

AN OPTIMIZATION APPROACH TO COMPUTING THE IMPLIED VOLATILITY OF AMERICAN OPTIONS

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ABSTRACT. We present a method to compute the implied volatility of American options as a mathematical program with equilibrium constraints. The formulation we present is new, as are the convergence results we prove. The algorithm holds the promise of being practical to implement, and we demonstrate some preliminary numerical results to this end.

1. INTRODUCTION

The implied volatility of a stock is considered to be an essential tool for risk management purposes. It is important both for calibrating option pricing models, and for its own sake to get an idea of how the markets rate the riskiness of a stock (or other appropriate asset).

For European options under the Black-Scholes model, calculating the implied volatility is a straightforward exercise since a closed-form solution exists for the price. However, in the case of American options, which generally do not have analytic solutions, the matter is more involved. A computational procedure must be reversed to recover the volatility from observed prices. If the goal is to derive the volatility from only one observed option price at a time, this too is not difficult to do by trial and error- usually an exact match can be found quite easily. But if one wants to find the volatility which best fits the model to a set of prices (a more statistically sound procedure), an exact fit is unlikely and trial and error is thus not as satisfactory, as there is then no way of knowing when to terminate the procedure. Moreover, trial and error will not work at all if the assumption of constant volatility is dropped and the problem becomes one of finding a volatility surface.

The problem of finding the implied volatility precisely is a difficult mathematical problem (see for instance [1], [2] and [12]). Practitioners

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tend to compute implied volatilities when they can, and otherwise rely on historical data (computing what are known as historical volatilities, see [11]). But this leaves a gap, as implied and historical volatilities provide different insights and serve different purposes. As American options are commonly traded in practice, the problem is one of real practical interest.

In this note, we address this question from an optimization perspective. Assuming the Black-Scholes framework, we demonstrate that if the American option pricing problem is formulated as a linear program, then the inverse problem of finding the implied volatility results in a particular kind of nonlinear program, known as a *mathematical program with equilibrium constraints* (MPEC). An approach involving a similar nonlinear program was taken in [10], but the derivation in that paper did not follow from a linear program as ours does, and therefore resulted in different theoretical properties. In particular we are able to apply some recent, powerful results about MPEC's (from [15]) to prove convergence for our method. Furthermore, while in this paper we assume the volatility is constant, our method extends readily to the non-constant case, which will be addressed in future work.

This paper is organized as follows: in section 2 the pricing of American options under the Black-Scholes model is reviewed. In section 3 we set up the problem of finding the implied volatility, and in section 4 we describe an algorithm for solving the problem and prove its convergence. Some preliminary numerical results are given in section 5.

2. AMERICAN OPTION PRICES UNDER THE BLACK-SCHOLES MODEL

In this paper we will consider American options under the standard Black-Scholes model. The Black-Scholes model of course postulates that stock prices follow a standard geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \geq 0,$$

on some probability space $\{\Omega, \mathcal{F}, P\}$ adapted to a filtration \mathcal{F}_t . We consider a put option on such a stock with maturity T and strike price K . We suppose that the constant risk-free interest rate is r . Then under the Black-Scholes paradigm, if the option were European (i.e. the holder can only exercise at time T), the price of the option at time 0 would be equal to

$$(1) \quad e^{-rT} \hat{E}[\max(K - S_T, 0)]$$

where \hat{E} represents the risk-neutral probability measure, under which S evolves according to the stochastic differential equation (SDE)

$$(2) \quad dS_t = rS_t dt + \sigma S_t dW_t.$$

The expectation (1) of course has an analytic solution, the Black-Scholes formula (see e.g. [11]). The situation, however, becomes considerably more complicated when the option is an American option. An American option gives the holder the right to exercise the option at any time up to time T , with the payoff at exercise time $t = \max(K - S_t, 0)$. Now the investor faces a decision problem: when to stop the stock process S_t so that his payoff is maximized (also known as an *optimal stopping* problem). The expected value of this maximized payoff discounted back to time 0 is the price of the American option. So the price of an American put option is

$$\max_{\tau \geq 0} e^{-r\tau} \hat{E}[\max(K - S_\tau, 0)]$$

where τ is a stopping time adapted to the filtration F_t . The price of an American put option does not have an analytic solution, and must be found numerically.

There are three general approaches to computing this answer- binomial trees, finite difference methods, and Monte Carlo simulation. Our focus in this paper will be on binomial trees. The idea of a binomial tree is very simple: let $Z_k = \sum_{i=1}^k Y_i$, where Y_i is a binomial random variable taking on each of the values 1 and -1 with probability .5, and $\Delta t = \frac{T}{n}$ is the time-step. It is a standard result in probability that $\sigma\sqrt{\Delta t}Z_k \rightarrow W_{k\frac{T}{n}}$ in distribution as $n \rightarrow \infty$ (see for instance [5]). Furthermore, one can easily show that the explicit solution to the SDE (2) is

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t}.$$

One can thus represent $S_{k\frac{T}{n}}$ as $S_0 e^{(r - \frac{\sigma^2}{2})k\frac{T}{n} + \sigma\sqrt{\Delta t}Z_k}$. It is again known that $S_0 e^{(r - \frac{\sigma^2}{2})k\frac{T}{n} + \sigma\sqrt{\Delta t}Z_k} \rightarrow S_{k\frac{T}{n}}$ in probability, and that in fact the expectations converge as well (see [7]). Thus one can approximate $\hat{E}[\max(K - S_t, 0)]$ for $t = k\frac{T}{n}$, where k is an integer between 0 and n , on a binomial tree with $\Delta t = \frac{T}{n}$, with the approximation converging to the continuous value as $n \rightarrow \infty$. Computing the approximate value for the American put option now becomes straightforward: a simple backward recursive process through the tree, from time T to time 0 produces the

price. For details on the dynamic programming approach see [7]. What matter for our purposes is that the dynamic programming formulation is completely equivalent to solving the following linear program:

$$(3) \quad \min \sum_{k=0}^n \sum_{j \in S(k)} v_{k,j}$$

$$s.t. \ v_{k,j} \geq \max(K - S_0 e^{(r-\frac{\sigma^2}{2})k\frac{T}{n} + \sigma\sqrt{\Delta t}j}, 0), \ k = 0, \dots, n, \ j \in S(k),$$

$$v_{k,j} \geq \frac{1}{2} e^{-r\frac{T}{n}} (v_{k+1,j+1} + v_{k+1,j-1}), \ k = 0, \dots, n-1, \ j \in S(k).$$

where $S(k)$ represents the net number of upward steps a random walk can take in k time-steps, so $j \in S(k)$ if and only if $|j| \leq k$ and $j \bmod 2 = k \bmod 2$. $v_{k,j}$ is the value function (i.e. the approximate value of the option) at time $k\frac{T}{n}$ with stock price $S = S_0 e^{(r-\frac{\sigma^2}{2})k\frac{T}{n} + \sigma\sqrt{\Delta t}j}$. Thus v_{0,S_0} is the price of the option at time 0. That solving this linear program indeed solves the (discretized) optimal stopping problem is shown in, for instance, [8]. We note that the standard way to apply binomial trees to American options is to let the probability p of an uptick in the tree be the risk-neutral probability, to ensure that no-arbitrage conditions hold even for each finite n . However, since the goal is really to approximate the continuous solution, the approach described here is completely valid, and has some advantages (see [11]). For this paper, the major advantage is that when σ is allowed to be a variable instead of a fixed parameter, all the functions in (3) remain continuously differentiable everywhere, which is important to ensure the good behavior of optimization methods. Done the other way there would be a discontinuity at $\sigma = 0$.

3. THE IMPLIED VOLATILITY OF AMERICAN OPTIONS

The inverse problem is to find the (implied) volatility which, given all other data, results in a given option price. As noted earlier, in the case of a European option, the problem is fairly straightforward, requiring that an analytic formula be inverted. For American put options, however, the issue is not so simple. As no closed-form solution exists, a computational procedure must be reversed. The problem which arises is as follows.

The computational procedure in the forward direction finds the option prices x by solving a linear program, $LP(\sigma)$, for given volatility σ .

Suppose now that σ is unknown, but instead we have a given set of prices x^* (corresponding to the nodes of our tree). What we want then is to find a σ such that x^* solves $\text{LP}(\sigma)$. Of course there is no guarantee that any volatility will fit the prices exactly, so what we really desire is to find (x, σ) which solve

$$\min \|x - x^*\|^2$$

$$s.t. \ x \text{ solves } \text{LP}(\sigma).$$

One fact which enables us to tackle this problem is that the first-order optimality conditions for a linear program are necessary and sufficient for determining a solution. Without loss of generality, $\text{LP}(\sigma)$ is a linear program of the form

$$\max c^T x$$

$$s.t. \ Ax \leq b(\sigma).$$

The dual of this problem is

$$\min b(\sigma)^T y$$

$$s.t. \ y^T A = c, y \geq 0.$$

The first-order conditions are

$$Ax \leq b(\sigma),$$

$$y^T A = c, y \geq 0,$$

$$y^T (b(\sigma) - Ax) = 0,$$

where we may replace the final “=” with a “ \leq ”. Thus our inverse pricing problem is to solve

$$(4) \quad \min \|x - x^*\|^2$$

$$s.t. \ Ax \leq b(\sigma),$$

$$y^T A = c, y \geq 0,$$

$$y^T (b(\sigma) - Ax) \leq 0.$$

This is in general a (nonconvex) nonlinear program (the set $\{(x, \sigma) : x \text{ solves } \text{LP}(\sigma)\}$ need not be convex), so finding the global minimum with certainty is not possible, but if local minima could be found, then by resolving the problem a few times, one might hope to get a reasonable estimate for a volatility which explains the data. But this problem has a particular structure which makes it difficult; it is known as a *mathematical program with equilibrium constraints* (MPEC). An MPEC is any nonlinear program whose constraints include those of the form

$$x^T F(x) \leq 0, x, F(x) \geq 0$$

for some mapping F . MPEC's pose difficulties for traditional nonlinear programming algorithms (see [3],[4] and [14]). Most significantly, until recently it was thought that interior-point methods are particularly prone to fail when applied to MPEC's. This would be disturbing, as interior-point methods are in general the most efficient means for solving practical, large-scale problems (such as the one we are dealing with here). However, in [4] we achieve considerable success in solving MPEC's using an interior-point method, and in [15] we prove some strong convergence results for interior-point methods applied to MPEC's. It is these results which we shall now use in the next section.

4. AN ALGORITHM FOR COMPUTING THE IMPLIED VOLATILITY

To remove some of the nonlinearity from (4), we write it as the equivalent problem

$$(5) \quad \begin{aligned} \min \quad & \|x - x^*\|^2 \\ \text{s.t.} \quad & Ax \leq b(\sigma), \\ & y^T A = c, y \geq 0, \\ & y^T b(\sigma) - c^T x \leq 0, \end{aligned}$$

where we have simply substituted the third line into the fourth. Note that while this reformulation makes the problem more linear, it does not eliminate the difficulty posed by MPEC's.

By now it has become clear, as indicated in [3] and [4], that the best way to solve (5) is to formulate the problem as

$$(6) \quad \min \|x - x^*\|^2 + \rho \zeta$$

$$\begin{aligned}
& s.t. \ Ax - b(\sigma) \leq \zeta, \\
& -\zeta \leq y^T A - c \leq \zeta, \ y \geq -\zeta, \\
& y^T b(\sigma) - c^T x \leq \zeta, \ \zeta \geq 0.
\end{aligned}$$

The idea is to relax the constraints and penalize the maximum of their violation in the objective function. The hope is that by making ρ sufficiently large, the violation ζ can be driven to 0, and hence a true solution to our problem be found. Usually for nonconvex problems it is not possible to guarantee this, in other words one cannot be sure that even a feasible point will be reached. But we shall now show that despite this problem being nonconvex, we can always find a solution with $\zeta = 0$, in other words we can always find a local minimum to (5).

We refer to (6) for fixed ρ as P_ρ . We define an *Interior Point Penalty Algorithm* (IPPM) as follows:

Assume we start with a starting value of ρ and a solution to P_ρ : $(x_\rho, \sigma_\rho, \zeta_\rho)$, $\zeta_\rho > 0$. Then increase ρ slightly and apply an interior-point method of the kind described in [9] or [13] to resolve the problem, using the previous solution as the new starting point.

Theorem 1. *If (6) satisfies some standard assumptions as laid out in [15], then the IPPM will converge to a solution of (6) with $\zeta = 0$ as $\rho \rightarrow \infty$.*

Proof. We make some standard assumptions about our problem. Specifically, we assume that at all solutions to P_ρ with $\zeta_\rho > 0$, active constraint gradients are linearly independent, second-order sufficiency conditions are satisfied, and Lagrange multipliers of active constraints are positive. We also need to assume that the feasible set for (5) is bounded. This can easily be ensured by placing appropriate upper and lower bounds on σ (though we will not write these explicitly). Under these conditions, theorem 1 in [15] shows that if ρ is increased sufficiently gradually, the IPPM will always converge to a point (x, σ, ζ) such that ζ (locally) minimizes the maximum constraint violation for (5) at (x, σ) . We now show that this must mean $\zeta = 0$. To do so, consider the problem without the objective function,

$$\begin{aligned}
(7) \quad & \min \ \zeta \\
& s.t. \ Ax - b(\sigma) \leq \zeta,
\end{aligned}$$

$$-\zeta \leq y^T A - c \leq \zeta, y \geq -\zeta,$$

$$y^T b(\sigma) - c^T x \leq \zeta, \zeta \geq 0,$$

that is, the problem of minimizing the maximum of the constraint violations of (5). If σ is a fixed constant, this is of course the problem of solving the optimality conditions of LP(σ), and it is linear. Thus the only local minimum is a global minimum, and it is shown in [8] that (3) always has a finite optimal solution. Thus the global minimum (when σ is fixed) must have $\zeta = 0$ (with x the solution to LP(σ)). Making σ a variable cannot change the situation: if (x, ζ) is not a local minimum for any fixed σ , clearly (x, σ, ζ) cannot be a local minimum for any σ , with σ variable. Thus the only solutions to (7) have $\zeta = 0$. This proves the theorem. \square

In the theorem we allowed ρ to go to ∞ . However, under some reasonable conditions, as described in [15], the IPPM will converge to a solution with $\zeta = 0$ for a finite ρ . The conditions amount to requiring in our case that if $(\hat{x}, \hat{\sigma})$ is a solution to (5), then LP($\hat{\sigma}$) be a nondegenerate LP with a unique optimal solution.. It is shown in [8] that (3) always has a unique primal solution, i.e. the optimal value function is always unique. LP($\hat{\sigma}$) is nondegenerate precisely when there are no redundant constraints at \hat{x} , which amounts to saying that at each node k, j in the binomial tree only one of the following two constraints should be tight:

$$v_{k,j} \geq \max(K - S_0 e^{(r - \frac{\sigma^2}{2})k \frac{T}{n} + \sigma \sqrt{\Delta t} j}, 0),$$

$$v_{k,j} \geq \frac{1}{2} e^{-r \frac{T}{n}} (v_{k+1,j+1} + v_{k+1,j-1}).$$

That is to say, in the solution to the discretized optimal stopping problem, at no point should the investor be indifferent between exercising and not exercising. In the original continuous model, of course, an optimal exercise boundary exists. But in the discretized version, the boundary need not intersect the tree at any node. This cannot be ruled out (the study of the exercise boundary for discretized optimal stopping problems is a nontrivial one), but in practice should rarely occur. Thus a solution should be reached quite quickly. Also, based on the theorem, one might worry that the penalty ρ will have to be increased slowly, but in practice most numerical evidence with MPEC's has shown that fast increases in ρ do in fact work (see [4]). Thus we

potentially have an algorithm that is quite practical to implement, and in the next section we provide some (preliminary) numerical results.

5. SOME NUMERICAL EXAMPLES

We tested our method on a simple example, using LOQO, an interior-point nonlinear solver (see [4] for some details). We considered an American put option on a stock with $S_0 = 12, K = 10, r = .05, T = 1$. We first supposed we had an observed value for the price at time 0. To get the “observed value” we set $\sigma = .2$, computed the approximate price (which turns out to be about .137) and perturbed it to be .16. We obtained the following results (where the last column gives the value of the penalty parameter needed to solve the MPEC):

n	observed price = .16		penalty
	implied σ	computed price	
5	.203968	.16	1
10	.206948	.16	10
20	.210402	.16	10
40	.217356	.174983	100

We note that for $n = 20$ we also found a solution with price = .193001, $\sigma = .225688$. Thus the problem is clearly nonconvex, and in general one might need some number of restarts to be confident that the best fit has been found.

To make things more interesting, we supposed we had observed prices not only for the current time, but the node in the tree representing one time-step forward and one downtick in the stock price (of course the derivation in this paper allows one to match prices corresponding to all the nodes in the tree, not just two). We let the two observed prices be .16 and .24:

n	observed prices = .16,.24		penalty
	implied σ	computed prices	
5	.196461	.139448, .252084	1
10	.20451	.153882, .244072	1
20	.213148	.165727, .235865	1
40	.217525	.175399, .227717	100

Again we obtained a fairly close fit (when fitting more than one price, an exact fit is not likely for a constant volatility). In general, prices for different options on the same underlying asset (i.e. with different strike prices, maturities, etc.) may be matched under our approach. Looking

forward, our method will even be able to handle a model which does not assume a constant volatility, but this will require an extension to the Black-Scholes model, which will be dealt with in future work.

6. CONCLUSION

In this paper we have presented an optimization approach to finding the implied volatility of American options, a problem of significant interest. We prove convergence of our method under general conditions, and provide some promising (if preliminary) numerical results. Future work will make the numerical implementation and experimentation more robust, and also consider extensions to the Black-Scholes model. Specifically, one can consider an asset whose price process S_t evolves according to the SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S, t) dW_t,$$

in other words a model in which the volatility is a function of state and time. Thus the inverse problem would be to find a true volatility surface, making the underlying model more consistent with the obvious fact that volatility is not constant. Such a price process was indeed considered in [10], but again, they did not prove convergence results. The point is that any Markov diffusion process can be approximated by a finite-state Markov chain, resulting in a linear program for solving the optimal stopping problem (see e.g. [6]). Thus the results in this paper would be directly applicable to the extended framework.

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