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Set Intersection Theorems and Existence of Optimal Solutions [1]

by

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Abstract

The question of nonemptiness of the intersection of a nested sequence of closed sets is fundamental in a number of important optimization topics, including the existence of optimal solutions, the validity of the minimax inequality in zero sum games, and the absence of a duality gap in constrained optimization. We introduce the new notion of an asymptotic direction of a sequence of closed sets, and the associated notions of retractive, horizon, and critical directions, and we provide several conditions that guarantee the nonemptiness of the corresponding intersection. We show how these conditions can be used to obtain simple proofs of some known results on existence of optimal solutions, and to derive some new results, including an extension of the Frank-Wolfe Theorem for (nonconvex) quadratic programming.

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1. INTRODUCTION

In this paper, we focus on the question whether a set intersection $\cap_{k=0}^{\infty} S_k$ is nonempty, where $\{S_k\}$ is a sequence of nonempty closed sets in $\Re^n$ with $S_{k+1} \subset S_k$ for all k. This is a fundamental issue in optimization, because it lies at the heart of a number of important questions, such as the following:

(a) Does a function $f : \Re^n \mapsto (-\infty, \infty]$ attain a minimum over a set X? This is true if and only if the intersection

$$\cap_{k=0}^{\infty} \big\{x \in X \mid f(x) \leq \gamma_k\big\}$$

is nonempty, where $\{\gamma_k\}$ is a scalar sequence with $\gamma_k \downarrow \inf_{x \in X} f(x)$.

(b) If C is a closed set and A is a matrix, is AC closed? To prove this, we may let $\{y_k\}$ be a sequence in AC that converges to some $\overline{y} \in \Re^n$, and then show that $\overline{y} \in AC$. If we introduce the sets

$$W_k = \big\{z \mid \|z - \overline{y}\| \leq \|y_k - \overline{y}\|\big\}, \qquad N_k = \{x \mid Ax \in W_k\},$$

and

$$S_k = C \cap N_k,$$

it is sufficient to show that the intersection $\cap_{k=0}^{\infty} S_k$ is nonempty.

(c) Given a function $F : \Re^{n+m} \mapsto (-\infty, \infty]$ that is closed (i.e., has a closed epigraph), is the function $f : \Re^n \mapsto [-\infty, \infty]$ defined by $f(x) = \inf_{z \in \Re^m} F(x, z)$ closed? It is known that this is a critical question in duality theory and minimax theory (see, e.g., [AuT03], [BNO03], [Roc70]). Properties of epi(f), the epigraph of f, can be inferred from properties of epi(F), the epigraph of F, by using the relation

$$P\big(\text{epi}(F)\big) \subset \text{epi}(f) \subset \text{cl}\Big(P\big(\text{epi}(F)\big)\Big),$$

where $\text{cl}(\cdot)$ denotes closure of a set and $P(\cdot)$ denotes projection on the space of (x, w), i.e., $P(x, z, w) = (x, w)$. [The left-hand side of this relation follows from the definition

$$\text{epi}(f) = \left\{(x, w) \;\Big|\; \inf_{z \in \Re^m} F(x, z) \leq w \right\}.$$

To show the right-hand side, note that for any $(x, w) \in \text{epi}(f)$ and every k, there exists a z_k such that $(x, z_k, w + 1/k) \in \text{epi}(F)$, so that $(x, w + 1/k) \in P\big(\text{epi}(F)\big)$ and $(x, w) \in \text{cl}\Big(P\big(\text{epi}(F)\big)\Big)$.] If F is closed and we can show that the projection $P(\cdot)$ preserves closedness [a special case of question (b) above], it follows that epi(f) is closed and f is closed.

If the sets S_k are compact, then $\cap_{k=0}^{\infty} S_k$ is nonempty and compact, a fact that underlies Weierstrass' Theorem (a closed function f attains a minimum over a compact set X); see the reasoning in (a) above. The special case where the sets S_k are convex has been the subject of much research, following the work of Helly [Hel21] and others (e.g., Fenchel [Fen51] and Rockafellar [Roc70]). A recent line of analysis that focuses on the issues (a)-(c) discussed above, is given in Section 1.5 of Bertsekas, Nedić, and Ozdaglar [BNO03], and is based on the notions of a direction of recession and lineality space. In this paper, we develop conditions that guarantee the nonemptiness of the intersection $\cap_{k=0}^{\infty} S_k$ in the general case where the sets S_k may not be bounded and may not be convex.

Our analysis is based on an extension of the notion of a direction of recession, the notion of an *asymptotic direction of the set sequence* $\{S_k\}$. This and the related notion of a *retractive direction* (see Section 2) are new in the form given here, but are closely related to ideas developed, principally within the context of optimization, by Auslender; see, for example, [Aus96], [Aus97], [Aus00], and the book by Auslender and Teboulle [AuT03]. These sources focus on asymptotic directions of sets and functions, rather than sequences of sets. It appears that our notion of asymptotic direction of a sequence of sets (rather than a set or a function) is simpler and often more convenient for the intended optimization applications. We also develop the notions of a *horizon* and *critical* directions, which are formulated here for the first time, for both cases of a single set and a sequence of sets. We show that the notions of asymptotic, retractive, horizon, and critical directions provide the basis for new set intersection theorems, new existence of optimal solutions results, and simpler proofs of known theorems.

We note that in the case where the sets S_k are convex, as well as closed, the set of asymptotic directions of $\{S_k\}$ is in effect the intersection of the recession cones of the sets S_k, and the set of retractive directions is related (but is not equal) to the intersection of the lineality spaces of the sets S_k. A horizon direction of $\{S_k\}$ is also somewhat related to common directions of recession of the sets S_k (see the discussion of Section 3).

We note also that the set of asymptotic directions, when specialized to a closed, possibly nonconvex, set (rather than a nested sequence of closed sets), is essentially the horizon cone described by Rockafellar and Wets [RoW98], and the asymptotic cone described by Auslender and Teboulle [AuT03]. These cones have been introduced in the works of Dedieu [Ded77], [Ded79], and have been the subject of considerable attention recently; see the references in [AuT03] and [RoW98].

We organize the material as follows. In Section 2, we introduce asymptotic directions, we develop some of their properties, and we prove our central results relating to the nonemptiness of a closed set intersection. In Section 3, we introduce horizon and critical directions, and we use them to derive additional set intersection theorems. Some of these theorems relate to level sets of bidirectionally flat functions, a class that includes convex quadratic and, more generally, convex polynomial functions. Finally, in Section 4 we extend some known results on the existence of optimal solutions, including a generalization of the Frank-Wolfe Theorem of (nonconvex) quadratic programming. While we do not discuss in this paper the application of our set intersection results to questions of preservation of closedness under linear transformation and partial minimization, our results can also be used for the analysis of these issues, as discussed earlier.

Throughout the paper, all analysis is done in the n-dimensional Euclidean $\Re^n$. Thus, unless otherwise specified, vectors and subsets are from $\Re^n$. All vectors are viewed as columns vectors, and a prime denotes transpose. The standard Euclidean norm, $\|x\| = \sqrt{x'x}$ is used throughout.

2. ASYMPTOTIC DIRECTIONS

We first introduce the basic notion of this paper in the following definition. A set sequence $\{S_k\}$ such that $S_{k+1} \subset S_k$ for all k, will be referred to as being *nested*.

Definition 2.1: Let $\{S_k\}$ be a nested sequence of nonempty closed sets. We say that a vector d is an *asymptotic direction of* $\{S_k\}$ if there exists a sequence $\{x_k\}$ such that

$$x_k \in S_k, \qquad k = 0, 1, \ldots,$$

and

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}.$$

A sequence $\{x_k\}$ associated with an asymptotic direction d as above is called an *asymptotic sequence corresponding to* d. An asymptotic direction d of $\{S_k\}$ is called *retractive* if, for every corresponding asymptotic sequence $\{x_k\}$ and every scalar $\overline{\alpha} > 0$, there exists an integer $\overline{k}$ such that

$$x_k - \alpha d \in S_k, \qquad \forall\ \alpha \in (0, \overline{\alpha}],\ k \geq \overline{k}.$$

The set sequence $\{S_k\}$ is called *retractive* if all its asymptotic directions are retractive.

Roughly speaking, an asymptotic direction is a direction along which we can escape towards ∞ through each of the sets S_k (see Fig. 2.1). In particular, $\{S_k\}$ *has an asymptotic direction if and only if all the sets S_k are unbounded*. An alternative and equivalent definition is that a

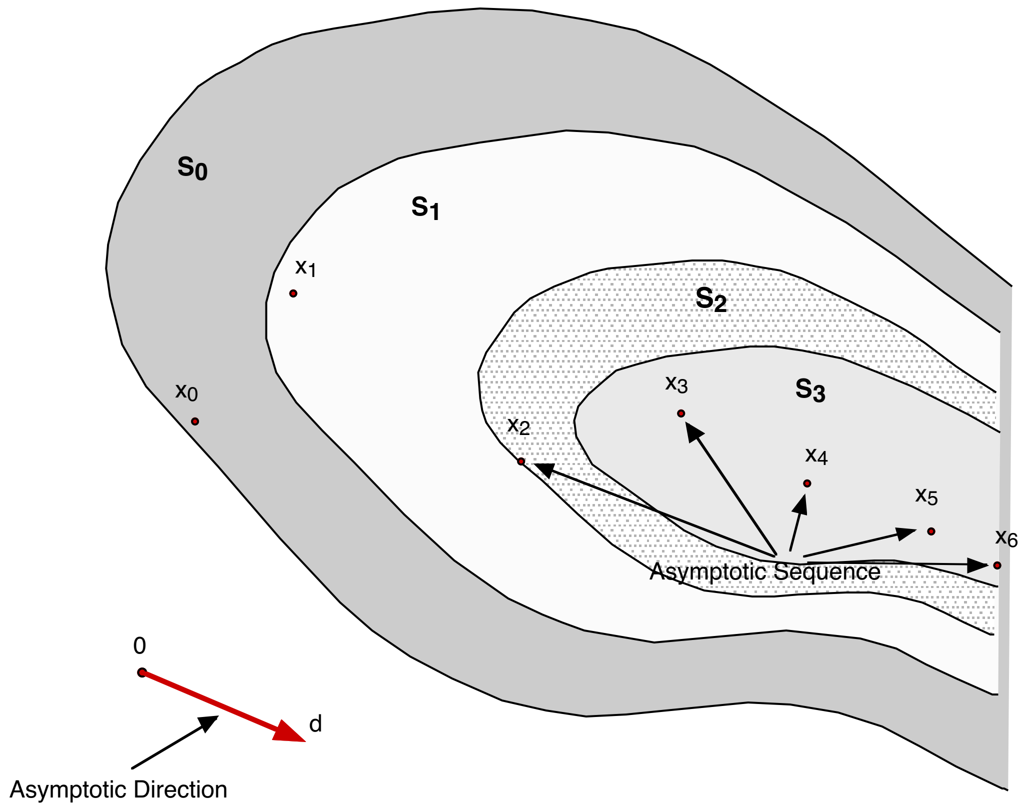

Figure 2.1. Illustration of an asymptotic direction of a sequence of nonconvex sets and a corresponding asymptotic sequence. The normalized direction sequence $x_k/\|x_k\|$ must converge to the normalized direction $d/\|d\|$.

vector $d \neq 0$ is an asymptotic direction of $\{S_k\}$ if there exists a sequence $\{x_k\}$ and a positive sequence $\{t_k\}$ with $x_k \in S_k$ for all k, $t_k \to \infty$, and

$$\frac{x_k}{t_k} \to d.$$

A retractive asymptotic direction d is one whose asymptotic sequences still belong to the corresponding sets S_k *when shifted in the opposite direction* $-d$ by any step α in some bounded interval of positive numbers (see Fig. 2.2). The importance of the notion of a retractive direction may not be apparent at first sight, but is motivated by the following proposition, a key result of this paper. The proof uses a minimum norm vector x_k from each set S_k, and involves two ideas:

(a) The intersection $\cap_{k=0}^{\infty} S_k$ is empty if and only if there is an unbounded sequence $\{x_k\}$ consisting of minimum norm vectors from the sets S_k.

(b) An asymptotic sequence $\{x_k\}$ consisting of minimum norm vectors from the sets S_k cannot correspond to a retractive direction, because such a sequence eventually (for large k) gets

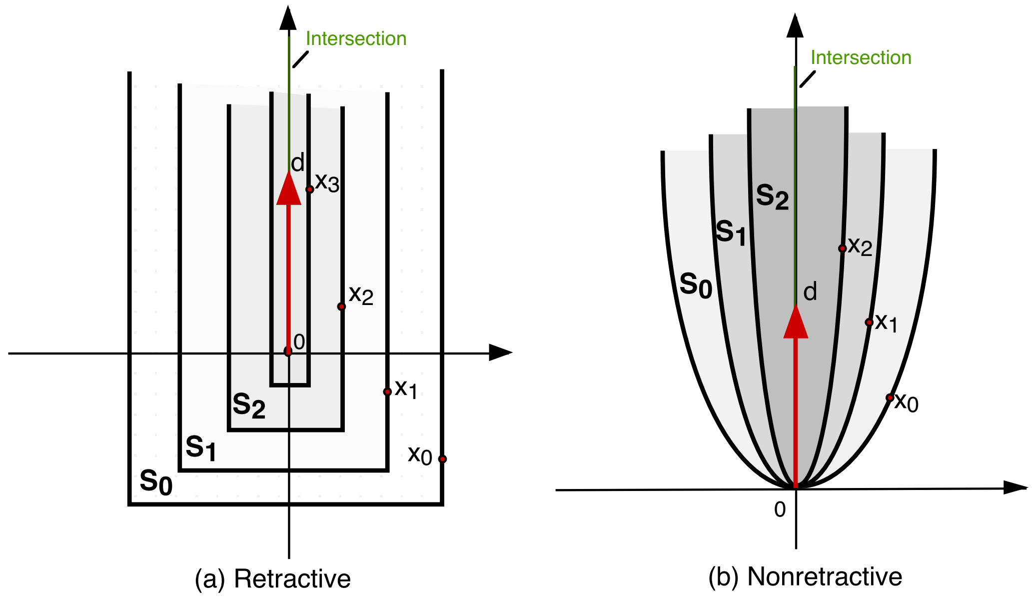

Figure 2.2. Illustration of retractive and nonretractive directions in $\Re^2$. In both cases the set of asymptotic directions is $\big\{(0,\beta) \mid \beta > 0\big\}$, and the intersection of the corresponding sequence is the set $\big\{(0,x_2) \mid x_2 \geq 0\big\}$. In (a), we have

$$S_k = \left\{(x_1,x_2) \;\Big|\; |x_1| \leq \frac{1}{k+1},\ x_2 \geq -\frac{1}{k+1}\right\},$$

and it can be seen that every asymptotic direction is retractive. In (b), we have

$$S_k = \left\{(x_1,x_2) \;\big|\; x_2 \geq (k+1)x_1^2\right\},$$

and it can be seen that all asymptotic directions are not retractive. As an example, for the asymptotic direction $(0,1)$, the corresponding asymptotic sequence $\big\{(k,(k+1)k^2)\big\}$ does not belong to S_k when shifted in the opposite direction $(0,-1)$.

closer to 0 when shifted in the opposite of the corresponding asymptotic direction d (see Fig. 2.3).

Proposition 2.1: A retractive nested sequence of nonempty closed sets has nonempty intersection.

Proof: Let $\{S_k\}$ be the given sequence. For each k, let x_k be a vector of minimum norm on the closed set S_k (such a vector exists by Weierstrass' theorem, since it can be obtained by minimizing $\|x\|$ over all x in the compact set $S_k \cap \{x \mid \|x\| \leq \|\overline{x}_k\|\}$, where $\overline{x}_k$ is any vector in S_k). It will be sufficient to show that a subsequence $\{x_k\}_{k\in\mathcal{K}}$ is bounded. Then, since $\{S_k\}$ is

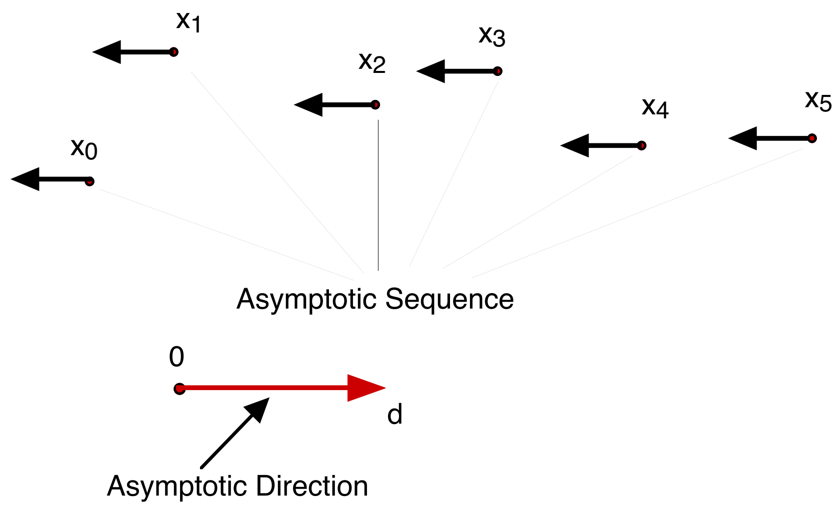

Figure 2.3. Geometric interpretation of the proof idea of Prop. 2.1. An asymptotic sequence $\{x_k\}$ corresponding to an asymptotic direction d eventually (for large k) gets closer to 0 when shifted in the opposite direction $-d$, so such a sequence cannot consist of vectors of minimum norm from S_k without contradicting the retractiveness assumption.

nested, for each m, we have $x_k \in S_m$ for all $k \in \mathcal{K}$, $k \geq m$, and since S_m is closed, each of the limit points of $\{x_k\}_{k\in\mathcal{K}}$ will belong to each S_m and hence also to $\cap_{m=0}^{\infty} S_m$, thereby showing the result. Thus, we will prove the proposition by showing that there is no subsequence of $\{x_k\}$ that is unbounded.

Indeed, assume the contrary, let $\{x_k\}_{k\in\mathcal{K}}$ be a subsequence such that

$$\lim_{k\to\infty,\, k\in\mathcal{K}} \|x_k\| = \infty,$$

and let d be the limit of a subsequence $\big\{x_k/\|x_k\|\big\}_{k\in\overline{\mathcal{K}}}$, where $\overline{\mathcal{K}} \subset \mathcal{K}$. For each $k = 0, 1, \ldots,$ define $y_k = x_m$, where m is the smallest index $m \in \overline{\mathcal{K}}$ with $k \leq m$. Then, since $y_k \in S_k$ for all k and $\lim_{k\to\infty}\{y_k/\|y_k\|\} = d$, we see that d is an asymptotic direction of $\{S_k\}$ and $\{y_k\}$ is an asymptotic sequence corresponding to d. Using the retractiveness assumption, let $\overline{\alpha} > 0$ and $\overline{k}$ be such that $y_k - \alpha d \in S_k$ for all $\alpha \in (0, \overline{\alpha}]$ and $k \geq \overline{k}$. We have $d'y_k \to \infty$, since $\|d\| = 1$ and $d'y_k/\|y_k\| \to 1$, so for all $k \geq \overline{k}$ with $2d'y_k > \overline{\alpha}$, we obtain

$$\|y_k - \overline{\alpha} d\|^2 = \|y_k\|^2 - \overline{\alpha}(2d'y_k - \overline{\alpha}) < \|y_k\|^2.$$

This is a contradiction, since for infinitely many k, y_k is the vector of minimum norm on S_k. **Q.E.D.**

For an example where the above proposition applies, consider the sequence $\{S_k\}$ of Fig. 2.2(a). Here the asymptotic directions, $(0, \beta)$, $\beta > 0$, are retractive, and indeed the intersection

$\cap_{k=0}^{\infty} S_k$ is nonempty. On the other hand, the condition for nonemptiness of $\cap_{k=0}^{\infty} S_k$ of the proposition is far from necessary. For example, the sequence $\{S_k\}$ of Fig. 2.2(b) has nonempty intersection, yet the asymptotic directions, $(0, \beta)$, $\beta > 0$, are not retractive.

We note that the conclusion of the preceding proposition holds also under a weaker definition of asymptotic direction, whereby d is called retractive if, for every corresponding asymptotic sequence $\{x_k\}$, there exists a bounded sequence of positive scalars $\{\alpha_k\}$ and an index $\overline{k}$ such that $x_k - \alpha_k d \in S_k$ for all $k \geq \overline{k}$. This definition does not work well, however, when we consider intersections of two or more sequences. By contrast, under the given Definition 2.1, it follows that the sequence obtained by intersection of two retractive sequences is retractive. In particular, we have the following proposition.

Proposition 2.2: Let $\{S_k^j\}$, $j = 1, \ldots, r$, be retractive nested sequences of nonempty closed sets. Consider the sets

$$S_k = S_k^1 \cap S_k^2 \cap \cdots \cap S_k^r, \qquad k = 0, 1, \ldots,$$

and assume that they are nonempty for all k. Then $\{S_k\}$ is retractive.

The definition of a retractive set sequence also implies that the sequence obtained by the union of two retractive sequences is retractive.

Proposition 2.3: Let $\{S_k^j\}$, $j = 1, \ldots, r$, be retractive nested sequences of nonempty closed sets. Then the sequence $\{S_k\}$, where

$$S_k = S_k^1 \cup S_k^2 \cup \cdots \cup S_k^r, \qquad k = 0, 1, \ldots,$$

is retractive.

Asymptotic Directions of Closed Sets

We now specialize the definitions of asymptotic directions and retractiveness to the case where all the sets in the sequence are the same. For this case, the notion of asymptotic direction was studied in the works of Dedieu [Ded77], [Ded79].

Definition 2.4: Given a nonempty closed set S, we say that d is an *asymptotic direction of* S if it is an asymptotic direction of the sequence $\{S_k\}$, where $S_k = S$ for all k, i.e., if $d \neq 0$ and there exists a sequence $\{x_k\} \subset S$, called an *asymptotic sequence corresponding to* d, such that $\|x_k\| \to \infty$, and

$$\frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}.$$

An asymptotic direction d is called *retractive* if, for every corresponding asymptotic direction $\{x_k\}$ and every scalar $\overline{\alpha} > 0$, there exists an integer $\overline{k}$ such that

$$x_k - \alpha d \in S, \qquad \forall\ \alpha \in (0, \overline{\alpha}],\ k \geq \overline{k}.$$

The set S is called *retractive* if all its asymptotic directions d are retractive.

We note that the definition of retractive closed sets was introduced by Auslender [Aus00], and Auslender and Teboulle [AuT03] (see [AuT03], p. 37; the definition given there is slightly different from ours).

Let us provide a characterization of asymptotic directions in the case where the sets are convex. We recall that given a nonempty convex set C, we say that a vector d is a *direction of recession* of C if $x + \alpha d \in C$ for all $x \in C$ and $\alpha \geq 0$ (see, e.g., [Roc70], [BNO03]). The set of all directions of recession of C, denoted by R_C, is the *recession cone* of C. The *lineality space* of C, denoted by L_C, is the set of directions of recession d whose opposite, $-d$, are also directions of recession:

$$L_C = R_C \cap (-R_C).$$

For a closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$, the *recession cone of* f and the *constancy space of* f, denoted R_f and L_f, respectively, are the (common) recession cone and lineality space of its nonempty level sets (see, e.g., [BNO03], [Roc70]).

The next proposition shows that the asymptotic directions of a closed convex set sequence are the nonzero common directions of recession of the sets in the sequence.

Proposition 2.4: Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. Then d is an asymptotic direction of $\{C_k\}$ if and only if $d \neq 0$ and $d \in \cap_{k=0}^{\infty} R_{C_k}$.

Proof: Let d be a nonzero vector in $\cap_{k=0}^{\infty} R_{C_k}$, and for each k, let z_k be a vector in C_k. Define

$$x_k = z_k + k(\|z_k\| + 1)d, \qquad k = 0, 1, \ldots$$

Then, by the convexity of C_k, since $d \in R_{C_k}$, we have $x_k \in C_k$ for all k. Furthermore, using the triangle inequality, we have

$$(k\|d\| - 1)\|z_k\| + k\|d\| \le \|x_k\| \le (k\|d\| + 1)\|z_k\| + k\|d\|,$$

from which it is seen that

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}.$$

Hence, d is an asymptotic direction of $\{C_k\}$.

Conversely, let d be an asymptotic direction of $\{C_k\}$, and let $\{x_k\}$ be a corresponding asymptotic sequence. In view of the convexity of C_k, to show that $d \in \cap_{k=0}^{\infty} R_{C_k}$, it will suffice to choose an arbitrary $\overline{k}$ and vector $z \in C_{\overline{k}}$, and to show that $z + d \in C_{\overline{k}}$. We assume without loss of generality that $z \neq x_k$ for all k, and we define

$$y_k = \frac{x_k - z}{\|x_k - z\|}\|d\|.$$

Since $\{x_k\}$ is unbounded, for sufficiently large k, the vector $z + y_k$ lies in the line segment connecting z and x_k, and since $x_k \in C_{\overline{k}}$ for $k \ge \overline{k}$, we have $z + y_k \in C_{\overline{k}}$. On the other hand, it is seen that since $x_k/\|x_k\| \to d/\|d\|$, we have $z + y_k \to z + d$. Since $C_{\overline{k}}$ is closed, it follows that $z + d \in C_{\overline{k}}$. **Q.E.D.**

As a special case of the preceding proposition, we obtain that for a closed convex set, the set of asymptotic directions coincides with the set of nonzero directions of recession. It can be seen also that for a closed convex set, the lineality space directions are retractive.

An important case of a retractive set is a polyhedral set, i.e., a set of the form

$$S = \{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\},$$

where $a_1, \ldots, a_r$ are vectors, and $b_1, \ldots, b_r$ are scalars. Then, the asymptotic directions of S are the nonzero vectors d that satisfy $a_j'd \le 0$ for all $j = 1, \ldots, r$, and all of them are retractive. This is a special case of the following proposition.

Proposition 2.5: Let S be a set which is the vector sum of a compact set and a polyhedral cone N. Then S is retractive and its asymptotic directions are the nonzero vectors in N.

Proof: Let $S = \overline{S} + N$, where $\overline{S}$ is compact, and N is the polyhedral cone

$$N = \{y \mid a_j'y \le 0,\ j = 1, \ldots, r\},$$

where $a_1, \ldots, a_r$ are some vectors. Any $x \in S$ can be written as $x = \overline{x} + y$, where $\overline{x} \in \overline{S}$ and $y \in N$, so for any $d \in N$, we have $x + \alpha d = \overline{x} + (y + \alpha d) \in S$ for all $\alpha \geq 0$. It follows that d is an asymptotic direction of S.

Conversely, let d be an asymptotic direction of S, and let $\{x_k\}$ be a corresponding asymptotic sequence. We can represent x_k as

$$x_k = \overline{x}_k + y_k, \qquad \forall\ k = 0, 1, \ldots,$$

where $\overline{x}_k \in \overline{S}$ and $y_k \in N$, so that

$$a_j' x_k = a_j'(\overline{x}_k + y_k), \qquad \forall\ k = 0, 1, \ldots,\ j = 1, \ldots, r.$$

Dividing both sides with $\|x_k\|$ and taking the limit as $k \to \infty$, we obtain

$$a_j' d = \lim_{k \to \infty} \frac{a_j' y_k}{\|x_k\|}.$$

Since $a_j' y_k \leq 0$ for all k and j, we obtain that $a_j' d \leq 0$ for all j, so that $d \in N$.

We will now show that d is retractive. Fix any $\overline{\alpha} > 0$. We consider two cases:

(1) $a_j' d = 0$. In this case, $a_j'(y_k - \overline{\alpha} d) \leq 0$ for all k, since $y_k \in N$ and $a_j' y_k \leq 0$.

(2) $a_j' d < 0$. In this case, we have

$$\frac{1}{\|x_k\|} a_j'(y_k - \overline{\alpha} d) = \frac{1}{\|x_k\|} a_j'(x_k - \overline{x}_k - \overline{\alpha} d),$$

so that since $\frac{x_k}{\|x_k\|} \to d$, $\{x_k\}$ is unbounded, and $\{\overline{x}_k\}$ is bounded, we obtain

$$\lim_{k \to \infty} \frac{1}{\|x_k\|} a_j'(y_k - \overline{\alpha} d) = a_j' d < 0.$$

Hence $a_j'(y_k - \overline{\alpha} d) < 0$ for k greater than some $\overline{k}$.

Thus, for $k \geq \overline{k}$ and $\alpha \in (0, \overline{\alpha}]$, we have $a_j'(y_k - \alpha d) \leq a_j'(y_k - \overline{\alpha} d) \leq 0$ for all j, so that $y_k - \alpha d \in N$ and $x_k - \alpha d \in S$. **Q.E.D.**

We recall that every polyhedral set can be written as the vector sum of a compact polyhedral set (the convex hull of a finite number of points) and a polyhedral cone (the Minkowski-Weyl representation, see, e.g., [Roc70], [BNO03]). Hence, the preceding proposition applies to the case where the set S is polyhedral. On the other hand, examples show that the polyhedrality assumption on the cone N in Prop. 2.5 cannot be easily relaxed.

Note that from Props. 2.2 and 2.5, it follows that the asymptotic directions of a nonempty set of the form $S = S^1 \cap \cdots \cap S^m$ such that each of the sets S^i, $i = 1, \ldots, m$, is the vector sum of a compact set and a polyhedral cone N^i, are the nonzero vectors in $\cap_{i=1}^m N^i$. Furthermore, S is retractive. The result of Prop. 2.5 can also be shown for a set S which is the vector sum of a compact set and the union of a finite number of polyhedral cones. Sets of this type have been studied within the class of asymptotically polyhedral sets introduced by Auslender and Teboulle [AuT03], who prove results that are similar to Prop. 2.5.

The following result does not seem to have been reported in the literature. It shows that closed level sets associated with concave functions are retractive.

Proposition 2.6: Let S be a nonempty closed set of the form

$$S = \{x \mid f(x) \geq 0\},$$

where $f : \Re^n \mapsto (-\infty, \infty]$ is a proper convex function. Then S is retractive.

Proof: Assume, to arrive at a contradiction, that there exists an asymptotic direction d of S that is not retractive, and let $\{x_k\}$ be a corresponding asymptotic sequence for which the definition of retractiveness is violated. Then there must exist a bounded sequence of positive scalars α_k with $f(x_k - \alpha_k d) < 0$ for an infinite subsequence of indexes k. Without loss of generality, assume that this is true for all indexes k. We have $x_k/\|x_k\| \to d/\|d\|$ and $\|x_k\| \to \infty$, so that $\|x_k - \alpha_k d\| \to \infty$ and $(x_k - \alpha_k d)/\|(x_k - \alpha_k d)\| \to d/\|d\|$. It follows that $\{x_k - \alpha_k d\}$ is an asymptotic sequence of the closure of the set $\{x \mid f(x) < 0\}$ and d is the corresponding asymptotic direction. By Prop. 2.4, d is a direction of recession of this closed convex set. Since $x_k - \alpha_k d$ lies in the interior of this set [$x_k - \alpha_k d$ belongs to $\{x \mid f(x) < 0\}$, which is the complement of S and hence an open set], x_k also lies in the interior of this set, i.e., $f(x_k) < 0$, contradicting the hypothesis $x_k \in S$. **Q.E.D.**

We finally provide some set intersection results involving convexity assumptions. Part (a) of the following proposition and the special case where X is a polyhedral set in part (b) are known (see [BNO03] and the references given there), but the proof given here is simpler than those found in the literature.

Proposition 2.7: Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. Denote

$$R = \cap_{k=0}^\infty R_{C_k}, \qquad L = \cap_{k=0}^\infty L_{C_k}.$$

(a) If $R = L$, then $\{C_k\}$ is retractive, and $\cap_{k=0}^{\infty} C_k$ is nonempty.

(b) Let X be a retractive closed set with set of asymptotic directions denoted by A. Assume that all the sets $S_k = X \cap C_k$ are nonempty, and that

$$A \cap R \subset L.$$

Then, $\{S_k\}$ is retractive, and $\cap_{k=0}^{\infty} S_k$ is nonempty.

Proof: (a) By Prop. 2.4, the asymptotic directions of $\{C_k\}$ are the nonzero directions in R, and by hypothesis, these are also directions in L. This implies that for an asymptotic direction d, we have $d \in L_{C_k}$ for all k, so for any corresponding asymptotic sequence $\{x_k\}$, we have $x_k - \alpha d \in C_k$ for all k and $\alpha \geq 0$. Hence d is retractive, and $\cap_{k=0}^{\infty} C_k$ is nonempty by Prop. 2.1.

(b) An asymptotic direction d of $\{S_k\}$ must belong to $A \cap R$, and hence also to L. Thus, for any corresponding asymptotic sequence $\{x_k\}$ we have $x_k \in C_k$ and hence $x_k - \alpha d \in C_k$ for all k and $\alpha \geq 0$. Since d is an asymptotic direction of X and X is retractive, this implies that d is retractive for $\{S_k\}$. From Prop. 2.1, it follows that $\cap_{k=0}^{\infty} S_k$ is nonempty. **Q.E.D.**

Proposition 2.7(b) applies to the case where X is the vector sum of a compact set and a polyhedral cone N (see Prop. 2.5), or to the case where X is specified by concave inequalities (see Prop. 2.6).

3. HORIZON DIRECTIONS AND ASSOCIATED INTERSECTION THEOREMS

We now focus on a key question: Given two set sequences $\{S_k^1\}$ and $\{S_k^2\}$ each with nonempty intersection by itself, and with

$$S_k^1 \cap S_k^2 \neq \emptyset, \qquad k = 0, 1, \ldots,$$

what causes the intersection sequence $\{S_k^1 \cap S_k^2\}$ to have an empty intersection? By sketching a few examples (see Fig. 3.1), one can see that the trouble lies with the existence of some "critical asymptotes." Roughly, these are asymptotic directions d, common to $\{S_k^1\}$ and $\{S_k^2\}$, and such that starting at $\cap_k S_k^2$ (or $\cap_k S_k^1$) and looking at the horizon along d, we do not meet $\cap_k S_k^1$ (or $\cap_k S_k^2$, respectively). With this in mind, we introduce a subset of asymptotic directions, called

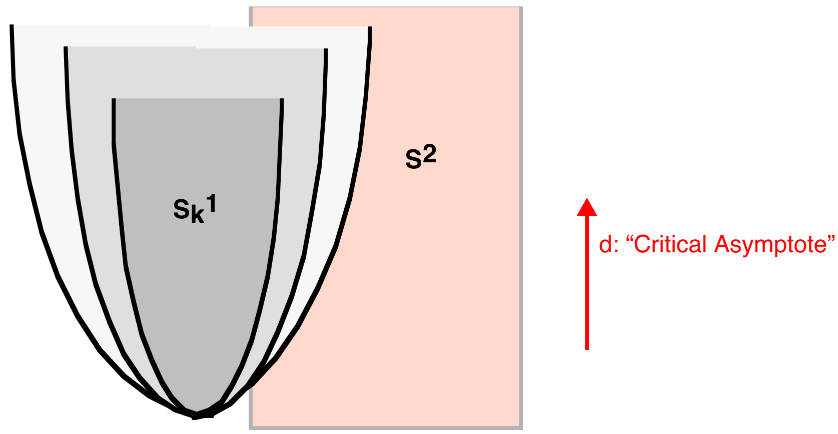

Figure 3.1. Illustration of a set sequence $\{S_k^1\}$ and a set S^2 with

$$S_k^1 \cap S^2 \neq \emptyset,$$

for all k, and $\cap_{k=0}^{\infty} S_k^1 \cap S^2 = \emptyset$. The problem is that there exists some "critical asymptote" along which the two sequences "asymptotically lose contact."

horizon directions, which we will subsequently use to make precise the meaning of a "critical asymptote."

Definition 3.1: Given a nested closed set sequence $\{S_k\}$ with nonempty intersection, we say that an asymptotic direction d of $\{S_k\}$ is a *horizon direction* with respect to a set G if, for every $x \in G$, there exists a scalar $\overline{a} \geq 0$ such that $x + \alpha d \in \cap_{k=0}^{\infty} S_k$ for all $\alpha \geq \overline{a}$. We say that d is a *global horizon direction* if $G = \Re^n$, and we say that it is a *local horizon direction* if $G = \cap_{k=0}^{\infty} S_k$.

Thus d is a horizon direction with respect to G if starting at any point of G and going along d we eventually enter and stay in $\cap_{k=0}^{\infty} S_k$. The definition of a horizon direction of a set sequence specializes naturally to the case of single closed set S by viewing the set as the constant sequence of sets $\{S_k\}$, where $S_k = S$ for all k. Thus, for example, the statement that d is a *horizon direction of S with respect to G* means that d is a horizon direction of the sequence $\{S_k\}$ with respect to G, where $S_k = S$ for all k.

It can be seen that if the sets S_k are convex, the set of local horizon directions is equal to the set of asymptotic directions, and also to the set of nonzero common directions of recession of all the sets S_k (see Prop. 2.4). The set of global horizon directions may be a strict subset of the set of asymptotic directions, even if the sets S_k are convex (take, for example, all the sets S_k to be equal to the same line on the plane).

Note also that if $\{S_k^1\}$ and $\{S_k^2\}$ are nested closed set sequences such that the sequence $\{S_k^1 \cap S_k^2\}$ has nonempty intersection, then a vector which is a horizon direction of both $\{S_k^1\}$ and $\{S_k^2\}$ with respect to a (common) set G is also a horizon direction of $\{S_k^1 \cap S_k^2\}$ with respect to G. However, the converse is not true, as simple examples indicate. On the other hand, the set of vectors that are global horizon directions of both $\{S_k^1\}$ and $\{S_k^2\}$ coincides with the set of global horizon directions of $\{S_k^1 \cap S_k^2\}$.

Here are some examples illustrating horizon directions.

Example 3.1

Let S be the complement of a bounded open set. Then all nonzero directions are asymptotic directions as well as global horizon directions.

Example 3.2

Let $f : \Re^n \to \Re$ be a convex function that is coercive, and let $S = \{x \mid f(x) \geq \gamma\}$ where γ is a scalar. Then S is closed and nonempty (since f is real-valued and hence continuous over $\Re^n$, as well as coercive), and the complement of S is bounded (since f is coercive). Hence all nonzero directions are asymptotic directions as well as global horizon directions of S.

Example 3.3 (Vector Sums of Compact Sets and Polyhedral Cones)

Let $S = X_1 \cap X_2 \cap \cdots \cap X_m$, where each X_i is the vector sum of a compact set and a polyhedral cone N_i. Then the set of asymptotic directions is $\cap_{i=1}^m N_i$ and is also equal to the set of local horizon directions. However, the set of global horizon directions may be strictly smaller, and in fact may be empty, even if $\cap_{i=1}^m N_i$ contains a nonzero direction (take for example $m = 1$ and X_1 to be a half-line on the plane).

Let us introduce a class of convex functions that includes convex quadratic and, more generally, convex polynomial functions. These functions are first introduced in Exercise 2.7 of [BNO03], and were shown to be interesting within the set intersection and existence of optimal solutions contexts. We recall that, for any closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$, a direction d belongs to the recession cone R_f if and only if, for every $x \in \text{dom}(f)$, we have

$$\lim_{\alpha \to \infty} \frac{f(x + \alpha d) - f(x)}{\alpha} \leq 0,$$

(see [Roc70], Theorems 8.5, 8.6).

Example 3.4 (Bidirectionally Flat Convex Functions)

Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function with the property that a direction d belongs to the constancy space L_f if and only if

$$\lim_{\alpha\to\infty} \frac{f(x+\alpha d)-f(x)}{\alpha} = 0, \qquad \forall\, x \in \mathrm{dom}(f). \tag{3.1}$$

Functions of this type will be referred to as *bidirectionally flat*. It is clear that convex polynomial functions, including convex quadratic functions, are bidirectionally flat, since they are polynomial along any direction, and hence constant along any direction d satisfying Eq. (3.1). Another class of bidirectionally flat functions, which also includes convex quadratic functions, have the form

$$f(x) = h(Ax) + c'x + b,$$

where A is an $m \times n$ matrix, c is a vector, b is a scalar, and $h : \Re^m \mapsto (-\infty, \infty]$ is a closed proper convex function satisfying

$$\liminf_{\|y\|\to\infty} \frac{h(y)}{\|y\|} = \infty.$$

In view of this property, we see that Eq. (3.1) is satisfied if and only if d is in the nullspace of A and $c'd = 0$, which is true if and only if $d \in L_f$.

Let $\{S_k\}$ be a set sequence defined by the level sets of f:

$$S_k = \{x \mid f(x) \le \gamma_k\},$$

where $\{\gamma_k\}$ is a scalar nonnegative sequence with $\gamma_k \downarrow 0$, and such that all the sets S_k are nonempty. Then for any nonzero $d \in R_f$, one of the following two cases holds:

(1) $d \in L_f$, in which case d is a local horizon direction that is retractive.

(2) $d \notin L_f$, in which case

$$\lim_{\alpha\to\infty} \frac{f(x+\alpha d)-f(x)}{\alpha} < 0, \qquad \forall\, x \in \Re^n.$$

In the latter case, we have $\lim_{\alpha\to\infty} f(x+\alpha d) = -\infty$ for all $x \in \mathrm{dom}(f)$, implying that $x+\alpha d \in \cap_{k=0}^\infty S_k$, for all sufficiently large α. Thus, d is a horizon direction with respect to $\mathrm{dom}(f)$.

Also, if there exists a direction d with $d \in R_f$ but $d \notin L_f$, then by the preceding argument, we must have $\inf_{x\in\Re^n} f(x) = -\infty$, so that $\cap_{k=0}^\infty S_k \neq \emptyset$. If on the other hand, we have $R_f = L_f$, then by Prop. 2.7(a), f attains its minimum over $\Re^n$, so again $\cap_{k=0}^\infty S_k \neq \emptyset$.

In conclusion, *if f is bidirectionally flat, every asymptotic direction of $\{S_k\}$ is either a horizon direction with respect to* $\mathrm{dom}(f)$*, or else it is a local horizon direction that is retractive. Furthermore, $\cap_{k=0}^\infty S_k \neq \emptyset$.*

As a special case of the preceding example, consider a set sequence $\{S_k\}$ defined by convex quadratic inequalities:

$$S_k = \{x \mid x'Qx + c'x + b \le \gamma_k\}.$$

For each asymptotic direction d, there are two possibilities:

(a) d is a global horizon direction that satisfies $Qd = 0$ and $c'd < 0$.

(b) d is a local horizon direction that is also a lineality direction (satisfies $Qd = 0$ and $c'd = 0$), and hence it is retractive.

Note that this property is not shared by nonconvex quadratic inequalities (unless the quadratic is strictly concave, see Example 3.2). As an example, for the subset of the plane

$$\big\{(x_1, x_2) \mid x_1 x_2 \geq 1\big\},$$

the set of asymptotic directions is

$$\big\{(d_1, d_2) \mid d_1 d_2 \geq 0\big\}.$$

The asymptotic direction $(0, 1)$ is not a global horizon direction. Furthermore, while it is a local horizon direction, it is not retractive because, for some of its corresponding asymptotic sequences [e.g., the sequence $\{(1/k, k)\}$], the requirement for retractiveness is not fulfilled.

Critical Directions

We now introduce another type of asymptotic direction whose character reflects some of the root causes of emptiness of set intersections. The idea is that if two nested sequences $\{S_k^1\}$ and $\{S_k^2\}$, each with nonempty intersection by itself ($\cap_{k=0}^{\infty} S_k^1 \neq \emptyset$ and $\cap_{k=0}^{\infty} S_k^2 \neq \emptyset$), and with nonempty intersection with the other ($S_k^1 \cap S_k^2 \neq \emptyset$, for all k), are combined to form an empty intersection $\cap_{k=0}^{\infty} (S_k^1 \cap S_k^2)$, then some of their common asymptotic directions must be "critical" in some sense. The following definition formulates this idea, and is motivated by properties of the asymptotic directions of level sets of bidirectionally flat functions.

Definition 3.2: Given a nested closed set sequence $\{S_k\}$ with nonempty intersection, we say that an asymptotic direction d of $\{S_k\}$ is a *critical direction* with respect to a set G if d is neither a horizon direction of $\{S_k\}$ with respect to G, nor a retractive local horizon direction of $\{S_k\}$. An asymptotic direction of $\{S_k\}$ is referred to as *noncritical* with respect to G if it is not critical with respect to G.

By convention, every asymptotic direction of $\{S_k\}$ is noncritical with respect to the empty set. In fact, Definitions 3.1 and 3.2 allow the possibility that the set G is empty, and are consistent with this convention.

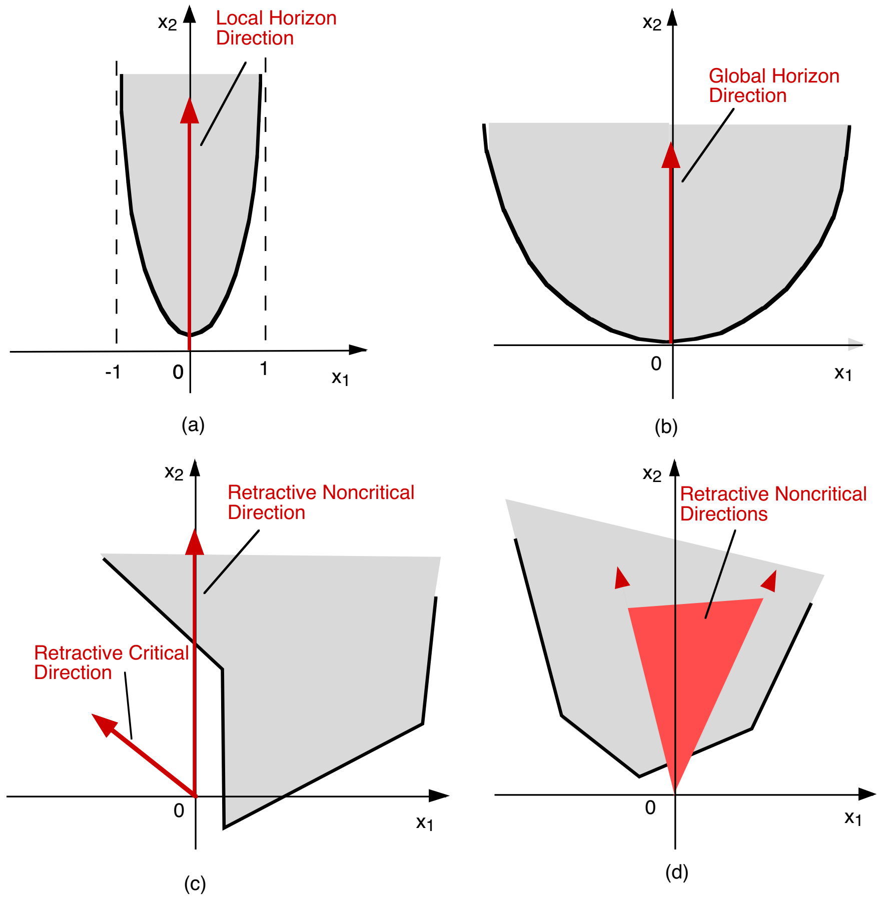

Figure 3.2. Examples of horizon, critical, and noncritical directions of various sets. In (a), the asymptotic direction $(0, \beta)$, $\beta > 0$, is a horizon direction and a noncritical direction with respect to any subset of the set $\big\{(x_1, x_2) \mid -1 < x_1 < 1\big\}$. It is not a horizon direction and it is a critical direction with respect to any other subset. In (b), the given set is $\big\{(x_1, x_2) \mid x_1^2 \le x_2\big\}$, and all the asymptotic directions $[(0, \beta),\ \beta > 0]$ are global horizon directions and hence noncritical directions with respect to any subset. The sets in (c) and (d) are retractive. In (c), some asymptotic directions [such as $(0, \beta)$, $\beta > 0$] are local horizon directions, while others are not and are therefore critical with respect to some sets. In (d), all asymptotic directions are local horizon directions and are therefore noncritical with respect to any set.

The definition of a critical and noncritical direction of a set sequence specializes naturally to the case of a single closed set S by viewing the set as the constant sequence of sets $\{S_k\}$, where $S_k = S$ for all k; see Fig 3.2. Note that all global horizon directions and all retractive local horizon directions are noncritical with respect to any nonempty set; see Fig. 3.3. Note also that all asymptotic directions of sequences of level sets of bidirectionally flat functions f, such as the ones considered in Example 3.4, are noncritical with respect to $\text{dom}(f)$. In particular, sequences of nonempty convex quadratic level sets $S_k = \{x \mid x'Qx + c'x + b \leq \gamma_k\}$, where $\gamma_k \downarrow 0$, have noncritical asymptotic directions with respect to $\Re^n$.

The significance of critical directions is illustrated by the following proposition, which shows that a sequence $\{S_k^1 \cap \cdots \cap S_k^r\}$ with empty intersection must have an asymptotic direction that is critical for at least one of its components $\{S_k^j\}$ with respect to some of the other components. In particular, this implies that if all the components $\{S_k^j\}$ have no critical directions, then the intersection $\cap_{k=0}^{\infty} (S_k^1 \cap \cdots \cap S_k^r)$ is nonempty. The following proposition is actually a special case of a more general proposition that we will prove shortly. We state the proposition separately because it is simpler and is still sufficient to show most of the results on existence of optimal solutions to be given in the next section.

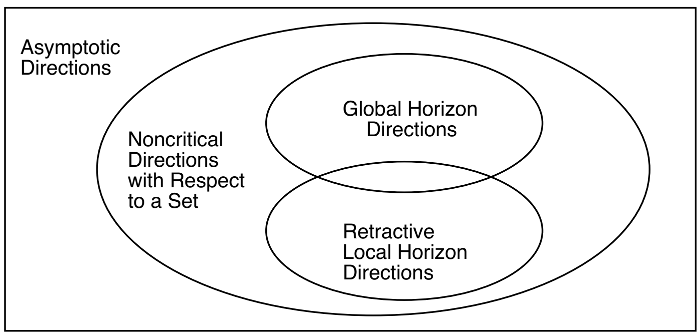

Figure 3.3. Relations between different types of asymptotic directions.

Proposition 3.1: Consider a set sequence $\{S_k\}$ of the form

$$S_k = S_k^1 \cap S_k^2 \cap \cdots \cap S_k^r,$$

where $\{S_k^j\}$, $j = 1, \ldots, r$, are nested sequences of nonempty closed sets such that $S_k \neq \emptyset$ for all k, and $\cap_{k=0}^{\infty} S_k^j \neq \emptyset$ for all j. If $\cap_{k=0}^{\infty} S_k = \emptyset$, there exists a nonempty index subset $J \subset \{1, \ldots, r\}$ such that $\cap_{k=0}^{\infty} \left(\cap_{j \in J} S_k^j\right) = \emptyset$, and an asymptotic direction of $\{\cap_{j \in J} S_k^j\}$ that for some $\bar{j} \in J$, is a critical direction of $\{S_k^{\bar{j}}\}$ with respect to $\cup_{j \in J - \{\bar{j}\}} \cap_{k=0}^{\infty} S_k^j$.

The proof of the preceding proposition will be obtained as a special case of the subsequent Prop. 3.2. The conclusion of the proposition cannot be replaced by the stronger conclusion that if $\cap_{k=0}^{\infty} S_k = \emptyset$, there exists an asymptotic direction of $\{S_k\}$ that for some $\bar{j} \in \{1, \ldots, r\}$, is a critical direction of $\{S_k^{\bar{j}}\}$ with respect to $\cup_{j \in \{1, \ldots, r\} - \{\bar{j}\}} \cap_{k=0}^{\infty} S_k^j$. This is shown by the following counterexample.

Example 3.5 (Counterexample)

Consider the following three sequences of nonempty closed sets:

$$S_k^1 = \big\{(x, y, z) | 0 \le z \le 1/k\big\}, \qquad k = 0, 1, \ldots,$$

$$S_k^2 = \big\{(x, y, z) | x > 0, 1/x \le z\big\}, \qquad k = 0, 1, \ldots,$$

$$S_k^3 = \big\{(x, y, z) | y \ge x^2\big\}, \qquad k = 0, 1, \ldots.$$

Each sequence has a nonempty intersection, and $S_k = S_k^1 \cap S_k^2 \cap S_k^3$ is nonempty for $k = 0, 1, \ldots$. The intersection of $\cap_{k=0}^{\infty} S_k^1$ and $\cap_{k=0}^{\infty} S_k^2$ is empty, and hence $\cap_{k=0}^{\infty} S_k$ is also empty. However, all asymptotic directions of $\{S_k\}$ are along the y axis, and are global horizon directions of $\{S_k^3\}$. These directions are also retractive local horizon directions for $\{S_k^1\}$ and $\{S_k^2\}$. Hence they are noncritical directions of $\{S_k^{\overline{j}}\}$ with respect to $\Re^3$ (and hence also with respect to the smaller set $\cup_{j \in \{1,2,3\} - \{\overline{j}\}} \cap_{k=0}^{\infty} S_k^j$) for all $\overline{j} \in \{1, 2, 3\}$.

As an illustration of how Prop. 3.1 may be applied, consider a sequence $\{S_k\}$ of the form

$$S_k = X \cap S_k^1 \cap S_k^2 \cap \cdots \cap S_k^r,$$

where for each j, S_k^j is the ellipsoid given by

$$S_k^j = \{x \mid x' Q_j x + c_j' x + b_j \le \gamma_k^j\},$$

and Q_j is a real symmetric positive semidefinite matrix, c_j is a vector, b_j is a scalar, and $\{\gamma_k^j\}$ is a scalar positive sequence with $\gamma_k^j \downarrow 0$. Furthermore, X is a nonempty closed set such that all its asymptotic directions are local horizon directions that are retractive. Then all the asymptotic directions of $\{S_k^j\}$ and X are noncritical with respect to $\Re^n$, Prop. 3.1 applies, and shows that $\cap_{k=0}^{\infty} S_k$ is nonempty, assuming the sets S_k are nonempty and $\cap_{k=0}^{\infty} S_k^j = \{x \mid x' Q_j x + c_j' x + b_j \le 0\}$ is nonempty for all j. Note that the purely quadratic case of this result ($X = \Re^n$) is given by Luo and Zhang [LuZ99].

The reasoning used above applies also in the more general case where the convex quadratic functions are replaced by any *real-valued* bidirectionally flat functions. However, Prop. 3.1 does not apply to set intersections involving *extended real-valued* bidirectionally flat functions (unless the domains of these functions are identical). The following proposition is a generalization of Prop. 3.1, which will allow us to deal with such situations (see Prop. 3.3 later in this section).

Proposition 3.2: Consider a set sequence $\{S_k\}$ of the form

$$S_k = X_k \cap S_k^1 \cap S_k^2 \cap \cdots \cap S_k^r,$$

where $\{X_k\}$ and $\{S_k^j\}$, $j = 1, \ldots, r$, are nested sequences of nonempty closed sets, such that $S_k \neq \emptyset$ for all k, $\cap_{k=0}^\infty X_k \neq \emptyset$, and $\cap_{k=0}^\infty S_k^j \neq \emptyset$ for all j. If $\cap_{k=0}^\infty S_k = \emptyset$, there exists a nonempty index subset $J \subset \{1, \ldots, r\}$ such that $\cap_{k=0}^\infty \big(X_k \cap (\cap_{j\in J} S_k^j)\big) = \emptyset$, and an asymptotic direction d of $\big\{X_k \cap (\cap_{j\in J} S_k^j)\big\}$ such that at least one of the following two holds:

(1) d is not a retractive local horizon direction of $\{X_k\}$.

(2) For some $\overline{j} \in J$, d is a critical direction of $\{S_k^{\overline{j}}\}$ with respect to the set $\cup_{j\in J-\{\overline{j}\}} \cap_{k=0}^\infty (X_k \cap S_k^j)$ (with the convention that this set equals $\cap_{k=0}^\infty X_k$ when $J = \{\overline{j}\}$).

Proof: We assume the contrary, i.e., that for every $J \subset \{1, \ldots, r\}$ such that $\cap_{k=0}^\infty \big(X_k \cap (\cap_{j\in J} S_k^j)\big) = \emptyset$, all asymptotic directions of $\big\{X_k \cap (\cap_{j\in J} S_k^j)\big\}$ are noncritical directions of each $\{S_k^{\overline{j}}\}$, $\overline{j} \in J$, with respect to $\cup_{j\in J-\{\overline{j}\}} \cap_{k=0}^\infty (X_k \cap S_k^j)$ (which by our convention equals $\cap_{k=0}^\infty X_k$ when $r = 1$), while they are also retractive local horizon directions of $\{X_k\}$.

Let A be the set of asymptotic directions of $\{S_k\}$ (which is nonempty since $\cap_{k=0}^\infty S_k = \emptyset$). Then, taking $J = \{1, \ldots, r\}$, we see that there must exist some $j_1 \in \{1, \ldots, r\}$ and some $d \in A$ that is a horizon direction of $\{S_k^{j_1}\}$ with respect to $\cup_{j\in J-\{j_1\}} \cap_{k=0}^\infty (X_k \cap S_k^j)$; otherwise each $d \in A$ would be retractive for $\{X_k\}$ and for all $\{S_k^j\}$, and hence also retractive for $\{S_k\}$, so, by Prop. 2.1, the hypothesis $\cap_{k=0}^\infty S_k = \emptyset$ would be contradicted.

Consider the sequence $\big\{S_k(1)\big\}$, obtained from $\{S_k\}$ when the sets $S_k^{j_1}$ are eliminated, i.e.,

$$S_k(1) = X_k \cap \big(\cap_{j\in J-\{j_1\}} S_k^j\big).$$

We argue by contradiction that $\cap_{k=0}^\infty S_k(1) = \emptyset$. Suppose that this is not so. Take any $x \in \cap_{k=0}^\infty S_k(1)$, and consider the direction $d \in A$ that is a horizon direction of $\{S_k^{j_1}\}$ with respect to $\cup_{j\in J-\{j_1\}} \cap_{k=0}^\infty (X_k \cap S_k^j)$. Since $d \in A$, d is also a retractive local horizon direction of $\{X_k\}$. Then we have

$$x + \alpha d \in \cap_{k=0}^\infty X_k, \qquad \forall\ \alpha \text{ sufficiently large}, \tag{3.2}$$

$$x + \alpha d \in \cap_{k=0}^\infty S_k^{j_1}, \qquad \forall\ \alpha \text{ sufficiently large}. \tag{3.3}$$

For the case where J has cardinality 2 or more, we show below that, for all $\overline{j} \in J - \{j_1\}$, we also have

$$x + \alpha d \in \cap_{k=0}^\infty S_k^{\overline{j}}, \qquad \forall\ \alpha \text{ sufficiently large}, \tag{3.4}$$

which, combined with Eqs. (3.2) and (3.3), contradicts the assumed emptiness of $\cap_{k=0}^{\infty} S_k$.

Indeed, for any $\overline{j} \in J - \{j_1\}$, under our working hypothesis, there are two possibilities:

(1) d is a retractive local horizon direction of $\{S_k^{\overline{j}}\}$, in which case, since $x \in \cap_{k=0}^{\infty} S_k^{\overline{j}}$, Eq. (3.4) holds.

(2) d is a horizon direction of $\{S_k^{\overline{j}}\}$ with respect to $\cup_{j \in J - \{\overline{j}\}} \cap_{k=0}^{\infty} (X_k \cap S_k^j)$, and since $j_1 \in J - \{\overline{j}\}$, d is a horizon direction of $\{S_k^{\overline{j}}\}$ with respect to $\cap_{k=0}^{\infty}(X_k \cap S_k^{j_1})$. Since, by Eqs. (3.2) and (3.3),

$$x + \alpha d \in \cap_{k=0}^{\infty}(X_k \cap S_k^{j_1})$$

for all α sufficiently large, this further implies that

$$(x + \alpha d) + \overline{\alpha} d \in \cap_{k=0}^{\infty} S_k^{\overline{j}}$$

for all $\overline{\alpha}$ sufficiently large. Thus, Eq. (3.4) holds in this case as well.

In conclusion, Eq. (3.4) holds for all $\overline{j} \in J - \{j_1\}$. Combining Eqs. (3.2)-(3.4), we see that $x + \alpha d \in \cap_{k=0}^{\infty} S_k$ for sufficiently large α. Thus, the contradiction argument showing that $\cap_{k=0}^{\infty} S_k(1) = \emptyset$ is complete.

We may now repeat this argument, with S_k replaced by $S_k(1)$ and J redefined as

$$J = \{1, \ldots, r\} - \{j_1\},$$

to obtain another index $j_2 \neq j_1$ such that $\cap_{k=0}^{\infty} S_k(2) = \emptyset$, where $S_k(2)$ is the set formed by intersection of all the sets S_k^j except $S_k^{j_1}$ and $S_k^{j_2}$, i.e.,

$$S_k(2) = X_k \cap \left(\cap_{j \in J - \{j_2\}} S_k^j\right).$$

Continuing the process, after r steps we conclude that $\cap_{k=0}^{\infty} S_k(r) = \cap_{k=0}^{\infty} X_k = \emptyset$, which contradicts the hypothesis. **Q.E.D.**

Note that Prop. 3.1 is obtained as the special case of Prop. 3.2 where the sets X_k are all equal to $\Re^n$. By combining Example 3.4 with the preceding proposition, we obtain a generalization of a result given as Exercise 2.7 of [BNO03] (the result of this exercise is the special case where $X = \Re^n$ in the following proposition).

Proposition 3.3: Consider a set sequence $\{S_k\}$ of the form

$$S_k = X_k \cap S^1 \cap S^2 \cap \cdots \cap S^r.$$

Here

$$X_k = X \cap S_k^0, \qquad k = 0, 1, \ldots,$$

where X is a closed set such that all its asymptotic directions are retractive local horizon directions. Furthermore, S_k^0 and S^j, $j = 1, \ldots, r$, are given by

$$S_k^0 = \{x \mid f_0(x) \leq \gamma_k\}, \quad k = 0, 1, \ldots, \qquad S^j = \{x \mid f_j(x) \leq 0\}, \quad j = 1, \ldots, r,$$

where $\{\gamma_k\}$ is a scalar nonnegative sequence with $\gamma_k \downarrow 0$, and for each $j = 0, 1, \ldots, r$, $f_j : \Re^n \mapsto (-\infty, \infty]$ is a closed proper convex function that is bidirectionally flat. Assume that S_k is nonempty for all k, and that $\cap_{k=0}^\infty X_k \subset \cap_{j=1}^r \mathrm{dom}(f_j)$. Then $\cap_{k=0}^\infty S_k$ is nonempty.

Proof: We first show that $\cap_{k=0}^\infty X_k \neq \emptyset$. Let A be the set of asymptotic directions of $\{X_k\}$. Then for all $d \in A$, we have $x + \alpha d \in X$ for all $x \in X$ and α sufficiently large, since the asymptotic directions of X are local horizon directions. Also, $d \in R_{f_0}$. There are two cases (see Example 3.4):

(1) $d \notin L_{f_0}$ for some $d \in A$. Then, since f_0 is closed, convex, and bidirectionally flat, $\lim_{\alpha \to \infty} f_0(x + \alpha d) = -\infty$ for all $x \in \mathrm{dom}(f_0)$, so that $x + \alpha d \in \cap_{k=0}^\infty X_k$ for all $x \in X \cap \mathrm{dom}(f_0)$ and all α sufficiently large. It follows that $\cap_{k=0}^\infty X_k \neq \emptyset$.

(2) $d \in L_{f_0}$ for all $d \in A$. Then, since f_0 is closed, convex, and bidirectionally flat, all $d \in A$ are retractive for $\{S_k^0\}$ as well as retractive for X. Hence, all $d \in A$ are retractive for $\{X_k\}$, and it follows that $\cap_{k=0}^\infty X_k \neq \emptyset$ by Prop. 2.1.

We will now prove that $\cap_{k=0}^\infty S_k \neq \emptyset$ by contradiction. In particular, we assume that $\cap_{k=0}^\infty S_k = \emptyset$, and we will verify that the conclusion of Prop. 3.2 does not hold. Indeed, consider an index subset $J \subset \{1, \ldots, r\}$ such that $\cap_{k=0}^\infty \big(X_k \cap (\cap_{j \in J} S_k^j)\big) = \emptyset$, and let d be an asymptotic direction of $\big\{X_k \cap (\cap_{j \in J} S_k^j)\big\}$. We will show that d is a noncritical direction of all $\{S_k^{\bar{j}}\}$, $\bar{j} \in J$, with respect to $\cap_{k=0}^\infty X_k$ [and hence also with respect to the smaller set $\cup_{j \in J - \{\bar{j}\}} \cap_{k=0}^\infty (X_k \cap S_k^j)$], while it is a retractive local horizon direction of $\{X_k\}$, thereby contradicting the conclusion of Prop. 3.2.

We first note that $d \in R_{f_j}$ for all $j \in J$. It follows that for each $j \in J$, either $d \in L_{f_j}$, in which case d is a horizon direction of S^j that is retractive, or $d \notin L_{f_j}$, in which case (since f_j is closed, convex, and bidirectionally flat) d is a horizon direction of S^j with respect to $\mathrm{dom}(f_j)$, and hence also a horizon direction of S^j with respect to $\cap_{k=0}^{\infty} X_k$ [since $\cap_{k=0}^{\infty} X_k \subset \mathrm{dom}(f_j)$ by assumption]. Thus, d is a noncritical direction of all S^j, $j \in J$, with respect to $\cap_{k=0}^{\infty} X_k$.

We also have that $d \in R_{f_0}$. Assume that $d \notin L_{f_0}$, and let x be any vector in $X \cap S_0^0 \cap S^1 \cap \cdots \cap S^r$. Then for all α sufficiently large, we have

$$x + \alpha d \in \cap_{k=0}^{\infty} S_k^0, \tag{3.5}$$

since $x \in \mathrm{dom}(f_0)$ and f_0 is closed, convex, and bidirectionally flat, so that $\lim_{\alpha \to \infty} f_0(x + \alpha d) = -\infty$. Furthermore, for all α sufficiently large, we have

$$x + \alpha d \in X \cap \big(\cap_{j \in J} S^j\big), \tag{3.6}$$

since d is a local horizon direction of X and a direction of recession of each S^j, $j \in J$. Equations (3.5) and (3.6) contradict the assumed emptiness of the intersection of $\big\{X_k \cap (\cap_{j \in J} S^j)\big\}$. Hence $d \in L_{f_0}$, from which by arguing as in case (2) above, we see that d is a retractive local horizon direction of $\{X_k\}$. Thus the conclusion of Prop. 3.2 is contradicted, and it follows that $\cap_{k=0}^{\infty} S_k$ is nonempty. **Q.E.D.**

Example 3.6 (A Counterexample for Bidirectionally Flat Functions)

To see that the assumption $\cap_{k=0}^{\infty} X_k \subset \cap_{j=1}^{r} \mathrm{dom}(f_j)$ is essential in Prop. 3.3, let $X = \Re^2$ and consider the following two bidirectionally flat functions f_0 and f_1 defined on $\Re^2$:

$$f_0(x_1, x_2) = x_1, \qquad f_1(x_1, x_2) = \phi(x_1) - x_2,$$

where the function $\phi : \Re \mapsto (-\infty, \infty]$ is convex, closed, and coercive with $\mathrm{dom}(\phi) = (0, 1)$ [for example, $\phi(t) = -\ln t - \ln(1 - t)$ for $0 < t < 1$]. Take also $\{\gamma_k\}$ to be any sequence in $(0, 1)$ with $\gamma_k \downarrow 0$, so

$$S_k = \big\{x \mid x_1 \le \gamma_k,\ \phi(x_1) - x_2 \le 0\big\}.$$

Then it can be verified that $S_k \ne \emptyset$ for every k [take $x_1 \downarrow 0$ and $x_2 \ge \phi(x_1)$], and we have

$$\cap_{k=0}^{\infty} X_k = \cap_{k=0}^{\infty} S_k^0 = \big\{x \mid f_0(x) \le 0\big\} = \{x \mid x_1 \le 0,\ x_2 \in \Re\},$$

and

$$\cap_{k=0}^{\infty} S_k^1 = \big\{x \mid f_1(x) \le 0\big\} = \{x \mid 0 < x_1 < 1,\ x_2 \in \Re\} = \mathrm{dom}(f_1).$$

The two sets are disjoint, so the conclusion of Prop. 3.3 is violated, and in particular we have

$$\cap_{k=0}^{\infty} S_k = \big\{x \mid f_0(x) \le 0,\ f_1(x) \le 0\big\} = \big(\cap_{k=0}^{\infty} X_k\big) \cap \big(\cap_{k=0}^{\infty} S_k^1\big) = \emptyset.$$

We now consider sets defined by a finite number of concave quadratic inequalities. We have seen that the asymptotic directions of such sets are retractive (see Prop. 2.6). We will delineate circumstances under which the asymptotic directions are also local horizon directions, so that they are noncritical.

Example 3.7 (Level Sets of Concave Quadratic Functions)

Consider a set of the form

$$S = \{x \mid x'Qx + c'x + b \geq 0\},$$

where Q is a positive semidefinite $n \times n$ matrix, c is a vector in $\Re^n$, and b is a scalar. We first derive the set of asymptotic directions of S, which we denote by A. We consider two cases:

(a) $Q = 0$. Then A is the set of nonzero directions of recession of the convex function $-(c'x + b)$, as discussed earlier:

$$A = \{d \mid c'd \geq 0,\, d \neq 0\}.$$

(b) $Q \neq 0$. Then we claim that the asymptotic directions of S are the nonzero vectors in $\Re^n$:

$$A = \{x \mid x \neq 0\}.$$

Indeed, take any $d \neq 0$ and any y such that

$$y'Qy + c'd > 0.$$

We will show that for sufficiently large k, the sequence of vectors

$$x_k = kd + \sqrt{k}y$$

is an asymptotic sequence of S that corresponds to d. We note that $\|x_k\| \to \infty$ and that $x_k/\|x_k\| \to d/\|d\|$. Furthermore, we have

$$x_k'Qx_k + c'x_k + b = k^2 d'Qd + 2k\sqrt{k}y'Qd + ky'Qy + kc'd + \sqrt{k}c'y + b. \tag{3.7}$$

If $d'Qd > 0$, clearly we have $x_k \in S$ for sufficiently large k. On the other hand, if $d'Qd = 0$ (equivalently, $Qd = 0$, since Q is positive semidefinite), then from Eq. (3.7),

$$x_k'Qx_k + c'x_k + b = k(y'Qy + c'd) + \sqrt{k}c'y + b.$$

Since $y'Qy + c'd > 0$, we again have $x_k \in S$ for sufficiently large k. Thus, for some integer $\overline{k}$, the subsequence of $\{x_k \mid k \geq \overline{k}\}$ fulfills the requirements for an asymptotic sequence of S corresponding to d, and it follows that d is an asymptotic direction.

We know from Prop. 2.6 that all asymptotic directions of S are retractive. However, some of these directions may be critical because they are not local horizon directions. For example, let

$$S = \left\{(x_1, x_2) \mid x_1 \leq x_2^2\right\}.$$

Then the vector $(1,0)$ is a retractive asymptotic direction (by the preceding analysis), but is not a local horizon direction. More generally, the set of asymptotic directions S that are not local horizon directions is the set

$$\overline{A} = \{d \mid Qd = 0,\ c'd < 0\}. \tag{3.8}$$

To see this, note that it is true in the case where $Q = 0$, where $A = \{x \mid x \neq 0\}$, and $\overline{A} = \emptyset$. In the case where $Q \neq 0$, note that for any $x \in S$, $d \in A$, and $\alpha \geq 0$, we have

$$(x + \alpha d)'Q(x + \alpha d) + c'(x + \alpha d) + b = x'Qx + c'x + b + \alpha^2 d'Qd + \alpha(2Qx + c)'d.$$

It follows that $x + \alpha d \in S$ for sufficiently large α if and only if either $d'Qd > 0$, or $Qd = 0$ and $c'd \geq 0$. This proves Eq. (3.8).

Consider now a set sequence $\{S_k\}$ defined by a finite number of concave quadratic inequalities:

$$S_k = P \cap \{x \mid x'Q_jx + c_j'x + b_j \geq \gamma_k^j,\ j = 1, \ldots, r\},$$

where $\{\gamma_k^j\}$ are scalar sequences with $\gamma_k^j \uparrow 0$, P is a polyhedral set, Q_j are nonzero positive semidefinite $n \times n$ matrices, c_j are vectors in $\Re^n$, and b_j are scalars. A slight extension of the preceding analysis, shows that the asymptotic directions of $\{S_k\}$ are a subset of the nonzero vectors in the recession cone R_P, and all of them are retractive. A sufficient condition for all asymptotic directions to be noncritical local horizon directions of $\{S_k\}$ is that $R_P \cap N(Q_j) = \{0\}$ for all $j = 1, \ldots, r$, where $N(Q_j)$ is the nullspace of Q_j. This is true in particular if all the matrices Q_j are positive definite.

Let us also introduce a class of nonconvex functions whose level sets have the essential property needed for application of Props. 3.1 and 3.2, so that they can be used in place of convex quadratic or real-valued bidirectionally flat functions to assert nonemptiness of a set intersection.

Example 3.8

Let $\{S_k\}$ be a level set sequence

$$S_k = \{x \mid f(x) \leq \gamma_k\},$$

defined by a function f of the form

$$f(x) = h(Ax) + c'x + b,$$

where A is an $m \times n$ matrix, $h : \Re^m \mapsto \Re$ is a closed proper function satisfying

$$\liminf_{\|y\| \to \infty} \frac{h(y)}{\|y\|} = \infty,$$

c is a vector, and b is a scalar. We assume that $\{\gamma_k\}$ is a scalar positive sequence with $\gamma_k \downarrow 0$, and that $\{x \mid f(x) \leq 0\} = \cap_{k=0}^{\infty} S_k$ is nonempty.

We have

$$f(x + \alpha d) = h(Ax + \alpha Ad) + c'(x + \alpha d) + b.$$

A vector $d \neq 0$ is a local horizon direction if and only if for every x with $f(x) \leq 0$, there exists $\overline{\alpha} \geq 0$ such that for all $\alpha \geq \overline{\alpha}$, we have $f(x + \alpha d) \leq 0$. In view of the coercivity property of h, this is true if and only if $Ad = 0$ and $c'd \leq 0$. A vector $d \neq 0$ is a global horizon direction if and only if, for every $x \in \Re^n$, there exists $\overline{\alpha} \geq 0$ such that for all $\alpha \geq \overline{\alpha}$, we have $f(x + ad) \leq 0$. This is true if and only if $Ad = 0$ and $c'd < 0$. Thus, every asymptotic direction of $\{S_k\}$ is either a global horizon direction, or else it is a local horizon direction that is retractive, i.e., it is noncritical with respect to $\Re^n$, and Prop. 3.1 applies.

4. EXISTENCE OF OPTIMAL SOLUTIONS

We will now consider the problem of minimizing a closed function $f : \Re^n \mapsto (-\infty, \infty]$ over a closed set $X \subset \Re^n$. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow \inf_{x \in X} f(x)$, and consider the (nonempty) level sets

$$V_k = \big\{x \mid f(x) \leq \gamma_k\big\}.$$

The set of vectors that minimize f over X is the intersection

$$X^* = \cap_{k=0}^{\infty} (X \cap V_k),$$

so to show existence of an optimal solution, we can use asymptotic directions, and the theory of Sections 2 and 3. In particular, it is sufficient to show that all asymptotic directions of the sequence $\{X \cap V_k\}$ are retractive (see Prop. 2.1). Also, if X is polyhedral or more generally, if it is the vector sum of a compact set and a polyhedral cone N, it is sufficient to show that all the asymptotic directions of $\{V_k\}$ that belong to N are retractive (see Prop. 2.2).

We first consider the case of a closed convex function f. The preceding analysis yields the following results.

Proposition 4.1: Let X be a closed subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed convex function such that $X \cap \mathrm{dom}(f) \neq \emptyset$. The set of minimizing points of f over X is nonempty under any one of the following two conditions:

(1) X is convex, and $R_X \cap R_f = L_X \cap L_f$.

(2) X is retractive and

$$A \cap R_f \subset L_f,$$

where A is the set of asymptotic directions of X.

Proof: Let

$$V_k = \{x \mid f(x) \le \gamma_k\},$$

where $\{\gamma_k\}$ is a scalar sequence such that $\gamma_k \downarrow \inf_{x\in X} f(x)$. We show that under each of the two conditions, the intersection $\cap_{k=0}^{\infty}(X \cap V_k)$ (which is the set of minimizing points) is nonempty.

Let condition (1) hold. The sets $X \cap V_k$ are nonempty, closed, convex, and nested. Furthermore, they have the same recession cone, $R_X \cap R_f$, and the same lineality space $L_X \cap L_f$, while by assumption, $R_X \cap R_f = L_X \cap L_f$. The result follows from Prop. 2.7(a).

Let condition (2) hold. The sets V_k are nested and the intersection $X \cap V_k$ is nonempty for all k. Furthermore, the sets V_k have the same recession cone, R_f, and the same lineality space, L_f, while by assumption, $A \cap R_f \subset L_f$. The result follows from Prop. 2.7(b). **Q.E.D.**

The arguments used in the preceding proof rely on the convexity and closedness of the level sets of the cost function. As a result, they apply also to the case of a closed function $f : \Re^n \mapsto (-\infty, \infty]$ that is *quasiconvex*, in the sense that all its level sets $\{x \mid f(x) \le \gamma\}$ are convex.

Proposition 4.2: (Quasiconvex Problems) Let X be a closed subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed quasiconvex function such that $X \cap \text{dom}(f) \neq \emptyset$. Let Γ be the set of all $\gamma > \inf_{x\in X} f(x)$, and denote

$$R_f = \cap_{\gamma\in\Gamma} R_\gamma, \qquad L_f = \cap_{\gamma\in\Gamma} L_\gamma,$$

where R_γ and L_γ are the recession cone and the lineality space of the level set $\{x \mid f(x) \le \gamma\}$, respectively. Then f attains a minimum over X if any one of the following conditions holds:

(1) X is convex, and $R_X \cap R_f = L_X \cap L_f$.

(2) X is retractive and

$$A \cap R_f \subset L_f,$$

where A is the set of asymptotic directions of X.

Proof: Similar to the proof of Prop. 4.1. **Q.E.D.**

We now give a result involving convex bidirectionally flat functions (see Example 3.4), which relies on the use of horizon directions. The following proposition extends an existence result given in [BNO03] as Exercise 2.7(c), and also a result of Belousov, which dates to 1977 (as discussed in [BeK02]) and deals with the case where the functions involved are convex polynomial functions (a special case of bidirectionally flat functions, as discussed in Example 3.4). The purely quadratic case of this result ($X = \Re^n$ and $f_j(x) = x'Q_jx + c_j'x + b_j$) was independently given by Terlaky [Ter85]; see also Luo and Zhang [LuZ99], who prove some additional results that involve in part nonconvex quadratic functions.

Proposition 4.3: (Bidirectionally Flat Functions) For $j = 0, 1, \ldots, r$, let $f_j : \Re^n \mapsto (-\infty, \infty]$ be closed proper convex functions that are bidirectionally flat, and let X be a closed set such that all its asymptotic directions are retractive local horizon directions. Assume that $\big(X \cap \mathrm{dom}(f_0)\big) \subset \cap_{j=1}^r \mathrm{dom}(f_j)$. Then the problem

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to} \quad x \in X, \quad f_j(x) \le 0, \; j = 1, \ldots, r, \end{aligned}$$

has at least one optimal solution if and only if its optimal value is finite.

Proof: Assume that f^*, the optimal value, is finite, and let $\{\gamma_k\}$ be a scalar sequence such that $\gamma_k \downarrow 0$. Consider the set sequence $\{S_k\}$ given by

$$S_k = X_k \cap S^1 \cap S^2 \cap \cdots \cap S^r,$$

where

$$X_k = X \cap S_k^0, \qquad S_k^0 = \{x \mid f_0(x) - f^* \le \gamma_k\}, \qquad k = 0, 1, \ldots,$$

$$S^j = \{x \mid f_j(x) \le 0\}, \qquad j = 1, \ldots, r.$$

Using Prop. 3.3, we have that $\cap_{k=0}^\infty S_k$, the optimal solution set, is nonempty. **Q.E.D.**

We now consider another type of direction, which when used in conjunction with horizon directions, yields some conditions that slightly improve on the conditions of Prop. 4.1. Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function, and let F_f be the set of all directions y such that for every $x \in \mathrm{dom}(f)$, the limit $\lim_{\alpha \to \infty} f(x + \alpha y)$ exists. We refer to F_f as the set of *directions along which f is flat*. Note that

$$L_f \subset F_f \subset R_f,$$

where L_f and R_f are the lineality space and recession cone of f, respectively. We have the following variant of Prop. 4.1.

Proposition 4.4: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function, and let X be a closed set such that all its asymptotic directions are local horizon directions that are retractive. Assume that

$$A \cap F_f \subset L_f,$$

where A is the set of asymptotic directions of X. Then the problem

$$\begin{aligned} &\text{minimize} \;\; f(x) \\ &\text{subject to} \;\; x \in X \end{aligned}$$

has at least one optimal solution if and only if its optimal value is finite.

Proof: Assume that the optimal value is finite. Then $X \cap \text{dom}(f) \neq \emptyset$. Let $d \in A \cap R_f$. If $d \notin F_f$, then we must have $\lim_{\alpha \to \infty} f(x + \alpha d) = -\infty$, for some $x \in \text{dom}(f) \cap X$. Since d is a local horizon direction of X, we have $x + \alpha d \in X$ for all $x \in X$ and sufficiently large α. It follows that $\inf_{x \in X} f(x) = -\infty$, a contradiction. Therefore, we must have $A \cap F_f = A \cap R_f$, so using the hypothesis, we obtain $A \cap R_f \subset L_f$. From Prop. 4.1, it follows that there exists at least one optimal solution. **Q.E.D.**

The following proposition extends a classical result, known as the *Frank-Wolfe Theorem* [FrW56], which states that *every (possibly nonconvex) quadratic programming problem has an optimal solution if and only if it has finite optimal value.* The following proposition becomes the Frank-Wolfe Theorem in the special case where the constraint set X is a polyhedral set.

Proposition 4.5: (Extended Frank-Wolfe Theorem I) Let Q be a real symmetric $n \times n$ matrix, and c be a vector in $\Re^n$. Let also X be a closed set such that all its asymptotic directions are retractive local horizon directions. Then the problem

$$\begin{aligned} &\text{minimize} \;\; x'Qx + c'x \\ &\text{subject to} \;\; x \in X \end{aligned}$$

has at least one optimal solution if and only if its optimal value is finite.

Proof: Assume that f^*, the optimal value, is finite. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, and denote the nonempty closed set

$$V_k = \{x \mid x'Qx + c'x \leq \gamma_k\}.$$

The set of optimal solutions is $\cap_{k=0}^{\infty} (X \cap V_k)$, so by Prop. 2.2, it will suffice to show that each asymptotic direction d of $\{X \cap V_k\}$ is retractive for $\{V_k\}$ (then since d is also retractive for X by assumption, d is retractive for $\{X \cap V_k\}$, and Prop. 2.1 applies).

Indeed, let $\{x_k\}$ be an asymptotic sequence corresponding to an asymptotic direction d of $\{X \cap V_k\}$. Since $x_k \in V_k$ for all k, we have $x_k'Qx_k + c'x_k \leq \gamma_k$. Denoting $d_k = x_k/\|x_k\|$ and dividing by $\|x_k\|^2$, we obtain

$$d_k'Qd_k + \frac{c'd_k}{\|x_k\|} \leq \frac{\gamma_k}{\|x_k\|^2}.$$

Taking the limit as $k \to \infty$, and using the fact $d_k \to d$ and $\|x_k\| \to \infty$, we see that $d'Qd \leq 0$.

For any $x \in X$, consider the vectors $\tilde{x}_k = x + kd$, $k = 0, 1, \ldots$. Since d is an asymptotic direction of $\{X \cap V_k\}$, it is also an asymptotic direction of X, and by the hypothesis, d is a local horizon direction of X. Thus, we have $\tilde{x}_k \in X$ for sufficiently large k, so the cost function value corresponding to $\tilde{x}_k$ is no less than f^*, and we have

$$\begin{aligned} f^* &\leq (x + kd)'Q(x + kd) + c'(x + kd) \\ &= x'Qx + c'x + k^2 d'Qd + k(c + 2Qx)'d \\ &\leq x'Qx + c'x + k(c + 2Qx)'d, \end{aligned}$$

where in the last inequality, we used the fact $d'Qd \leq 0$ shown earlier. From the finiteness of f^*, it follows that

$$(c + 2Qx)'d \geq 0, \qquad \forall\, x \in X.$$

Now consider the asymptotic sequence $\{x_k\}$ corresponding to the asymptotic direction d of $\{X \cap V_k\}$. For all sufficiently large k and $\alpha \geq 0$, the cost corresponding to $x_k - \alpha d$ satisfies

$$\begin{aligned} (x_k - \alpha d)'Q(x_k - \alpha d) + c'(x_k - \alpha d) &= x_k'Qx_k + c'x_k - \alpha(c + 2Qx_k)'d + \alpha^2 d'Qd \\ &\leq x_k'Qx_k + c'x_k \\ &\leq \gamma_k, \end{aligned}$$

where the first inequality follows from the facts $d'Qd \leq 0$ and $(c + 2Qx_k)'d \geq 0$ shown earlier. Thus for all sufficiently large k and $\alpha \geq 0$, we have $x_k - \alpha d \in V_k$, so that d is retractive for $\{V_k\}$. **Q.E.D.**

In the case where $X = X_1 \cap X_2 \cap \cdots \cap X_m$, with each X_i being the vector sum of a compact set and a polyhedral cone N_i (for example, when X is a polyhedral set), all asymptotic directions of X are retractive local horizon directions (see Example 3.3). Thus, the assumption on X of the preceding proposition is satisfied, and there exists an optimal solution when the optimal value is finite. This extended version of the Frank-Wolfe Theorem is credited to Kummer [Kum77] by Belousov and Klatte [BeK02]. The version of Frank-Wolfe Theorem given here is, of course, more general. For example, it applies to some situations where the constraint set is defined by concave inequalities (see Prop. 2.6, Example 3.7). In particular, a quadratic cost function attains a minimum over a set X defined by linear or strictly concave quadratic inequalities:

$$X = \{x \mid x'Q_jx + c_j'x + b_j \leq 0,\ j = 1, \ldots, r\},$$

where each matrix Q_j is either equal to 0 or is a negative definite matrix (see Example 3.7).

Note also that the preceding proof can be used to show the result under a slightly weaker assumption: one may assume that only the asymptotic directions d of X that satisfy $d'Qd \leq 0$ are retractive local horizon directions (rather than all asymptotic directions of X).

Finally, motivated by the argument of the preceding proof, we derive a further extension of the Frank-Wolfe Theorem, where the cost function may not be quadratic. The preceding proposition is obtained by verifying that in the special case where f is a quadratic function, assumption (2) of the following proposition is satisfied.

Proposition 4.6: (Extended Frank-Wolfe Theorem II) Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper function and let X be a closed set such that $X \cap \text{dom}(f) \neq \emptyset$. Assume that:

(1) All the asymptotic directions of X are retractive local horizon directions.

(2) For every decreasing scalar sequence $\{\gamma_k\}$ such that the sets

$$S_k = X \cap \big\{x \mid f(x) \leq \gamma_k\big\}, \qquad k = 0, 1, \ldots,$$

are nonempty, for every asymptotic direction d of $\{S_k\}$, and for each $x \in X$, we either have $\lim_{\alpha\to\infty} f(x + \alpha d) = -\infty$, or else $f(x + \alpha d)$ is a nondecreasing function of the scalar α.

Then the problem

$$\begin{aligned} &\text{minimize} \ \ f(x) \\ &\text{subject to} \ \ x \in X \end{aligned}$$

has at least one optimal solution if and only if its optimal value is finite.

Proof: The proof follows the line of the proof of Prop. 4.5. Assume that f^*, the optimal value, is finite. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, and denote

$$V_k = \big\{x \mid f(x) \leq \gamma_k\big\},$$

so that $S_k = X \cap V_k$. It will suffice to show that each asymptotic direction d of $\{S_k\}$ is retractive for $\{V_k\}$ (since d is also retractive for X by assumption, this shows that d is retractive for $\{S_k\}$, and Prop. 2.1 applies).

Indeed, if d is an asymptotic direction of $\{S_k\}$, then d is a local horizon direction of X, so that for each $x \in X$ and all α sufficiently large, $x + \alpha d \in X$ and hence $f(x + \alpha d) \geq f^*$. Thus, we cannot have $\lim_{\alpha\to\infty} f(x + \alpha d) = -\infty$, and from our assumptions, it follows that $f(x + \alpha d)$ is a monotonically nondecreasing function of α for all $x \in X$.

Now consider the asymptotic sequence $\{x_k\}$ corresponding to the asymptotic direction d of $\{S_k\}$. For all k, $f(x_k + \alpha d)$ is a monotonically nondecreasing function of α, so the cost corresponding to $x_k - \alpha d$, $\alpha \geq 0$, is no greater than $f(x_k)$. Since $x_k \in V_k$, this shows that $x_k - \alpha d \in V_k$ for all $\alpha \geq 0$ and hence d is retractive for $\{V_k\}$. Since d is also an asymptotic direction of X and hence is retractive for X, it follows that d is retractive for $\{S_k\}$. **Q.E.D.**

An example of a function that satisfies assumption (2) of the preceding proposition is a function of the form

$$f(x) = p(x'Qx) + c'x + b,$$

where Q is a positive semidefinite matrix, c is a vector, b is scalar, and $p(\cdot)$ is a polynomial. Indeed, assumption (2) clearly holds if p is a constant. If p is not a constant, for any asymptotic direction d of $\{S_k\}$, there are two cases: (1) $Qd \neq 0$, in which case the highest degree term in $p(\cdot)$ has a negative coefficient and hence $\lim_{\alpha\to\infty} f(x + \alpha d) = -\infty$; (2) $Qd = 0$, in which case either $\lim_{\alpha\to\infty} f(x + \alpha d) = -\infty$ or $f(x + \alpha d)$ is a nondecreasing function of α, depending on whether $c'd < 0$ or $c'd \geq 0$. Thus, assumption (2) again holds.

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