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Wavelength Assignment in Multi-Fiber WDM Networks by Generalized Edge Coloring

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Abstract

In this paper, we study wavelength assignment problems in multi-fiber WDM networks. We focus on the special case that all lightpaths have at most two links. This in particular holds in case the network topology is a star. As the links incident to a specific node in a meshed topology form a star subnetwork, results for stars are also of interest for general meshed topologies.

We show that wavelength assignment with at most two links per lightpath can be modeled as a generalized edge coloring problem. By this relation, we show that for a network with an even number of fibers at all links and at most two links per lightpath, all lightpaths can be assigned a wavelength without conversion. Moreover, we derive a lower bound on the number of lightpaths to be converted for networks with arbitrary numbers of fibers at the links.

A comparison with linear programming lower bounds reveals that the bounds coincide for problems with at most two links per lightpath. For meshed topologies, the cumulative lower bound over all star subnetworks equals the best known solution value for all realistic wavelength assignment instances available, by this proving optimality.

1 Introduction

As soon as the huge potential of optics as digital information carrier became clear, the problem of assigning wavelengths to ongoing optical connections (so-called *lightpaths*) in a communication network has gained the interest of researchers from areas as different as electrical engineering, theoretical computer science, and discrete mathematics. Especially the close relation to vertex coloring in graph theory has been considered by many authors, cf. Beauquier et al. [2] and Ferreira et al. [6]. Classical vertex coloring however models wavelength assignment in optical telecommunication networks in a very limited non-practical setting. In particular, the objective does not satisfy the needs for the design and operation of optical networks. Where in vertex coloring the objective is to minimize the number of colors used, in an optical network the number and availability of wavelengths (=colors)

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is fixed by the installed wavelength division multiplexing (WDM) systems. Moreover, the binary relations (=edges) typical for vertex coloring have to be generalized for wavelength assignment in multi-fiber WDM networks, ending up with a problem without much combinatorial structure (at first sight). In this paper, we therefore follow another approach: we show that the theory of (generalized) edge coloring provides a very appropriate tool for solving real-world wavelength assignment problems.

Wavelength assignment is of importance in both the design of an optical network and the operation of it. In the design or expansion of an optical network, we have to equip the nodes and links of a meshed topology with the capabilities to switch and transmit a set of lightpaths such that all (expected) demands for communication bandwidth can be realized. Moreover, in case of network failures, we still have to guarantee that a prespecified part of the demands survive. As the investment costs of optical communication equipment are high and the competitiveness of the telecommunication market depresses the profit margins on connections, minimizing the overall network cost is obviously in the interest of the network operator. For the same reasons the utilization of an operated network should be maximized.

The task to design a minimum cost optical network consists of three subtasks: dimensioning the hardware topology, routing a set of lightpaths, and assigning wavelengths to these lightpaths. The latter two are often combined to the Routing and Wavelength Assignment (RWA) problem (cf. [15, 16, 33]), whereas the dimensioning of the network has been studied less frequently (notably exceptions are Belotti [3], Brunetta et al. [4], Melian et al. [22, 23], and Zymolka et al. [35]). From a cost-oriented point of view however, dimensioning and routing are very close related to each other as the lightpaths consume the resources that are provided by the switching and transmission equipment. Therefore, Zymolka et al. [35] proposed to decompose the overall problem in a Dimensioning and Routing (DR) subproblem and a Wavelength Assignment (WA) subproblem. In the DR problem a hardware configuration consisting of optical fibers, WDM systems, and optical switches as well as a set of paths is determined that allows to route the traffic in normal operation and all failure states. In the WA problem, the paths are supplied with a wavelength at each of its links such that at each link the number of times a wavelength is assigned does not exceed its availability by the WDM systems. For the DR problem, experience from the design of other telecommunication networks can be explored, e.g., SDH network design [31]. Lower bounds on the cost of dimensioning and routing propagate to the overall network cost, and in case the WA subproblem can be solved without increasing the network cost, an optimal solution for the DR problem is optimal overall. In fact, computational experiments have shown that in many cases WA can be done without the installation of extra equipment, see [35] as well as Section 6.

The wavelength assignment problem remains an important subproblem as it decides whether or not additional equipment is necessary. Moreover, the problem shows interesting combinatorial properties. In case the available number of wavelengths does not suffice for an ongoing assignment of a single wavelength to each lightpath, the classical vertex coloring objective to minimize the number of wavelengths does not discriminate among solutions that are (ir)relevant for wavelength assignment. Therefore, instead of minimizing the number of wavelengths, two alternative objectives for wavelength assignment are considered in this paper. One option is to maximize the number of lightpaths that can be assigned an ongoing wavelength without conflict. Alternatively, a lightpath can be assigned different wavelengths at different parts of the path. For the technical realization of such an assignment, a so-call wavelength converter have to be installed in each intermediate node where the wavelength is exchanged. In this paper, we assume that wavelength converters can convert a single lightpath (see [15] for other converter models). In the first problem, we minimize the blocking number of lightpaths. In the second, we minimize the number of wavelength converters needed for a conflict-free assignment of the complete set of lightpaths.

The study of wavelength assignment concentrates on special network structures like line, star, tree, and ring networks. For the single fiber case, several results for minimizing the number of wavelengths in ring and tree networks have been derived, cf. Auletta et al. [1]. For multi-fiber line networks, Winkler and Zhang [32] proved that the minimum number of wavelengths needed equals the maximum load of the links. Stated otherwise, no wavelength conversion is needed in line networks. Li and Simha [20] studied ring networks as well as instances in which all lightpaths are restricted to at most two links. They showed that in case every link contains k fibers, k even, no wavelength conversion is necessary.

In this paper, we also focus on wavelength assignment with all lightpaths having at most two links, but do not limit to this setting. Lightpath sets with at most two links per lightpath will most likely not occur in practice, but the theoretical results are helpful to understand the problem in more general cases. In particular, the links incident to a specific node in a general meshed topology induce a star network. By definition, the lightpaths in a star network have at most two links, and thus all results in this paper apply to such networks. Moreover, a lower bound on the number of wavelength converters needed in the overall network can be computed by summing up the lower bounds over all stars.

For the two links per lightpath case, we apply results from (generalized) edge coloring to generalize the result of Li and Simha that no wavelength conversion is necessary by allowing link-individual, but even, number of fibers. For the case that the number of fibers is odd at some of the links we derive a lower bound on the number of wavelength converters needed. We show that in the setting of two links per lightpath, this lower bound equals a column generation based linear programming lower bound derived in [17, 19]. For general instances, a lower bound is derived in the way as described above and tested on a set of realistic test instances. Without exception, the lower bound turns out to be equal to the best known solution value provided by heuristic algorithms (cf. [18]), by this proving optimality.

The remaining of this article is structured as follows. In Section 2 we start with introducing the necessary notation and some preliminary results. Next, the case with at most two links per lightpath is studied in Section 3. The lower bound on the number of converters for general wavelength assignment instances is derived in Section 4. In Sections 5 and 6, we compare the lower bound in respectively theory and practice with alternative lower bounds as well as with the best known solution value for the minimum converter WA problem. Concluding remarks and suggestions for further research close this paper.

2 Preliminaries and notation

Optical networks and wavelength assignment can be modeled in many different ways. In this paper, we model it in such a way that the relevant features are incorporated and a distinction between graph theoretical notions and optical network notions is straightforward.

Graph theoretical notation. Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, we denote with $G[S]$ the subgraph induced by S. If $E(G)$ contains at most one edge between each pair of vertices, we call G *simple*; otherwise G is called a *multigraph*. The maximum number of parallel edges in G is denoted by $\mu(G)$.

The neighborhood of $v \in V(G)$ consist of all vertices adjacent to v and is denoted by $N_G(v)$. The closed neighborhood $N_G[v]$ is defined by $N_G(v) \cup \{v\}$. The edges incident to a vertex v are denoted by $\delta_G(v)$. Let $d_G(v) = |\delta_G(v)|$ be the degree of $v \in V(G)$ and let $\Delta(G) = \max_{v \in V(G)} d_G(v)$ denote the maximum degree of G. If the graph is clear from the context, the subscript in the above notation is suppressed.

The chromatic number $\chi(G)$ of a graph G is the minimum number of colors to be assigned to the vertices $V(G)$ such that adjacent vertices have different colors. It is well-known that deciding whether the chromatic number $\chi(G) \leq K$ for some $K \geq 3$ is $\mathcal{NP}$-hard (cf. [7]).

The chromatic index $\chi'(G)$ of a graph G is the minimum number of colors to be assigned to the edges $E(G)$ such that all edges incident to a vertex have different colors. For simple connected graphs, $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G) + 1$, cf. [30], but deciding the exact value is $\mathcal{NP}$-hard, see Holyer [13]. For multigraphs, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$.

Wavelength assignment notation. Let an undirected graph $\mathcal{N} = (N, L)$ define the physical topology of an optical network consisting of nodes N and links L. Links are assumed to be bidirectional, i.e., they provide the same capacity in both directions. For a link $\ell \in L$, κ_ℓ fibers with WDM systems are installed. If not explicitly mentioned, we assume all WDM systems to be equivalent, providing the available spectrum Λ once (i.e., each wavelength $\lambda \in \Lambda$ is available κ_ℓ times on link ℓ). Hence, in total $K_\ell = |\Lambda|\kappa_\ell$ optical channels are available on link $\ell \in L$. We assume that in each node $n \in N$ enough switching capacity is installed to switch all channels provided by the links incident to it.

For a path p in $\mathcal{N}$, we denote with d_p the number of lightpaths to be routed along this path. Let $L(p) \subseteq L$ denote the links of path p, whereas $N(p)$ denotes the intermediate nodes. Although lightpaths are considered to be bidirected (i.e., provide transmission capacity in both directions), for formulation purposes the path p is directed from a source ς^p to a target τ^p. All lightpaths that have to be assigned wavelengths are gathered in a multi-set $\mathcal{P}$ where each path p is contained d_p times. In total $d(\mathcal{P}) = \sum_{p \in \mathcal{P}} d_p$ lightpaths are considered. All lightpaths that share link $\ell \in L$ are subsumed in subset $\mathcal{P}_\ell \subset \mathcal{P}$, whereas $\mathcal{P}_n$ denotes all lightpaths that touch node $n \in N$. Throughout this paper, we assume that

$$\sum_{p \in \mathcal{P}_\ell} d_p \leq K_\ell \tag{1}$$

holds for all links $\ell \in L$, i.e., the installed channel capacities are sufficient to route the given set of lightpaths. Under this assumption, lightpaths consisting of a single link can always be assigned a wavelength independently from the other lightpaths. Therefore, such lightpaths are left out in our further considerations. We denote an instance of the wavelength assignment problem (WAP) by the quadruple $\mathcal{W} = (\mathcal{N}, \mathcal{P}, \Lambda, \kappa)$. Instances where all lightpaths are restricted to have at most two links are denoted by $\mathcal{W}_2$.

A wavelength converter is able to convert a single optical signal from any wavelength to any other wavelength. The assumption that it is possible to install an unlimited number of converters in a node guarantees that a feasible wavelength assignment exists. The quality of such a wavelength assignment is measured by the number of converters that are needed. With the *converter number* $\Upsilon(\mathcal{W})$ we denote the minimum number of converters needed in any feasible wavelength assignment.

Alternatively, WAP can be considered without wavelength conversion. In such a case the conflict-free assignment of a single wavelength to the whole lightpath can be limited to a subset of the lightpaths. Given a WAP instance $\mathcal{W}$, we denote with the *lightpath number* $\Psi(\mathcal{W})$ the maximum number of lightpaths that can be assigned a wavelength without wavelength conversion.

Preliminary results. Between both notions, the following relation holds for lightpaths with arbitrary number of links:

Lemma 1 $\Upsilon(\mathcal{W}) \geq d(\mathcal{P}) - \Psi(\mathcal{W})$ *for any instance* $\mathcal{W}$*.*

Proof: The minimum number of lightpaths that cannot be assigned a wavelength without conversion equals $d(\mathcal{P}) - \Psi(\mathcal{W})$. For each of these lightpaths, at least one converter is inevitable. □

Corollary 2 *For any instance* $\mathcal{W}$*,*

- $\Psi(\mathcal{W}) = d(\mathcal{P})$ *if and only if* $\Upsilon(\mathcal{W}) = 0$*, and*
- $\Psi(\mathcal{W}) = d(\mathcal{P}) - 1$ *if* $\Upsilon(\mathcal{W}) = 1$*.*

Note that the reverse of the second statement does not hold in general. It could be necessary to convert the wavelength more than once along a lightpath that could not be assigned a wavelength without conversion. In case $|L(p)| \leq 2$ however, there is only one intermediate node to convert the signal and thus the result can be strengthened further:

Corollary 3 $\Upsilon(\mathcal{W}_2) = d(\mathcal{P}) - \Psi(\mathcal{W}_2)$ *for any instance* $\mathcal{W}_2$*.*

In case $\kappa_\ell = 1$ for all links $\ell \in L$, the question whether or not a wavelength assignment without conversion exists is equivalent to the vertex coloring problem on the so-called *path conflict graph*. The path conflict graph $G_{\mathcal{P}}$ contains a vertex for every lightpath, and two vertices are adjacent if their lightpaths share a link.

Lemma 4 *For any instance $\mathcal{W}$ with $\kappa_\ell = 1$ for all $\ell \in L$, $\Upsilon(\mathcal{W}) = 0$ if and only if $\chi(G_\mathcal{P}) \leq |\Lambda|$.*

In case $|L(p)| \leq 2$ for all paths $p \in \mathcal{P}$, and $\kappa_\ell = 1$ for all links $\ell \in L$, the question whether or not a wavelength assignment without conversion exists is equivalent to the edge coloring problem in a graph G_L as well: Introduce for every link $\ell \in L$ a vertex v_ℓ in G_L. For a lightpath that goes along links ℓ_1 and ℓ_2, $\ell_1 \neq \ell_2$, we add an edge $\{v_{\ell_1}, v_{\ell_2}\}$. Note that parallel lightpaths result in parallel edges in the graph.

Lemma 5 *For any instance $\mathcal{W}_2$ with $\kappa_\ell = 1$ for all links $\ell \in L$, $\Upsilon(\mathcal{W}_2) = 0$ if and only if $\chi'(G_L) \leq |\Lambda|$.*

The relation between edge coloring and wavelength assignment has been considered before (cf. [5, 12, 18, 27]), particularly to prove that all variants of wavelength assignment are $\mathcal{NP}$-hard on star networks, even if $\kappa_\ell = 1$ for all $\ell \in L$.

3 Lightpaths with at most two links

In this section, we assume that $|L(p)| \leq 2$ for all $p \in \mathcal{P}$. We only state the results for $\Upsilon(\mathcal{W}_2)$ as the results for $\Psi(\mathcal{W}_2)$ can be derived by applying Corollary 3.

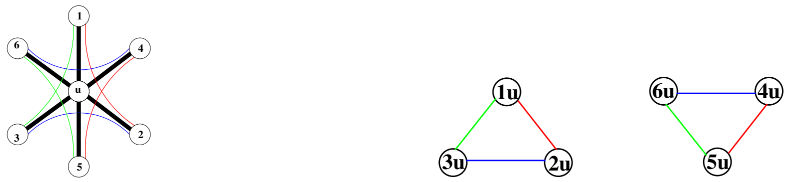
If $\kappa_\ell > 1$ for some link $\ell \in L$, a feasible assignment of wavelengths to the lightpaths is not equivalent to an edge coloring in G_L anymore, since two lightpaths that share link ℓ can be assigned the same wavelength. By its availability, we can in fact assign the same wavelength to at most κ_ℓ lightpaths that all share link ℓ. As the vertices of G_L correspond to the links whereas the edges correspond to the lightpaths, this implies that the same color can be used κ_ℓ times to color edges incident to vertex v_ℓ.

Introduced by Hakimi and Kariv [11], an *f-(edge)-coloring* of G is a coloring of the edges such that the number of identical colored edges incident to a vertex $v \in V(G)$ is limited by a vertex-specific bound $f_v \in \mathbb{Z}^+$. The minimum number of colors needed in an f-(edge)-coloring is the *f-chromatic index* $\chi'_f(G)$. Now, the following results can be easily verified.

Theorem 6 *Let $f_{v_\ell} = \kappa_\ell$ for all $\ell \in L$. For any instance $\mathcal{W}_2$, $\Upsilon(\mathcal{W}_2) = 0$ if and only if $\chi'_f(G_L) \leq |\Lambda|$.*

So, in order to determine whether or not wavelength conversion is necessary, we have to compute the f-chromatic index of G_L. Since $\chi'(G)$ is already $\mathcal{NP}$-hard to compute, $\chi'_f(G)$ is $\mathcal{NP}$-hard as well. Moreover, G_L need not to have any particular structure: In a star network, lightpaths can consist of any arbitrary set of two links resulting in any arbitrary edge in G_L. In fact, G_L need not to be connected, cf. the network and lightpaths in Figure 1.

Fortunately, lower and upper bounds on $\chi'_f(G)$ can be applied to bound $\Upsilon(\mathcal{W}_2)$. Let $d_f(v) = \frac{d(v)}{f_v}$ be the f-normalized degree of v and let $\Delta_f(G) = \max_{v \in V(G)} d_f(v)$ be the f-normalized maximum degree of G.



(a) Star network (b) Graph G_L

Figure 1: Optical network resulting in an f-edge-coloring problem with two components.

Lemma 7 (Hakimi and Kariv [11]) *Let G be a simple graph and $f_v > 0$ for all $v \in V(G)$. Then $\Delta_f(G) \le \chi'_f(G) \le \Delta_f(G) + 1$.*

Lemma 8 *Let $f_{v_\ell} = \kappa_\ell$ for all $\ell \in L$. For any instance $\mathcal{W}_2$ with $d_p = 1$ for all $p \in \mathcal{P}$, the converter number equals the minimum size of the smallest color class in a $|\Lambda| + 1$ f-edge-coloring.*

Proof: Since $\mathcal{W}_2$ does not contain parallel lightpaths, G_L does not contain parallel edges, and thus is simple. By (1), the normalized degree $d_f(v) \le |\Lambda|$ for all $v \in V(G)$. By Lemma 7, $\chi'(G_L) \le \Delta_f(G_L) + 1 \le |\Lambda| + 1$. Given a $|\Lambda| + 1$ f-edge-coloring, the number of converters is smallest by converting all lightpaths corresponding to edges in the smallest color class. Minimizing this size results in the minimum number of converters needed (note that if $\chi'(G_L) \le |\Lambda|$, the size of the smallest color class in a $|\Lambda| + 1$ f-edge-coloring is zero). □

For multigraphs it still holds that $\chi'_f(G_L) \ge \Delta_f(G_L)$. By (1), $\Delta_f(G_L) \le |\Lambda|$ and thus this bound does not provide us with a non-trivial lower bound on $\Upsilon(\mathcal{W}_2)$. An alternative lower bound for the f-chromatic index of multigraphs has been derived in [24]. This result can be adapted to our specific setting in which conversion has to be minimized, not colors. For a multigraph G and $f_v > 0$ for all $v \in V(G)$, let $\Gamma_f(G)$ be defined by

$$\Gamma_f(G) = \max\left\{0, \max_{S \subseteq V(G), f(S) \text{ odd}} |E(G[S])| - |\Lambda| \left\lfloor \tfrac{1}{2} f(S) \right\rfloor \right\}$$

where $f(S) := \sum_{v \in S} f_v$ denotes the sum over a subset $S \subseteq V(G)$ of vertex bounds. Note that by (1), $|E(G[S])| \le \frac{1}{2}|\Lambda| f(S)$ and thus subsets S with $f(S)$ even are left out of the maximum.

Theorem 9 *Let $f_{v_\ell} = \kappa_\ell$ for all $\ell \in L$. For any instance $\mathcal{W}_2$,*

$$\Upsilon(\mathcal{W}_2) \ge \Gamma_f(G_L) \qquad (2)$$

Proof: For a subset $S \subseteq V(G_L)$, at most $\lfloor \frac{1}{2} f(S) \rfloor$ of the edges can be colored with the same color (the maximum size of a so-called f-matching). So with $|\Lambda|$ colors available, at most $|\Lambda| \lfloor \frac{1}{2} f(S) \rfloor$ lightpaths can be colored without conversion. Hence, at least $|E(G_L[S])| -$

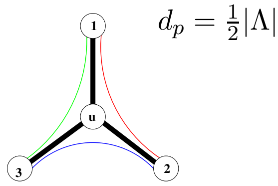


Figure 2: Star network for which $\Omega_f(G) = \frac{3}{2}|\Lambda|$.

$|\Lambda|\lfloor\frac{1}{2}f(S)\rfloor$ have to be converted. The result now follows by taking the maximum over all subsets S of G_L. □

For an instance $\mathcal{W}_2$, define $\Gamma_f(\mathcal{W}_2) := \Gamma_f(G_L)$. To see that $\Gamma_f(\mathcal{W}_2)$ is the key to a non-trivial lower bound on the number of converters, we bound $|E(G_L[S])|$ from above and show that this bound can be tight.

Proposition 10 *Let $S \subseteq V(G_L)$, $|S| \geq 3$. If $\kappa_\ell \geq k \geq 1$, for all $\ell \in L$, then*

$$|E(G_L[S])| \leq \frac{3k}{3k-1}|\Lambda| \left\lfloor \frac{1}{2}f(S) \right\rfloor \quad .$$

Moreover, if S corresponds to three (pairwise non-parallel) links incident to a central node, $\kappa_\ell = k = 1$ for all $v_\ell \in S$, and $d_p = \frac{1}{2}|\Lambda|$ for each path p corresponding to a pair of vertices in S (cf. Figure 2), the inequality is tight. In fact, $\Upsilon(\mathcal{W}_2) = \frac{1}{2}|\Lambda|$ for this star network.

Proof: A subset S of the vertices in the f-edge-coloring problem corresponds in WAP to a subset of the links $L_S \subseteq L$. Each induced edge corresponds to a lightpath p with $v_\ell \in S$ for both $\ell \in L(p)$. For a routing satisfying (1) it holds that

$$|E(G_L[S])| \leq \frac{1}{2}|\Lambda|f(S) = \frac{1}{2}|\Lambda| \sum_{v_\ell \in S} f_{v_\ell} = \frac{1}{2}|\Lambda| \sum_{v_\ell \in S} \kappa_\ell \quad . \tag{3}$$

By $|S| \geq 3$ and $\kappa_\ell \geq k \geq 1$ for all $\ell \in L$ we have

$$\frac{1}{2}\sum_{v_\ell \in S} \kappa_\ell \leq \frac{3k}{3k-1} \left\lfloor \frac{1}{2} \sum_{v_\ell \in S} \kappa_\ell \right\rfloor \quad . \tag{4}$$

In case of $|S| = 3$, $\kappa_\ell = k = 1$ for all $v_\ell \in S$ and $d_p = \frac{1}{2}|\Lambda|$ for each path p corresponding to a pair of vertices in S, inequalities (3) and (4) both turn into equalities and $\frac{3k}{3k-1} = \frac{3}{2}$. Now by Theorem 9, $\Upsilon(\mathcal{W}_2) \geq \frac{1}{2}|\Lambda|$. By converting all lightpaths for one pair of nodes we also have $\Upsilon(\mathcal{W}_2) \leq \frac{1}{2}|\Lambda|$. □

If the WDM systems installed at the fibers are not equivalent, lower bound $\Gamma_f(\mathcal{W}_2)$ can be adapted appropriately. Let κ_ℓ^λ denote the number of times wavelength $\lambda \in \Lambda$ is available on link $\ell \in L$ (by summing up over all WDM systems).

Theorem 11 *Let $f_{v_\ell}^\lambda = \kappa_\ell^\lambda$ for all $\ell \in L$, $\lambda \in \Lambda$. For any instance $\mathcal{W}_2$,*

$$\Upsilon(\mathcal{W}_2) \geq \max\left\{0, \max_{S \subseteq V(G)} |E(G[S])| - \sum_{\lambda \in \Lambda} \left\lfloor \tfrac{1}{2} f(S, \lambda) \right\rfloor \right\} \tag{5}$$

where $f(S,\lambda) := \sum_{v \in S} f_v^\lambda$.

Proof: Similar to the proof of Theorem 9. □

Back to the case where all WDM systems are equivalent, the upper bound of $\Delta_f(G) + 1$ for $\chi'_f(G)$ does not hold in case of multigraphs. In [11], several upper bounds for $\chi'_f(G)$ are derived as well. One of them is of special interest for the case that all capacities are even.

Lemma 12 (Hakimi and Kariv [11]) *Let G be a multigraph. Suppose $f_v > 1$ for all $v \in V(G)$. Then $\chi'_f(G) \leq \Delta'_f(G)$, where*

$$\Delta'_f(G) = \max_{v \in V(G)} \left\lceil \max\left(\frac{\lfloor \frac{1}{2} d(v) \rfloor}{\lfloor \frac{1}{2} f_v \rfloor}, \frac{\lceil \frac{1}{2} d(v) \rceil}{\lceil \frac{1}{2} f_v \rceil} \right) \right\rceil .$$

Theorem 13 *If κ_ℓ is even for all $\ell \in L$, then for any instance $\mathcal{W}_2$, $\Upsilon(\mathcal{W}_2) = 0$.*

Proof: Since $f_{v_\ell} = \kappa_\ell$ is even for all $v_\ell \in S$, $\lfloor \frac{1}{2} f_{v_\ell} \rfloor = \lceil \frac{1}{2} f_{v_\ell} \rceil = \frac{1}{2}\kappa_\ell$, and thus the maximum is determined by $\lceil \frac{1}{2} d(v_\ell) \rceil$ for some $v_\ell \in S$. For a proper routing, $d(v_\ell) \leq \kappa_\ell |\Lambda|$. Since κ_ℓ is even, $\lceil \frac{1}{2} \kappa_\ell |\Lambda| \rceil = \frac{1}{2} \kappa_\ell |\Lambda|$. Hence, $\chi'_f(G_L) \leq \Delta'_f(G) \leq |\Lambda|$ and the result follows. □

So, in case of an even number of fibers at all links and lightpaths restricted to at most two links, no wavelength conversion is necessary, whatever lightpaths have to be established.

4 General wavelength assignment instances

In real-life wavelength assignment instances the meshed topology of (optical) telecommunication networks prohibits the existence of wavelength assignment instances with at most two links per lightpath: the shortest path between two nodes often crosses three or more links, and thus any lightpath between these nodes will have more than two links. Even if the shortest path consists of two links, survivability requirements often imply that the demand have to be split among several paths. Finally, optimization of the network cost often leads to even longer paths to save equipment, see [35] for a discussion.

As the relation to generalized edge coloring depends on the condition that each lightpath has two links, it is unclear how to extend the results of the previous section to general wavelength assignment instances. In particular, the result that no conversion is necessary if κ_ℓ is even for all links seems to be difficult to generalize.

The lower bound on the converter number $\Upsilon(\mathcal{W}_2)$ given by (2) however can be used to determine a lower bound on $\Upsilon(\mathcal{W})$ in a general wavelength assignment instance $\mathcal{W}$. Consider a node $n \in N$. If we restrict WAP to the links incident to n, then all relevant lightpaths consist of one or two links. For this restricted instance we can apply (2) to derive a lower bound on the number of converters in node n. Summing up over all nodes we get the following result:

Theorem 14 *For any instance $\mathcal{W}$,*

$$\Upsilon(\mathcal{W}) \geq \sum_{n \in N} \Gamma_f(\mathcal{W}_n)$$

where $\mathcal{W}_n$ is defined by $(\mathcal{N}_n, \mathcal{P}_n, \Lambda, \kappa_n)$ with $\mathcal{N}_n = (N_{\mathcal{N}}[n], \delta_{\mathcal{N}}(n))$ and κ_n the fiber-vector restricted to the links $\delta_G(n)$.

Proof: From Theorem 9 it directly follows that $\Upsilon(\mathcal{W}_n) \geq \Gamma_f(\mathcal{W}_n)$ for all $n \in N$. Given a wavelength assignment for $\mathcal{W}$, it is straightforward to construct a wavelength assignment for $\mathcal{W}_n$ with the same number of converters in node $n \in N$ as in the original instance. Hence, $\Upsilon(\mathcal{W}) \geq \sum_{n \in N} \Upsilon(\mathcal{W}_n)$. □

Noteworthy, for the lightpath number $\Psi(\mathcal{W})$ such a result cannot be derived, since a lightpath can be converted in multiple nodes along the path. By the assumption that lightpaths are simple, i.e., do not visit nodes more than once, the best achievable is that converters in the same node convert different lightpaths:

Theorem 15 *For any instance $\mathcal{W}$, $\Psi(\mathcal{W}) \leq d(\mathcal{P}) - \max_{n \in N} \Gamma_f(\mathcal{W}_n)$ with $\mathcal{W}_n$ defined as in Theorem 14.*

In Section 6, the lower bound of Theorem 14 is computed for realistic WAP instances.

5 Theoretical comparison

In this section, we compare the derived bounds for $\Upsilon(\mathcal{W})$ and $\Psi(\mathcal{W})$ with other bounds theoretically. In [17], two integer linear programming formulations for the converter number $\Upsilon(\mathcal{W})$ have been developed (see also [19]). Both formulations can be easily adapted for the lightpath number $\Psi(\mathcal{W})$. By solving the linear relaxation of any of the formulations a lower respectively upper bound is computed.

The first formulation of the converter number problem is straightforward with variables for every combination of lightpath, link, and wavelength. In addition, variables to count the converters are introduced. In [17], it has been shown that if all WDM systems provide the same set of wavelengths, the value of the linear relaxation of this formulation always equals zero. This drawback is due to the symmetry in the (fractional) solutions: the exchange of any two wavelengths results in a solution with the same value.

To avoid the symmetry of solutions, a second formulation is presented in which variables are introduced for every set of lightpaths that can be assigned the same wavelength, without specifying the wavelength. The formulation generalizes those for the vertex coloring

(a) edge color graph G_L (=Petersen graph) (b) star network instance

Figure 3: Wavelength assignment instance with $2 = \Upsilon(\mathcal{W}_2) > \Upsilon^*(\mathcal{W}_2) = 0$ ($|\Lambda| = 3$).

problem [21] and for the edge coloring problem [25]. To compare this formulation with the lower bound of Theorem 14, we have to study the formulation in detail and need some more notation. For each $p \in \mathcal{P}$, let $\mathcal{S}_p$ denote the set of all subpaths s of p. Let $\mathcal{S} = \cup_{p\in\mathcal{P}}\mathcal{S}_p$ denote the set of all possible subpaths. A *path packing* ϕ is a multi-set of items of $\mathcal{S}$ such that all subpaths $s \in \phi$ can be assigned the same wavelength, i.e., for every link $\ell \in L$, at most κ_ℓ subpaths containing link ℓ are in the set ϕ. The multiplicity of each subpath $s \in \mathcal{S}$ in the path packing ϕ is denoted by t^s_ϕ. The collection of all multi-sets of $\mathcal{S}$ that are path packings is denoted by Φ.

For every path packing $\phi \in \Phi$ a general integer variable x_ϕ is introduced, denoting the number of wavelengths assigned to all subpaths $s \in \phi$. To specify the subpaths that are used to cover a path $p \in \mathcal{P}$ a second class of variables y^s_p is introduced, which denote the number of times subpath s is used to cover the lightpaths routed along path $p \in \mathcal{P}$. The converter number then is formulated as:

$$\Upsilon(\mathcal{W}) = \min \sum_{p\in\mathcal{P}} \sum_{s\in\mathcal{S}_p} y^s_p - \sum_{p\in\mathcal{P}} d_p \tag{6}$$

$$\text{s.t.} \quad \sum_{s\in\mathcal{S}_p:\ell\in L(s)} y^s_p = d_p \qquad \forall p \in \mathcal{P}, \ell \in L(p) \tag{7}$$

$$\sum_{\phi\in\Phi} t^s_\phi x_\phi = \sum_{p\in\mathcal{P}:s\in\mathcal{S}_p} y^s_p \qquad \forall s \in \mathcal{S} \tag{8}$$

$$\sum_{\phi\in\Phi} x_\phi \le |\Lambda| \tag{9}$$

$$y^s_p, x_\phi \in \mathbb{Z}^+_0 \tag{10}$$

For a lightpath the number of converters is given by the number of subpaths minus one. Summing up over all lightpaths gives the objective (6). For every link $\ell \in L(p)$ of a path $p \in \mathcal{P}$, (7) guarantees that d_p subpaths are selected. Constraints (8) model that each subpath $s \in \mathcal{S}$ has to be covered by path packings as often as it is needed for the lightpaths. Finally, constraint (9) restricts the number of selectable path packings to the size of the available spectrum Λ, and constraints (10) guarantee integrality.

Given this formulation, we define the *fractional converter number* $\Upsilon^*(\mathcal{W})$ as the value of the linear relaxation of (6)–(10). It is clear that $\Upsilon^*(\mathcal{W}) \leq \Upsilon(\mathcal{W})$ and there exist instances $\mathcal{W}_2$ for which $\Upsilon^*(\mathcal{W}_2) < \Upsilon(\mathcal{W}_2)$, cf. Figure 3. In contrast to the the value of the linear relaxation of the first formulation, $\Upsilon^*(\mathcal{W}) > 0$ for particular instances: Consider the example of Figure 2 one more time. Not only $\Gamma_f(\mathcal{W}_2) = \Upsilon(\mathcal{W}_2)$ but also $\Upsilon^*(\mathcal{W}_2) = \frac{1}{2}|\Lambda| = \Upsilon(\mathcal{W}_2)$.

The next theorem explains why $\Upsilon^*(\mathcal{W}_2) = \Gamma_f(\mathcal{W}_2)$ for the example of Figure 2 is not a coincidence.

Theorem 16 *For any instance* $\mathcal{W}_2$, $\Upsilon^*(\mathcal{W}_2) = \Gamma_f(\mathcal{W}_2)$.

Proof: For any instance $\mathcal{W}_2$, each lightpath $p \in \mathcal{P}$ consists of two links $\ell_1(p)$ and $\ell_2(p)$. Constraints (7) for subpath $s = \ell_1(p)$ and $s = \ell_2(p)$ imply that $y_p^{\ell_1(p)} = y_p^{\ell_2(p)}$. Moreover, $y_p^p = d_p - y_p^{\ell_1(p)}$ and thus the problem can be reformulated with only one instead of three variables. Without loss of generality we define $z_p = y_p^{\ell_1(p)} = y_p^{\ell_2(p)}$, the number of converters to be placed in the intermediate node of path p. After resubstitution the linear relaxation of (6)–(10) reads

$$\Upsilon^*(\mathcal{W}_2) = \min \sum_{p\in\mathcal{P}} z_p \tag{11}$$

$$\text{s.t.} \quad z_p \leq d_p \qquad \forall p \in \mathcal{P} \tag{12}$$

$$\sum_{\phi\in\Phi} t_\phi^p x_\phi = d_p - z_p \qquad \forall p \in \mathcal{P} \tag{13}$$

$$\sum_{\phi\in\Phi} t_\phi^\ell x_\phi = \sum_{p\in\mathcal{P}:\ell\in L(p)} z_p \qquad \forall \ell \in L \tag{14}$$

$$\sum_{\phi\in\Phi} x_\phi \leq |\Lambda| \tag{15}$$

$$z_p, x_\phi \geq 0 \tag{16}$$

where (8) is split into constraints (13) for the two link subpaths (i.e., the lightpaths) and constraints (14) for the single link subpaths (i.e., the links). Now, since the assignment of a wavelength to a single link does not cause a difficulty in a proper dimensioned networks, we relax the constraints (14) without loss of generality. Also constraints (12) are relaxed since the objective will keep z_p as low as possible. Moreover, constraints (13) are relaxed to greater than or equal constraints since covering a path with more than the demand does not affect the value of the linear program as long as the number of converters is minimized.

The path packings in the remaining linear program only differ in the number of times each path $p \in \mathcal{P}$ is taken. Since each path $p = \ell_1\ell_2$ corresponds to an edge $v_{\ell_1}v_{\ell_2} \in E(G_L)$, each path packing corresponds to an f-matching in G_L. Thus Φ now refers to the set of all f-matchings in G_L and t_ϕ^p denotes for $p = \ell_1\ell_2$ the number of edges between v_{ℓ_1} and v_{ℓ_2} taken in f-matching ϕ.

Next, we dualize the remaining linear program with variable sets π_p for the constraints (13) and π_Λ for constraint (15):

$$\Upsilon^*(\mathcal{W}_2) = \max \sum_{p\in\mathcal{P}} d_p\pi_p - |\Lambda|\pi_\Lambda \tag{17}$$

$$\text{s.t.} \quad \pi_p \leq 1 \qquad \forall p \in \mathcal{P} \tag{18}$$

$$\sum_{p\in\mathcal{P}} t^p_\phi \pi_p - \pi_\Lambda \leq 0 \qquad \forall \phi \in \Phi \tag{19}$$

$$\pi_p, \pi_\Lambda \geq 0 \tag{20}$$

In (17)–(20) we have a constraint (19) for every f-matching in G_L. In fact we only have to consider the maximal f-matchings, as the constraints for non-maximal f-matchings are dominated by the maximal f-matching constraints. The maximal f-matchings are exactly the extreme points of the f-matching polytope $P_{\mathcal{M}_f}(G)$. By applying Benders' reformulation [26] in a *reverse* way, we can obtain a formulation for $\Upsilon^*(\mathcal{W}_2)$ for which it is more easy to prove equality of $\Upsilon^*(\mathcal{W}_2)$ and $\Gamma_f(\mathcal{W}_2)$.

The f-matching polytope is completely described by (see [29, Chapter 31])

$$P_{\mathcal{M}_f}(G) = \left\{ \begin{array}{ll} y_e \geq 0 & \forall e \in E(G) \\ y(\delta_G(v)) \leq f_v & \forall v \in V(G) \\ y(E(G[S])) \leq \lfloor \frac{1}{2} f(S) \rfloor & \forall S \subseteq V(G) \text{ with } f(S) \text{ odd} \end{array} \right\}.$$

If we extend this description with an inequality for every subset $S \subseteq V(G)$, regardless $f(S)$ is odd or even, the system is totally dual integral (see [29, Chapter 31]). Given fixed values π_p for all $p \in \mathcal{P}$, (17)–(20) is maximized by the f-matching for which $\sum_{p\in\mathcal{P}} t^p_\phi \pi_p$ is maximized. This f-matching can be found by solving

$$\max \sum_{p\in\mathcal{P}} \pi_p y_p \tag{21}$$

$$\text{s.t.} \quad \sum_{p\in\mathcal{P}_\ell} y_p \leq f_{v_\ell} \qquad \forall \ell \in L \tag{22}$$

$$\sum_{\substack{\ell_1\ell_2\in\mathcal{P}: \\ v_{\ell_1}v_{\ell_2}\in E(G_L[S])}} y_{\ell_1\ell_2} \leq \lfloor \tfrac{1}{2} f(S) \rfloor \qquad \forall S \subseteq V(G_L) \tag{23}$$

$$y_p \geq 0 \tag{24}$$

or alternatively by solving its dual (with variables x_{v_ℓ} for constraints (22) and x_S for constraints (23))

$$\min \sum_{\ell\in L} f_{v_\ell} x_{v_\ell} + \sum_{S\subseteq V(G_L)} \lfloor \tfrac{1}{2} f(S) \rfloor x_S \tag{25}$$

$$\text{s.t.} \quad x_{v_{\ell_1}} + x_{v_{\ell_2}} + \sum_{\substack{S\subseteq V(G_L): \\ v_{\ell_1},v_{\ell_2}\in S}} x_S \geq \pi_p \qquad \forall p = \ell_1\ell_2 \in \mathcal{P} \tag{26}$$

$$x_{v_\ell}, x_S \geq 0 \tag{27}$$

to optimality. Since (22)–(24) is totally dual integral, we may assume without loss of generality that $x_{v_\ell} \in \mathbb{Z}_0^+$ and $x_S \in \mathbb{Z}_0^+$ for all $v_\ell \in V(G_L)$ and $S \subseteq V(G_L)$.

Since we minimize (25), the left hand side of constraints (26) will never exceed $\pi_p \leq 1$ in an optimal solution. Thus, we can restrict variables x_{v_ℓ} and x_S to be binary instead of general integer.

Given values π_p we define $S_\pi = \{p \in \mathcal{P} : \pi_p > 0\}$ to be the support set of paths. It is easy to verify that in this case the optimal solution of (25)–(27) is given by $x_{v_\ell} = 0$ for all $v_\ell \in G_L$, $x_{S_\pi} = 1$ and $x_S = 0$ for all $S \neq S_\pi$. Thus we neglect the variables x_{v_ℓ} without loss of generality. Moreover, we can add the constraint

$$\sum_{S \subseteq V(G_L)} x_S \leq 1 \tag{28}$$

Now, we can apply the reverse of a Benders' reformulation to (17)–(20) and obtain

$$\Upsilon^*(\mathcal{W}_2) = \max \sum_{p \in \mathcal{P}} d_p \pi_p - |\Lambda| \sum_{S \subseteq V(G_L)} \lfloor \tfrac{1}{2} f(S) \rfloor x_S \tag{29}$$

$$\text{s.t.} \quad \sum_{\substack{S \subseteq V(G_L): \\ v_{\ell_1}, v_{\ell_2} \in S}} x_S \geq \pi_p \qquad \forall p = \ell_1 \ell_2 \in \mathcal{P} \tag{30}$$

$$\sum_{S \subseteq V(G_L)} x_S \leq 1 \tag{31}$$

$$0 \leq \pi_p \leq 1, x_S \in \{0, 1\} \tag{32}$$

which exactly provides an integer linear programming formulation for $\Gamma_f(\mathcal{W}_2)$ and thus completes the proof. □

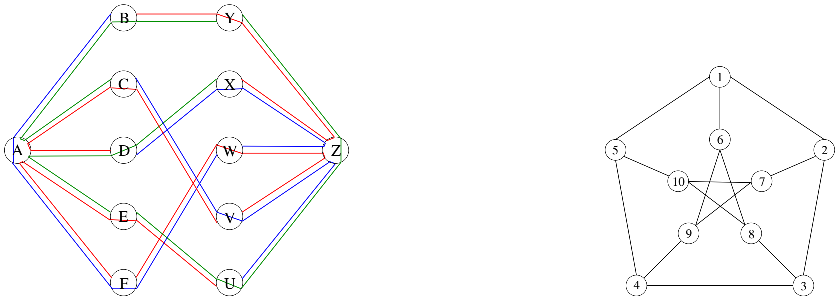
So, in case that every lightpath has at most two links the lower bounds $\Upsilon^*(\mathcal{W}_2)$ and $\Gamma_f(\mathcal{W}_2)$ are equivalent. For general instances $\mathcal{W}$ we have by Theorem 14 a combinatorial lower bound of $\sum_{n \in N} \Gamma_f(\mathcal{W}_n)$. To compare this bound with $\Upsilon^*(\mathcal{W})$, we have to compare in fact $\Upsilon^*(\mathcal{W})$ with $\sum_{n \in N} \Upsilon^*(\mathcal{W}_n)$. The following lemma, stating a rewriting of the objective, is helpful in determining this relation.

Lemma 17 *For any solution y satisfying the constraints* (7),

$$\sum_{p \in \mathcal{P}} \sum_{s \in \mathcal{S}_p} y_p^s - \sum_{p \in \mathcal{P}} d_p = \sum_{n \in N} \sum_{p \in \mathcal{P}: n \in N(p)} \sum_{s \in \mathcal{S}_p: \tau^s = n} y_p^s$$

Proof: The summation of the y-variables can be split into subpaths $s \in \mathcal{S}_p$ that reach the target node of path p and those that do not reach the target node:

$$\sum_{p \in \mathcal{P}} \sum_{s \in \mathcal{S}_p} y_p^s = \sum_{p \in \mathcal{P}} \sum_{s \in \mathcal{S}_p: \tau^s = \tau^p} y_p^s + \sum_{p \in \mathcal{P}} \sum_{s \in \mathcal{S}_p: \tau^s \neq \tau^p} y_p^s \quad .$$



(a) instance with $|\Lambda| = 2$ (b) path conflict graph $G_{\mathcal{P}}$ (=Petersen graph)

Figure 4: Wavelength assignment instance with $3 = \Upsilon^*(\mathcal{W}) > \sum_{n\in N} \Gamma_f(\mathcal{W}_n) = 2$.

By applying equation (7) for the last link of p, $\ell = L(p) \cap \delta_{\mathcal{N}}(\tau^p)$ we obtain

$$\sum_{p\in\mathcal{P}} \sum_{s\in\mathcal{S}_p} y_p^s - \sum_{p\in\mathcal{P}} d_p = \sum_{p\in\mathcal{P}} \sum_{s\in\mathcal{S}_p:\tau^s\neq\tau^p} y_p^s \quad .$$

Now the result follows by reordering the summed y-variables according to their target. □

Proposition 18 *For any instance $\mathcal{W}$, $\Upsilon^*(\mathcal{W}) \geq \sum_{n\in N} \Upsilon^*(\mathcal{W}_n)$ with $\mathcal{W}_n$ defined as in Theorem 14.*

Proof: Let $(\bar{x}, \bar{y})$ be a solution of the linear relaxation of (6)–(10) for the instance $\mathcal{W}$ with value $z(\bar{x}, \bar{y})$. For $n \in N$, we define a solution $(\bar{x}[n], \bar{y}[n])$ of the linear relaxation of (6)–(10) for the instance $\mathcal{W}_n$ as defined in Theorem 14. For $n \in N$, let for all $p \in \mathcal{P}_n$, $\bar{y}[n]_p^p := \sum_{s\in\mathcal{S}_p:n\in N(s)} \bar{y}_p^s$ (the sum over all subpaths that have n as an intermediate node) and for $\ell \in L(p)$, $\bar{y}[n]_p^\ell := d_p - \bar{y}[n]_p^p$. For every $\phi \in \Phi$ with $\bar{x}_\phi > 0$ we define a path packing $\phi[n] \in \Phi[n]$ with for all $p \in \mathcal{P}_n$, $t_{\phi[n]}^p := \sum_{s\in\mathcal{S}:p\subseteq s} t_\phi^s$ and for all $\ell \in \delta_G(n)$, $t_{\phi[n]}^\ell := \sum_{s\in\mathcal{S}:|s\cap\ell|=1} t_\phi^s$, and set $\bar{x}[n]_{\phi[n]} = \bar{x}_\phi$.

Now, it is easy to verify that $(\bar{x}[n], \bar{y}[n])$ is a feasible solution for the linear relaxation of (6)–(10) for the instance $\mathcal{W}_n$ and has value $z(\bar{x}[n], \bar{y}[n]) = \sum_{p\in\mathcal{P}_n} \bar{y}[n]_p^{\ell_1(p)}$.

By Lemma 17 we have $z(\bar{x}, \bar{y}) = \sum_{n\in N} z(\bar{x}[n], \bar{y}[n])$. If $(\bar{x}, \bar{y})$ is an optimal solution of the linear relaxation of (6)–(10) for instance $\mathcal{W}$, then it follows $\Upsilon^*(\mathcal{W}) = \sum_{n\in N} z(\bar{x}[n], \bar{y}[n]) \geq \sum_{n\in N} \Upsilon^*(\mathcal{W}_n)$. □

Corollary 19 *For any instance $\mathcal{W}$, $\Upsilon^*(\mathcal{W}) \geq \sum_{n\in N} \Gamma_f(\mathcal{W}_n)$ with $\mathcal{W}_n$ defined as in Theorem 14.*

Figure 4 shows an example for which $\Upsilon^*(\mathcal{W})$ (and $\Upsilon(\mathcal{W})$) equals three whereas the combinatorial lower bound by Theorem 14 is two. The links incident to node A and B define two stars, both with $\Gamma_f(\mathcal{W}_n) = 1$, whereas for all other nodes the bound equals zero. If we construct the path conflict graph $G_{\mathcal{P}}$, we obtain the well-known Petersen graph, for which it is known that $\chi(G_{\mathcal{P}}) = 3$. In fact, the size of the smallest color class in any 3-coloring equals three, and thus $\Upsilon(\mathcal{W}) = 3$. The linear relaxation in this case equals the optimum and thus is larger than the combinatorial bound.

6 Practical comparison

Corollary 19 states the theoretical relation between the combinatorial lower bound of Theorem 14 with the linear relaxation bound $\Upsilon^*(\mathcal{W})$. In this section, we compare these bounds for realistic wavelength assignment instances. For this purpose, we implemented a column generation algorithm for the linear relaxation, see [17] for details.

By Theorem 16, computing $\Gamma_f(\mathcal{W}_2)$ can be done via this column generation algorithm as well. Since the pricing problem in this case is equivalent to finding a maximum weighted f-matching, it follows that it can be computed in polynomial time by using a polynomial time algorithm for solving the linear relaxation (e.g., the ellipsoid method) [10]. However, $\Gamma_f(\mathcal{W}_2)$ can also be computed directly with a combinatorial algorithm: First note that the maximum is taken over all subsets with $f(S)$ odd and thus rounding always takes place. Moreover, for $f(S)$ to be odd, there must be an odd number of vertices $v \in S$ with f_v odd. Let $U = \{v \in V(G) : f_v \text{ odd}\}$. Now $\Gamma_f(G)$ can be restated as

$$\Gamma_f(G) = \max\left\{0, \tfrac{1}{2}|\Lambda| + \max_{S\subseteq V(G), |S\cap U| \text{ odd}} |E(G[S])| - \tfrac{1}{2}|\Lambda| f(S)\right\} \quad .$$

For $S \subseteq V(G)$, $|E(G[S])| = |E(G)| - |E(V, V \setminus S)|$ with $E(S,T) := \{vw \in E(G) : v \in S, w \in T\}$ and thus,

$$\Gamma_f(G) = \max\left\{0, \tfrac{1}{2}|\Lambda| + |E(G)| - \min_{S\subseteq V(G), |S\cap U| \text{ odd}} \gamma(S)\right\}$$

with

$$\gamma(S) = |E(V, V \setminus S)| + \tfrac{1}{2}|\Lambda| f(S) \quad .$$

The function $\gamma(S)$ is submodular (i.e., for all $S, T \subseteq V(G)$, $\gamma(S\cup T) \le \gamma(S)+\gamma(T)-\gamma(S\cap T)$ holds). Grötschel et al. [8, 9, 10] proved that submodular function minimization over all sets S with $|S \cap U|$ odd can be done in strongly polynomial time. At that time only the ellipsoid method was available for this. Independently, Schrijver [28] and Iwata et al. [14] gave a combinatorial strongly polynomial time algorithm for minimizing submodular functions, and hence $\Gamma_f(\mathcal{W}_2)$ can be computed combinatorially this way.

For our computations we use a much simpler algorithm that only considers a subcollection of all possible subsets S. Hence, the result is in fact a lower bound on $\Gamma_f(G)$. Our algorithm starts with $S = V(G)$ and repeatedly removes a vertex v from S such that $\gamma(S \setminus \{v\})$ is as large as possible. The algorithm reports the maximum $\gamma(S)$ encountered.

Our test set of wavelength assignment instances has 80 members that were generated in the context of an integer programming approach for optical network design, see [17] for

instance	$\|N\|$	$\|L\|$	$d(\mathcal{P})$	$\|\mathcal{P}\|$	UB	ILP LB	ILP time	Combinatorics LB	Combinatorics time
europe-up1	28	41	1008	353	2	2	24.19	2	0.00
europe-up3	28	41	1008	352	5	5	17.71	5	0.00
europe-up4	28	41	1008	349	1	1	21.50	1	0.00
germany-high2	17	26	836	75	4	4	0.28	4	0.00
germany-fp1	17	26	1193	64	8	8	0.39	8	0.00
germany+-up2	17	28	686	48	16	16	0.16	16	0.00
germany+-up3	17	28	686	47	12	12	0.26	12	0.00
germany+-up5	17	28	686	44	9	9	0.09	9	0.00
germany+-low2	17	28	699	82	9	9	0.49	9	0.00
germany+-fp5	17	28	1122	102	5	5	0.99	5	0.00

Table 1: Results for the wavelength assignment instances with non-zero best solution

further details. For each of four different network topologies, 20 instances were generated, all with $|\Lambda| = 40$. For 57 instances, a conversion-free solution was obtained by the heuristics described in [18, 19], and thus all lower bounds on $\Upsilon(\mathcal{W})$ will be zero as well. By recent advances in heuristic algorithms [34] another 13 instances turned out to have a conversion-free solution. For the remaining 10 instances, Table 1 shows the problem characteristics as well as the combinatorial lower bound, the linear programming lower bound, and the best known solution value (cf. [17, 34]). Besides the number of nodes $|N|$, the number of links $|L|$, and the number of lightpaths $d(\mathcal{P})$, the column $|\mathcal{P}|$ denotes the number of different lightpaths in $\mathcal{P}$ (note that $\mathcal{P}$ is a multi-set). The number of different paths $|\mathcal{P}|$ is an important measure for the performance of the column generation algorithm to solve the linear relaxation of the integer formulation. All computation times are in seconds on a Linux operated PC with 3.2 GHz Intel Pentium 4 HT processor. For the linear relaxation the fastest variant (cf. [17]) of the column generation algorithm has been taken.

Table 1 shows that without exemption the combinatorial lower bound equals the value of linear programming relaxation. Moreover, without any exception the bounds prove optimality of the best known solution. The combinatorial lower bound can be computed within a fraction of a second whereas the column generation algorithm is somewhat slower. Note that solving the linear relaxation of (6)–(10) for general instances is an $\mathcal{NP}$-hard problem since the pricing subproblem generalizes the set packing problem (cf. [17]), whereas computing the combinatorial lower bound can be done in polynomial time.

7 Concluding remarks

We have shown that in case all lightpaths have at most two links, wavelength assignment is strongly related to a generalized edge coloring problem known as f-edge-coloring. From bounds known for this problem, we have derived lower and upper bounds on the number of converters needed or the number of lightpaths establishable in a conflict-free assignment. In the special case that an even number of fibers is installed at every link, the lower and upper bound both equal zero and hence in such cases no wavelength conversion is necessary.

The practical relevance of these results lies in the fact that all links incident to a single node form a star network. If we restrict all lightpaths to the links of the star, we obtain an instance with at most two links per lightpath. Hence, we can compute a lower bound on the number of converters in the central node. Applying this algorithm to all nodes in the network gives us a lower bound on the number of converters in the overall network.

A theoretical comparison of the combinatorial lower bound with the lower bound provided by the linear relaxation value of a novel integer programming formulation of the converter problem reveals that both bounds coincide on instances with at most two links per lightpath. For general instances the linear relaxation bound can be better, but for all realistic instances we have shown that the bounds are equally good: Without exception, optimality of the best known solution could be proved as the lower bound equals the solution value.

The relation to the (generalized) edge coloring problem heavily relies on the restriction of at most two links per lightpath. If this restriction is lifted, we have to extend the graph G_L to a hypergraph. Unfortunately, not much is known about the (f-)coloring of the edges of a hypergraph. Further research in this direction could be considered. A related but different direction for further research is the investigation of the impact of the achieved results for special network structures that generalize the star, like spiders and trees.

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