On the Convergence of a Primal-Dual Second-Order Corrector Interior Point Algorithm for Linear Programming

Coralia Cartis*

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Abstract

The Primal-Dual Second Order Corrector (PDSOC) algorithm that we investigate computes on each iteration a corrector direction in addition to the direction of the standard primal-dual path-following interior point method [10] for Linear Programming (LP), in an attempt to improve performance. The corrector is multiplied by the square of the stepsize in the expression of the new iterate. While the outline of the PDSOC algorithm is known [28], we present a substantive theoretical interpretation of its construction. Further, we investigate its convergence and complexity properties, provided that a primal-dual strictly feasible starting point is available. Firstly, we use a new long-step linesearch technique suggested by M. J. D. Powell [1], and show that, when the centring parameters are bounded away from zero, the limit points of the sequence of iterates are primal-dual strictly complementary solutions of the LP problem. We consider also the popular choice of letting the centring parameters be of the same order as the duality gap of the iterates, asymptotically. A standard long-step linesearch is employed to prove that the sequence of iterates converges to a primal-dual strictly complementary solution of the LP problem, which may not be the analytic centre of the primal-dual solution set, as further shown by an example.

*Oxford University Computing Laboratory, Wolfson Building, Parks Road, Oxford, OX1 3QD, United Kingdom, (ccartis@comlab.ox.ac.uk). The author was supported through grant GR/S34472 from the Engineering and Physical Sciences Research Council of the UK.

1 Introduction

In the past fifteen years, Interior Point Methods (IPMs) have become highly successful in solving Linear Programming (LP) problems, especially large-scale ones, while enjoying good theoretical convergence and complexity properties (see [3, 4, 18, 22, 23, 24] for comprehensive reviews of the field of IPMs for LP). Examples of IPMs that are reliable both in theory and in practice include the Primal-Dual (PD) path-following method of Kojima et al. [10] with some *long-step* linesearch procedure [10, 24], and an *infeasible* formulation of this algorithm [9, 24]. The Primal-Dual Second-Order Corrector (PDSOC) algorithm that we consider in this paper computes on each iteration an additional direction, a corrector, to the direction of the PD algorithm, in an attempt to improve performance. Then the new iterate is constructed as a quadratic function of the steplength from the current iterate, where the PD direction and the corrector are multiplied by the stepsize and the square of the stepsize, respectively. A powerful reason for this choice of scaling of the corrector direction is given by the results in [1, 2]. There, we showed that if we scale the corrector only by the stepsize in the expression of the new iterates, the resulting algorithm—the Primal-Dual Corrector (PDC)—may fail to converge to a solution of the problem in both exact and finite arithmetic, regardless of the linesearch procedure that is employed, the cause of this failure being that the correctors exert too much influence on the direction in which the iterates move. Thus the construction of the iterates in the PDSOC algorithm attempts to reduce, in a natural way, the impact of the correctors. Moreover, we show in the present paper that the expression of each new iterate can be interpreted as a second-order Taylor approximation at the current iterate of a local path from the current iterate to a target point on the primal-dual central path of the problems. Therefore, the corrector direction provides curvature information of this path, and we take account of it in a theoretically rigorous way if we multiply it by the square of the stepsize in the expression of the new iterate.

Further, we investigate the convergence and complexity properties of the PDSOC algorithm. We are concerned with long-step versions of this algorithm, where the iterates belong to a large neighbourhood of the primal-dual central path, and we assume that a primal-dual strictly feasible starting point is available. Firstly, we introduce a new long-step linesearch, the $\sigma\beta$ procedure, that has been suggested by M. J. D. Powell in the context of the PD algorithm [1], and that offers certain advantages over the standard long-step linesearch [24] employed in the PD algorithm (see Subsection 3.3). We refer to the latter procedure as the γ stepsize technique (see again Subsection 3.3).

We employ the $\sigma\beta$ stepsize procedure in the PDSOC algorithm and show that the resulting variant, the PDSOC$^{\sigma\beta}$, has good global convergence and polynomial complexity properties.

In particular, we prove that, when the centring parameters are chosen to be bounded away from zero, the sequence of duality gaps of the iterates generated by the PDSOC$^{\sigma\beta}$ algorithm converges Q-linearly to zero (Theorem 4.4), thereby proving that all the limit points of the sequence of iterates are primal-dual strictly complementary solutions of the LP problem (Corollary 4.7). These results imply that the PDSOC$^{\sigma\beta}$ algorithm converges to the solution of the example problem on which the PDC algorithm [1, 2] failed to converge (see the first paragraph of this section and compare Theorem 4.4 with Corollary 4.3 in [2]). Theorem 4.4 also gives a worst-case complexity bound on the number of iterations required by the PDSOC$^{\sigma\beta}$ algorithm to generate an iterate with duality gap less than a prescribed tolerance, which is polynomial in the number of problem variables.

Besides keeping the centring parameter fixed to the same value on each iteration, or letting it decrease slowly in the course of the algorithm, some other choices have been suggested and have proved useful in practice. One such possibility is to let the centring parameters be of the same order as the duality gap of the iterates [21, 26, 27]. Thus, if convergence occurs, then the centring parameters tend to zero. Letting the centring parameters decrease to zero asymptotically has been observed in practice to speed up the convergence of the PD algorithm in the later iterations. Theoretical results in some cases confirm that such choices provide superlinear convergence of the duality gap to zero in the PD algorithm [27]. We show that the PDSOC$^{\gamma}$ algorithm (i.e., the PDSOC variant that employs the γ stepsize procedure) with such a choice of centring parameters generates a sequence of iterates that converges to a primal-dual strictly complementary solution of the problems being solved. This result extends an existing result for the PD algorithm [21] to second-order corrector methods.

It is well-known that, when the LP problem that we are solving has multiple solutions, the primal-dual central path of the problem converges to the *analytic centre* of the primal-dual solution set [25]. We consider in Section 6 whether the iterates generated by the PDSOC algorithm also converge to this particular solution, when the centring parameters in the algorithm are of the same order as the duality gap of the iterates. Then, we show that in exact arithmetic the sequence of iterates may have other limit points (than the analytic centre) in the solution set, which answers a conjecture of Tapia et al. [21] in the case of a second-order algorithm. Our results imply that such a choice of centring parameters may not be suitable in the PDSOC algorithm for applications that require computing the analytic centre of the solution set of the problems (see [5]).

We remark that other second- and higher-order primal-dual IPMs have been proposed for solving LP problems [13, 15, 16, 28]. The PDSOC differs from these algorithms in the choice of the local path that is being approximated on each iteration. Each of the algorithms in [15, 16], for example, computes Taylor approximations to local paths that start at the current iterate and end at the primal-dual solution set of the LP problem. The approximation of the local path in the PDSOC algorithm, however, attempts to bring the iterate closer to a target point on the primal-dual central path. Thus the PDSOC genuinely belongs to the class of primal-dual path-following IPMs.

The structure of the paper is as follows. Section 2 summarizes some LP duality and interior point method theory that is needed for the remainder of this article. In Section 3, we describe the PDSOC algorithm, motivate the construction of its iterates in terms of quadratic approximations of a local path, and introduce the γ and $\sigma\beta$ stepsize procedures. Section 4 investigates the convergence and complexity of the PDSOC$^{\sigma\beta}$ algorithm, while Section 5 presents some convergence properties of the PDSOC$^{\gamma}$ algorithm. Section 6 gives an example of the asymptotic behaviour of the PDSOC algorithm when the centring parameters tend to zero at the same rate as the duality gap of the iterates. In Section 7, we present the conclusions of the paper.

2 Some LP theory and terminology

Let the LP problem we are solving be given in the standard form

$$\min_{x\in\mathbb{R}^n} \quad c^\top x \quad \text{subject to} \quad Ax=b, \quad x\geq 0, \tag{P}$$

where $m<n$, $b\in\mathbb{R}^m$, $c\in\mathbb{R}^n$, and A is a real matrix of dimension $m\times n$. The dual problem corresponding to the primal problem (P) is

$$\max_{(y,s)\in\mathbb{R}^m\times\mathbb{R}^n} \quad b^\top y \quad \text{subject to} \quad A^\top y+s=c, \quad s\geq 0. \tag{D}$$

Let $\mathcal{F}_{PD}$ denote the set of primal-dual feasible points, i.e.,

$$\mathcal{F}_{PD} := \{w=(x,y,s)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n : \ Ax=b, \quad A^\top y+s=c, \quad x\geq 0, \quad s\geq 0\}, \tag{2.1}$$

and $\mathcal{S}_{PD}$, the primal-dual solution set, containing all triplets $w^*=(x^*,y^*,s^*)\in\mathcal{F}_{PD}$ such that x^* is a solution of (P) and (y^*,s^*), a solution of (D).

A triplet $(x^*,y^*,s^*)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$ belongs to $\mathcal{S}_{PD}$ if and only if it satisfies the optimality conditions

$$Ax^*=b, \qquad x^*\geq 0, \tag{2.2a}$$

$$A^\top y^*+s^*=c, \quad s^*\geq 0, \tag{2.2b}$$

$$x_i^*s_i^*=0, \qquad i=1,2,\ldots,n. \tag{2.2c}$$

Equivalently, a triplet $(x, y, s) \in \mathcal{F}_{PD}$ belongs to $\mathcal{S}_{PD}$ if and only if the duality gap $c^\top x - b^\top y$, which satisfies

$$c^\top x - b^\top y = x^\top s, \tag{2.3}$$

is zero. The points in the relative interior of $\mathcal{S}_{PD}$ are called *strictly complementary solutions* of (P) and (D). Such primal-dual solutions $w^\dagger = (x^\dagger, y^\dagger, s^\dagger)$ exist whenever the set $\mathcal{S}_{PD}$ is nonempty, and they are characterized by the property $x^\dagger + s^\dagger > 0$ (see for example, [1, 17]). Further, there exists a partition $(\mathcal{A}, \mathcal{I})$ of the index set $\{1, \ldots, n\}$, where one of the sets $\mathcal{A}$ and $\mathcal{I}$ may be empty, such that

$$\mathcal{S}_{PD} = \{w^* = (x^*, y^*, s^*) \in \mathcal{F}_{PD} : \ x^*_{\mathcal{A}} = 0 \quad \text{and} \quad s^*_{\mathcal{I}} = 0\}, \tag{2.4}$$

where $x^*_{\mathcal{A}} := (x^*_i \ : \ i \in \mathcal{A})$ and $s^*_{\mathcal{I}} := (s^*_j \ : \ j \in \mathcal{I})$. We call the sets $\mathcal{A}$ and $\mathcal{I}$ the *strict complementarity index sets*.

We assume that there exists a point $w^0 = (x^0, y^0, s^0)$ that satisfies

$$Ax^0 = b, \quad A^\top y^0 + s^0 = c, \quad x^0 > 0 \quad \text{and} \quad s^0 > 0, \tag{2.5}$$

and that the matrix A has full row rank. We refer to these assumptions as the **IPM conditions**. They are equivalent to requiring the sets $\mathcal{F}_{PD}$ and $\mathcal{S}_{PD}$ to be nonempty and bounded, respectively (see for example, Corollary 2.8 in [1]). The first condition implies that $\mathcal{S}_{PD}$ is nonempty.

Any point $w = (x, y, s)$ that satisfies (2.5) is called a *primal-dual strictly feasible point.* These points form the relative interior of the set $\mathcal{F}_{PD}$.

Subject to the IPM conditions, the perturbed system of optimality conditions [24] associated to (P) and (D)

$$F_\mu(w) := \begin{pmatrix} Ax - b \\ A^\top y + s - c \\ XSe - \mu e \end{pmatrix} = 0, \quad x > 0, \quad s > 0, \tag{2.6}$$

has a unique solution $w(\mu) = (x(\mu), y(\mu), s(\mu))$, for each $\mu > 0$ [12, 24], where in (2.6), XS is the diagonal matrix with diagonal elements $x_i s_i$, $i = \overline{1, n}$, and $e := (1, 1, \ldots, 1) \in \mathbb{R}^n$. As μ tends to zero, the points $w(\mu)$, $\mu > 0$, which form the *primal-dual central path*, converge to the *analytic centre* $w^c = (x^c, y^c, s^c)$ of the primal-dual solution set which is defined as the solution of the problem

$$\min_{(x,y,s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n} \quad -\sum_{i \in \mathcal{I}} \log x_i - \sum_{j \in \mathcal{A}} \log s_j \quad \text{subject to} \quad (x, y, s) \in \mathcal{S}_{PD}, \quad x_{\mathcal{I}} > 0, \quad s_{\mathcal{A}} > 0, \tag{2.7}$$

where $\mathcal{S}_{PD}$ is given in (2.4). We deduce from (2.4) and (2.7) that w^c is a strictly complementary solution of problems (P) and (D) [25].

3 The Primal-Dual Second-Order Corrector (PDSOC) algorithm

In Subsection 3.1, we describe the PDSOC algorithm in a way that establishes its clear connection to primal-dual path-following IPMs, namely, to the PD algorithm. Then, as promised in the introduction, Subsection 3.2 gives an interpretation of the quadratic search directions of the PDSOC algorithm in terms of Taylor approximations of a local path. Finally, Subsection 3.3 presents two ways of computing the stepsize in each iteration of the PDSOC algorithm.

3.1 Description of the algorithm

Let problems (P) and (D) satisfy the IPM conditions, and assume that a point $w^0 = (x^0, y^0, s^0)$ satisfying (2.5) is available as a starting point of the algorithm.

The PDSOC algorithm attempts to follow the primal-dual central path of (P) and (D) approximately to a solution of these problems, in a similar fashion to long-step primal-dual path-following IPMs.

At the current iterate $w^k = (x^k, y^k, s^k)$, $k \geq 0$, of the PDSOC algorithm, a parameter $\mu > 0$ is picked

$$\mu := \sigma^k \mu^k, \tag{3.1}$$

where $\mu^k := (x^k)^\top s^k/n$, and $\sigma^k \in (0,1)$ is a *centring parameter* that can be fixed at the start of the algorithm or computed on each iteration by some automatic procedure. Then we compute the Newton direction $dw^k = (dx^k, dy^k, ds^k)$ from w^k for the system $F_\mu(w) = 0$ in (2.6), i.e., dw^k is the solution of the linear system

$$F'_\mu(w^k)\, dw^k = -F_\mu(w^k), \tag{3.2}$$

where $F'_\mu(w^k)$ is the Jacobian of F_μ at w^k. The system (3.2) is equivalent to

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ S^k & 0 & X^k \end{pmatrix} \begin{pmatrix} dx^k \\ dy^k \\ ds^k \end{pmatrix} = - \begin{pmatrix} Ax^k - b \\ A^\top y^k + s^k - c \\ X^k S^k e - \sigma^k \mu^k e \end{pmatrix}. \tag{3.3}$$

Next, a *corrector* direction $dw^{k,c} = (dx^{k,c}, dy^{k,c}, ds^{k,c})$ is computed by solving the linear system

$$F'_\mu(w^k) dw^{k,c} = -F_\mu(w^k + dw^k). \tag{3.4}$$

The right-hand side of the system (3.4) represents the error that is introduced in the system $F_\mu(w) = 0$ (see (2.6)) by its linearization around w^k, and it has the explicit expression

$$F_\mu(w^k + dw^k) = \begin{pmatrix} A(x^k + dx^k) - b \\ A^\top(y^k + dy^k) + (s^k + ds^k) - c \\ (X^k + dX^k)(S^k + dS^k)e - \sigma^k \mu^k e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ dX^k dS^k e \end{pmatrix}, \qquad (3.5)$$

where the last equation depends on (3.3), and where dX^k and dS^k are the diagonal matrices with diagonal elements dx_i^k, $i = \overline{1,n}$, and ds_i^k, $i = \overline{1,n}$, respectively. It follows from (3.4) that the corrector direction attempts to correct this error, in order to position the new iterate closer to the primal-dual central path.

The new iterate w^{k+1} is the quadratic function of the steplength

$$w^{k+1} := w^k + \theta^k dw^k + (\theta^k)^2 dw^{k,c}, \qquad (3.6)$$

where the stepsize $\theta^k \in (0, 1]$ is chosen such that

$$x^{k+1} > 0 \quad \text{and} \quad s^{k+1} > 0. \qquad (3.7)$$

The strict inequalities (3.7), and those in (2.5), together with A having full row rank, imply that the Jacobian $F'_\mu(w^k)$ is nonsingular [24], and thus, the directions dw^k and $dw^{k,c}$, and the new iterate w^{k+1} are well-defined, for every $k \geq 0$.

An outline of the PDSOC algorithm is given next.

The PDSOC algorithm:

Assume that a point $w^0 = (x^0, y^0, s^0)$ is available that satisfies the strict feasibility conditions (2.5). Let $\epsilon > 0$ be a tolerance parameter. At the current iterate $w^k = (x^k, y^k, s^k)$, where $k \geq 0$, do:

Step 1: If $(x^k)^\top s^k \leq \epsilon$, STOP.

Step 2: Let $\mu^k := (x^k)^\top s^k / n$ and choose $\sigma^k \in (0, 1)$.
Compute the direction $dw^k = (dx^k, dy^k, ds^k)$ from the linear system (3.3).
Compute the corrector direction $dw^{k,c} = (dx^{k,c}, dy^{k,c}, ds^{k,c})$ from the system (3.4).

Step 3: Choose the stepsize $\theta^k \in (0, 1]$ such that the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, s^{k+1})$ defined by (3.6) satisfies (3.7).

Step 4: Let $k := k + 1$. Go to Step 1. $\diamond$

Since w^0 is feasible with respect to the primal-dual equality constraints, it follows inductively, from (3.3), (3.4) and (3.6), that all the iterates w^k remain feasible with respect to these constraints. This, (2.5) and (3.7) imply that the iterates w^k, $k \geq 0$, are primal-dual strictly feasible. Thus the only optimality condition that remains to be satisfied (asymptotically) by the iterates is the zero duality gap, i.e., $(x^k)^\top s^k = c^\top x^k - b^\top y^k \to 0$ as $k \to \infty$, which explains the termination criteria in Step 1. Moreover, since the iterates w^k satisfy the primal-dual equality constraints, the first $n + m$ equations of the systems (3.3) and (3.4) provide the orthogonality properties

$$(dx^k)^\top ds^k = 0, \ \ (dx^k)^\top ds^{k,c} = 0, \ \ (ds^k)^\top dx^{k,c} = 0, \ \ (dx^{k,c})^\top ds^{k,c} = 0, \ \ k \geq 0. \tag{3.8}$$

The relations (3.6), (3.8) and the last n equations of the systems (3.3) and (3.4) give the recurrence

$$(x^k(\theta))^\top s^k(\theta) = [1 - \theta(1 - \sigma^k)]\,(x^k)^\top s^k, \quad \theta \geq 0, \quad k \geq 0, \tag{3.9}$$

where w^k is the kth iterate of the PDSOC algorithm and $w^k(\theta)$ is defined by

$$w^k(\theta) := w^k + \theta dw^k + \theta^2 dw^{k,c}, \quad \theta \geq 0. \tag{3.10}$$

It follows from $\sigma^k \in (0, 1)$, $\theta^k \in (0, 1]$ and $(x^k)^\top s^k > 0$ that we have the strict reduction

$$(x^{k+1})^\top s^{k+1} < (x^k)^\top s^k, \quad k \geq 0. \tag{3.11}$$

When the centring parameters $\sigma^k \in (0, 1)$, $k \geq 0$, are bounded away from zero, equation (3.9) and the bound $\theta^k \leq 1$ imply that linear convergence of the duality gaps $(x^k)^\top s^k$ to zero is the best that can be achieved. A good rate of linear convergence of the duality gap to zero is, however, highly useful in practice. In the next section, we are going to prove Q-linear convergence of the duality gap to zero for the PDSOC algorithm when the $\sigma\beta$ stepsize procedure, to be defined in Subsection 3.3, is employed on each iteration.

We remark that if $dw^{k,c} := 0$, for each $k \geq 0$, the PDSOC algorithm coincides with the PD algorithm (see pages 8–9 of [24]).

The above description of the PDSOC algorithm does not in any way justify multiplying the corrector direction by the square of the stepsize in expression (3.6) of the new iterate w^{k+1}. Moreover, by analogy with composite Newton's method (see for example, [20], page 48 and the references therein) or with Mehrotra's highly popular predictor-corrector algorithm [11, 14], one may be inclined to believe that the algorithm should take *full* corrector directions, i.e., the new iterate should be chosen by a linesearch along the sum of the directions dw^k and $dw^{k,c}$. As already mentioned in the introduction, however, we showed in [1, 2] that such an approach may present severe disadvantages as the correctors may have too much impact on the direction in which the iterates move, causing the algorithm to fail to converge on some LP instances, in both exact and finite arithmetic, regardless of the choice of stepsize that is employed. In the next subsection, we give a substantive interpretation of the search (3.10).

3.2 A motivation for the construction of the PDSOC algorithm

Let $w^k = (x^k, y^k, s^k)$ be the kth iterate generated the PDSOC algorithm when applied to problems (P) and (D) that satisfy the IPM conditions. Let $\mu^k := (x^k)^\top s^k/n$ and $\sigma^k \in (0,1)$, $k \geq 0$. We establish a connection between the search directions (3.2) and (3.4), and the tangent and curvature at w^k of a local path of primal-dual strictly feasible points that starts at w^k and ends at the target point $w(\sigma^k\mu^k)$ of the primal-dual central path of the problems (see (2.6)). We consider the following perturbation to the system of optimality conditions of (P) and (D), depending on the parameter $\zeta \in [0,1]$,

$$\begin{cases} Ax &= b, \\ A^\top y + s &= c, \\ XSe &= (1-\zeta)X^k S^k e + \zeta \sigma^k \mu^k e. \end{cases} \tag{3.12}$$

For each $\zeta \in [0,1]$, the components of $v^k := (1-\zeta)X^k S^k e + \zeta\sigma^k\mu^k e$ are positive, since $x^k > 0$, $s^k > 0$, and $\sigma^k\mu^k > 0$. By analogy with the system (2.6), it follows that the system (3.12) has a unique solution $\underline{w}(\zeta) = (\underline{x}(\zeta), \underline{y}(\zeta), \underline{s}(\zeta))$ (see [19], pages 226–229 for a proof). These solutions define a local path

$$\mathcal{L}^k := \{\underline{w}(\zeta) = (\underline{x}(\zeta), \underline{y}(\zeta), \underline{s}(\zeta)) : \ \ \zeta \in [0,1]\}, \tag{3.13}$$

from $w^k = \underline{w}(0)$ to the point $w(\sigma^k\mu^k) = \underline{w}(1)$ of the primal-dual central path (see (2.6) with $\mu := \sigma^k\mu^k$). The path $\mathcal{L}^k$ is continuous and infinitely differentiable (the proof is similar to the ones in [12, 16]). By differentiating the system (3.12) with respect to ζ, we obtain

$$\begin{cases} A\dot{\underline{x}}(\zeta) &= 0, \\ A^\top \dot{\underline{y}}(\zeta) + \dot{\underline{s}}(\zeta) &= 0, \\ \underline{S}(\zeta)\,\dot{\underline{x}}(\zeta) + \underline{X}(\zeta)\,\dot{\underline{s}}(\zeta) &= -X^k S^k e + \sigma^k\mu^k e, \quad \zeta \in [0,1]. \end{cases} \tag{3.14}$$

Since $\underline{w}(0) = w^k$ and w^k is primal-dual strictly feasible, (3.14), (3.3) and the nonsingularity of the matrix of system (3.3) imply that

$$\dot{\underline{w}}(0) = dw^k. \tag{3.15}$$

Similarly, differentiating (3.14) with respect to ζ, we obtain

$$\begin{cases} A\ddot{\underline{x}}(\zeta) &= 0, \\ A^\top \ddot{\underline{y}}(\zeta) + \ddot{\underline{s}}(\zeta) &= 0, \\ \underline{S}(\zeta)\,\ddot{\underline{x}}(\zeta) + \underline{X}(\zeta)\,\ddot{\underline{s}}(\zeta) &= -2\dot{\underline{X}}(\zeta)\dot{\underline{s}}(\zeta), \quad \zeta \in [0,1], \end{cases} \tag{3.16}$$

which, together with $\underline{w}(0) = w^k$, (3.4) and (3.5), further implies

$$\frac{1}{2}\ddot{\underline{w}}(0) = dw^{k,c}. \tag{3.17}$$

From (3.15) and (3.17), it follows that $w^k(\theta)$ defined by (3.10) is the second order Taylor approximation of the nonlinear path $\mathcal{L}^k$, at w^k, and we saw that the PDSOC algorithm searches for the new iterate along $w^k(\theta)$. From (3.17), the corrector term $dw^{k,c}$ provides curvature information of the path $\mathcal{L}^k$, and we take account of it in a theoretically rigorous way if we multiply it by the square of the stepsize in w^{k+1}.

3.3 Stepsize procedures for the PDSOC algorithm

In this subsection, we specify how to perform Step 3 of the PDSOC algorithm.

The γ stepsize procedure We first investigate the possibility of employing a common and practical long-step linesearch procedure which is described for the PD algorithm on pages 84 and 96 of [24]. We refer to this stepsize technique as the γ stepsize procedure. In the context of the PDSOC algorithm, this translates into choosing the stepsize θ^k on the kth iteration of the PDSOC algorithm to be the largest $\overline{\theta} \in (0,1]$ that is allowed by the inequalities

$$x_i^k(\theta)s_i^k(\theta) \geq \gamma\mu^k(\theta), \quad 0 \leq \theta \leq \overline{\theta}, \quad i = 1, \ldots, n, \tag{3.18}$$

where $w^k(\theta) = (x^k(\theta), y^k(\theta), s^k(\theta))$ is the vector (3.10), where $\mu^k(\theta) := x^k(\theta)^\top s^k(\theta)/n$, and where $\gamma \in (0,1)$ is a parameter chosen at the start of the algorithm such that the first n constraints in (3.18) are satisfied at the starting point w^0, i.e.,

$$0 < \gamma \leq \frac{1}{\mu^0} \min\,(x_i^0 s_i^0 : \quad i = 1, \ldots, n). \tag{3.19}$$

The PDSOC algorithm with the γ stepsize procedure will be denoted by PDSOC$^\gamma$.

From (3.10), the left-hand side of expression (3.18) can be written as

$$\begin{aligned} x_i^k(\theta)s_i^k(\theta) = \; & x_i^k s_i^k + \theta(x_i^k ds_i^k + s_i^k dx_i^k) + \theta^2(x_i^k ds_i^{k,c} + s_i^k dx_i^{k,c} + dx_i^k ds_i^k) \\ & + \theta^3(dx_i^k ds_i^{k,c} + ds_i^k dx_i^{k,c}) + \theta^4 dx_i^{k,c} ds_i^{k,c}, \quad i = \overline{1,n}. \end{aligned} \tag{3.20}$$

Employing the $(m+n+i)$th equation of the systems (3.3), (3.4) and (3.5), the right-hand side of (3.20) takes the value

$$x_i^k(\theta)s_i^k(\theta) = x_i^k s_i^k + \theta(-x_i^k s_i^k + \sigma^k \mu^k) + \theta^3(dx_i^k ds_i^{k,c} + ds_i^k dx_i^{k,c}) + \theta^4 dx_i^{k,c} ds_i^{k,c}, \quad i = \overline{1,n}. \tag{3.21}$$

It follows from (3.9) that the inequalities (3.18) are equivalent to the conditions

$$\begin{aligned} \phi_i(\theta) := \; & dx_i^{k,c} ds_i^{k,c} \theta^4 + (dx_i^k ds_i^{k,c} + ds_i^k dx_i^{k,c})\theta^3 + [-x_i^k s_i^k + \gamma\mu^k + (1-\gamma)\sigma^k\mu^k]\theta \\ & + x_i^k s_i^k - \gamma\mu^k \geq 0, \quad i = \overline{1,n}. \end{aligned} \tag{3.22}$$

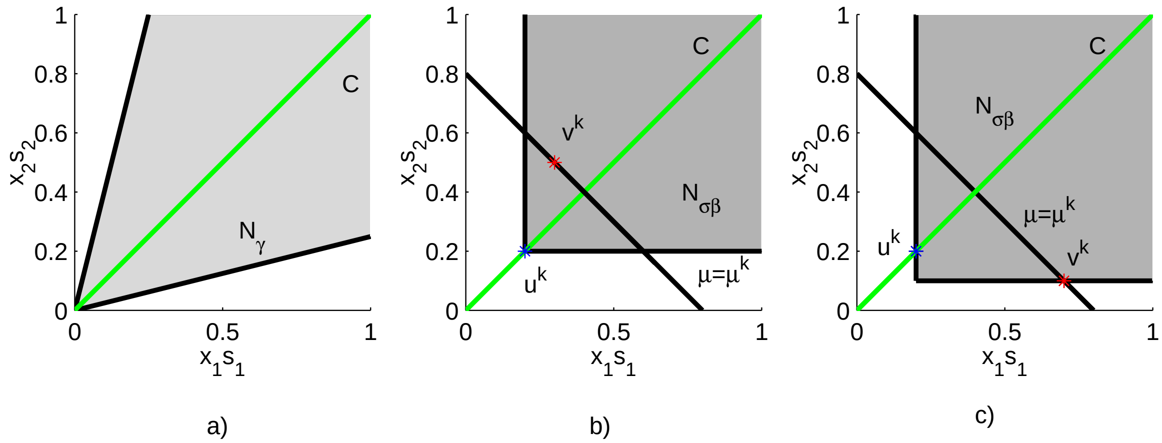

Figure 1: The neighbourhoods $\mathcal{N}_\gamma$ and $\mathcal{N}_{\sigma\beta}(w^k)$ of the primal-dual central path $\mathcal{C}$ in the space $\mathbb{R}^2$ of the complementarity products (x_1s_1, x_2s_2).

Condition (3.19) or the inequalities (3.18) on the previous iteration provide $\phi_i(0) \geq 0$, $i = \overline{1,n}$. If $\phi_i(0) = 0$, for some $i \in \{1, \ldots, n\}$, i.e. $x_i^k s_i^k = \gamma\mu^k$, then it follows from (3.22) that $\phi_i'(0) = (1-\gamma)\sigma^k\mu^k > 0$. Thus $\phi_i(\theta) \geq 0$, for sufficiently small and positive θ, for each $i \in \{1, \ldots, n\}$, which shows that the γ stepsize is well-defined for the PDSOC algorithm.

The inequalities (3.18) imply that the iterates w^k, $k \geq 0$, of the PDSOC$^\gamma$ algorithm belong to the set

$$\mathcal{N}_\gamma := \{(x, y, s) \in \mathcal{F}_{PD} : \; x_i s_i \geq \gamma \frac{x^\top s}{n}, \quad i = 1, 2, \ldots, n\}. \tag{3.23}$$

For any $\gamma \in (0, 1)$, the set (3.23) contains the primal-dual central path $\mathcal{C}$ defined by (2.6), and the primal-dual solution set $\mathcal{S}_{PD}$ defined by (2.2). The shaded region in Figure 1 a) represents an $\mathcal{N}_\gamma$ neighbourhood for $n = 2$ and $\gamma = 0.4$, in the space of the complementarity products (x_1s_1, x_2s_2). An explanation for Figure 1 b) and c) will be given shortly.

We remark that the constraints (3.19) may force the parameter γ to be inefficiently small. The parameter β of the $\sigma\beta$ procedure, however, can take any value in $(0, 1)$. Next we are going to describe this new linesearch technique.

The $\sigma\beta$ stepsize procedure tries to keep the iterates away from the boundaries of the feasible region, and was suggested by M. J. D. Powell for the PD algorithm with $\sigma^k = \sigma$, $k \geq 0$ [1].

The $\sigma\beta$ stepsize procedure At the start of the PDSOC algorithm, choose $\beta \in (0, 1)$. On every iteration $k \geq 0$ of the algorithm, compute the stepsize θ^k in Step 3 as the largest $\overline{\theta} \in (0, 1]$, such that the inequalities

$$x_i^k(\theta)s_i^k(\theta) \geq m_i^k := \min\,(x_i^k s_i^k, \sigma^k\beta\mu^k), \quad i = \overline{1,n}, \tag{3.24}$$

are satisfied for all $\theta \in [0, \overline{\theta}]$, the notation being taken from expression (3.10).

We denote by PDSOC$^{\sigma\beta}$, the variant of the PDSOC algorithm that employs the $\sigma\beta$ stepsize procedure.

To see that the stepsize procedure $\sigma\beta$ is well-defined, let i be any integer from $[1, n]$, and let ψ_i be the function

$$\begin{aligned} \psi_i(\theta) \ &:= x_i^k(\theta) s_i^k(\theta) - m_i^k \\ &= dx_i^{k,c} ds_i^{k,c} \theta^4 + (dx_i^k ds_i^{k,c} + ds_i^k dx_i^{k,c})\theta^3 + (-x_i^k s_i^k + \sigma^k \mu^k)\theta + x_i^k s_i^k - m_i^k, \end{aligned} \tag{3.25}$$

where the last equality depends on (3.21), and where $0 \leq \theta \leq 1$. The definition of m_i^k implies $\psi_i(0) \geq 0$, with equality if and only if $x_i^k s_i^k \leq \sigma^k \beta \mu^k$. Then, however, $\psi_i'(0) = -x_i^k s_i^k + \sigma^k \mu^k \geq (1-\beta)\sigma^k \mu^k > 0$ is obtained. Thus ψ_i is positive for sufficiently small $\theta > 0$. Since this is true for any $i \in \{1, \ldots, n\}$, we can always find a positive step $\theta \leq 1$ that satisfies $\psi_i(\theta) \geq 0$, $i = \overline{1, n}$, or the equivalent inequalities (3.24).

We remark that the presence of the term $x_i^k s_i^k$ in the expression (3.24) of m_i^k, $i = \overline{1, n}$, allows the parameter $\beta \in (0, 1)$ to be chosen independently of the starting point w^0.

For $k \geq 0$, we let

$$\mathcal{N}_{\sigma\beta}(w^k) := \{(x, y, s) \in \mathcal{F}_{PD} : \ \ x_i s_i \geq m_i^k = \min\left(x_i^k s_i^k, \sigma^k \beta \mu^k\right), \ \ i = \overline{1, n}\}, \tag{3.26}$$

represent the set defined by the linesearch conditions (3.24) on the kth iteration of the PDSOC$^{\sigma\beta}$ algorithm. The shaded regions in Figure 1 b) and c) display, in the space of the complementarity products $(x_1 s_1, x_2 s_2)$, the neighbourhood $\mathcal{N}_{\sigma\beta}(w^k)$ for $n = 2$, $\sigma^k = 0.9$, $\beta = 5/9$, and for two different points w^k with the same value of $\mu^k = 0.4$. Both of these displays also show the points $v^k := (x_1^k s_1^k, x_2^k s_2^k)$ and $u^k := \sigma^k \beta \mu^k (1, 1)$.

For $n \geq 2$, in the space of the complementarity products $(x_1 s_1, \ldots, x_n s_n)$, the set $\mathcal{N}_{\sigma\beta}(w^k)$ is the translation of the nonnegative orthant $\mathbb{R}^n_+$ to the point $(m_1^k, \ldots, m_n^k)$. The target point $w(\sigma^k \mu^k)$ on the primal-dual central path, which satisfies $x_i^k s_i^k = \sigma^k \mu^k$, $i = 1, 2, \ldots, n$, always belongs to $\mathcal{N}_{\sigma\beta}(w^k)$. We remark, however, that $\mathcal{N}_{\sigma\beta}(w^k)$ contains the primal-dual solution set $\mathcal{S}_{PD}$, which has $x_i s_i = 0$, $i = 1, 2, \ldots, n$, only possibly asymptotically as $k \to \infty$.

For convergence and complexity properties of the PD algorithm with the $\sigma\beta$ stepsize procedure, see Chapter 3 of [1].

4 On the convergence and complexity of the PDSOC$^{\sigma\beta}$ algorithm

Let the PDSOC$^{\sigma\beta}$ algorithm be applied to problems (P) and (D) that satisfy the IPM conditions. We let the centring parameters σ^k decrease monotonically to a strictly positive value. The main results of this section are Theorem 4.4 and Corollary 4.7. In the former, we show that the sequence of duality gaps of the iterates tends to zero, and that the termination criteria in Step 1 of the algorithm is satisfied in at most $\mathcal{O}(n^{3/4}\log((x^0)^\top s^0/\epsilon))$ iterations. In Corollary 4.7, we show that all the limit points of the sequence of iterates are primal-dual strictly complementary solutions of (P) and (D).

The next three lemmas are technical.

Lemma 4.1 *Let w^k, $k \geq 0$, be the sequence of iterates generated by the* PDSOC$^{\sigma\beta}$ *algorithm and let the sequence of centring parameters σ^k be monotonically decreasing. Then, for each $i \in \{1,\ldots,n\}$, the sequence $m_i^k/(\sigma^k\mu^k)$, $k \geq 0$, is monotonically increasing. This provides the property*

$$\frac{x_i^k s_i^k}{\sigma^k\mu^k} \geq \rho, \quad i = \overline{1,n}, \quad k \geq 0, \tag{4.1}$$

where

$$\rho := \frac{1}{\sigma^0\mu^0}\min\{m_i^0 : i = 1,2,\ldots,n\} \in (0,1). \tag{4.2}$$

Proof. Let $i \in \{1,\ldots,n\}$. From (3.24), we have

$$x_i^{k+1}s_i^{k+1} \geq m_i^k := \min\left(x_i^k s_i^k, \sigma^k\beta\mu^k\right), \tag{4.3}$$

which gives

$$\frac{x_i^{k+1}s_i^{k+1}}{\sigma^k\mu^k} \geq \min\left(\frac{x_i^k s_i^k}{\sigma^k\mu^k}, \beta\right), \quad k \geq 0. \tag{4.4}$$

Since σ^k decreases monotonically and μ^k decreases strictly as in (3.11), it follows that

$$\frac{x_i^{k+1}s_i^{k+1}}{\sigma^{k+1}\mu^{k+1}} \geq \min\left(\frac{x_i^k s_i^k}{\sigma^k\mu^k}, \beta\right), \quad k \geq 0. \tag{4.5}$$

Thus we obtain

$$\min\left(\frac{x_i^{k+1}s_i^{k+1}}{\sigma^{k+1}\mu^{k+1}}, \beta\right) \geq \min\left(\frac{x_i^k s_i^k}{\sigma^k\mu^k}, \beta\right), \quad k \geq 0, \tag{4.6}$$

which together with definition (3.24), implies

$$\frac{m_i^{k+1}}{\sigma^{k+1}\mu^{k+1}} \geq \frac{m_i^k}{\sigma^k\mu^k}, \quad k \geq 0. \tag{4.7}$$

The inequalities (4.1) follow from (4.7) and the definition of m_i^k, $i = \overline{1,n}$. This definition also implies that, for $k = 0$, we have $\rho \le \beta < 1$. □

Lemma 4.2 *Let problems (P) and (D) satisfy the* IPM *conditions and let* $w^k = (x^k, y^k, s^k)$ *be a primal-dual strictly feasible point. Let* $\mu^k := (x^k)^\top s^k/n$, $D^k := (X^k)^{1/2}(S^k)^{-1/2}$, *and let* $dw^k = (dx^k, dy^k, ds^k)$ *be defined by (3.3) for some* $\sigma^k \in (0,1)$. *Then we have the identity*

$$\|(D^k)^{-1}dx^k + D^k ds^k\|^2 = \|(D^k)^{-1}dx^k\|^2 + \|D^k ds^k\|^2. \tag{4.8}$$

Letting $\rho \in (0,1)$ *be such that*

$$\frac{x_i^k s_i^k}{\sigma^k \mu^k} \ge \rho, \quad i \in \{1, \ldots, n\}, \tag{4.9}$$

we obtain the bounds

$$\|(D^k)^{-1}dx^k + D^k ds^k\|^2 \le n\delta\mu^k, \tag{4.10a}$$

$$\|dX^k dS^k e\| \le \frac{1}{2} n\delta\mu^k, \tag{4.10b}$$

where $\delta := (1 + 1/\rho)$, *and* dX^k *and* dS^k *denote the diagonal matrices having as diagonal elements the components of the vectors* dx^k *and* ds^k, *respectively.*

Proof. The identity (4.8) follows from the first identity in (3.8).

The last n equations of the system (3.3) are

$$S^k dx^k + X^k ds^k = -X^k S^k e + \sigma^k \mu^k e. \tag{4.11}$$

To obtain (4.10a), we first multiply the equations (4.11) by $(X^k)^{-1/2}(S^k)^{-1/2}$ and obtain

$$(D^k)^{-1}dx^k + D^k ds^k = -(X^k)^{1/2}(S^k)^{1/2}e + \sigma^k \mu^k (X^k)^{-1/2}(S^k)^{-1/2}e, \tag{4.12}$$

which implies

$$\begin{aligned} &\|(D^k)^{-1}dx^k + D^k ds^k\|^2 \\ &\quad = \|(X^k)^{1/2}(S^k)^{1/2}e\|^2 - 2\sigma^k\mu^k e^\top e + (\sigma^k)^2(\mu^k)^2\|(X^k)^{-1/2}(S^k)^{-1/2}e\|^2 \\ &\quad = n\mu^k(1 - 2\sigma^k) + \sigma^k\mu^k \textstyle\sum_{i=1}^n \frac{\sigma^k\mu^k}{x_i^k s_i^k}. \end{aligned} \tag{4.13}$$

Employing (4.9) in the sum on the right-hand side of (4.13), we deduce

$$\|(D^k)^{-1}dx^k + D^k ds^k\|^2 \le n\mu^k(1 - 2\sigma^k) + \sigma^k\mu^k\frac{n}{\rho}, \tag{4.14}$$

which, together with $\sigma^k \in (0,1)$, yields (4.10a).

To show (4.10b), we employ

$$\begin{aligned}\|dX^k dS^k e\| = \|dX^k (D^k)^{-1} D^k dS^k e\| &\leq \|(D^k)^{-1} dx^k\| \cdot \|D^k ds^k\| \\ &\leq \tfrac{1}{2}\left(\|(D^k)^{-1} dx^k\|^2 + \|D^k ds^k\|^2\right),\end{aligned} \tag{4.15}$$

where the first inequality in the above expression follows from

$$\begin{aligned}\|dX^k (D^k)^{-1} D^k dS^k e\|^2 = \textstyle\sum_{i=1}^n \left(\frac{dx_i^k}{d_{ii}^k}\right)^2 (d_{ii}^k ds_i^k)^2 &\leq \textstyle\sum_{i=1}^n \left(\frac{dx_i^k}{d_{ii}^k}\right)^2 \sum_{i=1}^n (d_{ii}^k ds_i^k)^2 \\ &= \|(D^k)^{-1} dx^k\|^2 \|D^k ds^k\|^2.\end{aligned} \tag{4.16}$$

From (4.8) and (4.10a), we have

$$\|(D^k)^{-1} dx^k\|^2 + \|D^k ds^k\|^2 = \|(D^k)^{-1} dx^k + D^k ds^k\|^2 \leq n\mu^k \delta. \tag{4.17}$$

The bound (4.10b) follows from (4.15) and (4.17). □

It follows from (4.1) that condition (4.9) of Lemma 4.2 is satisfied by the PDSOC$^{\sigma\beta}$ algorithm. Thus relations (4.8) and the bounds (4.10) hold on each iteration k of the PDSOC$^{\sigma\beta}$ algorithm. In the next lemma we derive similar relations and bounds for the corrector direction $dw^{k,c}$. In addition to the assumptions in Lemma 4.1, we let the centring parameters σ^k be bounded away from zero, i.e. there exists $\underline{\sigma} \in (0,1)$ such that

$$\sigma^k \geq \underline{\sigma} \in (0,1), \quad k \geq 0. \tag{4.18}$$

Inequalities (4.1) and (4.18) imply

$$\frac{x_i^k s_i^k}{\mu^k} \geq \underline{\sigma}\rho, \quad i = \overline{1,n}, \quad k \geq 0, \tag{4.19}$$

which is used in the proof below.

Lemma 4.3 *Let problems (P) and (D) satisfy the* IPM *conditions. Let w^k, $k \geq 0$, be the sequence of iterates generated by the* PDSOC$^{\sigma\beta}$ *algorithm when applied to these problems. Assume that the sequence of centring parameters σ^k is monotonically decreasing and bounded below by $\underline{\sigma} > 0$. Then we have*

$$\|(D^k)^{-1} dx^{k,c} + D^k ds^{k,c}\|^2 = \|(D^k)^{-1} dx^{k,c}\|^2 + \|D^k ds^{k,c}\|^2, \quad k \geq 0, \tag{4.20}$$

where $D^k := (X^k)^{1/2}(S^k)^{-1/2}$, and the corrector $dw^{k,c}$ satisfies the bounds

$$\|(D^k)^{-1} dx^{k,c} + D^k ds^{k,c}\|^2 \leq n^2 \frac{\delta^2}{4\underline{\sigma}\rho}\,\mu^k, \tag{4.21}$$

$$\|dX^{k,c} dS^{k,c} e\| \leq n^2 \frac{\delta^2}{4\underline{\sigma}\rho}\,\mu^k, \quad \|dX^k dS^{k,c} e\| \leq n\sqrt{n}\frac{\delta\sqrt{\delta}}{2\sqrt{\underline{\sigma}\rho}}\,\mu^k, \quad \|dX^{k,c} dS^k e\| \leq n\sqrt{n}\frac{\delta\sqrt{\delta}}{2\sqrt{\underline{\sigma}\rho}}\,\mu^k, \tag{4.22}$$

where $\delta := (1 + 1/\rho)$ *and* $\rho \in (0, 1)$ *is the scalar (4.2). Here* $dX^{k,c}$ *and* $dS^{k,c}$ *are the diagonal matrices having as diagonal elements the components of the vectors* $dx^{k,c}$ *and* $ds^{k,c}$*, respectively.*

Proof. The identity (4.20) follows from the orthogonality property $(dx^{k,c})^\top ds^{k,c} = 0$ given in (3.8).

Let $k \geq 0$. Multiplying the last n equations of system (3.4), i.e.

$$S^k dx^{k,c} + X^k ds^{k,c} = -dX^k dS^k e, \tag{4.23}$$

by $(X^k)^{-1/2}(S^k)^{-1/2}$, we obtain

$$(D^k)^{-1} dx^{k,c} + D^k ds^{k,c} = -(X^k)^{-1/2}(S^k)^{-1/2} dX^k dS^k e. \tag{4.24}$$

Computing the Euclidean norm of both sides of (4.24), we find

$$\|(D^k)^{-1} dx^{k,c} + D^k ds^{k,c}\|^2 = \sum_{i=1}^{n} \frac{(dx_i^k ds_i^k)^2}{x_i^k s_i^k}. \tag{4.25}$$

Substituting (4.19) into (4.25), we deduce

$$\|(D^k)^{-1} dx^{k,c} + D^k ds^{k,c}\|^2 \leq \frac{1}{\underline{\sigma} \rho \mu^k} \|dX^k dS^k e\|^2. \tag{4.26}$$

We already remarked before the statement of this lemma that, due to (4.1), Lemma 4.2 holds for the PDSOC$^{\sigma\beta}$ algorithm. In particular, the bound (4.10b) holds. Substituting (4.10b) in (4.26), we obtain (4.21).

To show (4.22), we consider the length of the vector $dX^{k,p} dS^{k,q} e$, where p, $q \in \{0, c\}$, with $dx^{k,0} := dx^k$ and $ds^{k,0} := ds^k$. The following sequence of relations holds

$$\begin{aligned}
||dX^{k,p} dS^{k,q} e||^2 &= \textstyle\sum_{i=1}^n (dx_i^{k,p} ds_i^{k,q})^2 = \sum_{i=1}^n ((D^k)_{ii}^{-1} dx_i^{k,p})^2 (D_{ii}^k ds_i^{k,q})^2 \\
&\leq \left[\textstyle\sum_{i=1}^n ((D^k)_{ii}^{-1} dx_i^{k,p})^2\right] \cdot \left[\textstyle\sum_{i=1}^n (D_{ii}^k ds_i^{k,q})^2\right] = ||(D^k)^{-1} dx^{k,p}||^2 \cdot ||D^k ds^{k,q}||^2 \\
&\leq \left(||(D^k)^{-1} dx^{k,p}||^2 + ||D^k ds^{k,p}||^2\right) \cdot \left(||(D^k)^{-1} dx^{k,q}||^2 + ||D^k ds^{k,q}||^2\right).
\end{aligned}$$

Thus we deduce the inequalities

$$||dX^{k,c} dS^{k,c} e|| \leq ||(D^k)^{-1} dx^{k,c}||^2 + ||D^k ds^{k,c}||^2, \tag{4.27a}$$

$$||dX^k dS^{k,c} e|| \leq \sqrt{||(D^k)^{-1} dx^k||^2 + ||D^k ds^k||^2} \cdot \sqrt{||(D^k)^{-1} dx^{k,c}||^2 + ||D^k ds^{k,c}||^2}, \tag{4.27b}$$

$$||dX^{k,c} dS^k e|| \leq \sqrt{||(D^k)^{-1} dx^k||^2 + ||D^k ds^k||^2} \cdot \sqrt{||(D^k)^{-1} dx^{k,c}||^2 + ||D^k ds^{k,c}||^2}. \tag{4.27c}$$

From (4.27a), (4.20) and (4.21), we obtain the first inequality in (4.22). Substituting (4.8), (4.10a), (4.20) and (4.21) into (4.27b) and into (4.27c), we obtain the second and the third inequalities in (4.22), respectively. □

We are now ready to give the first main result of this section.

Theorem 4.4 *Let problems (P) and (D) satisfy the* IPM *conditions. Apply the* PDSOC$^{\sigma\beta}$ *algorithm to these problems, and choose the centring parameters σ^k, $k \geq 0$, to be monotonically decreasing and to satisfy (4.18). Then the stepsizes θ^k have the property*

$$\theta^k \geq \underline{\theta} := \underline{c}\,\frac{1}{n^{3/4}}, \quad k \geq 0, \tag{4.28}$$

for some $\underline{c} \in (0,1)$, independent of k and n. The sequence of duality gaps $(x^k)^\top s^k$, $k \geq 0$, of the iterates converges to zero, as $k \to \infty$. Moreover, given any $\epsilon > 0$, the algorithm takes at most $\mathcal{O}\left(n^{\frac{3}{4}} \log \frac{(x^0)^\top s^0}{\epsilon}\right)$ iterations to generate an iterate w^k such that $(x^k)^\top s^k < \epsilon$.

Proof. Let $i \in \{1, \ldots, n\}$ and $k \geq 0$. First, we deduce a lower bound on the product $x_i^k(\theta)s_i^k(\theta)$. Substituting the inequalities $x_i^k s_i^k \geq m_i^k := \min(x_i^k s_i^k, \sigma^k \beta \mu^k)$ and $dx_i^{k,p} ds_i^{k,q} \geq -\|dX^{k,p} dS^{k,q} e\|$, where $p, q \in \{0, c\}$ and $dw^{k,0} := dw^k$, into (3.21), we obtain the condition

$$x_i^k(\theta)s_i^k(\theta) \geq m_i^k(1-\theta)+\theta\sigma^k\mu^k-\theta^3\left(\|dX^k dS^{k,c}e\| + \|dX^{k,c}dS^k e\|\right)-\theta^4\|dX^{k,c}dS^{k,c}e\|, \tag{4.29}$$

for any $\theta \in [0,1]$. Furthermore, using $\sigma^k\beta\mu^k \geq m_i^k$, (4.29) becomes

$$x_i^k(\theta)s_i^k(\theta) \geq m_i^k+\theta\sigma^k(1-\beta)\mu^k-\theta^3\left(\|dX^k dS^{k,c}e\| + \|dX^{k,c}dS^k e\|\right)-\theta^4\|dX^{k,c}dS^{k,c}e\|, \tag{4.30}$$

for $\theta \in [0,1]$. The inequality (4.30) confirms that the conditions (3.24) allow the steplength θ^k to be positive. Substituting (4.18) and (4.22) in (4.30), we obtain the inequality

$$x_i^k(\theta)s_i^k(\theta) \geq m_i^k + \theta\underline{\sigma}(1-\beta)\mu^k - \theta^3 n\sqrt{n}\frac{\delta\sqrt{\delta}}{\sqrt{\underline{\sigma}\rho}}\,\mu^k - \theta^4 n^2 \frac{\delta^2}{4\underline{\sigma}\rho}\,\mu^k, \quad \theta \in [0,1]. \tag{4.31}$$

Thus if the conditions

$$m_i^k + \theta\underline{\sigma}(1-\beta)\mu^k - \theta^3 n\sqrt{n}\frac{\delta\sqrt{\delta}}{\sqrt{\underline{\sigma}\rho}}\,\mu^k - \theta^4 n^2 \frac{\delta^2}{4\underline{\sigma}\rho}\,\mu^k \geq m_i^k, \quad i = \{1, \ldots, n\}, \tag{4.32}$$

hold, then the inequalities (3.24) are satisfied. Since $\theta \geq 0$ and $\mu^k > 0$, the expressions (4.32) are equivalent to the condition

$$\underline{\sigma}(1-\beta) - \theta^2 n\sqrt{n}\frac{\delta\sqrt{\delta}}{\sqrt{\underline{\sigma}\rho}} - \theta^3 n^2 \frac{\delta^2}{4\underline{\sigma}\rho} \geq 0, \tag{4.33}$$

which is equivalent to the inequality

$$g(\theta) := 4\underline{\sigma}^2(1-\beta)\rho - 4\theta^2 n^{3/2}\delta(\delta\underline{\sigma}\rho)^{1/2} - \theta^3 n^2\delta^2 \geq 0. \tag{4.34}$$

Since $g(0) > 0$, the condition (4.34) allows $\theta > 0$. Let $\underline{c} \in (0,1)$ be the number that satisfies $4\underline{\sigma}^2(1-\beta)\rho - 4\underline{c}^2\delta(\delta\underline{\sigma}\rho)^{1/2} - \underline{c}^3\delta^2 = 0$, which is independent of k and n. The inequality

$\underline{c} < 1$ holds since $\underline{\sigma} < 1$, $\rho < 1$ and $\delta > 2$ provide $4\underline{\sigma}^2(1-\beta)\rho - 4\delta(\delta\underline{\sigma}\rho)^{1/2} - \delta^2 < 0$. Then $\theta \le \underline{c}/n^{3/4}$ implies $g(\theta) \ge g(\underline{c}/n^{3/4}) \ge 0$, because $n \ge 1$. Thus (4.28) is valid.

Next we substitute the bound (4.28) and $\sigma^k \le \sigma^0$ into the recurrence relation (3.9) for $\theta := \theta^k$, and obtain

$$(x^{k+1})^\top s^{k+1} \le \left(1 - \frac{\underline{c}(1-\sigma^0)}{n^{3/4}}\right)(x^k)^\top s^k, \quad k \ge 0, \tag{4.35}$$

which implies that the duality gap $(x^k)^\top s^k$ is decreased on each iteration by at least a constant factor in (0,1). Therefore the sequence of duality gaps tends to zero.

The inequalities (4.35) also provide the polynomial bound on the number of iterations. Specifically, we find

$$(x^k)^\top s^k \le \left(1 - \frac{r}{n^{3/4}}\right)^k (x^0)^\top s^0 \le e^{-\frac{rk}{n^{3/4}}}(x^0)^\top s^0, \tag{4.36}$$

where $r := \underline{c}(1-\sigma^0)$. Thus $(x^k)^\top s^k \le \epsilon$ holds, if

$$-k\frac{r}{n^{3/4}} \le \log\frac{\epsilon}{(x^0)^\top s^0}, \tag{4.37}$$

which implies that $(x^k)^\top s^k \le \epsilon$ is achieved for all $k \ge n^{3/4} r^{-1}\log((x^0)^\top s^0/\epsilon)$. □

The above proof implies that the constants that occur in the bound $\mathcal{O}(n^{3/4}\log((x^0)^\top s^0)/\epsilon)$ in Theorem 4.4 do not depend on the quantities (A, b, c) which constitute the data of our LP problem. Moreover, the worst-case iteration complexity bound for the PDSOC$^{\sigma\beta}$ algorithm given above is better than the worst-case iteration complexity bound for the PD algorithm with the $\sigma\beta$ stepsize procedure (see Theorem 3.6 in [1]).

The next lemma shows that after finitely many iterations, m_i^k is independent of i.

Lemma 4.5 *Let the conditions of Theorem 4.4 hold. There exists an iteration number* $k_0 \ge 0$ *such that*

$$m_i^k := \min(x_i^k s_i^k, \sigma^k\beta\mu^k) = \sigma^k\beta\mu^k, \quad i = \overline{1,n}, \quad \textit{for all} \quad k \ge k_0. \tag{4.38}$$

Proof. For any $i \in \{1,\ldots,n\}$, $\mu^k \to 0$ implies $x_i^k s_i^k \to 0$, as $k \to \infty$. Hence $x_i^{k_i+1}s_i^{k_i+1} < x_i^{k_i}s_i^{k_i}$ holds for some k_i, and then condition (3.24) gives $x_i^{k_i+1}s_i^{k_i+1} \ge \sigma^{k_i}\beta\mu^{k_i}$. Since both (σ^k) and (μ^k) are decreasing sequences, we further deduce that $m_i^{k_i+1} = \sigma^{k_i+1}\beta\mu^{k_i+1}$. By the same argument, this relation now holds inductively, for all $k \ge k_i + 1$. Letting $k_0 := \max\{k_i : i = \overline{1,n}\} + 1$, it follows that (4.38) holds for $k \ge k_0$. □

Subject to the conditions of Theorem 4.4, it follows immediately from (4.19) that the iterates w^k, $k \ge 0$, belong to $\mathcal{N}_\gamma$, defined in (3.23), with $\gamma := \underline{\sigma}\rho$. Thus we have established a connection between the neighbourhoods $\mathcal{N}_\gamma$ and $\mathcal{N}_{\sigma\beta}$. Moreover, this property and the next Lemma allow us to show some results concerning the limit points of the sequence of iterates w^k of the PDSOC$^{\sigma\beta}$ algorithm which we state in the Corollary following the Lemma.

Lemma 4.6 *[6, 24] Let problems (P) and (D) satisfy the* IPM *conditions and let* $w^k = (x^k, y^k, s^k)$, $k \geq 0$, *be a sequence of primal-dual strictly feasible points and* $\mu^k := (x^k)^\top s^k/n$, $k \geq 0$. *If*

$$\frac{x_i^k s_i^k}{\mu^k} \geq \gamma, \quad k \geq 0, \quad i = \overline{1,n}, \tag{4.39}$$

for some constant $\gamma > 0$, *and*

$$\mu^k \leq L, \quad k \geq 0, \tag{4.40}$$

for some $L > 0$, *then the following estimates hold*

$$x_i^k = \mu^k \Theta(1), \quad i \in \mathcal{A} \quad \text{and} \quad x_i^k = \Theta(1), \quad i \in \mathcal{I}, \tag{4.41a}$$

$$s_i^k = \Theta(1), \quad i \in \mathcal{A} \quad \text{and} \quad s_i^k = \mu^k \Theta(1), \quad i \in \mathcal{I}, \tag{4.41b}$$

where $\mathcal{A}$ *and* $\mathcal{I}$ *are the strict complementarity partition of the index set* $\{1, \ldots, n\}$ *as defined in* (2.4), *and* $\Theta(1)$ *denotes a term that is bounded above and below by positive constants that are independent of* k. *Additionally, if*

$$\mu^k \to 0, \quad k \to \infty, \tag{4.42}$$

then the sequence w^k *is bounded, thus having limit points, and every such limit point is a strictly complementary solution of (P) and (D).*

Proof. See, for example, the proof of Lemmas 2.10 and 2.14 in [1]. □

Corollary 4.7 *Let the conditions of Theorem 4.4 hold. Then the sequence of iterates* $w^k = (x^k, y^k, s^k)$, $k \geq 0$, *generated by the* PDSOC$^{\sigma\beta}$ *algorithm is bounded and all its limit points are primal-dual strictly complementary solutions. If (P) and (D) have a unique solution, the iterates* w^k *converge to this solution.*

Proof. Under the conditions of Theorem 4.4, the assumptions of Lemma 4.6 are satisfied by the sequence of iterates generated by the PDSOC$^{\sigma\beta}$ algorithm (see our remarks preceding the statement of Lemma). □

After we employed the PDSOC algorithm for correcting the behaviour of the algorithm that takes the full correctors, and analysed its convergence and complexity with the γ and the $\sigma\beta$ stepsize, we found out that the framework of the PDSOC algorithm is known [28]. The results in [28] address a more general setting; when applied to LP and to the PDSOC algorithm, one can deduce convergence of the duality gap to zero and polynomial complexity results for the PDSOC$^\gamma$ algorithm when $\sigma^k = \sigma \in (0,1)$, $k \geq 0$. In particular, it is reasonably straightforward to derive the following properties.

Theorem 4.8 *Let problems (P) and (D) satisfy the* IPM *conditions. Let w^k, $k \geq 0$, be the iterates generated by the* PDSOC$^\gamma$ *algorithm when applied to these problems with each σ^k in the interval*

$$0 < \underline{\sigma} \leq \sigma^k \leq \overline{\sigma} < 1, \quad k \geq 0. \tag{4.43}$$

Then $(x^k)^\top s^k \to 0$, $k \to \infty$, and all the limit points of w^k are primal-dual strictly complementary solutions. Moreover, given any $\epsilon > 0$, the PDSOC$^\gamma$ *algorithm takes at most $\mathcal{O}\left(n^{\frac{3}{4}} \log \frac{(x^0)^\top s^0}{\epsilon}\right)$ iterations to generate an iterate w^k such that $(x^k)^\top s^k < \epsilon$.*

Proof. See [28] or the proof of Theorem C.1 in Appendix C of [1]. □

No numerical experiments with the PDSOC algorithm are reported in [28] or elsewhere in the literature.

The next section presents a new convergence result for the PDSOC$^\gamma$ algorithm with a popular choice of centring parameters.

5 Convergence of the PDSOC$^\gamma$ algorithm when σ^k tends to zero at a certain rate

We now consider letting the centring parameters in the PDSOC$^\gamma$ algorithm be of the same order as the duality gap of the iterates asymptotically. The main result of this section is Theorem 5.7. To prove it, we need the following auxiliary lemmas.

Firstly, we recall a linear algebra result, namely, Hoffman's lemma [7]. We consider the polyhedron

$$\mathcal{P} := \{x \in \mathbb{R}^n : \ \tilde{A}x \leq \tilde{b}\}, \tag{5.1}$$

where $\tilde{b} \in \mathbb{R}^p$, $\tilde{A}$ is a real matrix of dimension $p \times n$, and p is a positive integer. We introduce the notation

$$a^+ := \max\{a, 0\}, \quad y^+ := (y_1^+, \ldots, y_p^+), \tag{5.2}$$

for any $a \in \mathbb{R}$ and any vector $y \in \mathbb{R}^p$. Hoffman's lemma for $\mathcal{P}$ is stated next.

Lemma 5.1 *[7] Let the polyhedron* (5.1) *be nonempty. Then there exists a positive constant h such that for any $x \in \mathbb{R}^n$, there exists a point $x^\dagger \in \mathcal{P}$ with*

$$\|x - x^\dagger\| \leq h\|(\tilde{A}x - \tilde{b})^+\|. \tag{5.3}$$

Proof. Let $F_n := \|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, $m := p$ and $F_m := \|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ in the Theorem of [7]. □

The generality of the next lemma will allow us to apply it in different contexts.

Lemma 5.2 *Let problems (P) and (D) satisfy the* IPM *conditions and let w^k be a primal-dual strictly feasible point. Let $\overline{w} = (\overline{x}, \overline{y}, \overline{s})$ be the solution of the linear system*

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ S^k & 0 & X^k \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{s} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, \tag{5.4}$$

where $z \in \mathbb{R}^n$. Then the following explicit expressions hold

$$\overline{x} = X^k(X^kS^k)^{-1/2}(I - P^k)(X^kS^k)^{-1/2}z, \tag{5.5a}$$

$$\overline{y} = -(AA^\top)^{-1}A\overline{s}, \tag{5.5b}$$

$$\overline{s} = S^k(X^kS^k)^{-1/2}P^k(X^kS^k)^{-1/2}z, \tag{5.5c}$$

where

$$P^k := D^kA^\top(M^k)^{-1}AD^k, \quad M^k := A(D^k)^2A^\top, \quad \textit{and} \quad D^k := (X^k)^{1/2}(S^k)^{-1/2}. \tag{5.6}$$

Proof. Multiplying the last n equations of (5.4) by $A(S^k)^{-1}$, and employing $A\overline{x} = 0$ and $\overline{s} = -A^\top\overline{y}$, we deduce

$$\overline{y} = -(M^k)^{-1}A(S^k)^{-1}z, \tag{5.7}$$

where M^k is defined in (5.6). Substituting this value into the last $2n$ equations of (5.4), we obtain

$$\overline{s} = A^\top(M^k)^{-1}A(S^k)^{-1}z, \tag{5.8a}$$

$$\overline{x} = [I - (S^k)^{-1}X^kA^\top(M^k)^{-1}A](S^k)^{-1}z. \tag{5.8b}$$

It can be verified that the expressions (5.8a) and (5.5c) are identical, and that the expressions (5.8b) and (5.5a) also coincide. The expression (5.5b) follows from the equations $A^\top\overline{y}+\overline{s} = 0$, and A having full row rank. □

The expressions (5.5a) and (5.5c) appear in [21], in the special case of the right-hand sides (3.3). The matrix P^k in (5.6) is the matrix of the orthogonal projection onto the range space of $D^kA^\top$. Thus it is symmetric, positive semidefinite and idempotent. These properties also hold for the matrix $I - P^k$, since it is the matrix of orthogonal projection onto the null space of AD^k. It follows that

$$|P^k_{ij}| \leq 1 \quad \text{and} \quad |(I - P^k)_{ij}| \leq 1, \quad i = \overline{1, n},\ j = \overline{1, n}. \tag{5.9}$$

Lemma 5.3 *Let the conditions and notations of Lemma 5.2 hold, and let* $\mu^k := (x^k)^\top s^k/n$. *Let*

$$(x^k)^\top s^k \le L \quad \textit{and} \quad x_i^k s_i^k \ge \gamma\mu^k, \quad i = \overline{1,n}, \tag{5.10}$$

also hold for some constants $L > 0$ *and* $\gamma \in (0,1)$, *independent of* k. *Then there exists a positive constant* q_0 *such that*

$$\max\{|\overline{x}_i|,\, |\overline{s}_i|\} \le \frac{q_0}{\mu^k}\sum_{j=1}^n |z_j|, \quad i = \overline{1,n}. \tag{5.11}$$

Proof. From (5.5a) and (5.5c), we deduce

$$\overline{x}_i = \frac{x_i^k}{\sqrt{x_i^k s_i^k}}\sum_{j=1}^n \frac{(I-P^k)_{ij}}{\sqrt{x_j^k s_j^k}} z_j \quad \text{and} \quad \overline{s}_i = \frac{s_i^k}{\sqrt{x_i^k s_i^k}}\sum_{j=1}^n \frac{P_{ij}^k}{\sqrt{x_j^k s_j^k}} z_j, \quad i = \overline{1,n}. \tag{5.12}$$

The first part of Lemma 4.6 applies due to the first condition in (5.10). Thus there exists a positive constant q_1 such that $x_i^k < q_1$ and $s_i^k < q_1$, $i = \overline{1,n}$. It follows from (5.9) that the bounds

$$\max\{|\overline{x}_i|,\, |\overline{s}_i|\} \le q_1 \frac{1}{\sqrt{x_i^k s_i^k}}\sum_{j=1}^n \frac{|z_j|}{\sqrt{x_j^k s_j^k}} \le \frac{q_1}{\gamma\mu^k}\sum_{j=1}^n |z_j|, \quad i = \overline{1,n}, \tag{5.13}$$

hold, where the last inequality depends on the last n expressions in (5.10). Thus we obtain (5.11), where $q_0 := q_1/\gamma$. □

These technical lemmas are employed in the next proof, which follows some ideas from the proof of Theorem 3.1 in [21].

Lemma 5.4 *Let problems (P) and (D) satisfy the* IPM *conditions, and let* w^k *be a primal-dual strictly feasible point satisfying* (5.10). *Let* dw^k *be the direction defined by the system* (3.3), *where* $\sigma^k \in (0,1)$. *Then there exist positive constants* q_2 *and* q_3, *independent of* k, *such that*

$$\|w^k + dw^k - w^*\| \le q_2\sigma^k + q_3\|w^k - w^*\|, \tag{5.14}$$

where $w^* = (x^*, y^*, s^*)$ *is any primal-dual solution.*

Proof. Let w^* be a primal-dual strictly complementary solution. If (5.14) holds for any such solution, then it holds for any primal-dual solution, by continuity.

Since w^k is primal-dual strictly feasible, the difference $w^k - w^*$ is the solution of the system (5.4) for $z := S^k(x^k - x^*) + X^k(s^k - s^*)$, and dw^k is the solution of (5.4) for $z := -X^kS^ke + \sigma^k\mu^ke$. Thus letting $v^k := w^k + dw^k - w^*$, we deduce that v^k is the solution of (5.4) for

$$z := X^kS^ke + \sigma^k\mu^ke - S^kx^* - X^ks^*, \tag{5.15}$$

so Lemma 5.3 provides the bounds

$$\max\{|x_i^k+dx_i^k-x_i^*|,\ |s_i^k+ds_i^k-s_i^*|\} \leq \frac{q_0}{\mu^k}\sum_{j=1}^{n}|x_j^k s_j^k+\sigma^k\mu^k - s_j^k x_j^* - x_j^k s_j^*|, \quad i=\overline{1,n}. \tag{5.16}$$

Since w^* is a strictly complementary solution, it follows from (2.4) that $x_j^* = 0$, $j \in \mathcal{A}$, and $s_j^* = 0$, $j \in \mathcal{I}$. Moreover, the index sets $\mathcal{A}$ and $\mathcal{I}$ form a partition of $\{1,\ldots,n\}$. Thus the bounds (5.16) become

$$\max\{|x_i^k+dx_i^k-x_i^*|,\ |s_i^k+ds_i^k-s_i^*|\} \leq nq_0\sigma^k + q_0\sum_{j\in\mathcal{A}}\frac{x_j^k}{\mu^k}|s_j^k-s_j^*| + q_0\sum_{j\in\mathcal{I}}\frac{s_j^k}{\mu^k}|x_j^k-x_j^*|, \quad i=\overline{1,n}. \tag{5.17}$$

Due to the assumptions (5.10), the first part of Lemma 4.6 applies, which gives the estimates (4.41). Thus the terms x_j^k/μ^k, $j\in\mathcal{A}$ and s_j^k/μ^k, $j\in\mathcal{I}$ in (5.17) are bounded above by positive constants independent of k. Therefore (5.17) implies

$$\max\{|x_i^k+dx_i^k-x_i^*|,\ |s_i^k+ds_i^k-s_i^*|\} \leq nq_0\sigma^k + \tilde{q}_0\|w^k-w^*\|, \tag{5.18}$$

for some positive constant $\tilde{q}_0$. Now (5.5b) implies that there exists a constant $c_0 > 0$ such that $\|\overline{y}\| \leq c_0\|\overline{s}\|$, which becomes

$$\|y^k+dy^k-y^*\| \leq c_0\|s^k+ds^k-s^*\|, \tag{5.19}$$

in the present case $\overline{w} = v^k$. The bounds (5.18) and (5.19) give the inequalities (5.14). □

We are now ready to deduce a bound on the length of dw^k.

Lemma 5.5 *[21] Under the conditions of Lemma 5.4, the direction dw^k has the property*

$$\|dw^k\| \leq q_2\sigma^k + q_4(x^k)^\top s^k, \tag{5.20}$$

where q_2 and q_4 are positive constants.

Proof. [21] From the triangle inequality and (5.14), we obtain

$$\begin{aligned}\|dw^k\| &\leq \|w^k+dw^k-w^*\| + \|w^k-w^*\| \\ &\leq q_2\sigma^k + (q_3+1)\|w^k-w^*\|,\end{aligned} \tag{5.21}$$

for any primal-dual solution w^*. In order to pick a suitable w^*, we obtain from (2.3) that the optimality conditions (2.2) of (P) and (D) are equivalent to the system

$$Ax=b, \quad A^\top y+s=c, \quad c^\top x-b^\top y=0, \quad x\geq 0, \quad s\geq 0, \tag{5.22}$$

which we write in the form $\tilde{A}w \le \tilde{b}$, where $w = (x, y, s)$,

$$\tilde{A} := \begin{pmatrix} A & 0 & 0 \\ -A & 0 & 0 \\ 0 & A^\top & I \\ 0 & -A^\top & -I \\ c^\top & -b^\top & 0 \\ -c^\top & b^\top & 0 \\ -I & 0 & 0 \\ 0 & 0 & -I \end{pmatrix} \quad \text{and} \quad \tilde{b} := \begin{pmatrix} b \\ -b \\ c \\ -c \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.23}$$

It follows from Lemma 5.1 that there exists a positive constant h such that, for any primal-dual strictly feasible point w^k, there exists a primal-dual solution $w^{*,k}$ that satisfies

$$\|w^k - w^{*,k}\| \le h\|(\tilde{A}w^k - \tilde{b})^+\| = h(c^\top x^k - b^\top y^k) = h(x^k)^\top s^k, \tag{5.24}$$

the last equation being given by (2.3). The required relation (5.20) now follows from (5.21) and (5.24), with $q_4 := h(q_3 + 1)$. □

The next lemma provides a bound on the corrector direction $dw^{k,c}$, under the assumption that σ^k is bounded above by a multiple of the duality gap $(x^k)^\top s^k$, for sufficiently large k.

Lemma 5.6 *Let problems (P) and (D) satisfy the* IPM *conditions, and let w^k, $k \ge 0$, be the sequence of iterates generated by the* PDSOC$^\gamma$ *algorithm. If $\sigma^k \in (0, 1)$ satisfies*

$$\sigma^k \le q\,(x^k)^\top s^k, \quad k \ge k_0, \tag{5.25}$$

for some $k_0 \ge 0$ and $q > 0$, then the inequalities

$$\|dw^k\| \le q_5\,(x^k)^\top s^k \quad \textit{and} \quad \|dw^{k,c}\| \le q_6\,(x^k)^\top s^k, \quad k \ge k_0, \tag{5.26}$$

hold, where q_5 and q_6 are positive constants.

Proof. It follows from (3.11) and (3.18) that the conditions (5.10) are satisfied by w^k for all $k \ge 0$. Thus the bound (5.20) holds for $k \ge 0$. Further, from (5.25), the bound on $\|dw^k\|$ in (5.26) is valid for $k \ge k_0$, with $q_5 := q_2 q + q_4$.

To establish the bound (5.26) on $dw^{k,c}$, we remark that, due to (3.4), $\overline{w} := dw^{k,c}$ is defined by the system (5.4) for $z := -dX^k dS^k e$, $k \ge 0$. Hence Lemma 5.3 and the first part of (5.26) provide

$$\max\{|dx_i^{k,c}|, |ds_i^{k,c}|\} \le \frac{q_0}{\mu^k}\sum_{j=1}^n |dx_j^k ds_j^k| \le \frac{q_0\, q_5^2}{\mu^k}[(x^k)^\top s^k]^2, \quad i = \overline{1,n}, \quad k \ge k_0. \tag{5.27}$$

It follows from $\mu^k = (x^k)^\top s^k/n$ that

$$\max\{\|dx^{k,c}\|, \|ds^{k,c}\|\} \leq n^{3/2}\, q_0\, q_5^2 (x^k)^\top s^k, \quad k \geq k_0. \tag{5.28}$$

From (5.5b) with $\overline{y} = dy^{k,c}$ and $\overline{s} := ds^{k,c}$, we deduce

$$\|dy^{k,c}\| \leq c_0 \|ds^{k,c}\|, \quad k \geq 0, \tag{5.29}$$

which is analogous to (5.19). The required bound on $\|dw^{k,c}\|$ follows from (5.28) and (5.29). □

We remark that Lemma 5.6 gives $\mathcal{O}([(x^k)^\top s^k]^2)$ bounds on the lengths of $dX^k dS^{k,c}e$, $dX^{k,c}dS^k e$, and $dX^{k,c}dS^{k,c}e$, which are stronger than the bounds (4.22) on the length of the same vectors. We are now ready to prove the convergence of the PDSOC$^\gamma$ algorithm for a particular choice of the centring parameters σ^k.

Theorem 5.7 *Let problems (P) and (D) satisfy the* IPM *conditions. Let w^k, $k \geq 0$, be the sequence of iterates generated by the* PDSOC$^\gamma$ *algorithm when applied to these problems. Let $\sigma^k \in (0,1)$ satisfy*

$$\sigma^k = q\,(x^k)^\top s^k, \quad k \geq k_0, \tag{5.30}$$

for some $k_0 \geq 0$ and positive constant q. Then the duality gap $(x^k)^\top s^k$ of the iterates converges to zero and the iterates w^k converge to a primal-dual strictly complementary solution of (P) and (D).

Some suitable ways to choose the centring parameters σ^k, $k \geq 0$, to satisfy (5.30) are given after the proof of the theorem.

Proof. Conditions (3.19) and (3.18) imply

$$x_i^k s_i^k \geq \gamma \mu^k, \quad i = \overline{1,n}, \quad k \geq 0. \tag{5.31}$$

We first show that the stepsize θ^k is bounded away from zero. From (3.21) and (5.31), the left-hand side of the inequalities (3.18) is bounded below by

$$x_i^k(\theta) s_i^k(\theta) \geq (1-\theta)\gamma\mu^k + \theta\,\sigma^k\mu^k - \theta^3(\|dX^k dS^{k,c}e\| + \|dX^{k,c}dS^k e\|) - \theta^4\|dX^{k,c}dS^{k,c}e\|, \tag{5.32}$$

where $i = \overline{1,n}$, $\theta \in [0,1]$ and $k \geq 0$. Therefore Lemma 5.6 with the elementary inequalities

$$\begin{gathered} \|dX^k dS^{k,c}e\| \leq \|dx^k\| \cdot \|ds^{k,c}\|, \quad \|dX^{k,c}dS^k e\| \leq \|dx^{k,c}\| \cdot \|ds^k\|, \\ \|dX^{k,c}dS^{k,c}e\| \leq \|dx^{k,c}\| \cdot \|ds^{k,c}\|, \quad k \geq 0, \end{gathered} \tag{5.33}$$

yields

$$x_i^k(\theta) s_i^k(\theta) \geq (1-\theta)\gamma\mu^k + \sigma^k\mu^k\,\theta - [(x^k)^\top s^k]^2\,(2q_5q_6\,\theta^3 + q_6^2\,\theta^4), \quad i = \overline{1,n}, \quad \theta \in [0,1], \tag{5.34}$$

for $k \geq k_0$. Thus when $k \geq k_0$, if the inequality

$$(1-\theta)\gamma\mu^k + \sigma^k\mu^k\,\theta - [(x^k)^\top s^k]^2\,(2q_5q_6\,\theta^3 + q_6^2\,\theta^4) \geq \gamma\mu^k(\theta), \tag{5.35}$$

holds, then the conditions (3.18) are satisfied. From (3.9), we have

$$\mu^k(\theta) := \frac{1}{n}x^k(\theta)^\top s^k(\theta) = (1-\theta+\theta\,\sigma^k)\mu^k, \quad \theta \geq 0, \quad k \geq 0, \tag{5.36}$$

so (5.35) becomes

$$\sigma^k\mu^k(1-\gamma)\theta - [(x^k)^\top s^k]^2\,(2q_5q_6\,\theta^3 + q_6^2\,\theta^4) \geq 0, \quad k \geq k_0. \tag{5.37}$$

Further, (5.30) and $\mu^k := (x^k)^\top s^k/n$ imply that (5.37) is equivalent to

$$\{q(1-\gamma) - 2nq_5q_6\,\theta^2 - nq_6^2\,\theta^3\}\,[(x^k)^\top s^k]^2\theta \geq 0, \quad k \geq k_0. \tag{5.38}$$

Therefore θ^k, $k \geq k_0$, is bounded below by the positive constant $\underline{\theta} := \min\{\hat{\theta}, 1\}$, where $\hat{\theta}$ is the positive number that satisfies $q(1-\gamma) - 2nq_5q_6\,\hat{\theta}^2 - nq_6^2\,\hat{\theta}^3 = 0$. Further, because every steplength is positive, we let $\tilde{\theta} \in (0,1)$ be a constant lower bound on θ^k, $k \geq 0$. We have $\sigma^k \leq \overline{\sigma} := \sup\{\sigma^k : k \geq 0\}$. Moreover, $\overline{\sigma}$ is strictly less than one since $\sigma^k < 1$, $k \geq 0$, and $\sigma^{k+1} < \sigma^k$, $k \geq k_0$ (see (3.11)). Then it follows from (3.9) that

$$(x^{k+1})^\top s^{k+1} \leq \delta(x^k)^\top s^k, \quad k \geq 0, \tag{5.39}$$

where the constant $\delta := (1-\tilde{\theta}+\tilde{\theta}\overline{\sigma})$ is strictly less than one. Condition (5.39) provides

$$(x^k)^\top s^k \leq \delta^k(x^0)^\top s^0, \quad k \geq 0, \tag{5.40}$$

which implies $(x^k)^\top s^k \to 0$, as $k \to \infty$.

Next we show that w^k, $k = 0, 1, 2, \ldots$, is a Cauchy sequence, in order to infer that it is convergent, $\mathbb{R}^p$ being a complete metric space, $p \geq 1$.

Let $k \geq k_0$ and $r \geq 1$. From (3.6), we obtain

$$w^{k+r} - w^k = \sum_{i=1}^{r-1}[\theta^{k+i}dw^{k+i} + (\theta^{k+i})^2 dw^{k+i,c}], \tag{5.41}$$

which with $\theta^{k+i} \leq 1$, $i \geq 0$, implies

$$\|w^{k+r} - w^k\| \leq \sum_{i=1}^{r-1}(\|dw^{k+i}\| + \|dw^{k+i,c}\|). \tag{5.42}$$

It follows from the bounds (5.26) and (5.40) that

$$\begin{aligned}\|w^{k+r} - w^k\| &\leq (q_5 + q_6) \sum_{i=1}^{r-1} (x^{k+i})^\top s^{k+i} \\ &\leq (q_5 + q_6) \left[(x^0)^\top s^0\right] \sum_{i=1}^{r-1} \delta^{k+i} = (q_5 + q_6) \left[(x^0)^\top s^0\right] \delta^{k+1} \frac{1-\delta^{r-1}}{1-\delta}.\end{aligned} \tag{5.43}$$

Since (5.43) holds for any $k \geq k_0$ and $r \geq 1$, and since $\delta \in (0,1)$, we deduce from (5.43) that $\|w^{k+r} - w^k\| \to 0$, as k, $r \to \infty$. Thus w^k is a Cauchy sequence and is convergent.

Since $(x^k)^\top s^k \to 0$, $k \to \infty$, and since (5.31) holds, the limit point of the sequence of iterates w^k is a primal-dual strictly complementary solution, according to Lemma 4.6. □

It is a consequence of the above theorem that, if the centring parameters σ^k are given the values (5.30) for sufficiently large k, then they converge to zero as $k \to \infty$, the rate of this convergence being the same as that of the duality gap. However, in contrast to the results in the previous section for the PDSOC$^{\sigma\beta}$ algorithm, the above proofs and results do not yield a polynomial complexity bound on the number of iterations required by the PDSOC$^\gamma$ algorithm to generate an iterate with duality gap less than some tolerance $\epsilon > 0$. A reason for this shortcoming arises from the employment of Hoffman's Lemma to provide the constant h of inequality (5.24). This constant is very difficult to compute explicitly [7] and its dependence on the problem dimensions m and n seems impossible to determine precisely. On the other hand, a favourable feature of the analysis is that it ensures that the sequence of iterates has a unique limit point, even when (P) and (D) have multiple solutions.

To ensure condition (5.30), we may set $\sigma^k \in (0,1)$ to be

$$\sigma^k \in (0,1), \quad \text{for } k < k_0 \quad \text{and} \quad \sigma^k := q\,(x^k)^\top s^k, \quad \text{for } k \geq k_0, \tag{5.44}$$

for some $k_0 \geq 0$, where $q > 0$ satisfies $q\,(x^{k_0})^\top s^{k_0} < 1$. Then $\sigma^k \in (0,1)$, for all $k \geq k_0$ since μ^k, $k \geq 0$, is strictly decreasing as in (3.11). If the user prefers not to choose an iteration number k_0 from which point onwards σ^k is of order μ^k, then the following method is suitable [21]. Let $\sigma \in (0,1)$ and $q > 0$ be provided by the user, and set

$$\sigma^k := \min(\sigma,\, q\,(x^k)^\top s^k), \quad k \geq 0. \tag{5.45}$$

Next we show that the parameters (5.45) satisfy (5.30).

Corollary 5.8 *Let problems (P) and (D) satisfy the* IPM *conditions. Let $\sigma \in (0,1)$ and $q > 0$. Let the* PDSOC$^\gamma$ *algorithm be applied to these problems with the centring parameters σ^k, $k \geq 0$, defined in* (5.45). *Then condition* (5.30) *holds for some $k_0 \geq 0$, and Theorem 5.7 applies.*

Proof. Let us assume that σ^k defined in (5.45) does not satisfy (5.30). Due to (3.11), this is equivalent to assuming that $\sigma^k = \sigma$, $k \geq 0$. Then, conforming to Theorem 4.8, the duality gaps $(x^k)^\top s^k$ converge to zero as $k \to \infty$. Thus given any $q > 0$, there exists an index k_0 such that $(x^k)^\top s^k < \sigma/q$, $k \geq k_0$, which together with (5.45) provides the contradiction $\sigma = \sigma^k < \sigma$, $k \geq k_0$. □

We end this section with the remark that when σ^k satisfies (5.30), the $\sigma\beta$ stepsize inequalities (3.24) no longer provide a lower bound on the sequences $x_i^k s_i^k / \mu^k$, $i = \overline{1, n}$, $k \geq k_0$. This is the main obstacle that has frustrated our attempts to prove convergence of the PDSOC$^{\sigma\beta}$ algorithm for such choices of centring parameters.

The results of the next section imply that subject to the conditions of Theorem 5.7 and to (P) and (D) having multiple solutions, the primal-dual strictly complementary solution that is the limit point of the sequence of iterates generated by the PDSOC$^\gamma$ algorithm may not be the analytic centre of the primal-dual solution set.

6 An example of the asymptotic behaviour of the PDSOC algorithm

Let problems (P) and (D) satisfy the IPM conditions. Let the centring parameters σ^k in the PDSOC algorithm be chosen to satisfy (5.30), and the stepsizes θ^k to be bounded away from zero, i.e.,

$$\theta^k \geq \tilde{\theta}, \quad k \geq 0, \quad \text{for some constant } \tilde{\theta} \in (0, 1]. \tag{6.1}$$

Subject to the conditions of Theorem 5.7, the properties (5.30) and (6.1) are satisfied by the PDSOC$^\gamma$ algorithm.

The purpose of this section is to demonstrate that, for any variant of the PDSOC algorithm satisfying (5.30) and (6.1), the limit of its sequence of iterates may not be the analytic centre w^c of the primal-dual solution set.

We consider the LP problem

$$\min_{x \in \mathbb{R}^3} \; x_1 \quad \text{subject to} \quad x_2 + x_3 = 2, \;\; x = (x_1, x_2, x_3) \geq 0, \tag{6.2}$$

and its dual

$$\max_{(y,s) \in \mathbb{R} \times \mathbb{R}^3} \; 2y \quad \text{subject to} \quad s_1 = 1, \;\; y + s_2 = 0, \;\; y + s_3 = 0, \;\; s = (s_1, s_2, s_3) \geq 0. \tag{6.3}$$

The primal-dual solution set is

$$\mathcal{S}_{PD} = \{(x^*, y^*, s^*) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 : \; x^* = (0, \alpha, 2-\alpha), \; \alpha \in [0,2], \quad y^* = 0, \; s^* = (1,0,0)\}. \tag{6.4}$$

We see that the IPM conditions hold.

It follows from (6.4) and (2.7) that the primal problem (6.2) has multiple solutions and the analytic centre $w^c = (x^c, y^c, s^c)$ of $\mathcal{S}_{PD}$ is

$$x^c = (0,1,1), \quad y^c = 0, \quad s^c = (1,0,0). \tag{6.5}$$

Any primal-dual strictly feasible point $w^k = (x^k, y^k, s^k)$ of (6.2) and (6.3) is characterised by the conditions

$$x_2^k + x_3^k = 2, \;\; s_1^k = 1, \;\; s_2^k = s_3^k = -y^k > 0, \;\; x_1^k > 0, \; x_2^k > 0, \; x_3^k > 0. \tag{6.6}$$

Let λ and x_2^0 be any numbers from $(0,1)$, and let the other components of $w^0 = (x^0, y^0, s^0)$ have the values

$$x_1^0 = s_2^0 = s_3^0 = -y^0 := \lambda, \quad s_1^0 := 1, \quad x_3^0 := 2 - x_2^0. \tag{6.7}$$

It follows from (6.6) that w^0 is primal-dual strictly feasible for the current LP problem.

We consider the PDSOC algorithm with the centring parameters defined by (5.30) with $q = 1/3$ and $k_0 = 0$, i.e.,

$$\sigma^k := \mu^k = \frac{(x^k)^\top s^k}{3} > 0, \quad k \geq 0. \tag{6.8}$$

Firstly, we claim that σ^k defined in (6.8) is less than one for $k \geq 0$. For $k = 0$, it follows from (6.8) and (6.7) that $\sigma^0 = \mu^0 = \lambda < 1$. The inequality (3.11) provides the relation $\mu^{k+1} < \mu^k$ for $k \geq 0$. It follows from (6.8) that $\sigma^k < \sigma^0 < 1$, $k \geq 0$.

For $k \geq 0$, the system (3.3) that defines the direction dw^k from w^k becomes

$$\begin{cases} dx_2^k + dx_3^k = 0, \\ ds_1^k = 0, \\ ds_2^k + dy^k = 0, \\ ds_3^k + dy^k = 0, \\ x_i^k ds_i^k + s_i^k dx_i^k = -x_i^k s_i^k + \sigma^k \mu^k, \;\; i = \overline{1,3}, \end{cases} \tag{6.9}$$

which together with (6.6) yields the following explicit expression for the components of dw^k

$$dx^k = \left(-x_1^k + \sigma^k \mu^k, \; \frac{\sigma^k \mu^k}{s_2^k}(1 - x_2^k), \; \frac{\sigma^k \mu^k}{s_3^k}(1 - x_3^k) \right) \tag{6.10a}$$

$$ds^k = (0, \; \sigma^k \mu^k - s_2^k, \; \sigma^k \mu^k - s_3^k), \quad dy^k = -ds_2^k. \tag{6.10b}$$

The corrector direction $dw^{k,c}$ from w^k is defined by the system

$$\begin{cases} dx_2^{k,c} + dx_3^{k,c} = 0, \\ ds_1^{k,c} = 0, \\ dy^{k,c} + ds_2^{k,c} = 0, \\ dy^{k,c} + ds_3^{k,c} = 0, \\ s_1^k dx_1^{k,c} + x_1^k ds_1^{k,c} = -dx_1^k ds_1^k, \\ s_2^k dx_2^{k,c} + x_2^k ds_2^{k,c} = -dx_2^k ds_2^k, \\ s_3^k dx_3^{k,c} + x_3^k ds_3^{k,c} = -dx_3^k ds_3^k. \end{cases} \tag{6.11}$$

Thus $dw^{k,c}$ has the components

$$dx_1^{k,c} = 0, \quad dx_2^{k,c} = \frac{-dx_2^k ds_2^k}{s_2^k}, \quad dx_3^{k,c} = -dx_2^{k,c}, \tag{6.12a}$$

$$ds_1^{k,c} = 0, \quad ds_2^{k,c} = ds_3^{k,c} = dy^{k,c} = 0. \tag{6.12b}$$

From (6.6), (3.6), (6.10) and (6.12), we obtain that the iterates are defined recursively by

$$x_1^{k+1} = (1-\theta^k)x_1^k + \theta^k \sigma^k \mu^k, \tag{6.13a}$$

$$x_2^{k+1} = x_2^k + \theta^k \sigma^k \mu^k \frac{1-x_2^k}{s_2^k} - (\theta^k)^2 \sigma^k \mu^k \frac{(1-x_2^k)(\sigma^k\mu^k - s_2^k)}{(s_2^k)^2}, \quad x_3^{k+1} = 2 - x_2^{k+1}, \tag{6.13b}$$

$$s_1^{k+1} = s_1^k = 1, \quad s_2^{k+1} = s_3^{k+1} = -y^{k+1} = (1-\theta^k)s_2^k + \theta^k\sigma^k\mu^k, \quad k \geq 0. \tag{6.13c}$$

From $x_1^0 = s_2^0 = s_3^0$, (6.13a) and (6.13c), we obtain inductively

$$x_1^k = s_2^k = s_3^k = -y^k, \quad k \geq 0. \tag{6.14}$$

It follows from $s_1^k = 1$ and $x_2^k + x_3^k = 2$ that

$$\mu^k := \frac{1}{3}(x_1^k s_1^k + x_2^k s_2^k + x_3^k s_3^k) = x_1^k, \quad k \geq 0. \tag{6.15}$$

Substituting (6.14) and (6.15) into the expressions (6.13a) and (6.13b), we obtain

$$x_1^{k+1} = \left(1 - \theta^k + \theta^k\sigma^k\right) x_1^k, \tag{6.16a}$$

$$x_2^{k+1} = x_2^k + \theta^k\sigma^k(1-x_2^k) + (\theta^k)^2\sigma^k(1-\sigma^k)(1-x_2^k), \quad x_3^{k+1} = 2 - x_2^{k+1}, \quad k \geq 0. \tag{6.16b}$$

The next lemma presents the promised result concerning the asymptotic behaviour of the sequence of iterates generated by the PDSOC algorithm.

Lemma 6.1 *Let the* PDSOC *algorithm satisfy the conditions* (6.1) *and* (6.8). *Let* w^k *be the sequence of iterates generated by the algorithm when applied to problems* (6.2) *and* (6.3), *starting from* w^0 *that satisfies* (6.7). *Then* w^k, $k \geq 0$, *converges to a primal-dual strictly complementary solution of the problems which is not the analytic centre* w^c *of the solution set.*

Proof. Employing (6.1) and $\sigma^k \le \sigma^0$ in (6.16a), we obtain the bound

$$x_1^{k+1} \le (1 - \tilde{\theta} + \tilde{\theta}\sigma^0)\, x_1^k, \quad k \ge 0. \tag{6.17}$$

The factor $(1 - \tilde{\theta} + \tilde{\theta}\sigma^0)$ is less than one due to $\tilde{\theta} > 0$ and $\sigma^0 < 1$. Thus the inequality (6.17) and the equations (6.14) provide the limits

$$x_1^k \to 0, \quad s_2^k \to 0, \quad s_3^k \to 0, \quad y^k \to 0, \quad \text{as} \quad k \to \infty. \tag{6.18}$$

If the choice (6.8) is made, then (6.17) and (6.15) imply, inductively, the bound

$$\sigma^k = \mu^k \le \mu^0 (1 - \tilde{\theta} + \tilde{\theta}\sigma^0)^k, \quad k \ge 0. \tag{6.19}$$

Now since the limits (6.18) hold and (6.6) implies $s_1^k = 1$, $k \ge 0$, it remains to consider the behaviour of x_2^k and x_3^k as $k \to \infty$.

The first equation in (6.16b) implies

$$1 - x_2^{k+1} = (1 - x_2^k)\left[1 - \theta^k\sigma^k - (\theta^k)^2\sigma^k(1 - \sigma^k)\right], \quad k \ge 0. \tag{6.20}$$

Since $\theta^k > 0$ and $0 < \sigma^k < 1$, we have $\eta^k := 1 - \theta^k\sigma^k - (\theta^k)^2\sigma^k(1-\sigma^k) < 1$, $k \ge 0$. Moreover, from $\theta^k \le 1$, we deduce that $\eta^k \ge (1 - \sigma^k)^2 > 0$. Thus, due to $x_2^0 < 1$, relations (6.20) imply, inductively, the inequalities $x_2^k < x_2^{k+1} < 1$, $k \ge 0$, which further give that x_2^k is convergent. Since $x_3^k = 2 - x_2^k$, the iterates x_3^k are also convergent. The remarks so far establish that w^k converges to a primal-dual solution, say w^*, of (6.2) and (6.3). Moreover, the iterates satisfy the condition (4.39) since $x_1^k s_1^k/\mu^k = 1$, $x_2^k s_2^k/\mu^k = x_2^k \ge x_2^0 > 0$, and $x_3^k s_3^k/\mu^k = x_3^k > 1$ for $k \ge 0$. Thus Lemma 4.6 implies that w^* satisfies the strict complementarity condition.

Convergence to the analytic centre w^c occurs if and only if $x_2^k \to 1$. The equations (6.20) provide

$$1 - x_2^{k+1} = (1 - x_2^0)\prod_{i=0}^{k}\left[1 - \theta^i\sigma^i - (\theta^i)^2\sigma^i(1 - \sigma^i)\right], \quad k \ge 0. \tag{6.21}$$

Thus convergence to w^c fails if and only if $x_2^0 \ne 1$ (which holds in our case), and if the choices of θ^k and σ^k have the property that the product $\prod_{k=0}^{\infty}[1 - \theta^k\sigma^k - (\theta^k)^2\sigma^k(1 - \sigma^k)]$ is finite and nonzero, which is equivalent to

$$\sum_{k=0}^{\infty} \sigma^k\left[\theta^k + (\theta^k)^2(1 - \sigma^k)\right] < \infty. \tag{6.22}$$

Since $0 < \tilde{\theta} \le \theta^k \le 1$ and $0 < \sigma^k \le \sigma^0 < 1$, we deduce

$$0 < \tilde{\theta} + \tilde{\theta}^2(1 - \sigma^0) \le \theta^k + (\theta^k)^2(1 - \sigma^k) \le 2. \tag{6.23}$$

Thus condition (6.22) is equivalent to $\sum_{k=0}^{\infty} \sigma^k < \infty$. In particular, convergence to w^c occurs when σ^k is bounded away from zero. On the other hand, if the choice (6.8) is made, then, since $0 < 1 - \tilde{\theta} + \tilde{\theta}\sigma^0 < 1$, the bound (6.19) implies that the series $\sum_{k=0}^{\infty} \sigma^k$ is finite. Thus $w^* \neq w^c$. □

When the centring parameters σ^k, $k \geq 0$, are bounded away from zero and (P) has one equality constraint and multiple solutions, we showed in [1] that the iterates generated by the PDSOC algorithm converge to the analytic centre of the primal-dual solution set of the problem. The difficulty of the analysis involved, however, has prevented us from extending this result to the general case when (P) has multiple equality constraints and solutions.

7 Conclusions

In this paper we considered computing an additional direction, a corrector, to augment the direction of the PD algorithm, and multiplying this corrector by the square of the stepsize in the expression of the new iterate. We saw that such a construction can be viewed as a quadratic approximation of a local path from the current iterate to a target point on the primal-dual central path. Moreover, the scaling of the corrector can be justified as an attempt to improve the bad behaviour of the PDC algorithm [1, 2] that multiplies the corrector direction only by the stepsize in the expression of the new iterate. Indeed, we found that the PDSOC algorithm has very good convergence (and in some cases even complexity) properties for some practical choices of the centring parameters σ^k and the stepsize. These results imply, for example, that the PDSOC$^{\sigma\beta}$ algorithm converges on the example problems on which the PDC algorithm failed to do so.

A new long-step linesearch technique, the $\sigma\beta$ procedure, was analysed whose parameter β does not dependent on the starting point of the algorithm. When the centring parameters are monotonically decreasing to a positive value in the course of the algorithm, Q-linear rate of convergence of the duality gap to zero was proved for the PDSOC$^{\sigma\beta}$ algorithm which may be very useful in practice. We believe this result extends to *infeasible* variants of the PDSOC algorithm that allow the starting point to be infeasible with respect to the primal-dual equality constraints.

We also proved convergence of the PDSOC algorithm when the centring parameters have the same order as the duality gap of the iterates asymptotically, and the stepsize is computed by the popular long-step γ procedure. Then, the parameters σ^k tend to zero as $k \to \infty$. Though it allows the possibility of superlinear convergence of the duality gap to zero, we remark that, in our experience, such a choice of centring parameters may lose efficiency of the algorithm on early iterations.

Our preliminary numerical experiments with the PDSOC algorithm are very encouraging.

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