PERTURBATIONS AND METRIC REGULARITY

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Abstract

A point x is an approximate solution of a generalized equation $b \in F(x)$ if the distance from the point b to the set $F(x)$ is small. "Metric regularity" of the set-valued mapping F means that, locally, a constant multiple of this distance bounds the distance from x to an exact solution. The smallest such constant is the "modulus of regularity", and is a measure of the sensitivity or conditioning of the generalized equation. We survey recent approaches to a fundamental characterization of the modulus as the reciprocal of the distance from F to the nearest irregular mapping. We furthermore discuss the sensitivity of the regularity modulus itself, and prove a version of the fundamental characterization for mappings on Riemannian manifolds.

Key words: metric regularity, set-valued mapping, condition number, Eckart-Young theorem, distance to ill-posedness, Riemannian manifold, regularity modulus.

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1 Introduction

When we seek to invert a mapping, a central aspect is the sensitivity of the answer to small changes in data, or in other words, the "conditioning" of the mapping. For example, consider a smooth map F between Euclidean spaces, and suppose that $F(0) = 0$ and that the derivative $\nabla F(0)$ is surjective. Then from the classical inverse function theorem one can easily obtain the existence of a constant $\kappa > 0$ such that, for all small vectors b, the set $F^{-1}(b)$ contains vectors of size less than $\kappa\|b\|$. Here b measures the "residual" of how much the equation $F(0) = 0$ is not satisfied and the distance from 0 to the set of solutions of the equation $b = F(x)$ is bounded by the constant κ times the residual. Usually, the residual is easy to compute or estimate while finding a solution might be considerably more difficult. The surjectivity of the derivative $\nabla F(0)$ gives information about solutions based on the value of the residual; in particular, if we know the rate of convergence of the residual to zero, then we will obtain the rate of convergence of approximate solutions to an exact one.

This fundamental, linear relationship between data change and solution error is known as metric regularity. Given two metric spaces (X, d_X) and (Y, d_Y), a set-valued mapping $F : X \rightrightarrows Y$, and points $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$, we call F *metrically regular at* $\bar{x}$ *for* $\bar{y}$ if there exists a constant $\kappa > 0$ such that

$$d_X(x, F^{-1}(y)) \leq \kappa d_Y(y, F(x)) \text{ for all } (x, y) \text{ close to } (\bar{x}, \bar{y}) \ . \tag{1.1}$$

Here, the inverse mapping $F^{-1} : Y \rightrightarrows X$ is defined as

$$F^{-1}(y) = \{x \in X : y \in F(x)\},$$

and the distance notation means

$$d_X(x, S) = \inf\{d_X(x, s) : s \in S\}$$

for any set $S \subset X$, interpreted as $+\infty$ when $S = \emptyset$. The *regularity modulus* $\operatorname{reg} F(\bar{y}|\bar{z})$ is the infimum of all $\kappa > 0$ satisfying the above condition. By convention, $\operatorname{reg} F(\bar{y}|\bar{z}) = +\infty$ when F is not metrically regular at $\bar{y}$ for $\bar{z}$.

When F is a linear and bounded mapping acting between Banach spaces X and Y, denoted $F \in \mathcal{L}(X, Y)$, the quantity $\operatorname{reg} F$ is the same for every point. Further, $\operatorname{reg} F < \infty$ if and only if F is surjective. In particular, when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, the $m \times n$ matrix F must have full rank or, equivalently, the columns of F must be linearly independent. When $m = n$ the matrix F is then invertible. When F is a smooth map as above, $\operatorname{reg} F(\bar{x}|F(\bar{x}))$ is equal to the regularity modulus of the derivative $\nabla F(\bar{x})$ thus reducing its computation to the linear case.

In this paper we are concerned with the problem of how the metric regularity of a mapping depends on perturbations acting on the mapping. There is a large body of literature on this subject written in the last century and some of it is surveyed here. In this first introductory section we collect a number of results, many supplied with proofs, on the question "how much" a metrically regular mapping can be perturbed before losing its regularity. In recent years much progress has been made in this direction by establishing that the "distance" from a given regular mapping to the set of nonregular mappings is equal to the reciprocal of the regularity modulus. This remarkably simple and surprising relationship not only complements some results in classical analysis, but also gives insights on how to define "conditioning" of set-valued mappings.

In Section 2 we focus on local analysis of the effect of perturbations on metric regularity by estimates for the sensitivity of the regularity modulus. Section 3 presents a radius theorem for mappings acting on Riemannian manifolds.

We start our presentation with a classical result which describes the effect of perturbations on the invertibility of linear mappings. Henceforth in this section, unless otherwise stated, X and Y are Banach spaces.

Theorem 1.1. (Banach lemma) *If $F \in \mathcal{L}(X,Y)$ is invertible and $B \in \mathcal{L}(X,Y)$ satisfies $\|B\| \le \|F^{-1}\|^{-1}$, then*

$$\|(F+B)^{-1}\| \le (\|F^{-1}\|^{-1} - \|B\|)^{-1}.$$

The Banach lemma has far-reaching generalizations which we present next. First we consider positively homogeneous set-valued mappings $F : X \rightrightarrows Y$, that is, mappings F whose graphs are cones. For such set-valued mappings, the standard concept of operator norm is replaced by certain quantities that are also called "norms" although these mappings do not form a vector space. The *outer* and the *inner* norms are defined as

$$\|F\|^+ = \sup_{x\in \mathbb{B}} \sup_{y\in F(x)} \|y\| \quad \text{and} \quad \|F\|^- = \sup_{x\in \mathbb{B}} \inf_{y\in F(x)} \|y\|,$$

where $\mathbb{B}$ denotes the closed unit ball. Next, we give an outer norm version of the Banach lemma supplied with an elementary proof, which in particular covers the classical Banach lemma when the mapping F is invertible:

Theorem 1.2. (Banach lemma for the outer norm, [21]) *Let $F : X \rightrightarrows Y$ be positively homogeneous and let $B \in \mathcal{L}(X,Y)$. If $[\|F^{-1}\|^+]^{-1} \ge \|B\|$ then*

$$\|(F+B)^{-1}\|^+ \le ([\|F^{-1}\|^+]^{-1} - \|B\|)^{-1}.$$

Proof. If $\|F^{-1}\|^+ = +\infty$ there is nothing to prove. On the other hand, $\|F^{-1}\|^+ = 0$ if and only if $F(x) = \emptyset$ for all nonzero x in which case $\|F+H\|^+ = 0$ and the result follows in this case, too. Suppose $0 < \|F^{-1}\|^+ < +\infty$. If the result fails, there exists $B \in \mathcal{L}(X,Y)$ with $\|B\| \le \|F^{-1}\|^+$ such that

$$\|(F+B)^{-1}\|^+ > ([\|F^{-1}\|^+]^{-1} - \|B\|)^{-1}.$$

Then by definition there exist $y \in Y$ with $\|y\| \le 1$ and $x \in X$ with $x \in (F+B)^{-1}(y)$ such that

$$\|x\| > ([\|F^{-1}\|^+]^{-1} - \|B\|)^{-1}$$

which is the same as

$$\frac{1}{\|x\|^{-1} + \|B\|} > \|F^{-1}\|^+. \tag{1.2}$$

But then $y - B(x) \in F(x)$ and

$$\|y - Bx\| \le \|y\| + \|B\|\|x\| \le 1 + \|B\|\|x\|. \tag{1.3}$$

If $y = Bx$ then $0 \in F(x)$ which obviously yields $x = 0$, a contradiction. Hence $\alpha := \|y - Bx\| > 0$ and by the positive homogeneity, $(\alpha x, \alpha(y - Bx)) \in \operatorname{gph} F$ and $\alpha\|y - Bx\| = 1$ which implies, by definition,

$$\|F^{-1}\|^{+} \geq \frac{\|x\|}{\|y - Bx\|}.$$

Combining the inequality with (1.2) and (1.3), we have

$$\|F^{-1}\|^{+} \geq \frac{\|x\|}{\|y - Bx\|} \geq \frac{\|x\|}{1 + \|B\|\|x\|} = \frac{1}{\|x\|^{-1} + \|B\|} > \|F^{-1}\|^{+},$$

which is a contradiction, and the proof is complete. □

Positively homogeneous mappings with infinite outer norm are called *singular*. For linear and continuous $F : X \to Y$, nonsingularity in this sense coincides with the traditional notion when $\dim X = \dim Y < \infty$, but it means in general that F^{-1} is single-valued relative to $\operatorname{rge} F$, its domain.

We move on to the study of perturbations of the inner norm $\|F^{-1}\|^{-}$ by the simple expedient of taking adjoints [22]. With respect to the spaces X^* and Y^* dual to X and Y, the *upper adjoint* of a positively homogeneous mapping $F : X \rightrightarrows Y$ is the mapping $F^{*+} : Y^* \rightrightarrows X^*$ defined by

$$(y^*, x^*) \in \operatorname{gph} F^{*+} \quad \Leftrightarrow \quad \langle x^*, x\rangle \leq \langle y^*, y\rangle \text{ for all } (x, y) \in \operatorname{gph} F,$$

whereas the *lower adjoint* is the mapping $F^{*-} : Y^* \rightrightarrows X^*$ defined by

$$(y^*, x^*) \in \operatorname{gph} F^{*-} \quad \Leftrightarrow \quad \langle x^*, x\rangle \geq \langle y^*, y\rangle \text{ for all } (x, y) \in \operatorname{gph} F.$$

The graphs of both F^{*+} and F^{*-} correspond to the closed convex cone in $X^* \times Y^*$ that is polar to $\operatorname{gph} F$, except for permuting (x^*, y^*) to (y^*, x^*) and introducing certain changes of sign. Furthermore,

$$\|F\|^{+} = \|F^{*+}\|^{-} = \|F^{*-}\|^{-} \quad \text{and} \quad \|F\|^{-} = \|F^{*+}\|^{+} = \|F^{*-}\|^{+}.$$

For a mapping $B \in \mathcal{L}(X, Y)$ both adjoints coincide with the usual notion B^*, and a simple calculation shows that

$$(F + B)^{*+} = F^{*+} + B^*,$$

and analogously for $(F + B)^{*-}$. When the mapping F is *sublinear*, that is, its graph is not only a cone, but also a convex cone, this allows us to pass via adjoints from the outer norm to the inner norm. In this case $\|F^{-1}\|^{-} < \infty$ if and only if F is *surjective*, or, equivalently, F^{*+} is nonsingular. Moreover, $\|F^{-1}\|^{-} = \operatorname{reg} F(0\,|\,0)$. The Banach lemma for the inner norm is completely analogous to that for the outer norm, with the difference that now the graph of the mapping F must be a convex and closed cone:

Theorem 1.3. (Banach lemma for the inner norm, [17]) *Let $F : X \rightrightarrows Y$ be sublinear with closed graph and let $B \in \mathcal{L}(X, Y)$. If $[\|F^{-1}\|^{-}]^{-1} \geq \|B\|$ then*

$$\|(F + B)^{-1}\|^{-} \leq ([\|F^{-1}\|^{-}]^{-1} - \|B\|)^{-1}.$$

In order to extend these results to general mappings we need localizations of the properties involved and quantitative characteristics to replace the inner and outer norms. Here metric regularity enters the stage as a natural localization of surjectivity. Also, the perturbations now are allowed to be nonlinear and to measure their magnitude, we use "calmness" and lipschitz moduli. We move briefly to more abstract spaces to define quantities used later in the paper. Given a metric space (X, d) and a linear normed space $(Y, \|\cdot\|)$, the *calmness modulus* of a function $f : X \to Y$ at a point $\bar{x} \in \operatorname{dom} f$ is defines as

$$\operatorname{clm} f(\bar{x}) = \limsup_{\substack{x \in \operatorname{dom} f, d(x,\bar{x}) \to 0, \\ d(x,\bar{x})>0}} \frac{\|f(x) - f(\bar{x})\|}{d(x, \bar{x})}.$$

The *lipschitz modulus* of f at a point $\bar{x} \in \operatorname{int} \operatorname{dom} f$ is defined as

$$\operatorname{lip} f(\bar{x}) = \limsup_{\substack{d(x,\bar{x}) \to 0, d(x',\bar{x}) \to 0 \\ d(x,x')>0}} \frac{\|f(x') - f(x)\|}{d(x', x)}.$$

so $\operatorname{lip} f(\bar{x}) < \infty$ signals that for any $\kappa > \operatorname{lip} f(\bar{x})$ there exists a neighborhood U of $\bar{x}$ such that f is lipschitz continuous on U with a constant κ.

The calmness and lipschitz moduli have some of the properties of a norm such as positive homogeneity and the triangle inequality. For X and Y Banach spaces, the Fréchet derivative $\nabla f(\bar{x})$ of a function $f : X \to Y$ at $\bar{x}$ can be defined as a mapping from $\mathcal{L}(X, Y)$ such that $\operatorname{clm}[f - \nabla f(\bar{x})](\bar{x}) = 0$, in which case $\operatorname{clm} f(\bar{x}) = \|\nabla f(\bar{x})\|$. Analogously, f is strictly differentiable at $\bar{x}$ exactly when $\operatorname{lip}[f - \nabla f(\bar{x})](\bar{x}) = 0$. In general, one has $\operatorname{clm} f(\bar{x}) \le \operatorname{lip} f(\bar{x})$ where the strict inequality is possible (even with an infinite gap).

The first extension of the Banach lemma (and, in fact, of the Banach open mapping principle) to nonlinear mappings goes back to the work of Graves[1]. Up to some adjustments in notation, Graves' result is as follows:

Theorem 1.4. (Graves theorem, [10]) *Let a function $f : X \to Y$ be continuous near $\bar{x}$ and let $F \in \mathcal{L}(X, Y)$ be surjective, that is $\operatorname{reg} F < \infty$. If $\operatorname{lip}(f - F)(\bar{x}) < (\operatorname{reg} F)^{-1}$, then*

$$\operatorname{reg} f(\bar{x}\,|\,f(\bar{x})) \le \Big((\operatorname{reg} F)^{-1} - \operatorname{lip}(f - F)(\bar{x})\Big)^{-1}. \tag{1.4}$$

The Banach lemma follows for $f = F + B$.

Long after the publication of Graves theorem, it was observed by Milyutin [7] that, in order to obtain an estimate of the form (1.4) the mapping F does not need to be linear. In fact, the perturbation $f - F$ of F to obtain f can be any function with sufficiently small lipschitz modulus. Apparently, Milyutin was the first to fully understand the metric character of the metric regularity[2]. In the last two decades, Graves theorem was further generalized in various ways. We give here ia result for set-valued mappings which fits exactly into the format of the Banach lemma:

Theorem 1.5. (Banach lemma for set-valued mappings, [8]) *Consider a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed and $0 < \operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \infty$. Then for any $g : X \to Y$ such that $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \cdot \operatorname{lip} g(\bar{x}) < 1$, one has*

$$\operatorname{reg}(g + F)(\bar{x}\,|\,\bar{y} + g(\bar{x})) \;\le\; (\operatorname{reg} F(\bar{x}\,|\,\bar{y})^{-1} - \operatorname{lip} g(\bar{x}))^{-1}.$$

[1]A related result was obtained much earlier by Lyusternik [18] by using a similar iterative procedure resembling the Newton method. In the Lyusternik theorem, however, the mapping A is the (continuous) Fréchet derivative of f and then the result does not give a perturbation estimate as in the Banach lemma.

[2]As remarked by A. Ioffe.

In the original Banach lemma, F and g are linear and bounded mappings. Theorem 1.5 is equivalent to the following result:

Theorem 1.6. (lipschitz perturbations, [8]) *Consider a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed and a function $g : X \to Y$. If $\operatorname{reg} F(\bar{x}|\bar{y}) < \kappa < \infty$ and $\operatorname{lip} g(\bar{x}) < \delta < \kappa^{-1}$, then*

$$\operatorname{reg}(g + F)(\bar{x}|g(\bar{x}) + \bar{y}) \le (\kappa^{-1} - \delta)^{-1}.$$

Variants of Theorem 1.6 are proved in the literature by employing iterative procedures similar to the one used in the proofs of Lyusternik and Graves. Here we present a different, strikingly simple proof of this theorem, based on the Ekeland principle, which is a shortcut through some recent results of Ioffe [12]. There is a price to pay though, and it is an additional assumption which always holds, e.g., in finite dimensions. We put the nontrivial part of this proof in a lemma:

Lemma 1.7. *Consider a set-valued mapping $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed and suppose that there exists a neighborhood W of $\bar{x}$ such that for each y near $\bar{y}$ the function $x \mapsto d(y, F(x))$ is lower semicontinuous on W. Also, consider the following condition:*

$$\text{(1.5)} \qquad \begin{cases} \text{there is a constant } \mu > 0 \text{ and neighborhoods } U \text{ of } \bar{x} \text{ and } V \text{ of } \bar{y} \text{ such that} \\ \text{for every } (x, y) \in U \times V \text{ with } d(y, F(x)) > 0 \text{ there exists } u \in U \setminus x \text{ satisfying} \\ \qquad d(y, F(u)) + \mu\|u - x\| \le d(y, F(x)). \end{cases}$$

If condition (1.5) holds, then F is metrically regular at $\bar{x}$ for $\bar{y}$ with a constant $\kappa = \mu^{-1}$. Conversely, if F is metrically regular at $\bar{x}$ for $\bar{y}$, then for every $\kappa > \operatorname{reg} F(\bar{x}|\bar{y})$ condition (1.5) holds with $\mu = \kappa^{-1}$.

Proof. Let the condition (1.5) hold. With no loss of generality, we can suppose the neighborhood W is closed and contained in U. Given any $y \in V$, we will prove

$$\text{(1.6)} \qquad d(x, F^{-1}(y)) \le \mu^{-1} d(y, F(x)) \quad \text{for all } x \in W,$$

thus obtaining that $\operatorname{reg} F(\bar{x}|\bar{y}) \le \mu^{-1}$. If $\varphi(x) = 0$ we have $x \in F^{-1}(y)$ and (1.6) holds automatically. Assume $\varphi(x) > 0$. We apply the Ekeland principle (see for example [5, Thm 7.5.1]) for the function

$$f(x) = \begin{cases} \varphi(x) & \text{if } x \in W, \\ +\infty & \text{otherwise,} \end{cases}$$

obtaining that there exists $w \in W$ such that

$$\text{(1.7)} \qquad \|w - x\| \le \frac{\varphi(x)}{\mu} \ \text{ and } \ \varphi(w) < \varphi(z) + \mu\|z - w\| \ \text{ for every } \ z \in W, z \ne w.$$

If $\varphi(w) > 0$ then, according to the assumed condition (1.5) with $x = w$, there exists a corresponding $u \in W$ which, when put in (1.7) in place of z, gives us

$$\varphi(u) + \mu\|u - w\| \le \varphi(w) < \varphi(u) + \mu\|u - w\|,$$

a contradiction. Hence $\varphi(w) = 0$, that is $w \in F^{-1}(y)$, and from (1.7) we get $\|x - w\| \le \mu^{-1}\varphi(x)$. Taking the infimum on the left with respect to $w \in F^{-1}(y)$ and returning to the original notation we obtain (1.6).

To prove the converse statement, let F be metrically regular and let $\kappa > \operatorname{reg} F(\bar{x}\,|\,\bar{y})$. Choose κ' such that $\kappa > \kappa' > \operatorname{reg} F(\bar{x}\,|\,\bar{y})$ and let the neighborhoods U of $\bar{x}$ and V of $\bar{y}$ correspond to the constant κ' in the definition of metric regularity. Let $y \in V$; then, by the metric regularity of F, we have that $F^{-1}(y) \neq \emptyset$, so let $x \in U$ be such that $d(y, F(x)) > 0$ (otherwise there is nothing to prove). Then there is a δ satisfying $0 < \delta < (\kappa - \kappa')d(y, F(x))$ and $u \in F^{-1}(y)$ such that $0 < \|u - x\| \leq d(x, F^{-1}(y)) + \delta$. Then $u \neq x$ and

$$\|u - x\| \leq d(x, F^{-1}(y)) + \delta \leq \kappa' d(y, F(x)) + \delta \leq \kappa d(y, F(x)).$$

Taking into account that $d(u, F^{-1}(y)) = 0$ we obtain (1.5) with constant $\mu = (\kappa)^{-1}$. □

Next we give a proof of Theorem 1.6 based on Lemma 1.7 under the assumption of the lemma that, for some neighborhood W of $\bar{x}$, the distance function $x \mapsto F(y, x)$ is lower semicontinuous on W for all y near $\bar{y}$. This holds automatically in finite dimensional spaces when $\operatorname{gph} F$ is closed locally around $(\bar{x}, \bar{y})$.

Proof of Theorem 1.6. Let $\kappa > \operatorname{reg} F(\bar{x}\,|\,\bar{y})$ and $\operatorname{lip} g(\bar{x}) < \delta$ such that $\kappa\delta < 1$ and let U and V be neighborhoods of $\bar{x}$ and $\bar{y}$, respectively, where the metric regularity condition (1.1) holds for F and δ is a lipschitz constant of g in U. Let a and b be positive constants such that $\mathbb{B}_a(\bar{x}) \subset U$ and $\mathbb{B}_{b+\delta a}(\bar{y}) \subset V$ and also the set $\operatorname{gph} F \cap (\mathbb{B}_a(\bar{x}) \times \mathbb{B}_{b+\delta a}(\bar{y} + g(\bar{x})))$ is closed. Then for every $x \in \mathbb{B}_a(\bar{x})$ and $y \in \mathbb{B}_b(\bar{y})$ we have $y - g(x) \in V$.

Let $x \in \mathbb{B}_a(\bar{x})$ and $y \in \mathbb{B}_b(\bar{y} + g(\bar{x}))$. If $d(y, (g + F)(x)) = 0$ then the metric regularity condition for $g + F$ is automatically satisfied with any constant. Let $d(y, (g + F)(x)) > 0$. Then $d(y - g(x), F(x)) > 0$ and hence from the metric regularity of F at $\bar{x}$ for $\bar{y}$ with a constant κ and Lemma 1.7 there exists $u \in \mathbb{B}_a(\bar{x})$, $u \neq x$ such that

$$d(y - g(x), F(u)) + \kappa^{-1}\|u - x\| \leq d(y - g(x), F(x)).$$

Adding and subtracting $d(y - g(u), F(u))$ in the left-hand side of this inequality and using the lipschitz continuity with constant 1 of the distance function $d(\cdot, F(u))$ and the lipschitz continuity of g in $\mathbb{B}_a(\bar{x})$ we obtain

$$d(y - g(u), F(u)) + (\kappa^{-1} - \delta)\|u - x\| \leq d(y - g(x), F(x)).$$

But this means that the condition (1.5) holds for $g + F$ with a constant $\kappa^{-1} - \delta$ and by Lemma 1.7 we conclude that $g + F$ is metrically regular at $\bar{x}$ for $\bar{y} + g(\bar{x})$ with $\operatorname{reg}(g + F)(\bar{x}\,|\,\bar{y} + g(\bar{x})) \leq (\kappa^{-1} - \delta)^{-1}$ as claimed. □

In the preceding lines we gave an estimate of how much a metrically regular mapping can be perturbed before losing its regularity. Next, we study the problem of how sharp our estimate is, that is, the problem of finding the size of the least perturbation that destroys the metric regularity. This problem is ultimately related to the idea of conditioning of mappings.

It has been long recognized by numerical analysts that the "closer" a matrix is to the set of singular matrices, the "harder" it is to invert it. A qualitative measure for nonsingularity of a matrix A is its *absolute condition number* $\operatorname{cond}(A) = \|A^{-1}\|$. This distance property of the absolute condition number for matrix inversion is captured by the classical Eckart-Young theorem:

$$\inf\{\, \|B\| \mid A + B \text{ not invertible } \} = \|A^{-1}\|^{-1} = 1/\operatorname{cond}(A).$$

This equality is also valid for invertible linear and bounded mappings acting in Banach spaces.

Results of the type of the Eckart-Young theorem are sometimes called "distance theorems" and also "condition number theorems": for an extended discussion of distances to ill-posedness of various problems in numerical analysis, see Demmel [6]. In [8] the term "radius of regularity" was proposed, expanding the idea to general mappings $F : X \rightrightarrows Y$ in order to measure, with respect to a pair $(\bar{x}, \bar{y})$ where regularity holds, how far F can be perturbed before regularity may be lost. The formal definition given in [8] for metric regularity is as follows:

Radius of metric regularity. *For any mapping $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, the radius of metric regularity at $\bar{x}$ for $\bar{y}$ is the value*

$$\operatorname{rad} F(\bar{x}\,|\,\bar{y}) \;=\; \min_{G:X\to Y}\Big\{ \operatorname{lip} G(\bar{x}) \,\Big|\, F + G \textit{ not metrically regular at } \bar{x} \textit{ for } \bar{y} + G(\bar{x}) \Big\}.$$

Following the logic of the first part of this section, we present first a radius theorem for nonsingularity of positively homogeneous mappings:

Theorem 1.8. (radius theorem for nonsingularity, ([8]) *For any $F : X \rightrightarrows Y$ that is positively homogeneous,*

$$\inf_{B\in\mathcal{L}(X,Y)} \{ \|B\| \mid F + B \textit{ singular } \} = \frac{1}{\|F^{-1}\|^{+}}. \tag{1.8}$$

Proof. The Banach lemma for nonsingularity gives us the inequality $\le$. Let $r > 1/\|F^{-1}\|^{+}$. There exists $(\hat{x}, \hat{y}) \in \operatorname{gph} F$ with $\|\hat{y}\| = 1$ and $\|\hat{x}\| > 1/r$. Let $x^* \in X^*$ satisfy $x^*(\hat{x}) = \|\hat{x}\|$ and $\|x^*\| = 1$. The linear and bounded mapping

$$B(x) = -\frac{x^*(x)\hat{y}}{\|\hat{x}\|}$$

satisfies $\|B\| = 1/\|\hat{x}\| < r$ and also $(F + B)(\hat{x}) = F(\hat{x}) - \hat{y} \ni 0$. Then the nonzero $\hat{x} \in (F + B)^{-1}(0)$ and hence $\|F + B\|^{+} = \infty$ which by definition means that $F + G$ is singular. Hence the infimum in (1.8) is less than r. Appealing to the choice of r we conclude that the infimum in (1.8) cannot be more than $1/\|F^{-1}\|^{+}$ and we are done. □

By using duality as outlined above, an Eckart-Young theorem for surjectivity of a sublinear mapping with closed graph can be obtained in a completely analogous format:

Theorem 1.9. (radius theorem for surjectivity, [17]) *Let $F : X \rightrightarrows Y$ be sublinear and with closed graph. Then*

$$\inf_{B\in\mathcal{L}(X,Y)} \{ \|B\| \,|\, F + B \textit{ not surjective } \} = \frac{1}{\|F^{-1}\|^{-}}.$$

In finite dimensions, a radius equality holds for metric regularity of a general set-valued mapping; the only requirement is that the graph of the mapping be closed near the reference point:

Theorem 1.10. (radius theorem for metric regularity, [8]) *For a mapping $F : X \rightrightarrows Y$ with $\dim X < \infty$ and $\dim Y < \infty$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed,*

$$\operatorname{rad} F(\bar{x}\,|\,\bar{y}) \;=\; \frac{1}{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}\;.$$

The proof in [8] uses the characterization of the metric regularity of a mapping through the nonsingularity of its coderivative and then applied the radius theorem for nonsingularity (Theorem 1.8). Ioffe [14] gave recently a direct proof of a sharper result without using generalized differentiation; in this paper he also derived a radius theorem in infinite dimensions when the mapping F is single-valued and the perturbations are of the class of locally lipschitz functions. In another paper [14] Ioffe also showed that in infinite dimensions, the radius equality does not hold, in general, for linear perturbations. On the bright side, a radius theorem was obtained in [19] for mappings acting from Asplund to finite-dimensional spaces. Such a theorem was also recently derived in [3] for a specific mapping describing a semi-infinite constraint system. The question of whether or not a radius theorem is valid for metric regularity of a general set-valued mappings under lipschitz perturbations in infinite dimensions remains open.

Is there a radius theorem valid when the perturbation g is a set-valued mapping having a suitable lipschitz-type property? The answer to this question turns out to be *no, in general,* as the following example shows:

Example 1.11. Consider $F : I\!R \rightrightarrows I\!R$ and $G : I\!R \rightrightarrows I\!R$ of the form

$$F(x) = \{-2x, 1\} \qquad \text{and} \qquad G(x) = \{x^2, -1\}, \quad x \in I\!R.$$

Then F is metrically regular while G is lipschitz continuous with respect to the Pompeiu-Hausdorff metric on the whole $I\!R$ and moreover, when restricted to a neighborhood of the origin, G is just a quadratic function. We have $\operatorname{reg} F(0|0) = 1/2$ while the lipschitz modulus of the single-valued localization of G at zero for zero is zero. The mapping

$$(F + G)(x) = \{x^2 - 2x, x^2 + 1, -2x - 1, 0\}, \quad x \in I\!R$$

is not metrically regular at the origin. Indeed, if (x, y) is close to zero and $y > 0$, then $(F + G)^{-1}(y) = 1 - \sqrt{1+y}$, so that $d(x, (F+G)^{-1}(y)) = |x - 1 + \sqrt{1+y}|$, but also $d(y, (F+G)(x)) = \min\{|x^2 - 2x - y|, y\}$. Take $x = \varepsilon > 0$ and $y = \varepsilon^2$. Then, since $(\varepsilon - 1 + \sqrt{1+\varepsilon^2})/\varepsilon^2 \to \infty$, the mapping $F + G$ is not metrically regular at zero for zero.

Thus, the radius of metric regularity under set-valued perturbation is zero, in general.

To complete the picture, in the recent paper [9] other regularity properties of set-valued mappings have been studied and radius theorems were obtained for some of these properties while for others it was shown that such results do not hold, in general. A radius theorem of a different type when the perturbations are restricted to positive definite symmetric linear mappings is given in [23].

In Section 2 we go a step further in the analysis of the effect of perturbations on metric regularity. By using the observation that the regularity modulus is the reciprocal of a distance function, we obtain a simple formula for the lipschitz modulus of the regularity modulus for a mapping which is subject to lipschitz perturbations.

Curiously, the statement of the radius theorem needs no more than the metric structure on the domain space suggesting the natural question of the theorem's validity for other metric spaces. A study of some analogous questions appears in [13]. In Section 3 we prove a radius theorem when the domain space is a Riemannian manifold. Our proof technique is conceptually simple. We make no recourse to recent work developing variational analysis on Riemannian manifolds [16]. Instead, we appeal to the Nash embedding theorem to work on a submanifold of a Euclidean space, endowed with geodesic distance, and then apply the basic radius theorem (Theorem 1.9).

2 Sensitivity of the regularity modulus

If a radius theorem holds for a mapping F, then its regularity modulus is the reciprocal to a distance function. In this section, we give a simple argument to show that in any reasonable metric space X, or even in a "pseudo-metric" space (where $d(x, y) = 0$ does not necessarily imply that $x = y$), the distance function d_S to a subset S of X satisfies

$$\operatorname{clm} d_S^{-1}(\bar{u}) = \operatorname{lip} d_S^{-1}(\bar{u}) = d_S^{-2}(\bar{u}). \tag{2.1}$$

By "reasonable", we just mean that the distance can be defined via the length of paths in the space: examples include all normed spaces and Riemannian manifolds.

An illuminating example is the absolute condition number of a matrix, which not only measures the changes of the solution due to perturbations of data but also indicates, by the Eckart-Young theorem, how far a nonsingular matrix is from the set of singular matrices. Hence, cond(A) is a reciprocal distance to which the equality (2.1) can be applied, obtaining

$$\operatorname{clm}(\operatorname{cond}(A)) = \limsup_{B \to 0} \frac{\|(A+B)^{-1}\| - \|A^{-1}\|}{\|B\|} = \|A^{-1}\|^2 = [\operatorname{cond}(A)]^2.$$

Inasmuch as cond(A) is the *normalized* calmness modulus of the inverse A^{-1} itself and, on the other hand, the larger the condition number is, the harder it is to invert a matrix, this equality illustrates the common situation in numerical analysis, see e.g. Demmel [6], that the condition number has the property of being no easier to compute than the solution itself.

Consider a set X equipped with a *pseudo-metric*, that is, a function $d : X \times X \to I\!R$ which obeys the symmetry condition $d(x, y) = d(y, x)$ and the triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$ and also $x = y$ yields $d(x, y) = 0$, but $d(x, y) = 0$ need not imply that $x = y$. For a given $\bar{x} \in I\!R^n$, such a space is the space of functions from $I\!R^n$ to $I\!R^m$ with $d(f, g) = \operatorname{clm}(f - g)(\bar{x})$. Clearly, we can consider the calmness and lipschitz moduli of functions acting in pseudo-metric spaces.

The pseudo-metric d is called *intrinsic* if the "distance" between any two points coincides with the infimum of the lengths of rectifiable paths between the points [1]. One can check that this is equivalent to the property that, given any two points $u, v \in X$ and any scalar $\varepsilon > 0$, there exists a third point $w \in X$ with the distances $d(u, w)$ and $d(w, v)$ both less than $\varepsilon + d(u, v)/2$. In fact, more generally, for any scalar λ in the interval $(0, 1)$, there then exists a point w satisfying

$$d(u, w)) < \lambda(d(u, v) + \varepsilon) \quad \text{and} \quad d(w, v) < (1 - \lambda)(d(u, v) + \varepsilon).$$

Spaces with an intrinsic metric are also known as "length spaces" or "path metric spaces". Clearly, in any normed space, the induced metric is intrinsic. The distance associated with any Riemannian manifold is intrinsic by definition [11].

Consider the space of functions from $I\!R^n$ to $I\!R^m$ equipped with the pseudo-metric $d(f, g) = \operatorname{lip}(f - g)(\bar{x})$ for some fixed $\bar{x} \in I\!R^n$. This pseudo-metric is intrinsic. Indeed, given functions u and v, from the very definition of the lipschitz modulus, the function $w = (u + v)/2$ satisfies the "path" condition required. The same conclusion of valid if the pseudo-metric is defined by the calmness modulus.

Given a pseudo-metric space (X, d), the distance function $d_S : X \to I\!R_+$ associated with a nonempty set $S \subset X$ is defined by

$$d_S(u) = \inf\{d(u, v) : v \in S\}.$$

We begin by collecting some well-known lipschitz properties of distance functions.

Proposition 2.1. *Given a pseudo-metric space (X, d) and a nonempty set $S \subset X$, the lipschitz and calmness moduli of the distance function d_S satisfy*

$$0 \leq \operatorname{clm} d_S(u) \leq \operatorname{lip} d_S(u) \leq 1 \quad \text{for all} \quad u \in X. \tag{2.2}$$

If the pseudo-metric d is intrinsic, then

$$\operatorname{clm} d_S(u) = \operatorname{lip} d_S(u) = 1 \quad \text{for all } u \text{ with } d_S(u) > 0. \tag{2.3}$$

Proof. The inequalities (2.2) are a straightforward application of the triangle inequality. Now suppose the metric d is intrinsic and $d_S(u) > 0$. For each integer $n = 1, 2, \ldots$, there exists a point $y_n \in S$ satisfying

$$d(y_n, u) < (1 + n^{-2}) d_S(u).$$

Since d is intrinsic, there exists a point $u_n \in X$ satisfying

$$\begin{aligned} d_S(u_n) \leq d(y_n, u_n) &< (1 - n^{-1})(1 + n^{-2}) d_S(u) \\ d(u_n, u) &< n^{-1}(1 + n^{-2}) d_S(u). \end{aligned}$$

Hence we deduce

$$\begin{aligned} \limsup_{v \to u} \frac{|d_S(v) - d_S(u)|}{d(v, u)} &\geq \limsup_{n \to \infty} \frac{d_S(u) - d_S(u_n)}{d(u, u_n)} \\ &\geq \limsup_{n \to \infty} \frac{d_S(u) - (1 - n^{-1})(1 + n^{-2}) d_S(u)}{n^{-1}(1 + n^{-2}) d_S(u)} \\ &= \limsup_{n \to \infty} \frac{n^2 - n + 1}{n^2 + n} \\ &= 1, \end{aligned}$$

so equation (2.3) now follows. ◻

In Euclidean spaces the above result can be refined in the following way:

Proposition 2.2. *Suppose that (X, d) is a Euclidean space (with the induced metric), and that $u \notin \operatorname{int} S$. Then*

$$\operatorname{lip} d_S(u) = 1, \tag{2.4}$$

and if, furthermore, S is Clarke regular at u, then

$$\operatorname{clm} d_S(u) = 1. \tag{2.5}$$

Proof. It suffices to prove equation (2.4) when the point u lies in the boundary of the set S, and in this case the normal cone $N_S(u)$ contains a unit vector y [22, Ex. 6.19]. We can measure the lipschitz modulus of the distance function via its subdifferential, by [22, Thm 9.13]:

$$\operatorname{lip} d_S(u) = \sup \Big\{ \|w\| : w \in \partial d_S(u) \Big\}.$$

Since, by [22, Ex. 8.53], we know $y \in \partial d_S(u)$, equation (2.4) now follows.

Suppose, in addition, that the set S is Clarke regular at the point u, and choose a unit normal vector $y \in N_S(u)$. By [22, Ex. 8.53], the subderivative of d_S at u in the direction y is the distance from y to the tangent cone $T_S(u)$, which by Clarke regularity equals 1. ☐

The equality (2.5) can fail in the absence of Clarke regularity. For example, the set

$$S = \{(x, y) \in \mathbb{R}^2 : |y| \geq x^2\}$$

has the origin in its boundary, and yet $\operatorname{clm} d_S(0,0) = 0$.

To move from distance functions to their reciprocals, the following chain rule is helpful.

Proposition 2.3. *Consider a pseudo-metric space X, a point $\bar{u} \in X$, a function $g : X \to \mathbb{R}$, and a function $f : \mathbb{R} \to [-\infty, +\infty]$.*

(i) *If the modulus of calmness* $\operatorname{clm} g(\bar{u})$ *is finite and f is differentiable at $g(\bar{u})$, then*

$$\operatorname{clm}(f \circ g)(\bar{u}) = |f'(g(\bar{u}))| \cdot \operatorname{clm} g(\bar{u}).$$

(ii) *If g is lipschitz around $\bar{u}$ and f is strictly differentiable at $g(\bar{u})$, then*

$$\operatorname{lip}(f \circ g)(\bar{u}) = |f'(g(\bar{u}))| \cdot \operatorname{lip} g(\bar{u}).$$

Proof. The differentiability of the function f implies that the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(y) = \begin{cases} \dfrac{f(y) - f(g(\bar{u}))}{y - g(\bar{u})} & (y \neq g(\bar{u})) \\[2ex] f'(g(\bar{u})) & (y = g(\bar{u})) \end{cases}$$

is continuous at $g(\bar{u})$. Since, for all points u with $d(u, \bar{u})$,

$$\frac{|f(g(u)) - f(g(\bar{u}))|}{d(u, \bar{u})} = |h(g(u))| \cdot \frac{|g(u) - g(\bar{u})|}{d(u, \bar{u})},$$

part (i) now follows by taking the $\limsup$ as $d(u, \bar{u}) \to 0$.

Part (ii) is similar. The strict differentiability of f implies that the function $p : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$p(y, z) = \begin{cases} \dfrac{f(y) - f(z)}{y - z} & (y \neq z) \\[2ex] f'(g(\bar{u})) & (y = z) \end{cases}$$

is continuous at the point $(g(\bar{u}), g(\bar{u}))$. Since, for all distinct points $u, v \in X$,

$$\frac{|f(g(u)) - f(g(v)|}{d(u, v)} = |p(g(u), g(v))| \cdot \frac{|g(u) - g(v)|}{d(u, v)},$$

part (ii) now follows by taking the $\limsup$ of each side as $d(u, \bar{u}) \to 0$. ☐

The following example shows that the differentiability assumptions in Proposition 2.3 cannot be relaxed to lipschitz continuity.

Example 2.4. Take $X = I\!R$ and f and g of the form

$$f(x) = \begin{cases} x & \text{for } x \le 0, \\ 2x & \text{for } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3x & \text{for } x \le 0, \\ x & \text{for } x > 0 \end{cases}.$$

Then

$$f(g(x)) = \begin{cases} 3x & \text{for } x \le 0, \\ 2x & \text{for } x > 0 \end{cases}$$

and we have $\operatorname{clm} f(g(0)) = \operatorname{clm} f(0) = 2$, $\operatorname{clm} g(0) = 3$; that is,

$$3 = \operatorname{clm}(f \circ g)(0) < \operatorname{clm} f(g(0)) \cdot \operatorname{clm} g(0) = 6.$$

The calmness and the lipschitz moduli are the same, hence the same strict inequality holds for clm replaced by lip. □

Our basic result about reciprocal distances now follows. An analogous result was recently announced independently in [4].

Theorem 2.5. *Consider a pseudo-metric space (X, d), a nonempty set $S \subset X$, and any point $\bar{u} \in X$ with $d_S(u) > 0$. Then the lipschitz and calmness moduli for the reciprocal distance satisfies*

$$\operatorname{clm} d_S^{-1}(\bar{u}) \le \operatorname{lip} d_S^{-1}(\bar{u}) \le d_S^{-2}(\bar{u}).$$

Equality holds throughout if the metric d is intrinsic (and hence in particular if X is a normed space or a Riemannian manifold).

Proof. Apply Proposition 2.3 with the function $f(x) = 1/x$ and use Proposition 2.1. □

We provide next several illustrations of the above result in the context of moduli of regularity of set-valued mappings:

Corollary 2.6. (conditioning of surjectivity) *Any closed sublinear surjective set-valued mapping between Banach spaces $F : X \rightrightarrows Y$ satisfies*

$$\operatorname{clm}(\|F^{-1}\|^{-})(F) = \limsup_{H \to 0 \text{ in } L(X,Y)} \frac{\left| \|(F+H)^{-1}\|^{-} - \|F^{-1}\|^{-} \right|}{\|H\|} = (\|F^{-1}\|^{-})^2.$$

Proof. We apply Theorem 2.5 to the metric space $U = L(X, Y)$ and the set

$$S = \{ H \in L(X, Y) \mid F + H \text{ is not surjective} \}.$$

From the radius theorem for surjectivity (Theorem 1.9), we have $d_S^{-1}(0) = \|F^{-1}\|^{-}$, and hence, more generally, for any map $H \in L(X, Y)$, we obtain

$$d_S^{-1}(H) = \|(F + H)^{-1}\|^{-}.$$

The result now follows by direct substitution. □

A parallel result holds for the conditioning of injectivity for a positively homogeneous mapping, where the inner norm is replaced by the outer norm. For variety, we give an expression for the lipschitz modulus of the outer norm:

$$\mathrm{lip}(\|F^{-1}\|^{+})(F) = \limsup_{H,H'\to 0 \text{ in } L(X,Y)} \frac{\left|\|(F+H)^{-1}\|^{+} - \|(F+H')^{-1}\|^{+}\right|}{\|H-H'\|} = (\|F^{-1}\|^{+})^2.$$

Turning to the more general, inhomogeneous setting, we have the following result.

Theorem 2.7. (conditioning of regularity modulus) *Consider a set-valued mapping $F : X \rightrightarrows Y$ between finite dimensional metric spaces which is metrically regular at $\bar{x}$ for $\bar{y}$ and let* $\mathrm{gph}\, F$ *be locally closed at $(\bar{x}, \bar{y})$. Then*

$$\mathrm{clm}(\mathrm{reg}\, F(\bar{x}\,|\,\bar{y}))(F) = \limsup_{\substack{\mathrm{lip}\, g(\bar{x})\to 0\\ g(\bar{x})=0}} \frac{\left|\mathrm{reg}(F+g)(\bar{x}\,|\,\bar{y}) - \mathrm{reg}\, F(\bar{x}\,|\,\bar{y})\right|}{\mathrm{lip}\, g(\bar{x})} = (\mathrm{reg}\, F(\bar{x}|\bar{y}))^2.$$

Proof. We again apply Theorem 2.4 to the pseudo-metric space of functions $g : I\!R^n \to I\!R^m$ with $g(\bar{x}) = 0$ with the pseudo-metric $d(u, v) = \mathrm{lip}(u - v)(\bar{x})$ and the set S given by

$$\{\, g : I\!R^n \to I\!R^m, g(\bar{x}) = 0 \mid F + g \;\text{ not metrically regular at } \bar{x} \text{ for } \bar{y}\}.$$

According to Theorem 1.10, $d_S(0) = [\mathrm{reg}\, F(\bar{x}|\bar{y})]^{-1}$. Hence, more generally, for any function $g : I\!R^n \to I\!R^m$ satisfying $g(\bar{x}) = 0$ we have

$$d_S^{-1}(g) = \mathrm{reg}(F + g)(\bar{x}|\bar{y}).$$

The result now follows. □

3 A Radius Theorem on Riemannian manifold

A Riemannian manifold M is, briefly, a smooth (C^∞) manifold associated with a smoothly varying inner product on each tangent space. The distance $d_M(m_1, m_2)$ between two points $m_1, m_2 \in M$ is the infimum of the lengths of connecting paths: if M is connected, then it becomes a metric space with this induced "intrinsic" metric. A good basic reference is [2].

For our present purpose, we are interested only in the metric structure of the manifold M. We may therefore appeal to the Nash embedding theorem [20], which shows that, as a metric space, M is isometric to a smooth submanifold of a Euclidean space E with the induced intrinsic metric. We denote the (smooth) isometry by $\iota : M \to E$ and the metric induced by the norm in E by d_E. (In fact, since we are only interested in local properties of the manifold, we only need the easier, local version of the embedding result.)

Consider a point $\bar{m} \in M$. Elementary properties of submanifolds of E ensure that the projection $P_{\iota M} : E \to E$ is smooth on a neighborhood of $\bar{m}$, with derivative at $\bar{m}$ just the projection onto the tangent space to ιM at $\bar{m}$. Since this projection therefore has norm one, we immediately deduce

$$\mathrm{lip}\, P_{\iota M}(\bar{m}) = 1. \tag{3.1}$$

Proposition 3.1. (comparing metrics) *Near any point $\bar{m}$ in the Riemannian manifold M, the intrinsic distance d_M and the distance d_E on the embedding ιM are related by*

$$\lim_{\substack{m_1, m_2 \to \bar{m} \\ m_1 \neq m_2}} \frac{d_M(m_1, m_2)}{d_E(\iota m_1, \iota m_2)} = 1.$$

Consequently, both the embedding $\iota : M \to E$ and its inverse $\iota^{-1} : \iota M \to M$ have Lipschitz modulus 1 (at $\bar{m}$ and $\iota\bar{m}$ respectively).

Proof. By the isometry, $d_M(m_1, m_2)$ is the infimum of the Euclidean length of paths between ιm_1 and ιm_2 in the submanifold $\iota M \subset E$. These lengths are bounded below by $\|\iota m_1 - \iota m_2\|$. On the other hand, since the projection $P_{\iota M}$ is smooth near $\iota\bar{m}$ with $\|\nabla P_{\iota M}(\iota\bar{m})\| = 1$ given any $\varepsilon > 0$, for all points $e \in E$ close to ιm we have $\|\nabla P_{\iota M}(e)\| < 1 + \varepsilon$. Hence for points $m_1, m_2 \in M$ near $\bar{m}$, as the scalar t ranges from 0 to 1, the point $P_{\iota M}(\iota m_1 + t(\iota m_2 - \iota m_1))$ traces a smooth path from ιm_1 to ιm_2 with length

$$\begin{aligned}
&\int_0^1 \Big\| \frac{d}{dt} P_{\iota M}(\iota m_1 + t(\iota m_2 - \iota m_1)) \Big\| \, dt \\
&\qquad = \int_0^1 \Big\| \nabla P_{\iota M}(\iota m_1 + t(\iota m_2 - \iota m_1))(\iota m_2 - \iota m_1) \Big\| \, dt \\
&\qquad \leq \int_0^1 \Big\| \nabla P_{\iota M}(\iota m_1 + t(\iota m_2 - \iota m_1)) \Big\| \cdot \|\iota m_2 - \iota m_1\| \, dt \\
&\qquad \leq (1 + \varepsilon)\|\iota m_2 - \iota m_1\|.
\end{aligned}$$

The result now follows. □

We next extend this result to distances to sets.

Proposition 3.2. (comparing distances to sets) *Consider any sequence of sets $F_r \subset M$ (for $r = 1, 2, \ldots$). As $r \to \infty$, the following conditions are equivalent:*

(i) $d_M(\bar{m}, F_r) \to 0$;

(ii) $d_E(\iota\bar{m}, \iota F_r) \to 0$.

Assuming these conditions hold, consider any sequence of points $m_r \to \bar{m}$ in M with $m_r \notin F_r$ for each r. Then

$$\frac{d_M(m_r, F_r)}{d_E(\iota m_r, \iota F_r)} \to 1.$$

Proof. If condition (i) holds, then there exists a sequence of points $m_r \in F_r$ such that $d_M(\bar{m}, m_r) \to 0$. Proposition 3.1 (comparing metrics) shows $d_E(\iota\bar{m}, \iota m_r) \to 0$, so condition (ii) follows. The converse is analogous.

For the last claim, there exists sequences $u_r \in F_r$ and $v_r \in \iota F_r$ satisfying

$$\begin{aligned}
d_M(m_r, u_r) &< (1 + r^{-1}) d_M(m_r, F_r) \\
d_E(\iota m_r, v_r) &< (1 + r^{-1}) d_E(\iota m_r, \iota F_r).
\end{aligned}$$

Consequently we have the inequalities

$$\frac{(1+r^{-1})^{-1}d_M(m_r,u_r)}{d_E(\iota m_r,\iota u_r)} \quad < \quad \frac{d_M(m_r,F_r)}{d_E(\iota m_r,\iota F_r)} \quad < \quad \frac{d_M(m_r,\iota^{-1}v_r)}{(1+r^{-1})^{-1}d_E(\iota m_r,v_r)}.$$

The claim now follows, since both left and right hand sides approach 1 as $r \to \infty$, by Proposition 3.1. $\square$

Consider a connected Riemannian manifold M, a finite-dimensional normed space Z, and a set-valued mapping $F : M \rightrightarrows Z$. Given points $\bar{m} \in M$ and $\bar{z} \in F(\bar{m})$, the regularity modulus of F at $\bar{m}$ for $\bar{z}$ is the infimum of all $\kappa > 0$ such that

$$d_M(m, F^{-1}(z)) \le \kappa d_Z(z, F(m)) \text{ for all } (m,z) \text{ close to } (\bar{m},\bar{z}). \tag{3.2}$$

As above, we consider an isometric embedding $\iota : M \to E$ of M into a Euclidean space E.

Our next step in deriving a radius theorem is to reinterpret the regularity modulus in terms of the embedding.

Lemma 3.3. (regularity after embedding) *The regularity modulus of F at $\bar{m}$ for $\bar{z}$ is the infimum of all $\kappa > 0$ such that*

$$d_E(e, \iota(F^{-1}(z))) \le \kappa d_Z(z, F(\iota^{-1}e)) \textit{ for all } (e,z) \textit{ close to } (\iota\bar{m},\bar{z}) \textit{ with } e \in \iota M. \tag{3.3}$$

Proof. Denote the above infimum by $\bar{\kappa}$, and let $\hat{\kappa} = \operatorname{reg} F(\bar{m}|\bar{z})$. Since any two points $m_1, m_2 \in M$ satisfy

$$d_M(m_1,m_2) \ge d_E(\iota m_1, \iota m_2),$$

property (3.2) implies property (3.3), so we deduce $\hat{\kappa} \ge \bar{\kappa}$. If $\bar{\kappa} = +\infty$, then there is nothing more to prove, so suppose $\bar{\kappa} < +\infty$, and $\hat{\kappa} > \bar{\kappa}$: we shall derive a contradiction.

Choose real κ', κ'' satisfying $\bar{\kappa} < \kappa' < \kappa'' < \hat{\kappa}$. By the definition of $\hat{\kappa}$, there exist sequences of points $m_r \to \bar{m}$ in the manifold M and $z_r \to \bar{z}$ in the space Z satisfying

$$d_M(m_r, F^{-1}(z_r)) > \kappa'' d_Z(z_r, F(m_r)) \tag{3.4}$$

(and hence in particular $z_r \notin F(m_r)$) for all $r = 1, 2, \ldots$. By the definition of $\bar{\kappa}$,

$$d_E(\iota\bar{m}, \iota(F^{-1}(z_r))) < \kappa' d_Z(z_r, F(\bar{m})) \tag{3.5}$$

and

$$d_E(\iota m_r, \iota(F^{-1}(z_r))) < \kappa' d_Z(z_r, F(m_r)) \tag{3.6}$$

for all large r. Since $z_r \to \bar{z} \in F(\bar{m})$, the right hand side of inequality (3.5) approaches zero, and hence so does the left. We can now apply Proposition 3.2 (comparing distances to sets) to deduce

$$\frac{d_M(m_r, F^{-1}(z_r))}{d_E(\iota m_r, \iota F^{-1}(z_r))} \to 1.$$

But inequalities (3.4) and (3.6) imply, for all large r, that the left hand side is at least $\kappa''/\kappa' > 1$, which is the desired contradiction. $\square$

In the above result, the inverse of the embedding ι was applied only at points in the embedded submanifold $\iota M \subset E$: since ι is injective, its inverse is a well-defined function on the submanifold. However, we can also interpret ι as a set-valued mapping, in which case its inverse $\iota^{-1} : E \to M$ is given by

$$\iota^{-1}(e) = \begin{cases} \{m\}, & \text{if } \iota m = e \\ \emptyset, & \text{if } e \notin \iota M. \end{cases}$$

With this notation, we can reinterpret Lemma 3.3 (regularity after embedding) as evaluating the regularity modulus of the mapping $F \circ \iota^{-1}$, as follows.

Lemma 3.4. (regularity after composition)

$$\operatorname{reg} F(\bar{m}|\bar{z}) = \operatorname{reg}(F \circ \iota^{-1})(\iota\bar{m}|\bar{z}). \tag{3.7}$$

Proof. The right hand side is infimum of all $\kappa > 0$ such that

$$d_E(e, \iota(F^{-1}(z))) \leq \kappa d_Z(z, F(\iota^{-1}e)) \text{ for all } (e,z) \text{ close to } (\iota\bar{m}, \bar{z}).$$

But for points $e \notin \iota M$, we have $F(\iota^{-1}e) = \emptyset$ and hence $d_Z(z, F(\iota^{-1}e)) = +\infty$, so the condition above holds automatically for all such e. Hence this condition is equivalent to condition (3.3), and the result now follows by Lemma 3.3. □

To study the radius of regularity, it is helpful first to compare Lipschitz functions on the manifold M and its embedding ιM.

Lemma 3.5. (comparing Lipschitz moduli I) *Consider any function $g : E \to Z$ satisfying $g(\iota\bar{m}) = 0$. Then*

$$\begin{aligned} \operatorname{lip}(g \circ \iota)(\bar{m}) &\leq \operatorname{lip} g(\iota\bar{m}) \quad \textit{and} \\ \operatorname{reg}(F + (g \circ \iota))(\bar{m}|\bar{z}) &= \operatorname{reg}((F \circ \iota^{-1}) + g)(\iota\bar{m}|\bar{z}). \end{aligned}$$

Proof. The first inequality follows from Proposition 3.1 (comparing metrics). The second follows from the observation

$$(F + (g \circ \iota)) \circ \iota^{-1} = (F \circ \iota^{-1}) + g,$$

after applying Lemma 3.4 (regularity after composition). □

Lemma 3.6. (comparing Lipschitz moduli II) *Given any function $h : M \to Z$ satisfying $h(\bar{m}) = 0$, the function $h \circ \iota^{-1} \circ P_M$ is single-valued near $\iota\bar{m}$, and*

$$\begin{aligned} \operatorname{lip}(h \circ \iota^{-1} \circ P_M)(\iota\bar{m}) &= \operatorname{lip} h(\bar{m}) \quad \textit{and} \\ \operatorname{reg}((F \circ \iota^{-1}) + (h \circ \iota^{-1} \circ P_M))(\iota\bar{m}|\bar{z}) &= \operatorname{reg}(F + h)(\bar{m}|\bar{z}). \end{aligned}$$

Proof. Observe

$$\operatorname{lip}(h \circ \iota^{-1} \circ P_M)(\iota\bar{m}) \leq \operatorname{lip} h(\bar{m}) \cdot \operatorname{lip}(\iota^{-1})(\iota\bar{m}) \cdot \operatorname{lip} P_M(\iota\bar{m}) = \operatorname{lip} h(\bar{m}),$$

by equation (3.1) and Proposition 3.1 (comparing metrics). On the other hand,

$$h \circ \iota^{-1} \circ P_M \circ \iota = h,$$

so, again by Proposition 3.1,

$$\operatorname{lip} h(\bar{m}) \leq \operatorname{lip}(h \circ \iota^{-1} \circ P_M)(\iota\bar{m}) \cdot \operatorname{lip} \iota(\bar{m}) = \operatorname{lip}(h \circ \iota^{-1} \circ P_M)(\iota\bar{m}),$$

and the first equation now follows. The second follows from the observation

$$(F + h) \circ \iota^{-1} = (F \circ \iota^{-1}) + (h \circ \iota^{-1} \circ P_M),$$

after applying Lemma 3.4 (regularity after composition). □

Lemma 3.7. (radius after composition)

$$\operatorname{rad} F(\bar{m}|\bar{z}) = \operatorname{rad}(F \circ \iota^{-1})(\iota\bar{m}|\bar{z}).$$

Furthermore, if the infimum defining either radius of regularity is attained, then so is the other.

Proof. Consider any real $\gamma > \operatorname{rad}(F \circ \iota^{-1})(\iota\bar{m}|\bar{z})$. By definition, there exists a function $g : E \to Z$ satisfying $g(\iota\bar{m}) = 0$ and $\operatorname{lip} g(\iota\bar{m}) < \gamma$, and such that the mapping $(F \circ \iota) + g$ is not metrically regular at $\iota\bar{m}$ for $\bar{z}$. By Lemma 3.5 (comparing Lipschitz moduli I), the mapping $F + (g \circ \iota)$ is not metrically regular at $\bar{m}$ for $\bar{z}$, and $\operatorname{lip}(g \circ \iota)(\bar{m}) < \gamma$, so $\operatorname{rad} F(\bar{m}|\bar{z}) < \gamma$.

On the other hand, consider any real $\gamma > \operatorname{rad} F(\bar{m}|\bar{z})$. By definition, there exists a function $h : M \to Z$ satisfying $h(\bar{m}) = 0$ and $\operatorname{lip} h(\bar{m}) < \gamma$, and such that the mapping $F + h$ is not metrically regular at $\bar{m}$ for $\bar{z}$. We now apply Lemma 3.6 (comparing Lipschitz moduli II). The mapping $h \circ \iota^{-1} \circ P_M$ is single-valued near the point $\iota\bar{m}$, with

$$\operatorname{lip}(h \circ \iota^{-1} \circ P_M)(\iota\bar{m}) = \operatorname{lip} h(\bar{m}) < \gamma,$$

and the mapping $(F \circ \iota^{-1}) + (h \circ \iota^{-1} \circ P_M)$ is not metrically regular at $\iota\bar{m}$ for $\bar{z}$. A standard construction [22] gives a Lipschitz function $g : E \to Z$ agreeing with $h \circ \iota^{-1} \circ P_M$ near $\iota\bar{m}$, so now we have $g(\iota\bar{m}) = h(\bar{m}) = 0$ and $\operatorname{lip} g(\iota\bar{m}) < \gamma$, and furthermore, $(F \circ \iota^{-1}) + g$ is not metrically regular at $\iota\bar{m}$ for $\bar{z}$. Hence $\operatorname{rad}(F \circ \iota^{-1})(\iota\bar{m}|\bar{z}) < \gamma$.

The claimed equality of radii now follows. Furthermore, the constructions in the preceding two paragraphs show how attainment in either radius definition implies attainment in the other. □

The main result of this section now follows quickly.

Radius theorem on manifolds. *If M is a Riemannian manifold, Z is a finite-dimensional normed space, consider a mapping $F : M \rightrightarrows Z$. Then for any $(\bar{m}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed,*

$$\operatorname{rad} F(\bar{m}\,|\,\bar{y}) \;=\; \frac{1}{\operatorname{reg} F(\bar{m}\,|\,\bar{y})}\,. \tag{3.8}$$

Moreover, the calmness modulus of the regularity modulus with respect to lipschitz additive perturbations is

$$\operatorname{clm}(\operatorname{reg} F(\bar{x}\,|\,\bar{y}))(F) = (\operatorname{reg} F(\bar{x}\,|\,\bar{y}))^2. \tag{3.9}$$

Proof. The equality (3.8) follows immediately from Lemma 3.4 (regularity after composition), Lemma 3.7 (radius after composition), and the Theorem 1.10 (radius theorem). To obtain (3.9) we use Theorem 2.7 (conditioning of the regularity modulus) and Lemmas 3.4 again. □

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