i^k)_j (\mathbf{f}_i)_j - \mathbf{b} \right\| + \alpha^2 \left\| AQ^k S \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \right\| \\
& \quad + \alpha^2 \|AQ^k\| \exp(\alpha \|S\|) \|S\|^2 \left\| \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\lambda_i^k)_j + \alpha \Delta(\lambda_i)_j\right] (\mathbf{f}_i)_j \right\| \\
& \leq (1-\alpha)\epsilon_p + 2\alpha^2 \eta.
\end{aligned}$$

The first inequality is due to (5a) and (19); the last one is because of the definition of η, orthogonal transformations don't change the norm of a vector, and $\|\mathbf{r}_p^k\| \le \epsilon_p$. Therefore,

$$\alpha \le \min \left\{ \frac{\sigma_1 \epsilon^*}{\eta + 2\gamma_p \eta}, \frac{\epsilon_p}{2\eta} \right\}.$$

Now consider $g_d(\alpha)$.
When $\|\mathbf{r}_d^k\| > \epsilon_d$:

$$\begin{aligned} g_d(\alpha) \ge (1-\alpha){\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k + \alpha\sigma_1 {\boldsymbol{\lambda}^k}^T \boldsymbol{\omega}^k + \alpha^2 \Delta\boldsymbol{\lambda}^T \Delta\boldsymbol{\omega} \\ - \gamma_d (1-\alpha)\|\mathbf{r}_d^k\| - \alpha^2 \gamma_d \left\| S \sum_{i=1}^n \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j \right\| \\ - \alpha^2 \gamma_d \exp(\alpha\|S\|)\|S\|^2 \left\| \sum_{i=1}^n \sum_{j=1}^{r_i} \left[(\omega_i^k)_j + \alpha\Delta(\omega_i)_j \right] (\mathbf{f}_i)_j \right\| \\ \ge \alpha\sigma_1\epsilon^* - \alpha^2\eta - 2\alpha^2\gamma_d\eta. \end{aligned}$$

The first inequality is due to (19), (5c), and (5b); the second inequality is from the fact that the kth iterate is in the neighborhood $\mathcal{N}$ and the definitions of ϵ^* and η.
When $\|\mathbf{r}_d^k\| \le \epsilon_d$:

$$\begin{aligned} \left\| A^*(\mathbf{y}^k + \alpha\Delta\mathbf{y}) + Q^k \exp(\alpha S) \sum_{i=1}^n \sum_{j=1}^{r_i} \left[(\omega_i^k)_j + \alpha\Delta(\omega_i)_j \right] (\mathbf{f}_i)_j - \mathbf{c} \right\| \\ \le (1-\alpha) \left\| A^*\mathbf{y}^k + Q^k \sum_{i=1}^n \sum_{j=1}^{r_i} (\omega_i^k)_j (\mathbf{f}_i)_j - \mathbf{c} \right\| + \alpha^2 \left\| Q^k S \sum_{i=1}^n \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j \right\| \\ + \alpha^2 \exp(\alpha\|S\|)\|S\|^2 \left\| \sum_{i=1}^n \sum_{j=1}^{r_i} \left[(\omega_i^k)_j + \alpha\Delta(\omega_i)_j \right] (\mathbf{f}_i)_j \right\| \le (1-\alpha)\epsilon_d + 2\alpha^2\eta. \end{aligned}$$

The first inequality is due to (5b) and (19); the last one is because of the definition of η and $\|\mathbf{r}_d^k\| \le \epsilon_d$. Therefore,

$$\alpha \le \min \left\{ \frac{\sigma_1 \epsilon^*}{\eta + 2\gamma_d \eta}, \frac{\epsilon_d}{2\eta} \right\}.$$

By the same arguments as those in [9]:

$$\begin{aligned} f_{ij}(\alpha) \ge \sigma_1 \frac{\epsilon^*}{\nu}(1-\gamma_c)\alpha - \eta\alpha^2, \\ h(\alpha) \ge (\sigma_2 - \sigma_1)\epsilon^*\alpha - \eta\alpha^2. \end{aligned}$$

So

$$\hat{\alpha}^k = \min \left\{ 1, \frac{\sigma_1\epsilon^*}{\eta + 2\gamma_p\eta}, \frac{\epsilon_p}{2\eta}, \frac{\sigma_1\epsilon^*}{\eta + 2\gamma_d\eta}, \frac{\epsilon_d}{2\eta}, \frac{\sigma_1\epsilon^*}{\nu\eta}(1-\gamma_c), (\sigma_2 - \sigma_1)\frac{\epsilon^*}{\eta} \right\}.$$

By Lemma 2.2, F^k is a bijection for regular iterates. Assume the initial point is regular. Then the iterates can always be made regular by perturbation of stepsizes. Furthermore, the stepsizes can always be at least half of the original ones.

Now we can prove the following convergence result.

Theorem 3.1 *Suppose there exists $d > 0$ such that $\forall N > 0$, $\exists n \geq N$ so that for all unit length vector $\mathbf{w}$, $\|F^n \mathbf{w}\| \geq d$. Then algorithm KMM must stop after finite steps.*

Proof: Assume the algorithm doesn't stop after finite iterations. Then for each $k > 0$, we have

$$\boldsymbol{\lambda}^{k^T} \boldsymbol{\omega}^k \geq \epsilon^* \quad \text{and} \quad \left\| (\boldsymbol{\lambda}^k, \boldsymbol{\omega}^k) \right\|_1 \leq \Upsilon,$$

because otherwise, the iteration will terminate due to the stopping criteria. Boundedness of $\mathbf{y}^k$ is due to the dual feasible constraint. Also observe that Q^k is orthogonal; so its norm is 1. If the conditions of the theorem are satisfied, there must exist a subsequence $\{(\boldsymbol{\lambda}^{m_i}, \boldsymbol{\omega}^{m_i}, \mathbf{y}^{m_i}, Q^{m_i})\}_{i=1}^{\infty}$ such that for all m_i, $(F^{m_i})^{-1}$ is upper bounded. The right-hand side of (5) depends continuously on $(\boldsymbol{\lambda}^{m_i}, \boldsymbol{\omega}^{m_i}, \mathbf{y}^{m_i}, Q^{m_i})$. So the norm of the right-hand side of (5) is upper bounded. Therefore, a uniform upper bound on the solutions to (5) for the subsequence $\{m_i\}$ exists.

By the analysis above this theorem, there's a lower bound α^* for $\hat{\alpha}^{m_i}$. After the perturbations of step sizes to ensure the regularity of $\mathbf{x}$ and $\mathbf{z}$, the lower bound on $\hat{\alpha}^k$ is at least $\frac{\alpha^*}{2}$. The algorithm imposes the decrease of the sequence $\left\{ \boldsymbol{\lambda}^{j^T} \boldsymbol{\omega}^j \right\}_{j=1}^{\infty}$. So for each m_i in the subsequence, by $h(\alpha) \geq 0$, we see

$$\begin{aligned}
&\boldsymbol{\lambda}^{m_i+1^T} \boldsymbol{\omega}^{m_i+1} \\
&\quad \leq \left[1 - \frac{\alpha^*}{2}(1 - \sigma_3) \right] \boldsymbol{\lambda}^{m_i^T} \boldsymbol{\omega}^{m_i} \leq \left[1 - \frac{\alpha^*}{2}(1 - \sigma_3) \right] \boldsymbol{\lambda}^{m_{i-1}+1^T} \boldsymbol{\omega}^{m_{i-1}+1} \\
&\qquad \leq \left[1 - \frac{\alpha^*}{2}(1 - \sigma_3) \right]^2 \boldsymbol{\lambda}^{m_{i-1}^T} \boldsymbol{\omega}^{m_{i-1}} \leq \cdots \leq \left[1 - \frac{\alpha^*}{2}(1 - \sigma_3) \right]^i \boldsymbol{\lambda}^{m_1^T} \boldsymbol{\omega}^{m_1}.
\end{aligned}$$

That means the whole sequence $\{\boldsymbol{\lambda}^{j^T} \boldsymbol{\omega}^j\}_{j=1}^{\infty}$ converges to 0, which contradicts the assumption.
■

3.3 Boundedness of Iterates

In the analysis above, the KMM algorithm may abort due to unboundedness of variables. To make sure that each iterate is bounded, we modify the algorithm in the previous section based on [7], which is also for LP. We first describe the algorithm, then give convergence analysis.

Algorithm description.

Suppose $(\hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ is an interior feasible solution to (3) and the eigenvalues of $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ satisfy $\delta_p \mathbf{1} \leq \hat{\boldsymbol{\lambda}} \leq \chi_p \mathbf{1}$, $\delta_d \mathbf{1} \leq \hat{\boldsymbol{\omega}} \leq \chi_d \mathbf{1}$.

Since A is continuous and surjective, there exist $\zeta_p > 0$ and $\zeta_d > 0$, such that $\forall \|\tilde{\mathbf{b}}\| \leq \zeta_p$, $\|\tilde{\mathbf{c}}\| \leq \zeta_d$, the system

$$\begin{aligned} A\mathbf{x} &= \tilde{\mathbf{b}} \\ A^* \mathbf{y} + \mathbf{z} &= \tilde{\mathbf{c}} \end{aligned} \tag{20}$$

has a solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \tilde{\mathbf{y}})$ with its primal and dual eigenvalues satisfy $\left\| \tilde{\boldsymbol{\lambda}} \right\|_\infty \leq \frac{1}{2}\delta_p$ and $\|\tilde{\boldsymbol{\omega}}\|_\infty \leq \frac{1}{2}\delta_d$.

For instance, let A^+ denote the Moore-Penrose generalized inverse of A. Let $\mathbf{h} \stackrel{\text{def}}{=} A^+ \mathbf{b}$. Let $\boldsymbol{\lambda}_h$ be the eigenvalues of $\mathbf{h}$. Then

$$\|\boldsymbol{\lambda}_h\|_\infty \leq \|\boldsymbol{\lambda}_h\|_2 = \|\mathbf{h}\| \leq \|A^+\| \|\tilde{\mathbf{b}}\|.$$

Obviously, $(\mathbf{h}, \mathbf{0}, \tilde{\mathbf{c}})$ is a solution to (20). So one can set $\zeta_p = \frac{1}{2} \frac{\delta_p}{\|A^+\|}$, $\zeta_d = \frac{1}{2}\delta_d$.

If $\epsilon_p > \zeta_p$, we replace ϵ_p with ζ_p; if $\epsilon_d > \zeta_d$, we replace ϵ_d with ζ_d. The neighborhood $\tilde{\mathcal{N}}$ is defined as the following.

$$\begin{aligned}\tilde{\mathcal{N}} \overset{\text{def}}{=} \Big\{(\boldsymbol{\lambda}, \boldsymbol{\omega}, \mathbf{y}, Q) \colon & \boldsymbol{\lambda} \in \mathbb{R}^{\nu}, \boldsymbol{\omega} \in \mathbb{R}^{\nu}, \mathbf{y} \in \mathbb{R}^{m}, Q \in K, \boldsymbol{\lambda} > 0, \boldsymbol{\omega} > 0; \\ & (\boldsymbol{\lambda}_i)_j (\boldsymbol{\omega}_i)_j \geq \gamma_c \frac{\boldsymbol{\lambda}^T \boldsymbol{\omega}}{\nu} \ (j = 1, 2; i = 1, \dots, n); \\ & \boldsymbol{\lambda}^T \boldsymbol{\omega} \geq \gamma_p \|A\mathbf{x} - \mathbf{b}\| \text{ and } \|A\mathbf{x} - \mathbf{b}\| \leq \zeta_p, \\ & \text{or } \|A\mathbf{x} - \mathbf{b}\| \leq \epsilon_p; \\ & \boldsymbol{\lambda}^T \boldsymbol{\omega} \geq \gamma_d \|A^*\mathbf{y} + \mathbf{z} - \mathbf{c}\| \text{ and } \|A^*\mathbf{y} + \mathbf{z} - \mathbf{c}\| \leq \zeta_d, \\ & \text{or } \|A^*\mathbf{y} + \mathbf{z} - \mathbf{c}\| \leq \epsilon_d. \Big\}\end{aligned}$$

Other parts of the algorithm is the same as that stated in the previous section.

Bounds on step sizes.

Next we give a lower bound on the stepsize α.

Assume $\|\mathbf{r}_p^k\| \leq \zeta_p$. Then

$$\begin{aligned}&\left\| AQ^k \exp(\alpha S) \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\lambda_i^k)_j + \alpha \Delta(\lambda_i)_j \right] (\mathbf{f}_i)_j - \mathbf{b} \right\| \\ &\quad \leq (1-\alpha) \left\| AQ^k \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i^k)_j (\mathbf{f}_i)_j - \mathbf{b} \right\| + \alpha^2 \left\| AQ^k S \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\lambda_i)_j (\mathbf{f}_i)_j \right\| \\ &\quad + \alpha^2 \|AQ^k\| \exp(\alpha\|S\|) \|S\|^2 \left\| \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\lambda_i^k)_j + \alpha \Delta(\lambda_i)_j \right] (\mathbf{f}_i)_j \right\| \\ &\quad \leq (1-\alpha)\zeta_p + 2\alpha^2 \eta.\end{aligned}$$

The first inequality is due to (5a) and (19); the last one is because of the definition of η and $\|\mathbf{r}_p^k\| \leq \zeta_p$
.

Analogously, assume $\|\mathbf{r}_p^k\| \leq \zeta_p$:

$$\begin{aligned}&\left\| A^*(\mathbf{y}^k + \alpha \Delta \mathbf{y}) + Q^k \exp(\alpha S) \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\omega_i^k)_j + \alpha \Delta(\omega_i)_j \right] (\mathbf{f}_i)_j - \mathbf{c} \right\| \\ &\quad \leq (1-\alpha) \left\| A^*\mathbf{y}^k + Q^k \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\omega_i^k)_j (\mathbf{f}_i)_j - \mathbf{c} \right\| + \alpha^2 \left\| Q^k S \sum_{i=1}^{n} \sum_{j=1}^{r_i} \Delta(\omega_i)_j (\mathbf{f}_i)_j \right\| \\ &\quad + \alpha^2 \exp(\alpha\|S\|) \|S\|^2 \left\| \sum_{i=1}^{n} \sum_{j=1}^{r_i} \left[(\omega_i^k)_j + \alpha \Delta(\omega_i)_j \right] (\mathbf{f}_i)_j \right\| \leq (1-\alpha)\zeta_d + 2\alpha^2 \eta.\end{aligned}$$

The first inequality is due to (5b) and (19); the last one is because of the definition of η and $\|\mathbf{r}_d^k\| \leq \zeta_d$
.

Let α^* be the lower bound of the stepsize of the KMM algorithm in the previous section. Then

$$\alpha^{**} \overset{\text{def}}{=} \min\left(\frac{\zeta_p}{2\eta}, \frac{\zeta_d}{2\eta}, \alpha^* \right)$$

is a lower bound for the modified algorithm of this section. So the analysis in the previous section can be carried on to the algorithm of this section.

Iterates bounds

Next, we show the boundedness of each iterate.

Consider the perturbed system:

$$\begin{aligned} \mathbf{z} + A^T\mathbf{y} &= \mathbf{c} + \tilde{\mathbf{c}} \\ A\mathbf{x} &= \mathbf{b} + \tilde{\mathbf{b}} \\ \mathbf{x} &\geq_{Sq} \mathbf{0} \\ \mathbf{z} &\geq_{Sq} \mathbf{0}. \end{aligned} \tag{21}$$

Assume $\|\tilde{\mathbf{b}}\| \leq \zeta_p$, $\|\tilde{\mathbf{c}}\| \leq \zeta_d$. Then (20) has a solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \tilde{\mathbf{y}})$ with its primal and dual eigenvalues satisfy $\left\|\tilde{\boldsymbol{\lambda}}\right\|_\infty \leq \frac{1}{2}\delta_p$ and $\|\tilde{\boldsymbol{\omega}}\|_\infty \leq \frac{1}{2}\delta_d$.

Let

$$\breve{\mathbf{x}} \stackrel{\text{def}}{=} \hat{\mathbf{x}} + \tilde{\mathbf{x}}, \quad \breve{\mathbf{y}} \stackrel{\text{def}}{=} \hat{\mathbf{y}}, \quad \breve{\mathbf{z}} \stackrel{\text{def}}{=} \hat{\mathbf{z}} + \tilde{\mathbf{z}}.$$

Write the eigenvalue decomposition of $\mathbf{e}$ with respect to the Jordan frame of $\hat{\mathbf{x}}$. Note that an element is in Sq iff all its eigenvalues are nonnegative. Because $\delta_p\mathbf{1} \leq \hat{\boldsymbol{\lambda}} \leq \chi_p\mathbf{1}$, one sees

$$\hat{\mathbf{x}} - \delta_p\mathbf{e} \geq_{Sq} \mathbf{0}, \qquad \chi_p\mathbf{e} - \hat{\mathbf{x}} \geq_{Sq} \mathbf{0}.$$

Similarly, write the eigenvalue decomposition of $\mathbf{e}$ with respect to the Jordan frame of $\tilde{\mathbf{x}}$. Then because $|\tilde{\boldsymbol{\lambda}}| \leq \frac{1}{2}\delta_p\mathbf{1}$, one gets

$$\frac{1}{2}\delta_p\mathbf{e} + \tilde{\mathbf{x}} \geq_{Sq} \mathbf{0}, \qquad \frac{1}{2}\delta_p\mathbf{e} - \tilde{\mathbf{x}} \geq_{Sq} \mathbf{0}.$$

Since Sq is a convex cone, the sum of any two elements in Sq is still in Sq. So

$$(\hat{\mathbf{x}} - \delta_p\mathbf{e}) + \left(\frac{1}{2}\delta_p\mathbf{e} + \tilde{\mathbf{x}}\right) \geq_{Sq} \mathbf{0}, \qquad (\chi_p\mathbf{e} - \hat{\mathbf{x}}) + \left(\frac{1}{2}\delta_p\mathbf{e} - \tilde{\mathbf{x}}\right) \geq_{Sq} \mathbf{0}.$$

Write the eigenvalue decomposition of $\mathbf{e}$ with respect to the Jordan frame of $\breve{\mathbf{x}}$. Therefore,

$$\frac{1}{2}\delta_p\mathbf{1} \leq \breve{\boldsymbol{\lambda}} \leq \chi_p\mathbf{1} + \frac{1}{2}\delta_p\mathbf{1}.$$

Analogously,

$$\frac{1}{2}\delta_d\mathbf{1} \leq \breve{\boldsymbol{\omega}} \leq \chi_d\mathbf{1} + \frac{1}{2}\delta_d\mathbf{1}.$$

Then we have the following lemma.

Lemma 3.1 *Each iterate in $\tilde{\mathcal{N}}$ is bounded.*

Proof: The kth iterate is a solution to

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} + \mathbf{r}_b^k \\ A^*\mathbf{y} + \mathbf{z} &= \mathbf{c} + \mathbf{r}_c^k. \end{aligned}$$

Since $\|\mathbf{r}_b^k\| \leq \delta_p$, $\|\mathbf{r}_c^k\| \leq \delta_d$, by the analysis above, there exists $(\breve{\mathbf{x}}, \breve{\mathbf{y}}, \breve{\mathbf{z}})$ satisfying the above perturbed constraints with

$$\frac{1}{2}\delta_p\mathbf{1} \leq \breve{\boldsymbol{\lambda}} \leq \chi_p\mathbf{1} + \frac{1}{2}\delta_p\mathbf{1}, \qquad \frac{1}{2}\delta_d\mathbf{1} \leq \breve{\boldsymbol{\omega}} \leq \chi_d\mathbf{1} + \frac{1}{2}\delta_d\mathbf{1}.$$

For briefness, we omit the superscript k in the remaining proof, let $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ denote the kth iterate. So

$$A(\mathbf{x} - \breve{\mathbf{x}}) = \mathbf{0}, \quad A^*(\mathbf{y} - \breve{\mathbf{y}}) + \mathbf{z} - \breve{\mathbf{z}} = \mathbf{0}.$$

Hence

$$\langle \mathbf{x} - \breve{\mathbf{x}}, \mathbf{z} - \breve{\mathbf{z}} \rangle = -\langle \mathbf{x} - \breve{\mathbf{x}}, A^*(\mathbf{y} - \breve{\mathbf{y}}) \rangle = -\langle A(\mathbf{x} - \breve{\mathbf{x}}), \mathbf{y} - \breve{\mathbf{y}} \rangle = 0.$$

Therefore,

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{z} \rangle + \langle \breve{\mathbf{x}}, \breve{\mathbf{z}} \rangle &= \langle \mathbf{x}, \breve{\mathbf{z}} \rangle + \langle \breve{\mathbf{x}}, \mathbf{z} \rangle \\
&= \langle \mathbf{x}, \frac{1}{2}\delta_d \mathbf{e} \rangle + \langle \mathbf{x}, \breve{\mathbf{z}} - \frac{1}{2}\delta_d \mathbf{e} \rangle + \langle \frac{1}{2}\delta_p \mathbf{e}, \mathbf{z} \rangle + \langle \breve{\mathbf{x}} - \frac{1}{2}\delta_p \mathbf{e}, \mathbf{z} \rangle \\
&\geq \frac{1}{2}\delta_d \langle \mathbf{x}, \mathbf{e} \rangle + \frac{1}{2}\delta_p \langle \mathbf{e}, \mathbf{z} \rangle \\
&= \frac{1}{2}\delta_d \left\| \boldsymbol{\lambda} \right\|_1 + \frac{1}{2}\delta_p \left\| \boldsymbol{\omega} \right\|_1 .
\end{aligned}$$

The inequality is because $\breve{\mathbf{z}} - \frac{1}{2}\delta_d \mathbf{e} \geq_{Sq} \mathbf{0}$, $\breve{\mathbf{x}} - \frac{1}{2}\delta_p \mathbf{e} \geq_{Sq} \mathbf{0}$, and Sq is self-dual.

On the other hand,

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{z} \rangle + \langle \breve{\mathbf{x}}, \breve{\mathbf{z}} \rangle &\leq \boldsymbol{\lambda}^T \boldsymbol{\omega} + \|\breve{\mathbf{x}}\| \cdot \|\breve{\mathbf{z}}\| \\
&\leq {\boldsymbol{\lambda}^0}^T \boldsymbol{\omega}^0 + n \left(\chi_p + \frac{1}{2}\delta_p \right) \left(\chi_d + \frac{1}{2}\delta_d \right).
\end{aligned}$$

The last inequality is due to the fact that the duality gap is reduced at each iteration and the bounds on the eigenvalues of $\breve{\mathbf{x}}$, $\breve{\mathbf{z}}$. Combine the two inequalities:

$$\frac{1}{2}\delta_d \left\| \boldsymbol{\lambda} \right\|_1 + \frac{1}{2}\delta_p \left\| \boldsymbol{\omega} \right\|_1 \leq {\boldsymbol{\lambda}^0}^T \boldsymbol{\omega}^0 + n \left(\chi_p + \frac{1}{2}\delta_p \right) \left(\chi_d + \frac{1}{2}\delta_d \right).$$

■

4 A Newton Type Algorithm

In this section, we give a Newton type algorithm for the Q method. We first describe the algorithm and then prove that the algorithm is good for "warm staring".

The Algorithm

By Lemma 2.1, when $\mathbf{x}$ and $\mathbf{z}$ are in Sq, $\langle \mathbf{x}, \mathbf{z} \rangle = 0$ iff there is a Jordan frame $\mathbf{f}_1, \ldots, \mathbf{f}_r$ that simultaneously diagonalize $\mathbf{x}$ and $\mathbf{z}$; furthermore, the corresponding products of eigenvalues are zero. As before, we fix a Jordan frame $\{(\mathbf{f}_i)_j : j = 1, \ldots, r_i;\ i = 1, \ldots, n\}$, write $\mathbf{x}$ and $\mathbf{z}$ in their polar decompositions:

$$\mathbf{x} = Q \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\lambda_i)_j (\mathbf{f}_i)_j, \quad \mathbf{z} = Q \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\omega_i)_j (\mathbf{f}_i)_j.$$

The complementarity conditions –

$$(\lambda_i)_j (\omega_i)_j = 0, \quad (\lambda_i)_j \geq 0, \quad (\omega_i)_j \geq 0$$

– are further formulated by some complementarity function equality $\varphi = 0$ such as

$$\min\left[(\lambda_i)_j, (\omega_i)_j\right] = 0, \quad \sqrt{(\lambda_i)_j^2 + (\omega_i)_j^2} - (\lambda_i)_j - (\omega_i)_j = 0. \tag{22}$$

For different i and j, φ_{ij} is not necessarily the same. So the optimum conditions are the following:

$$AQ\sum_{i=1}^{n}\sum_{j=1}^{r_i}(\lambda_i)_j(\mathbf{f}_i)_j = \mathbf{b}$$

$$A^*\mathbf{y} + Q\sum_{i=1}^{n}\sum_{j=1}^{r_i}(\omega_i)_j(\mathbf{f}_i)_j = \mathbf{c}$$

$$\varphi_{ij}\left[(\lambda_i)_j,(\omega_i)_j\right] = 0 \quad (j=1,\dots,r_i;\ i=1,\dots,n).$$

We denote $\mathbf{w} \stackrel{\text{def}}{=} (\boldsymbol{\lambda},\boldsymbol{\omega},Q)^T$, and use $E(\mathbf{w})$ to represent the mapping of the left-hand-side of the above function evaluated at $\mathbf{w}$.

Then apply Newton's method to the resulting system. The transformation $Q_i \in K_i$ is replaced by $Q_i(I+S_i)$ (for $S_i \in \mathfrak{l}_i$). Map both sides of the dual feasible equation by Q^*. Write

$$\mathbf{r}_p = \mathbf{b} - A\mathbf{x}, \quad \mathbf{r}_d = Q^*\left(\mathbf{c} - A^*\mathbf{y} - \mathbf{z}\right), \quad (\mathbf{r}_{ci})_j = -\varphi_{ij}\left[(\lambda_i)_j,(\omega_i)_j\right].$$

Write $B = AQ$ and $(p_{ij},q_{ij})^T \in \partial\varphi_{ij}\left[(\lambda_i)_j,(\omega_i)_j\right]$. Then the search direction is the solution to the following system.

$$BS\sum_{i=1}^{n}\sum_{j=1}^{r_i}(\lambda_i)_j(\mathbf{f}_i)_j + B\sum_{i=1}^{n}\sum_{j=1}^{r_i}\Delta(\lambda_i)_j(\mathbf{f}_i)_j = \mathbf{r}_p, \tag{23a}$$

$$B^*\Delta\mathbf{y} + S\sum_{i=1}^{n}\sum_{j=1}^{r_i}(\omega_i)_j(\mathbf{f}_i)_j + \sum_{i=1}^{n}\sum_{j=1}^{r_i}\Delta(\omega_i)_j(\mathbf{f}_i)_j = \mathbf{r}_d, \tag{23b}$$

$$p_{ij}\Delta(\lambda_i)_j + q_{ij}\Delta(\omega_i)_j = (\mathbf{r}_{ci})_j\,. \tag{23c}$$

As that of (5), the left-hand-side of (23) maps a linear space into a same dimensional linear space.

Suppose the strict complementarity condition is satisfied at optimum. For $i=1,\dots,n$, split the index set $L_i \stackrel{\text{def}}{=} \{1,\dots,r_i\}$ into two parts:

$$L_{i\lambda} \stackrel{\text{def}}{=} \{i \in L_i \colon (\lambda_i)_j = 0,\ (\omega_i)_j \neq 0\},$$
$$L_{i\omega} \stackrel{\text{def}}{=} \{i \in L_i \colon (\lambda_i)_j \neq 0,\ (\omega_i)_j = 0\},$$

"Warm Staring"

In this part, we show that starting from a solution to the old problem, the Newton type algorithm finds a solution to the perturbed problem Q quadratically.

Assume each complementarity function φ_{ij} has the following property:

$$p_{ij}, q_{ij} \in \mathbb{R}, \text{ and}$$
$$\begin{cases} p_{ij} \neq 0 \\ q_{ij} = 0 \end{cases} (j \in L_{i\lambda}), \quad \begin{cases} p_{ij} = 0 \\ q_{ij} \neq 0 \end{cases} (j \in L_{i\omega}). \tag{I}$$

Property I is satisfied by both equations in (22).

Lemma 4.1 *Suppose the complementarity functions φ_{ij} in* (23) *satisfy Property I. Also assume strict complementarity, primal-dual nondegeneracy, and conditions* (8) *of optimal* $\mathbf{x}$ *and* $\mathbf{z}$*. Then the mapping defined by the left-hand-side of* (23) *is one-to-one at optimum.*

Proof: Since the structure of (23) is the same as that of (5), the proof is similar to that of Lemma 2.3. ■

The mapping defined by the left-hand-side of (23) also satisfies Lipschitz condition. So Newton's method applied to E has local Q-quadratic convergence rate.

To consider perturbations of data, we further assume each complementarity function φ_{ij} has the following property.

(II) At a neighborhood of a strict complementarity solution to φ_{ij}, φ'_{ij} is Lipschitz continuous.

Both functions in (22) satisfy property II. Under property II, the linear mapping defined by the left-hand-side of (23) is the Fréchet derivative of E and E' is Lipschitz continuous at optimum. Therefore, pure Newton's iterates:

$$\mathbf{w}^{k+1} = \mathbf{w}^k - \left[E'(\mathbf{w}^k)\right]^{-1} E(\mathbf{w}^k)$$

have local Q-quadratic convergence rate.

Let $\mathbf{w}^{\text{old}}$ be a solution to the symmetric cone program. Now the data of the problem are perturbed. Denote the perturbation as $\Delta\mathbf{b}$, $\Delta\mathbf{c}$, ΔA. We use E^{old} to represent the linear transformation before perturbation and E to represent the linear transformation after perturbation. Let U denote the open unit ball in the proper space and $\operatorname{cl} U$ its closure.

We will first consider perturbations of $\mathbf{b}$ and $\mathbf{c}$.

In this case, only the right-hand-side of the system of equations are changed by $-\left(\Delta\mathbf{b}, \Delta\mathbf{c}, \mathbf{0}\right)^T$. Suppose the assumptions in lemma 4.1 are satisfied. Then, $E'(\mathbf{w}^{\text{old}})$ is one-to-one. By [3, p. 253, lemma 1, chapter 7], for some positive r and δ, $\mathbf{w} \in \mathbf{w}^{\text{old}} + rU$ implies E is differentiable and $E'(\mathbf{w})\mathbf{v}$ is distance at least δ from 0 for any unit vector $\mathbf{v}$. Therefore $\|E'(\mathbf{w})^{-1}\| \leq \frac{1}{\delta}$.
By [3, p. 254, lemma 2, chapter 7], if $\mathbf{w}$ and $\mathbf{v}$ lie in $\mathbf{w}^{\text{old}} + r \operatorname{cl} U$,

$$\|E(\mathbf{w}) - E(\mathbf{v})\| \geq \delta \|\mathbf{w} - \mathbf{v}\|. \tag{24}$$

Similar to [3, p. 254, lemma 3, chapter 7], we can prove the following lemma.

Lemma 4.2 *For any* $0 \leq l \leq 1$, $E(\mathbf{w}^{old} + lrU)$ *contains* $E(\mathbf{w}^{old}) + (\frac{1}{2} lr\delta)U$.

Proof: Given any $\mathbf{v} \in E(\mathbf{w}^{\text{old}}) + (\frac{1}{2} lr\delta)U$, let $\mathbf{w}$ be the minimum of $\|\mathbf{v} - E(\cdot)\|^2$ over $\mathbf{w}^{\text{old}} + lr \operatorname{cl} U$. Then $\mathbf{w}$ must belong to $\mathbf{w}^{\text{old}} + lrU$. Otherwise, by (24),

$$\begin{aligned}
\frac{1}{2} lr\delta &> \|\mathbf{v} - E(\mathbf{w}^{\text{old}})\| \geq \|E(\mathbf{w}) - E(\mathbf{w}^{\text{old}})\| - \|\mathbf{v} - E(\mathbf{w})\| \\
&\geq \delta \|\mathbf{w} - \mathbf{w}^{\text{old}}\| - \|\mathbf{v} - E(\mathbf{w}^{\text{old}})\| \\
&\geq l\delta r - \frac{1}{2} l\delta r = \frac{1}{2} l\delta r.
\end{aligned}$$

The third inequality is also because that $\mathbf{w}$ is a minimum. Thus $\mathbf{w}$ is a local minimum for $\|\mathbf{v} - E(\cdot)\|^2$ and consequently its Gâteaux derivative is zero:

$$\nabla \|\mathbf{v} - E(\mathbf{w})\|^2 = \mathbf{0}.$$

By the chain rule,

$$\|\mathbf{v} - E(\mathbf{w})\| \cdot E'(\mathbf{w}) = \mathbf{0}.$$

But [3, p. 253, lemma 1, chapter 7] implies $E'(\mathbf{w}) \neq \mathbf{0}$. Thus

$$E(\mathbf{w}) = \mathbf{v}.$$

■

Because E' is Lipschitz continuous at $\mathbf{w}^{\text{old}}$, for some positive l' and ρ, $\mathbf{w}$ and $\mathbf{v}$ in $\mathbf{w}^{\text{old}} + l'U$ implies

$$\|E'(\mathbf{w}) - E'(\mathbf{v})\| \leq \rho\|\mathbf{w} - \mathbf{v}\|.$$

Set $l^* = \min\left(2\frac{\delta}{\rho r}, \frac{1}{r}, \frac{1}{2}, \frac{l'}{2r}\right)$. Suppose $\|(\Delta\mathbf{b}, \Delta\mathbf{c}, \mathbf{0})^T\| < \frac{1}{2}l^*r\delta$. Then the new problem E has a solution, designated as $\mathbf{w}^{\text{new}}$, contained in $\mathbf{w}^{\text{old}} + l^*rU$. We will use induction to prove the Q-quadratic convergence of the Newton sequence to $\mathbf{w}^{\text{new}}$ from $\mathbf{w}^{\text{old}}$.

Apparently, $\left\|\mathbf{w}^{\text{old}} - \mathbf{w}^{\text{new}}\right\|_2 < l^*r$. Assume $\left\|\mathbf{w}^k - \mathbf{w}^{\text{new}}\right\|_2 < l^*r$. Then

$$\left\|\mathbf{w}^k - \mathbf{w}^{\text{old}}\right\|_2 \leq \left\|\mathbf{w}^k - \mathbf{w}^{\text{new}}\right\|_2 + \left\|\mathbf{w}^k - \mathbf{w}^{\text{old}}\right\|_2 < 2l^*r.$$

Because $l^* \leq \frac{1}{2}$ and $l^* \leq \frac{l'}{2r}$, $\mathbf{w}^k \in \left(\mathbf{w}^{\text{old}} + l'U\right) \cap \left(\mathbf{w}^{\text{old}} + rU\right)$. By [4, p. 75, lemma 4.1.12],

$$\begin{aligned}
\|\mathbf{w}^{k+1} - \mathbf{w}^{\text{new}}\| &= \|\mathbf{w}^k - \left[E'(\mathbf{w}^k)\right]^{-1} E(\mathbf{w}^k) - \mathbf{w}^{\text{new}}\| \\
&= \left[E'(\mathbf{w}^k)\right]^{-1} \left[E(\mathbf{w}^{\text{new}}) + E'(\mathbf{w}^k)\left(\mathbf{w}^k - \mathbf{w}^{\text{new}}\right) - E(\mathbf{w}^k)\right] \\
&\leq \frac{\rho}{2\delta}\|\mathbf{w}^k - \mathbf{w}^{\text{new}}\|^2.
\end{aligned}$$

By induction,

$$\|\mathbf{w}^{k+1} - \mathbf{w}^{\text{new}}\| \leq \frac{\rho}{2\delta}\|\mathbf{w}^k - \mathbf{w}^{\text{new}}\|^2 < \frac{\rho}{2\delta}(l^*r)^2 \leq l^*r.$$

We have proved the Q-quadratic convergence of the sequence.

Now we add perturbation of A. Since $A\mathbf{x} = (A + \Delta A)\mathbf{x} - \Delta A\mathbf{x}$, change of the function value is

$$E(\mathbf{w}^{\text{old}}) - E^{\text{old}}(\mathbf{w}^{\text{old}}) = \left(\Delta A\mathbf{x}^{\text{old}} - \Delta\mathbf{b}; -\Delta\mathbf{c} - \Delta A^*\mathbf{y}^{\text{old}}; \mathbf{0}\right).$$

Note that perturbations may only modify the Lipschitz constant ρ for E', and have no effects on l'. Also observe that only ΔA may change ρ, and ρ depends linearly on A. So there exists $\nu_1 > 0$, such that when $\|\Delta A\| \leq \nu_1$, we have $\rho^{\text{new}} \leq 2\rho^{\text{old}}$.
Because E' is uppersemicontinuous (see [3]), according to perturbation lemma, there exists a positive number ν_2, so that when $\|\Delta A\| \leq \nu_2$, for any $\mathbf{w} \in \mathbf{w}^{\text{old}} + \frac{r}{2}U$, we have $\|E^{-1}\| \leq \frac{2}{\delta}$.
Therefore, $E(\mathbf{w}^{\text{old}} + \frac{1}{2}lrU)$ contains $E(\mathbf{w}^{\text{old}}) + \frac{1}{8}lr\delta U$.
Let $\nu = \min(\nu_1, \nu_2)$. Assume $\|\Delta A\| \leq \nu$. Then as the proof above, let $l^* \leq \min\left(\frac{\delta}{\rho^{\text{old}}r}, \frac{1}{r}, \frac{1}{2}, \frac{1}{l'}\right)$. Suppose $\left\|\left(\Delta A\mathbf{x}^{\text{old}} - \Delta\mathbf{b}; -\Delta\mathbf{c} - \Delta A^*\mathbf{y}^{\text{old}}; \mathbf{0}\right)\right\| \leq \frac{1}{8}l^*r\delta$. We can get Q-quadratic convergence rate for the Newton sequence starting from $\mathbf{w}^{\text{old}}$ to $\mathbf{w}^{\text{new}}$.

5 Examples

In this section, we give two concrete examples of the Q method for SDP and SOCP.

5.1 Symmetric Semi-definite Matrices

Let $\mathcal{S}^m$ denote the space of real symmetric matrices of order m. Then $\mathcal{S}^m$ equipped with the Jordan product

$$X \circ Y = \frac{1}{2}(XY + YX)$$

is a Jordan algebra. The symmetric cone associated with this Jordan algebra is the cone of positive definite real symmetric matrices. The rank is m, the trace and determinant are the usual ones, and the inverse is also the usual ones (see [5, p. 31, example 1]).

Let $M_{m,n}(\mathbb{F})$ be the set of all m-by-n matrices over the field $\mathbb{F}$. For every matrix $A = [a_{ij}] \in M_{m,n}$, let $\operatorname{vec} A$ be the mn-dimensional vector associated with it:

$$\operatorname{vec} A \stackrel{\text{def}}{=} (a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn})^T.$$

Let $A \otimes B$ denote the Kronecker product of $A = [a_{ij}] \in M_{m,n}(\mathbb{F})$ and $B = [b_{ij}] \in M_{p,q}(\mathbb{F})$:

$$A \otimes B \stackrel{\text{def}}{=} \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Let I be the identity matrix. Then (see [8, chapter 4.3])

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \tag{25}$$

$$\operatorname{vec}(AB) = (I \otimes A) \operatorname{vec} B = (B^T \otimes I) \operatorname{vec} A. \tag{26}$$

Use (26), it's easy to check that the linear mapping defined by X can be written as

$$L(X) = \frac{1}{2}(I \otimes X + X \otimes I).$$

Let $E_{ij} \in \mathcal{S}^m$ denote the matrix whose (i, j) entry is 1 and the remaining entries are 0. Use standard basis. Choose the fixed Jordan frame as $E_{11}, \ldots, E_{mm}$. By [5, p. 63, example 1],

$$V(f_i, \frac{1}{2}) = \begin{pmatrix} & & & u_1 & & & \\ & & & \vdots & & & \\ & & & u_{i-1} & & & \\ u_1 & \ldots & u_{i-1} & 0 & u_{i+1} & \ldots & u_m \\ & & & u_{i+1} & & & \\ & & & \vdots & & & \\ & & & u_m & & & \end{pmatrix}.$$

Therefore,

$$V_{ij} = \{aE_{ij} + aE_{ji} \colon a \in \mathbb{R}\}.$$

So for any $U_{ij} \in V_{ij}$, by (25), we have

$$[L(E_{ii}), L(U_{ij})] = \frac{1}{4} I \otimes (u_{ij}E_{ij} - u_{ij}E_{ji}) + \frac{1}{4} (u_{ij}E_{ij} - u_{ij}E_{ji}) \otimes I.$$

Hence

$$\mathfrak{l} = \left\{ \left(I \otimes S - S^T \otimes I \right) : S^T = -S \right\}.$$

The corresponding Q method is in [1].

5.2 Second-order Cones

Let indices of a vector start from 0. We use standard basis. Define a product "$\circ$" on $\mathbb{R}^{n+1}$ as

$$\mathbf{x} \circ \mathbf{y} = \begin{pmatrix} \mathbf{x}^T \mathbf{y} \\ x_0 y_1 + y_0 x_1 \\ \vdots \\ x_0 y_n + y_0 x_n \end{pmatrix}.$$

Then $\mathbb{R}^{n+1}$ equipped with "$\circ$" is a Jordan algebra. The symmetric cone associated with this Jordan algebra is the Loréntz cone (see [5, p. 48, example 2]):

$$\left\{(x_0, x_1, \dots, x_n)\colon x_0 > \sqrt{x_1^2 + \cdots + x_n^2}\right\}.$$

It is also known as second-order cone, quadratic cone, or ice-cream cone.

For an element $\mathbf{x} \in \mathbb{R}^{n+1}$, let $L(\mathbf{x})$ be the linear mapping of $\mathbb{R}^{n+1}$, whose matrix is

$$L(\mathbf{x}) = \operatorname{Arw} \mathbf{x} \overset{\text{def}}{=} \begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ x_1 & x_0 & & \\ \vdots & & \ddots & \\ x_n & & & x_0 \end{pmatrix}.$$

Then, it is easy to verify that $\forall\, \mathbf{y} \in \mathbb{R}^{n+1}$, $L(\mathbf{x})\mathbf{y} = \mathbf{x} \circ \mathbf{y}$ and $\left[L(\mathbf{x}), L(\mathbf{x}^2)\right] = 0$, where, $[\cdot]$ is the Lie bracket.

It is proved in [5, p. 63] that for this Jordan algebra, the identity element is $\mathbf{e} \overset{\text{def}}{=} (1; \mathbf{0})$; and the only possible idempotents are $\mathbf{e}$ and $(\frac{1}{2}; \bar{\mathbf{f}})$ with $\bar{\mathbf{f}}^T \bar{\mathbf{f}} = \frac{1}{4}$. The number

$$\min\left\{k > 0\colon (\mathbf{e}, \mathbf{x}, \mathbf{x}^2, \dots, \mathbf{x}^k) \text{ are linearly dependent}\right\}$$

is no greater than 2; so the rank of this Jordan algebra is 2. Therefore, the set of primitive idempotents of this Jordan algebra is $\left\{(\frac{1}{2}; \bar{\mathbf{f}})\colon \bar{\mathbf{f}}^T \bar{\mathbf{f}} = \frac{1}{4}\right\}$.

Observe the spectral decomposition second version of any $\mathbf{x} \in \mathbb{R}^{n+1}$ can be written as

$$\mathbf{x} = (x_0 + \|\bar{\mathbf{x}}\|_2)\mathbf{f}_{\mathbf{x}} + (x_0 - \|\bar{\mathbf{x}}\|_2)(\mathbf{e} - \mathbf{f}_{\mathbf{x}}),$$

where

$$\mathbf{f}_{\mathbf{x}} = \begin{cases} (\frac{1}{2}; \frac{\bar{\mathbf{x}}}{2\|\bar{\mathbf{x}}\|_2}) & \bar{\mathbf{x}} \neq \mathbf{0} \\ (\frac{1}{2}; \frac{1}{2}; \mathbf{0}) & \bar{\mathbf{x}} = \mathbf{0}. \end{cases}$$

Hence, the two eigenvalues of $\mathbf{x}$ are $(x_0 + \|\bar{\mathbf{x}}\|_2)$ and $(x_0 - \|\bar{\mathbf{x}}\|_2)$.

By [5, p. 63, example 2],

$$V(\mathbf{f}, \frac{1}{2}) = \left\{(0; \bar{\mathbf{u}})\colon \bar{\mathbf{u}}^T \bar{\mathbf{f}} = 0\right\}.$$

The polar decomposition of a variable $\mathbf{x}$ with respect to $\mathbf{f}$ is

$$\mathbf{x} = \left(4\bar{\mathbf{x}}^T \bar{\mathbf{f}} + x_0 - 2\bar{\mathbf{x}}^T \bar{\mathbf{f}}\right) \mathbf{f} + \left(x_0 - 2\bar{\mathbf{x}}^T \bar{\mathbf{f}}\right) \left(\frac{1}{2}; -\bar{\mathbf{f}}\right) + \left(0; \bar{\mathbf{x}} - (4\bar{x}^T \bar{\mathbf{f}})\bar{\mathbf{f}}\right).$$

The fixed Jordan frame we choose is $\{\mathbf{f} = (\frac{1}{2}; \frac{1}{2}; \mathbf{0}),\ \mathbf{e} - \mathbf{f} = (\frac{1}{2}; -\frac{1}{2}; \mathbf{0})\}$. Therefore, $V_{12} = \{(0; 0; \mathbf{s})\colon \mathbf{s} \in \mathbb{R}^{n-1}\}$. After simple calculation, we find $\forall S \in \mathfrak{l}$, its matrix can be written as

$$\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & s_2 & \cdots & s_n \\ 0 & -s_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -s_n & 0 & \cdots & 0 \end{pmatrix},$$

since $S = [L(\mathbf{f}),\ L(0, 0, s_2, \dots, s_n)]$.

It is possible to use other Jordan frames - denoted as $\{\tilde{\mathbf{f}}, e - \tilde{\mathbf{f}}\}$. In order to have the most zeros in the matrix representation, $\tilde{\mathbf{f}}$ should be $(\frac{1}{2}; \frac{1}{2}; \mathbf{e}_k)$, where $\mathbf{e}_k$ is the kth standard basis in $\mathbb{R}^{n-1}$. Then the matrix of $\tilde{S}$ is obtained by interchaging the $2nd$ row and the kth row, the $2nd$ column and the kth column of that of S.

See [2] for the interior point algorithm and [11] for the Newton type algorithm.

6 Conclusion

The Q method of [1] is generized to the symmetric cone programming. Two algorithms –an infeasible interior point algorithm and a Newton type algorithm– are proposed for the resulting system. The infeasible interior point method can be started from an arbitrary point, but the local convergence rate is linear. On the other hand, the iterates of the Newton type method converge Q-quadratically locally, but don't have global convergence property. Hence for cold start problems we suggest using the infeasible interior point method and switch to the Newton type method when the iterates are close to the optimum. And for "warm starting" problems, the Newton type method is better.

A Appendix

A.1 Basics of Jordan Algebra

For completeness, this section cites some results mostly from [5] that are used in the paper.

Definition A.1 *(Jordan Algebra [5, p. 24, chapter II.1]) Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. An algebra V over $\mathbb{F}$ is called a Jordan Algebra if, $\forall\, \mathbf{x}, \mathbf{y} \in V$:*

$$\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x},$$
$$\mathbf{x}(\mathbf{x}^2\mathbf{y}) = \mathbf{x}^2(\mathbf{x}\mathbf{y}).$$

For an element $\mathbf{x} \in V$, define

$$m(\mathbf{x}) \stackrel{\text{def}}{=} \min\left\{k > 0\colon (\mathbf{e}, \mathbf{x}, \mathbf{x}^2, \ldots, \mathbf{x}^k) \text{ are linearly dependent.}\right\}$$

The rank of V is defined as (see [5, p. 28, chapter II.2])

$$r = \max\left\{m(\mathbf{x})\colon \mathbf{x} \in V\right\}.$$

Definition A.2 *(Complete system of orthogonal idempotents [5, p. 43, chapter III.1].) If*

$$\mathbf{f}_i^2 = \mathbf{f}_i,$$
$$\mathbf{f}_i\mathbf{f}_j = \mathbf{0} \quad (i \neq j),$$
$$\mathbf{f}_1 + \cdots + \mathbf{f}_k = \mathbf{e}.$$

Then $\mathbf{f}_1, \ldots, \mathbf{f}_k$ is called a complete system of orthogonal idempotents.

Let $\mathbb{F}[\mathbf{x}]$ denote the algebra over $\mathbb{F}$ of polynomials in one variable with coefficients in $\mathbb{F}$.

Theorem A.1 *(Spectral theorem, first version [5, p. 43, theorem III 1.1].) For $\mathbf{x}$ in V there exist unique real numbers $\lambda_1, \ldots, \lambda_k$, all distinct, and a unique complete system of orthogonal idempotents $\mathbf{f}_1, \ldots, \mathbf{f}_k$ such that*

$$\mathbf{x} = \lambda_1\mathbf{f}_1 + \cdots + \lambda_k\mathbf{f}_k.$$

The numbers λ_j are called the eigenvalues of $\mathbf{x}$. Furthermore, for $j = 1, \ldots, k$, $\mathbf{f}_j \in \mathbb{R}[\mathbf{x}]$.

An idempotent is said to be primitive if it is non-zero and cannot be written as the sum of two (necessarily orthogonal) non-zero idempotents.

Definition A.3 *(Jordan frame.) A complete system of orthogonal primitive idempotents is called a Jordan frame.*

Theorem A.2 *(Spectral theorem, second version [5, p. 44, theorem III.1.2]) For* $\mathbf{x} \in V$*, there exist a Jordan frame* $\mathbf{f}_1, \ldots, \mathbf{f}_r$ *and real numbers* $\lambda_1, \ldots, \lambda_r$ *such that*

$$\mathbf{x} = \lambda_1 \mathbf{f}_1 + \cdots + \lambda_r \mathbf{f}_r.$$

Definition A.4 *(Inverse [5, p. 30, chapter II.2]) An element* $\mathbf{x}$ *is said to be invertible if there exists an element* $\mathbf{y} \in \mathbb{F}[\mathbf{x}]$*, such that* $\mathbf{xy} = \mathbf{e}$*. Since* $\mathbb{F}[\mathbf{x}]$ *is associative,* $\mathbf{y}$ *is unique. It is called the inverse of* $\mathbf{x}$*, and is denoted by* $\mathbf{x}^{-1}$*.*

By [5, p. 31, proposition II 2.4], an element $\mathbf{x}$ is invertible if and only if $\det(\mathbf{x}) \neq 0$, i.e. $\mathbf{x}$ has no zero eigenvalues.

Definition A.5 *(Symmetric cone [5, p. 4, chapter I.1]) An open convex cone* Sy *is said to be homogeneous if* $G(Sy)$ *is transitive on* Sy*, i.e. for any* $\mathbf{x}, \mathbf{y} \in Sy$*, there exists* $g \in G(Sy)$ *such that* $g\mathbf{x} = \mathbf{y}$*. And* Sy *is said to be symmetric if it is self-dual and homogeneous.*

A cone is self-dual implies that it is proper, i.e. $\operatorname{cl} Sy \cap (-\operatorname{cl} Sy) = \{0\}$.

Definition A.6 *(Peirce decomposition [5, p. 62]) For any idempotent* $\mathbf{f} \in V$*, by [5, p. 45, proposition III.1.3], the only possible eigenvalues of* $L(\mathbf{f})$ *are* 1*,* $\frac{1}{2}$ *and* 0*. The space* V *is the direct sum of the corresponding subspaces* $V(\mathbf{f}, 1)$*,* $V(\mathbf{f}, \frac{1}{2})$ *and* $V(\mathbf{f}, 0)$*. The decomposition*

$$V = V(\mathbf{f}, 1) + V(\mathbf{f}, \frac{1}{2}) + V(\mathbf{f}, 0)$$

is called the Peirce decomposition with respect to the idempotent $\mathbf{f}$*.*

The decomposition is orthogonal with respect to any associative symmetric bilinear form.

Given a Jordan frame $\{(\mathbf{f}_i)_j : j = 1, \ldots, r_i;\ i = 1, \ldots, n\}$, consider the subspaces

$$V_{i(jj)} \stackrel{\text{def}}{=} V\left((\mathbf{f}_i)_j, 1\right) = \mathbb{R}(\mathbf{f}_i)_j, \tag{27}$$

$$V_{i(jk)} \stackrel{\text{def}}{=} V\left((\mathbf{f}_i)_j, \frac{1}{2}\right) \cap V\left((\mathbf{f}_i)_k, \frac{1}{2}\right). \tag{28}$$

Theorem A.3 *[5, p. 68, theorem IV.2.1] Any subspace* V_i *of* V *decomposes as the following orthogonal direct sums:*

$$V_i = \bigoplus_{j \leq k} V_{i(jk)}.$$

Furthermore,

$$V_{i(jk)} \cdot V_{i(jk)} \subset V_{i(jj)} + V_{i(kk)},$$
$$V_{i(jk)} \cdot V_{i(pq)} = \{\mathbf{0}\}, \ \textit{if } \{j, k\} \cap \{p, q\} = \emptyset.$$

Therefore, any $\mathbf{x} \in V$ can be written as

$$\mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{r_i} x_{ij} + \sum_{i=1}^{n} \sum_{j<k} x_{ijk}$$

with $x_{ij} \in \mathbb{R}(\mathbf{f}_i)_j$, $x_{ijk} \in V_{i(jk)}$.

Proposition A.1 *[5, p. 65, proposition IV.1.4] Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two orthogonal primitive idempotents. If* $\mathbf{x} \in V\left(\mathbf{a}, \frac{1}{2}\right) \cap V\left(\mathbf{b}, \frac{1}{2}\right)$*, then*

$$\mathbf{x}^2 = \frac{1}{2}\|\mathbf{x}\|^2(\mathbf{a} + \mathbf{b}).$$

Let $Aut(V)_0$ denote the identity component of $Aut(V)$

Theorem A.4 *[5, p. 55, Theorem III.5.1]*

$$Aut(V)_0 = K, \qquad Der(V) = \mathfrak{k}.$$

Theorem A.5 *[5, p. 71, IV.2.5] If* $\{\mathbf{c}_1, \dots, \mathbf{c}_k\}$ *and* $\{\mathbf{d}_1, \dots, \mathbf{d}_k\}$ *are two Jordan frames on* V_i*, then there exists an automorphism* $A \in K_i$ *such that*

$$A\mathbf{c}_j = \mathbf{d}_j \quad (1 \le j \le k).$$

Theorem A.6 *(Polar decomposition [5, p. 102, VI.2]) For a given Jordan frame* $\{(\mathbf{f}_i)_j\}_{j=1}^{r_i}$ *in* V_i*, since* K_i *is already transitive on the set of Jordan frames, any* $\mathbf{x}_i$ *in* V_i *can be written as*

$$\mathbf{x}_i = Q_i\mathbf{a}_i \ \left(Q_i \in K_i;\ a_i = (\lambda_i)_1(\mathbf{f}_i)_1 + \cdots + (\lambda_i)_{r_i}(\mathbf{f}_i)_{r_i};\ (\lambda_i)_j \in \mathbb{R},\ j = 1, \dots, r_i\right).$$

The numbers $(\lambda_i)_j$ are the eigenvalues of $\mathbf{x}$; so they are unique without considering orders. Note that the eigenvalues of $\mathbf{x}^2$ are $(\lambda_i)_j^2$. Obviously, $\mathbf{x} \in Sq$ iff $(\lambda_i)_j \ge 0$ for $(j = 1, \dots, r_i,\ i = 1, \dots, n)$; $\mathbf{x} \in \operatorname{Int} Sq$ iff $(\lambda_i)_j > 0$.

Proposition A.2 *([5, p. 103, proposition VI.2.1]) Let* $\mathbf{a} \in W_i$ *and* $S \in \mathfrak{l}_i$*. Then* $S\mathbf{a}$ *is orthogonal to* W_i*. If* $\mathbf{a}$ *is regular, i.e.* $a_j \neq a_k$ *for* $j \neq k$*. Then the mapping*

$$\mathfrak{l}_i \mapsto W_i^{\perp}:\ S \mapsto Sa$$

is an isomorphism.

References

[1] Farid Alizadeh, Jean-Pierre A. Haeberly, and Michael L. Overton. Primal-dual interior-point methods for semidefinite programming: convergence rates, stability and numerical results. *SIAM J. Optim.*, 8(3):746–768 (electronic), 1998.

[2] Farid Alizadeh and Yu Xia. The Q method for second-order cone programming. Technical Report AdvOl-Report No. 2004/15, McMaster University, 2004.

[3] Frank H. Clarke. *Optimization and nonsmooth analysis.* John Wiley & Sons Inc., New York, 1983. A Wiley-Interscience Publication.

[4] John E. Dennis, Jr. and Robert B. Schnabel. *Numerical methods for unconstrained optimization and nonlinear equations.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1983.

[5] Jacques Faraut and Adam Korányi. *Analysis on symmetric cones.* The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.

[6] Leonid Faybusovich. Linear systems in Jordan algebras and primal-dual interior-point algorithms. *J. Comput. Appl. Math.*, 86(1):149–175, 1997. Special issue dedicated to William B. Gragg (Monterey, CA, 1996).

[7] Roland W. Freund, Florian Jarre, and Shinji Mizuno. Convergence of a class of inexact interior-point algorithms for linear programs. *Math. Oper. Res.*, 24(1):50–71, 1999.

[8] Roger A. Horn and Charles R. Johnson. *Topics in matrix analysis.* Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original.

[9] Masakazu Kojima, Nimrod Megiddo, and Shinji Mizuno. A primal-dual infeasible-interior-point algorithm for linear programming. *Math. Programming*, 61(3, Ser. A):263–280, 1993.

[10] Irvin J. Lustig. Feasibility issues in a primal-dual interior-point method for linear programming. *Math. Programming*, 49(2, (Ser. A)):145–162, 1990/91.

[11] Yu Xia. An algorithm for perturbed second-order cone programs. Technical Report AdvOl-Report No. 2004/17, McMaster University, 2004.