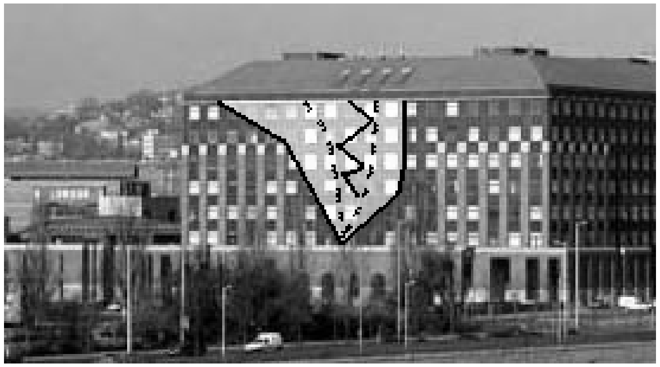
Operations Research
Report 2005-01



On a closedness theorem

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Marc 2005

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ISSN 1215 - 5918

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Abstract

In [5] several conditions are described which imply the closedness of linear images of convex sets. Here we combine two such theorems into a common generalization. We give a proof of the general theorem which is simpler than the proof obtained by combining the proofs of the two theorems. The paper is almost self-contained as we give transparent proofs of the separation theorems used. Also we present an application of our general closedness theorem in the theory of duality in convex programming.

Keywords: separation theorems, closedness theorems, duality in convex programming

1 Introduction

It is well-known that the linear image of a closed convex set is not necessarily closed. For example consider the following closed convex set:

$$C_0 := \left\{(x_1, x_2) \in \mathcal{R}^2 : x_1 > 0,\, x_2 \geq 1/x_1\right\}.$$

Project C_0 to the x_1- or x_2-axis, the resulting image set will not be closed. However, if we project the convex set C_0 to a line with positive slope, then we get as the image set a closed half of the line. The reason for this is that in the latter case a certain regularity condition is satisfied: the line (the image space of the projection) contains at least one point with positive coordinates (i.e. a point from the interior of the barrier cone of C_0), as is explained in Theorem 1.1.

Thus generally for the linear image to be closed we need regularity conditions, even if the set to be transformed is closed and convex. In [5] this

problem is studied extensively, and several closedness theorems, among ot ers Theorem 1.1 and Theorem 1.2, are derived. The former theorem describ regularity conditions which imply the closedness of the convex set

$$AC := \{Ax : x \in C\},$$

where A is an m by n real matrix and C is a closed convex set in $\mathcal{R}^n$. T latter theorem deals with the closedness of the convex set

$$C + P := \{x + z : x \in C,\, z \in P\},$$

where C is a closed convex set in $\mathcal{R}^m$ and P is a polyhedron in $\mathcal{R}^m$.

Though Theorem 1.1 and Theorem 1.2 are stated in terms of sets, the pro in [5] for Theorem 1.2 uses deep results from the theory of convex functions. this paper we consider a common generalization of Theorem 1.1 and Theore 1.2. Our main result, Theorem 1.3 describes regularity conditions which imp the closedness of the set $(AC) + P$ where A is an m by n real matrix, C a closed convex set in $\mathcal{R}^n$ and P is a polyhedron in $\mathcal{R}^m$. Then the spec cases when $P = \{0\}$ resp. when $A = E$ (E denotes the identity matri give back Theorem 1.1 and Theorem 1.2. Moreover we present here proofs Theorem 1.3 which "stay on the set level" and thus avoid the usage of t deep function-theoretical results mentioned above.

In what remains from the introduction we state Theorems 1.1, 1.2 and 1 and examine the relations between them. First we fix some notations.

In what follows $\operatorname{Im} A$ resp. $\operatorname{Ker} A$ denotes the range space (i.e. the ima space) and the null space (i.e. the kernel) of the m by n real matrix A. is well-known that $(\operatorname{Im} A)^\perp = \operatorname{Ker}(A^T)$ and $(\operatorname{Ker} A)^\perp = \operatorname{Im}(A^T)$ where denotes transpose and $^\perp$ denotes orthogonal complement ([8]).

For a K convex cone in $\mathcal{R}^d$ let K^* denote the *dual cone* of K, i.e. let

$$K^* := \left\{a \in \mathcal{R}^d : a^T x \geq 0 \ (x \in K)\right\}.$$

Then K^* is a closed convex cone. For example if C is a nonempty convex s in $\mathcal{R}^d$ then the *recession cone* and the *barrier cone* of C, i.e.

$$\operatorname{rec} C := \left\{z \in \mathcal{R}^d : x + \lambda z \in C \ (x \in C,\, \lambda \geq 0)\right\},$$
$$\operatorname{bar} C := \left\{a \in \mathcal{R}^d : \inf\{a^T x : x \in C\} > -\infty\right\},$$

are convex cones. If K is a convex cone then $\operatorname{rec} K = K$ and $\operatorname{bar} K = K$ Easy separation argument shows that if C is a nonempty closed convex set $\mathcal{R}^d$ then $(\operatorname{bar} C)^* = \operatorname{rec} C$. (Note that the latter equation implies that $\operatorname{rec} C$ closed.) Specially the dual of the dual of a K convex cone is the closure of th cone, i.e. $K^{**} = \mathrm{cl}\, K$. This implies also that if C is a nonempty closed convex set then $(\mathrm{rec}\, C)^* = \mathrm{cl}\,(\mathrm{bar}\, C)$. (The barrier cone is not closed generally, so the closure sign can not be left out.)

For a $C \subseteq \mathcal{R}^d$ convex set $\mathrm{ri}\, C$ denotes the relative interior of C. Properties of the relative interior of convex sets are studied in [5]. We list some results from [5] that will be important for us:

– The relative interior of a nonempty convex set is nonempty.

– If C is a convex set, $x_0 \in \mathrm{ri}\, C$ and $x_1 \in \mathrm{cl}\, C$, then $x_0 + \lambda(x_1 - x_0) \in \mathrm{ri}\, C$ for every $0 \leq \lambda < 1$.

– If C is a convex set, then $\mathrm{cl}\,(\mathrm{ri}\, C) = \mathrm{cl}\, C$ and $\mathrm{ri}\,(\mathrm{cl}\, C) = \mathrm{ri}\, C$.

– Let C be a convex set in $\mathcal{R}^n$, and let A be an m by n real matrix. Then $\mathrm{ri}\,(AC) = A(\mathrm{ri}\, C)$.

– Let A be an m by n real matrix. Let C be a convex set in $\mathcal{R}^m$ such that $A^{-1}(\mathrm{ri}\, C) \neq \emptyset$. Then

$$\mathrm{ri}\,(A^{-1}C) = A^{-1}(\mathrm{ri}\, C) \text{ and } \mathrm{cl}\,(A^{-1}C) = A^{-1}(\mathrm{cl}\, C). \tag{1}$$

(Here $A^{-1}(C) := \{x : Ax \in C\}$.)

We will use these results without proof. Now we can state Theorems 1.1, 1.2 and 1.3.

The following theorem is known, see Theorem 9.1 and Lemma 16.2 in [5]. The proof in [5] uses consequences of the Bolzano-Weierstrass theorem, for example Cantor's intersection theorem.

Theorem 1.1. *Let A be an m by n real matrix, and let C be a closed convex set in $\mathcal{R}^n$. Then between the statements*

a) $\mathrm{Im}\,(A^T) \cap \mathrm{ri}\,(\mathrm{bar}\, C) \neq \emptyset$*,*

b) $(\mathrm{Ker}\, A) \cap (\mathrm{rec}\, C) \subseteq -\mathrm{rec}\, C$*,*

c) AC is closed, and $\mathrm{rec}\,(AC) = A\mathrm{rec}\, C$*,*

hold the following logical relations: a) is equivalent to b); a) or b) implies c).

Suppose that we have proved already that a) implies the closedness of AC. Substituting the closed convex set C with the closed convex cone $\mathrm{rec}\, C$ we get that a) implies the closedness of $A\mathrm{rec}\, C$ also. Hence to prove that $A\mathrm{rec}\, C = \mathrm{rec}\,(AC)$ it is enough to show that the dual cones are equal, i.e. $A^{-T}(\mathrm{cl}\,\mathrm{bar}\, C) = \mathrm{cl}\,\mathrm{bar}\,(AC)$. (Here A^{-T} stands for $(A^T)^{-1}$.) By (1) a) implies that $A^{-T}(\mathrm{cl}\,\mathrm{bar}\, C) = \mathrm{cl}\, A^{-T}(\mathrm{bar}\, C)$, and trivially the latter set equals $\mathrm{cl}\,\mathrm{bar}\,(AC)$ (even without taking closures). Thus we have proved that a) implies $A\mathrm{rec}\, C = \mathrm{rec}\,(AC)$ if a) implies the closedness of AC.

The following theorem also can be found in [5], see Theorem 20.3. T
proof in [5] uses deep results from the theory of convex functions concerni
conjugation, infimal convolution.

Theorem 1.2. *Let P be a polyhedron in $\mathcal{R}^m$, and let C be a closed conv
set in $\mathcal{R}^m$. Then between the statements*

a) $(\mathrm{bar}\, P) \cap \mathrm{ri}\,(\mathrm{bar}\, C) \neq \emptyset$*,*

b) $(-\mathrm{rec}\, P) \cap (\mathrm{rec}\, C) \subseteq -\mathrm{rec}\, C$*,*

c) $C + P$ is closed,

hold the following logical relations: a) is equivalent to b); a) or b) implies

In [5] the theorem is proved in the form above, the fact that a) or
implies not only the closedness of $C + P$ but even the equation $\mathrm{rec}\,(C + P)$
$(\mathrm{rec}\, C) + (\mathrm{rec}\, P)$ is not mentioned there. This deficiency can be eliminated
the same way as we have proved that in the case of Theorem 1.1 a) impli
the equation $A\mathrm{rec}\, C = \mathrm{rec}\,(AC)$ provided that a) implies the closedness
AC.

Now suppose that we have proved already that in the case of Theore
1.2 a) implies the closedness of $C + P$. Then substituting $P = \mathrm{Ker}\, A$
can obtain the similar implication in the case of Theorem 1.1. This is
immediate consequence of Abrams' theorem ([1]), which theorem states th
AC is closed if and only if $C+ \mathrm{Ker}\, A$ is closed (even if C is not supposed to
closed and convex but is an arbitrary set). Thus the fact that in the case
Theorem 1.1 a) implies the closedness of AC is exactly the $P = \mathrm{Ker}\, A$ spec
case of the similar fact in the case of Theorem 1.2.

Instead of Theorem 1.1 and Theorem 1.2 we will consider their comm
generalization,

Theorem 1.3. *Let A be an m by n real matrix, let P be a polyhedron
$\mathcal{R}^m$, and let C be a closed convex set in $\mathcal{R}^n$. Then between the statements*

a) $(A^T \mathrm{bar}\, P) \cap \mathrm{ri}\,(\mathrm{bar}\, C) \neq \emptyset$*,*

b) $A^{-1}(-\mathrm{rec}\, P) \cap (\mathrm{rec}\, C) \subseteq -\mathrm{rec}\, C$*,*

c) $(AC) + P$ *is closed, and* $\mathrm{rec}\,((AC) + P) = (A\mathrm{rec}\, C) + \mathrm{rec}\, P$*,*

hold the following logical relations: a) is equivalent to b); a) or b) implies

Suppose that we have proved (for example in the way described in [
that in the case of Theorem 1.1 a) implies the closedness of AC and in t
case of Theorem 1.2 a) implies the closedness of $C + P$. Statement a)
Theorem 1.3 implies statement a) in Theorem 1.1, and thus the closedness
AC. Trivially $\mathrm{ri}\,\mathrm{bar}\,(AC) = \mathrm{ri}\, A^{-T}(\mathrm{bar}\, C)$ (this equation holds even witho

the ri sign). By (1) the latter set equals $A^{-T}(\operatorname{ri}\operatorname{bar} C)$. From this can be seen easily that statement a) in Theorem 1.3 implies statement a) in Theorem 1.2 for the closed convex set AC and the polyhedron P. Consequently $(AC)+P$ is closed. We obtained a proof of the fact that the closedness of the set $(AC)+P$ follows from statement a) in Theorem 1.3. Then the proof of the fact that in the case of Theorem 1.3 a) implies c) can be finished in a similar way as in the case of Theorem 1.1.

This way we have proved the larger part of Theorem 1.3, but the resulting proof is unnecessarily complicated due to the fact that the proof of Theorem 1.2 uses deep results from the theory of convex functions. A proof of Theorem 1.3 can be given that is about as complex as the proof of Theorem 1.1 in [5] based on the Bolzano-Weierstrass theorem.

This proof will be presented in §3 after we list (and in part prove) some homogenization and separation lemmas in §2. §3 contains also the sketch of a second proof for Theorem 1.3, and further closedness theorems. In §4 we describe an application of these closedness theorems in the theory of duality in convex programming.

2 Homogenization and separation

In this section we list the homogenization and separation lemmas which will be used in the proof of the main theorem in §3.

For a $K \subseteq \mathcal{R}^{d+1}$ convex cone resp. for a $C \subseteq \mathcal{R}^d$ convex set let us define $C(K)$ and $K(C)$ as follows:

$$C(K) := \left\{x \in \mathcal{R}^d : (1,x) \in K\right\},\ K(C) := \operatorname{cone}(\{1\} \times C).$$

(Here cone denotes convex conical hull.) Then $C(K)$ is a convex set in $\mathcal{R}^d$ and $K(C)$ is a convex cone in $\mathcal{R}^{d+1}$. It can be easily seen that $C = C(K(C))$ and $\operatorname{cl} C = C(\operatorname{cl} K(C))$ for any C convex set in $\mathcal{R}^d$. Thus all of the (closed) convex sets in $\mathcal{R}^d$ can be obtained if we apply the $C(.)$ operation to the (closed) convex cones in $\mathcal{R}^{d+1}$. This makes possible that to prove certain statements about (closed) convex sets it is enough to prove the statement for sets with more structure: (closed) convex cones. This method is called homogenization.

We summarize some properties of the $K(.)$ operation in the following useful lemma. (For a proof see Corollary 6.8.1 and Theorem 8.2 in [5].)

Lemma 2.1. *Let C be a nonempty convex set in $\mathcal{R}^d$. Then*

$$\operatorname{ri} K(C) = \{(\lambda, \lambda x) : \lambda > 0,\ x \in \operatorname{ri} C\}.$$

If C is closed also then

$$\operatorname{cl} K(C) = K(C) \cup \{(0,z) : z \in \operatorname{rec} C\}.$$

A set in $\mathcal{R}^n$ is a *polyhedron* if a matrix $A \in \mathcal{R}^{m\times n}$ and a vector $b \in \mathcal{R}^m$ c
be found such that $P = \{x : Ax \geq b\}$. Then trivially $\operatorname{rec} P = \{x : Ax \geq 0$
and

$$\operatorname{cl} K(P) = \{(\lambda, x) : Ax - \lambda b \geq 0,\ \lambda \geq 0\}$$

is a polyhedron as well as a cone (in other words $\operatorname{cl} K(P)$ is a *polyhedral con*
This observation is used in the proof of Motzkin's theorem in [6].

Motzkin's theorem states that P is a polyhedron if and only if there ex
a polytope Q and a finitely generated cone R such that $P = Q + R$. (
set $C \subseteq \mathcal{R}^d$ is said to be a *polytope* resp. a *finitely generated cone* if the
exists a set $S \subseteq \mathcal{R}^d$ such that S has finite elements and C is the conv
hull resp. the convex conical hull of S.) Specially the polyhedral cones a
exactly the finitely generated cones. Note that if $P = Q + R$ where Q is
polytope and R is a finitely generated cone then $\operatorname{bar} P = \operatorname{bar} R = R^*$, a
thus $\operatorname{rec} P = R^{**} = R$.

To emphasize further the usefulness of the homogenization ideas we gi
transparent proofs of important separation theorems that are based on the
ideas.

The following theorem is well-known. A simple proof can be found in [3], s
Theorem 4.1.1 and Corollary 4.1.3. If S_1 and S_2 are subsets of $\mathcal{R}$ then we w
use the $S_1 \leq S_2$ notation to denote that $x_1 \leq x_2$ for every $x_1 \in S_1$, $x_2 \in S$
Similarly in the case of the $<$ and $=$ relations.

Theorem 2.1.

a) Let C be a nonempty closed convex set in $\mathcal{R}^d$. Let $x \in \mathcal{R}^d$ such th
$x \notin C$. Then there exists a vector $a \in C + \{-x\}$ and a positive number ε su
that $\{a^T x + \varepsilon\} < a^T C$.

b) Let C and C' be nonempty convex sets in $\mathcal{R}^d$. Suppose that C is close
C' is compact and $C \cap C' = \emptyset$. Then there exists a vector $a \in C + (-C')$ a
a positive number ε such that $a^T C' + \{\varepsilon\} < a^T C$.

The following two theorems can be found in [5], see Theorem 11.3 a
Theorem 20.2. A convex set C is said to be *relatively open* if $C = \operatorname{ri} C$.

Theorem 2.2.

a) Let L be a subspace in $\mathcal{R}^d$, and let K be a convex cone in $\mathcal{R}^d$. Suppo
that $L \cap \operatorname{ri} K = \emptyset$. Then there exists $a \in \mathcal{R}^d$ such that $a^T L = 0 < a^T \operatorname{ri} K$.

b) Let C and C' be disjoint nonempty relatively open convex sets in $\mathcal{R}^d$. Then there exists $a \in \mathcal{R}^d$ such that $a^T C' < a^T C$.

Proof. a) Let $x_k \in \text{ri}\, K$ $(k = 1, 2, \ldots)$ such that $x_k \to 0$ $(k \to \infty)$. It can be easily seen that $(\text{cl}\, K) + (\text{ri}\, K) = \text{ri}\, K$. Hence the closed convex sets $\{x_k\} + \text{cl}\, K$ are subsets of $\text{ri}\, K$ and so are disjoint from L. Specially they are disjoint from the compact convex set $O := \{x \in L : ||x|| \leq 1\}$. By Theorem 2.1 there exist $a_k \in K + L$ $(k = 1, 2, \ldots)$ such that $a_k^T O < a_k^T (\{x_k\} + \text{cl}\, K)$. We can suppose that $||a_k|| = 1$, and by the Bolzano-Weierstrass theorem that $a_k \to a$ $(k \to \infty)$ where $a \in \text{cl}\,(K + L)$, $||a|| = 1$. We will show that $a^T L = 0 < a^T \text{ri}\, K$.

If $0 \neq x \in L$ then $x/||x|| \in O$, and trivially $0 \in \text{cl}\, K$. This implies that $a_k^T (x/||x||) < a_k^T (x_k + 0)$ $(k = 1, 2, \ldots)$. Taking limits we get $a^T x \leq 0$. This means that $a \in -L^* = L^\perp$. The fact that $a \in K^*$ can be proved similarly. Now we can prove that $0 < a^T \text{ri}\, K$. Suppose indirectly that there exist $x \in \text{ri}\, K$ such that $a^T x = 0$. This together with $a \in K^*$ would imply that $a \in K^\perp$. But then $a \in L^\perp \cap K^\perp \subseteq (L + K)^\perp$, a contradiction as $a \in \text{cl}\ (L + K)$ also and $a \neq 0$.

b) Apply part a) to the subspace $L := \mathcal{R} \times \{0\} \subseteq \mathcal{R}^{d+1}$ and the convex cone $K := K(C + (-C')) \subseteq \mathcal{R}^{d+1}$! $\square$

The proof which is described above relies on the Bolzano-Weierstrass theorem. A proof of Theorem 2.2 which does not use the Bolzano-Weierstrass theorem can be found in [5]. Note that part a) of Theorem 2.1 is an immediate consequence of Theorem 2.2. Really, for the vector $x \notin C$ an open ball O can be found such that x is the centre of O and O does not intersect with C. By Theorem 2.2 there exists a vector a such that $a^T O < a^T \text{ri}\, C$. From this the conclusion of part a) of Theorem 2.1 follows easily.

We remark without proof that part b) of Theorem 2.1, in the special case when $C' = Q$ is a polytope, also can be derived as a consequence of Theorem 2.2, and thus can be proved without using the Bolzano-Weierstrass theorem. We only sketch a possible approach here:

1. Using homogenization we can suppose that $C = K$ is a closed convex cone. (If not, consider $\{1\} \times Q$ and $\text{cl}\, K(C)$ instead of Q and C!) Let $S := \{a : a^T Q < 0\}$. We have to show that S meets K^*.

2. Prove that S is nonempty. This can be done using (a version of) the Farkas Lemma ([6]).

3. Deduce that $\text{cl}\, S = -R^*$ where R is the finitely generated cone $\text{cone}\,(Q)$.

4. If S would not meet K^* then by Theorem 2.2 a vector $x \neq 0$ would exist such that $x^T S < x^T \text{ri}\,(K^*)$. This would imply $x^T S \leq 0 \leq x^T (K^*)$. Then x would be an element of $R^{**} = R$ and $K^{**} = K$, a contradiction as Q and K are disjoint sets.

The proof of the following theorem is based on Theorem 2.2, and thus c
be proved also without using the Bolzano-Weierstrass theorem.

Theorem 2.3.

a) Let R be a polyhedral cone in $\mathcal{R}^d$, and let K be a convex cone in $\mathcal{R}$
Suppose that $R \cap \text{ri}\, K = \emptyset$. Then there exists $a \in \mathcal{R}^d$ such that $a^T R \leq 0$
$a^T \text{ri}\, K$.

b) Let P be a nonempty polyhedron in $\mathcal{R}^d$, and let C be a nonempty r
atively open convex set in $\mathcal{R}^d$. Suppose that $P \cap C = \emptyset$. Then there exi
$a \in \mathcal{R}^d$ such that $a^T P < a^T C$.

Proof. a) We prove this statement by induction on the dimension of $R \cup$
By Theorem 2.2 b) there exists $a_1 \in \mathcal{R}^d$ such that $a_1^T \text{ri}\, R < a_1^T \text{ri}\, K$. Th
$a_1^T R \leq 0 \leq a_1^T K$. If $\text{ri}\, K$ is not a subset of $a_1^\perp$ then $a_1^T R \leq 0 < a_1^T \text{ri}\, K$ and t
proof is finished. Thus we can suppose that $a_1^T R \leq 0 = a_1^T K$. Let P' be t
intersection of R and the hyperplane $-a_1 + a_1^\perp$. Then P' is a polyhedron, a
it is not empty as $a_1^T \text{ri}\, R < 0$. By Motzkin's theorem there exist a polyto
Q' and a finitely generated cone R' such that $P' = Q' + R'$. Then $R' = R \cap$
is a subset of R and thus disjoint from $\text{ri}\, K$. As the dimension of $R' \cup K$ is le
than the dimension of $R \cup K$, by the induction hypothesis there exists $a_2 \in$
such that $a_2^T R' \leq 0 < a_2^T \text{ri}\, K$. Choose $\lambda \geq 0$ such that $(a_2 + \lambda a_1)^T Q' \leq$
holds ($a_1^T Q' = -||a_1||^2 < 0$ so this can be done). Let $a := a_2 + \lambda a_1$, th
$a^T R' \leq 0$ holds also, thus $a^T P' \leq 0$. From this $a^T R \leq 0 < a^T \text{ri}\, K$ follo
easily.

b) Apply part a) to the polyhedral cone $R := \text{cl}\, K(P) \subseteq \mathcal{R}^{d+1}$ and t
convex cone $K := K(C) \subseteq \mathcal{R}^{d+1}$! $\square$

Now an equivalent form of Theorem 2.3 a) can be derived easily. Lemm
2.2 is known in the special case when $R = L$ is a subspace (see Lemma 16
in [5]).

Lemma 2.2. *Let R be a polyhedral cone in $\mathcal{R}^d$, and let K be a conv*
cone in $\mathcal{R}^d$. Then $R \cap \text{ri}\, K \neq \emptyset$ if and only if $(-R^*) \cap (K^*) \subseteq -K^*$. $\square$

In the next section Lemma 2.1 and Lemma 2.2 will be used in the proof
the main theorem.

3 Proof of the main theorem

In this section we describe two proofs of Theorem 1.3 and further closedne
theorems.

The **first proof** of Theorem 1.3 consists of seven steps. In Step 1 we prove the equivalence of statements a) and b).

1. For any K convex cone (specially for the convex cone $\operatorname{bar} P$) $(A^T K)^* = A^{-1}(K^*)$ holds trivially. Also for a C nonempty closed convex set the equation $(\operatorname{bar} C)^* = \operatorname{rec} C$ holds. Hence the equivalence of statements a) and b) is the immediate consequence of Lemma 2.2.

In Steps 2-5 we prove that statement b) implies the closedness of the set $AC + P$. We start with a special case, and generalize this special case step by step until we obtain the implication in the general case as well.

2. First consider the special case when $C = K$ and $P = R$ are convex cones, and K is supposed to be pointed also, i.e. $K \cap (-K) = \{0\}$ holds. Choose a matrix B such that $R = \{z : Bz \leq 0\}$. Then statement b) can be written succinctly in the following form: $\{x \in K : BAx \geq 0\} = \{0\}$. Let $x_k \in K$, $z_k \in R$ $(k = 1, 2, \ldots)$, and suppose that $Ax_k + z_k \to y$ $(k \to \infty)$. We have to prove that $y \in AK + R$.

If the set $\{x_k : k = 1, 2, \ldots\}$ is bounded then by the Bolzano-Weierstrass theorem we can suppose that the corresponding sequence converges to a point from the set K, i.e. $x_k \to x$ $(k \to \infty)$ where $x \in K$. Then $Ax_k \to Ax$ $(k \to \infty)$ and thus $z_k \to y - Ax$ $(k \to \infty)$. By the closedness of R, $y - Ax \in R$, and $y \in AK + R$ follows.

We show that the set $\{x_k : k = 1, 2, \ldots\}$ can not be unbounded. Otherwise we can suppose that $||x_k|| \to \infty$ $(k \to \infty)$. Let $x'_k := x_k/||x_k||$ $(k = 1, 2, \ldots)$, then $||x'_k|| = 1$ $(k = 1, 2, \ldots)$, and by the Bolzano-Weierstrass theorem we can suppose that $x'_k \to x'$ $(k \to \infty)$ where $x' \in K$, $||x'|| = 1$. We reach contradiction if we show that $BAx' \geq 0$ as then only $x' = 0$ would be possible contradicting the fact that $||x'|| = 1$. As

$$||x_k||Ax'_k + z_k \to y \ (k \to \infty),$$

so

$$||x_k||BAx'_k + Bz_k \to By \ (k \to \infty).$$

Here $||x_k|| \to \infty$, $BAx'_k \to BAx'$ $(k \to \infty)$, and $Bz_k \leq 0$ $(k = 1, 2, \ldots)$. Hence $BAx' \geq 0$ follows easily.

3. The case, when as in the previous step $C = K$ and $P = R$ are convex cones, but K is not supposed to be pointed, can be easily reduced to the the case already dealt with. Let $L := K \cap (-K)$ then $K = L + (L^\perp \cap K)$. Here $L^\perp \cap K$ is already pointed. Moreover the sum of finitely generated cones is also a finitely generated cone, thus by Motzkin's theorem the set $(AL) + R$ is a polyhedral cone. By the result obtained in the previous step, the set

$$AK + R = A(L^\perp \cap K) + (AL + R)$$

is closed if

$$Ax \in -(AL + R),\ x \in L^\perp \cap K \text{ implies } x = 0.$$

But this latter implication can be seen easily to be a consequence of stat ment b). Thus statement b) implies the closedness of $AK + R$.

4. In this step let $P = R$ be a convex cone as in the previous step, but t set C is not restricted to be a convex cone. This case can be reduced to t result in the previous step using homogenization. By Lemma 2.1

$$AC + R = C\left(\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \operatorname{cl} K(C) + \begin{pmatrix} 0 \\ R \end{pmatrix}\right).$$

This set (as we have proved in Step 3) is closed if

$$\left.\begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} \lambda \\ x \end{pmatrix} \in -\begin{pmatrix} 0 \\ R \end{pmatrix}, \\ \begin{pmatrix} \lambda \\ x \end{pmatrix} \in \operatorname{cl} K(C) \end{array}\right\} \text{ implies } \begin{pmatrix} \lambda \\ x \end{pmatrix} \in -\operatorname{cl} K(C).$$

Again by Lemma 2.1 the latter implication is exactly statement b) in t special case when $P = R$. Thus statement b) implies the closedness of $AC+$ in this special case also.

5. In the general case by Motzkin's theorem we can choose a polytope Q and polyhedral cone R such that $P = Q + R$. Then $R = \operatorname{rec} P$. By the compactne of Q the set

$$AC + P = Q + (AC + R)$$

is closed if the set $AC + R$ is closed. On the other hand, as we have prov in Step 4, the closedness of the set $AC + R$ follows from statement b). Th statement b) implies the closedness of the set $AC + P$ in the general case well.

Finally in Steps 6-7 we prove that statement b) implies that $\operatorname{rec}(AC+P)$ $(A\operatorname{rec} C)+\operatorname{rec} P$.

6. Consider again the special case when $P = R$ is a convex cone. As we ha seen in Step 4, statement b) implies the closedness of the set

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \operatorname{cl} K(C) + \begin{pmatrix} 0 \\ R \end{pmatrix}.$$

On the other hand it can be easily verified that

$$\begin{aligned}
&\operatorname{cl}\left(\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \operatorname{cl} K(C) + \begin{pmatrix} 0 \\ R \end{pmatrix}\right) = \\
&\quad = \operatorname{cl}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & E \end{pmatrix} \operatorname{cl} \begin{pmatrix} K(C) \\ R \end{pmatrix}\right) = \\
&\quad = \operatorname{cl}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & E \end{pmatrix} \begin{pmatrix} K(C) \\ R \end{pmatrix}\right) = \\
&\quad = \operatorname{cl}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & E \end{pmatrix} K \begin{pmatrix} C \\ R \end{pmatrix}\right) = \\
&\quad = \operatorname{cl} K(AC+R).
\end{aligned}$$

Hence

$$\operatorname{cl} K(AC+R) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \operatorname{cl} K(C) + \begin{pmatrix} 0 \\ R \end{pmatrix}.$$

Intersecting the set on the left hand side resp. the set on the right hand side of this equation with the hyperplane $\{0\} \times \mathcal{R}^m$, by Lemma 2.1 we get the set $\{0\} \times \operatorname{rec}(AC+R)$ resp. the set $\{0\} \times ((A \operatorname{rec} C) + R)$. Thus statement b) implies $\operatorname{rec}(AC+R) = (A \operatorname{rec} C) + R$ in the special case when $P = R$.
7. In the general case we can apply Motzkin's theorem again. Let Q and R be the same sets as in Step 5. Then by the result obtained in the previous step

$$\begin{aligned}
\operatorname{rec}(AC+P) &= \operatorname{rec}(Q+AC+R) = \\
&= \operatorname{rec}(AC+R) = (A \operatorname{rec} C) + R = (A \operatorname{rec} C) + \operatorname{rec} P.
\end{aligned}$$

(Here we used also that adding a compact convex set to a closed convex set does not change the barrier cones and thus dually the recession cones.) Thus statement b) implies the equation $\operatorname{rec}(AC+P) = (A \operatorname{rec} C) + \operatorname{rec} P$ in the general case as well. □

Thus we have concluded the first proof of the main theorem. Note that the proof could not have been applied in the case of Theorem 1.1. Really, in Step 3 above we needed the stronger version of Step 2, where the polyhedral cone R is added to the convex cone AK. Thus the first proof is an example of the following principle: Sometimes proving the more general form of a theorem makes possible new proofs which can not be applied in the case of the weaker form of the theorem. (We heard this principle from András Frank.)

We scetch yet another approach here, a **second proof** of Theorem 1.3.

We prove the equivalence of statements a) and b) the same way as in Step 1 above. Then we verify that statement a) implies statement c). Due to the remarks made in the introduction we have to prove the similar implication only in the case of Theorem 1.2. This can be done in four steps as follows:

1. First we can prove that if R is a polyhedral cone in $\mathcal{R}^d$ and K is a conv
cone in $\mathcal{R}^d$ then $R \cap \operatorname{ri} K \neq \emptyset$ implies that $(R \cap K)^* = R^* + (K^*)$. The spec
case of this result when $R = L$ is a subspace is Krein's theorem. A proof
this special case based on part a) of Theorem 2.2 can be found in [4]. Th
using part a) of Theorem 2.3 instead of part a) of Theorem 2.2 we can eas
generalize the proof of Krein's theorem and obtain a proof of the gener
case. Given a vector $c \in (R \cap K)^*$ the proof first deals with the case wh
$c^\perp \cap R \cap \operatorname{ri} K \neq \emptyset$. In this case we can easily derive that $c \in R^* + (K^\perp$
On the other hand, in the case when the polyhedral cone $c^\perp \cap R$ does n
intersect with $\operatorname{ri} K$, we can use part a) of Theorem 2.3 and conclude th
$c \in R^* + (K^*)$. This way we have proved the nontrivial inclusion in t
equation $(R \cap K)^* = R^* + (K^*)$.
2. For a convex set C let C° denote the *dual set* of C, i.e. let $C^\circ := C(K(C)$
Then C° is a closed convex set containing the origin. For a K convex co
$K^\circ = K^*$ holds. Applying the result obtained in Step 1 to the polyhedr
cone $\mathcal{R} \times R \subseteq \mathcal{R}^{d+1}$ and the convex cone $K(C) \subseteq \mathcal{R}^{d+1}$ we can easily deri
the following more general result: If R is a polyhedral cone in $\mathcal{R}^d$ and C is
convex set in $\mathcal{R}^d$ then $R \cap \operatorname{ri} C \neq \emptyset$ implies the equation $(R \cap C)^\circ = R^* + (C$
(and specially the closedness of the set $R^* + (C^\circ)$).
3. A simple separation argument based on part a) of Theorem 2.1 shows th
for a $C \subseteq \mathcal{R}^d$ closed convex set $0 \in C$ implies that $(C^\circ)^\circ = C$. Thus applyi
the result described in the previous step to the dual sets R^* and C° inste
of R and C we obtain the following result: Let R be a polyhedral cone in $\mathcal{R}$
Let C be a closed convex set in $\mathcal{R}^d$ such that $0 \in C$. If R^* intersects with t
relative interior of C° then the set $R + C$ is closed. Due to the fact that R^*
a convex cone, we can substitute C° with the convex cone $\operatorname{cone}(C^\circ) = \operatorname{bar}$
This way we obtain the following result: Let R be a polyhedral cone in $\mathcal{R}$
and let C be a closed convex set in $\mathcal{R}^d$. Then $R^* \cap \operatorname{ri}(\operatorname{bar} C) \neq \emptyset$ impli
the closedness of the set $R + C$. (We do not need the $0 \in C$ assumption
translation does not change the barrier cone and the closedness of a set.)
4. Now we are ready to prove that statement a) implies statement c) in t
case of Theorem 1.2. By Motzkin's theorem there exist a polytope Q and
polyhedral cone R such that $P = Q + R$. Then $\operatorname{bar} P = R^*$. By the resu
obtained in the previous step statement a) implies the closedness of the s
$R + C$ which in turn (by compactness of the polytope Q) implies the closedne
of the set $P + C$.

We concluded the (sketch of the) proof of Theorem 1.2 and thus the seco
proof of Theorem 1.3 as well. To fill in the gaps in the second proof is left
the reader.

The main difference between the two proofs described above is that wh the first proof is based on the Bolzano-Weierstrass theorem, the second proof can be carried through without using this theorem. To see this we have to prove without using the Bolzano-Weierstrass theorem (!) for example that the convex set $Q+C$ is closed if Q is a polytope and C is a closed convex set.

This can be done in the following way: By part a) of Theorem 2.1 the closed convex set $C \subseteq \mathcal{R}^d$ is the intersection of all the special closed convex sets of the form $\{x : a^T x \leq \beta\}$ where $a \in \mathcal{R}^d$, $\beta \in \mathcal{R}$ and $a^T C \leq \{\beta\}$. The set Q is a polytope so there exists a finite set S such that Q is the convex hull of S. Then (using part b) of Theorem 2.1 in the special case when C' is a polytope) it can be easily seen that the set $Q+C$ is the intersection of all the special closed convex sets of the form

$$\left\{x : a^T x \leq \beta + \max(a^T S)\right\}$$

where $a \in \mathcal{R}^d$, $\beta \in \mathcal{R}$ and $a^T C \leq \{\beta\}$. Consequently the set $Q+C$ is closed.

Due to the remarks made in §2, this proof (and thus the second proof of Theorem 1.3) can be carried through without using the Bolzano-Weierstrass theorem.

Now we describe further closedness theorems, consequences of Theorem 1.3 (also consequences of Theorem 1.1 and Theorem 1.2 respectively).

Theorem 3.1. *Let A be an m by n real matrix. Let C_1 and C_2 be closed convex sets in $\mathcal{R}^n$ and $\mathcal{R}^m$ respectively. Then between the statements*

a) $(A^T \mathrm{ri}\,(\mathrm{bar}\, C_2)) \cap \mathrm{ri}\,(\mathrm{bar}\, C_1) \neq \emptyset$,
b) $A^{-1}(-\mathrm{rec}\, C_2) \cap (\mathrm{rec}\, C_1) \subseteq A^{-1}(\mathrm{rec}\, C_2) \cap (-\mathrm{rec}\, C_1)$,
c) $(AC_1) + C_2$ *is closed, and* $\mathrm{rec}\,((AC_1) + C_2) = (A \mathrm{rec}\, C_1) + \mathrm{rec}\, C_2$,

hold the following logical relations: a) is equivalent to b); a) or b) implies c).

Proof. Apply Theorem 1.3 to the matrix (A, E), the closed convex set $C_1 \times C_2$ and the polyhedron $\{0\}$! $\square$

Theorem 3.2. *Let A be an m by n real matrix, let P be a polyhedron in $\mathcal{R}^n$, and let C be a closed convex set in $\mathcal{R}^m$. Then between the statements*

a) $(A^{-T}(\mathrm{bar}\, P)) \cap \mathrm{ri}\,(\mathrm{bar}\, C) \neq \emptyset$,
b) $A(-\mathrm{rec}\, P) \cap (\mathrm{rec}\, C) \subseteq -\mathrm{rec}\, C$,
c) $(AP) + C$ *is closed, and* $\mathrm{rec}\,((AP) + C) = (A \mathrm{rec}\, P) + \mathrm{rec}\, C$,

hold the following logical relations: a) is equivalent to b); a) or b) implies c).

Proof. Apply Theorem 1.3 to the matrix E, the closed convex set C a
the polyhedron AP! $\square$

In the next section we present an application of Theorems 1.3, 3.1 and 3
in the theory of duality in convex programming.

4 Duality theorems

In this section we describe a duality theorem similar to Rockafellar's duali
theorem in [7]. We will see also that the two theorems have important comm
special cases.

We will use the terminology and notations of [7] here. Let $f : \mathcal{R}^n$
$\mathcal{R} \cup \{+\infty\}$ be a convex function, and let $g : \mathcal{R}^m \to \mathcal{R} \cup \{-\infty\}$ be a conca
function. Let $A \in \mathcal{R}^{m \times n}$, $b \in \mathcal{R}^m$, $c \in \mathcal{R}^n$. We will consider the followi
pair of programs from [7]:

$$\begin{array}{ll} (P): & \inf f(x) - g(Ax - b) + c^T x,\ x \in \mathcal{R}^n, \\ (D): & \sup g^c(y) - f^c(A^T y - c) + b^T y,\ y \in \mathcal{R}^m. \end{array}$$

Here f^c resp. g^c denotes the *convex conjugate function* of f resp. the *conca*
conjugate function of g, i.e. let

$$f^c(a) := \sup\left\{a^T x - f(x) : x \in \mathcal{R}^n\right\},\ g^c(y) := \inf\left\{y^T z - g(z) : z \in \mathcal{R}^m\right\}$$

Let $F(f)$ resp. $F(g)$ denote the domain of finiteness of the function f a
g respectively, i.e. let

$$F(f) := \{x \in \mathcal{R}^n : f(x) < +\infty\},\ F(g) := \{z \in \mathcal{R}^m : g(z) > -\infty\}.$$

The points of the set

$$\mathbf{P} := F(f) \cap \{x : Ax - b \in F(g)\}$$

are called the *feasible solutions* of program (P). We denote by v_P the *optim*
value of program (P), i.e. let

$$v_P := \inf\left\{f(x) - g(Ax - b) + c^T x : x \in \mathbf{P}\right\}.$$

For the program (D) the set $\mathbf{D}$ and the value v_D can be defined similarly.

Specially, let $K_1 \subseteq \mathcal{R}^m$, $K_2 \subseteq \mathcal{R}^n$ be convex cones, and define f and g
follows:

$$f(x) := \begin{cases} 0, & \text{if } x \in K_2, \\ +\infty & \text{otherwise,} \end{cases} \quad g(z) := \begin{cases} 0, & \text{if } z \in K_1, \\ -\infty & \text{otherwise.} \end{cases}$$

Then (P) and (D) form the following pair of *cone-linear programs*:

$$\inf c^T x,\ Ax \geq_{K_1} b,\ x \geq_{K_2} 0,\ \sup b^T y,\ A^T y \leq_{K_2^*} c,\ y \geq_{K_1^*} 0,$$

where $z_1 \geq_K z_2$ denotes that $z_1 - z_2 \in K$.

Weak duality (i.e. the inequality $v_P \geq v_D$) follows from the definitions. To obtain equality here (i.e. for strong duality theorems) generally we have to assume certain regularity conditions. An example of such theorems is Rockafellar's duality theorem in [7], where strong duality is the consequence of stability conditions.

Here we present a duality theorem which is similar to Rockafellar's duality theorem, only the stability conditions are replaced with closedness conditions. Though the significance of closedness conditions was realized early in the history of cone-linear programs (see [2]), in the generality of programs (P) and (D), to the best of the author's knowledge, the following three theorems are new. For the strong duality theorem to hold we will need the following

Primal closedness assumption: Suppose that the set

$$C_P := \begin{pmatrix} A & 0 \\ c^T & 1 \end{pmatrix} [f] + (-[g])$$

is closed.

Here $[f]$ resp. $[g]$ denotes the *epigraph* of f, resp. the *hypograph* of g, i.e.

$$[f] := \{(x, \mu) \in \mathcal{R}^{n+1} : f(x) \leq \mu\},\ [g] := \{(z, \nu) \in \mathcal{R}^{m+1} : g(z) \geq \nu\}.$$

We will deduce our strong duality theorem from the following convex Farkas theorem, similarly as in the theory of linear programming.

Theorem 4.1. *Suppose that the primal closedness assumption is satisfied, and* $\mathbf{P} \cup \mathbf{D} \neq \emptyset$. *Then for every* $\delta \in \mathcal{R}$ *the following statements are equivalent:*
a) there exists a vector $x \in \mathcal{R}^n$ *such that* $f(x) - g(Ax - b) + c^T x \leq \delta$ *holds;*
b) for every $y \in \mathcal{R}^m$ *the inequality* $g^c(y) - f^c(A^T y - c) + b^T y \leq \delta$ *holds.*

Proof. First, statement a) implies statement b), this can be proved similarly as the weak duality theorem.

On the other hand we show that statement b) implies statement a). Suppose indirectly that b) holds while a) does not.

It can be proved easily that the vector (b, δ) is an element of the set C_P if and only if there exists a vector $x \in \mathbf{P}$ such that $f(x) - g(Ax - b) + c^T x \leq \delta$. As a) does not hold, $(b, \delta) \notin C_P$. By part a) of Theorem 2.1, there exist a

vector $a \in \mathcal{R}^n$ and a number $\alpha \in \mathcal{R}$ such that with the $\hat{a} := (a, \alpha)$ notati
$\hat{a}^T(b, \delta) < \inf \hat{a}^T C_P$, i.e.

$$a^T b + \alpha\delta < \inf((a^T A + \alpha c^T, \alpha)[f]) - \sup((a^T, \alpha)[g]). \qquad ($$

It can be seen easily that here $\alpha \geq 0$.

If $\alpha = 0$ then by (2)

$$a^T b < \inf(a^T A F(f)) - \sup(a^T F(g)).$$

Consider first the case when $\mathbf{P} \neq \emptyset$, and let $x_0 \in \mathbf{P}$. Then $x_0 \in F(f)$, a
$Ax_0 - b \in F(g)$, so we obtain the inequality

$$a^T b < a^T A x_0 - a^T(Ax_0 - b),$$

a contradiction. Suppose now that $\mathbf{D} \neq \emptyset$, and let $y_0 \in \mathbf{D}$. Let ε deno
the positive number that we get subtracting the left hand side of (2) fro
the right hand side. Let λ be a large enough positive number such that t
inequality

$$\delta < g^c(y_0) - f^c(A^T y_0 - c) + b^T y_0 + \lambda\varepsilon$$

holds. As the right hand side of this inequality is at most $g^c(y) - f^c(A^T y$
$c) + b^T y$ where $y := y_0 - \lambda a$, we reached contradiction with b).

If $\alpha > 0$, then we can suppose that $\alpha = 1$. Then (2) implies the followi
inequality:

$$\delta < g^c(-a) - f^c(A^T(-a) - c) + b^T(-a),$$

contradicting b) again. □

If the functions f and g are closed also, then $f^{cc} = f$ and $g^{cc} = g$. Th
the following theorem is an immediate consequence of Theorem 4.1.

Dual closedness assumption: Suppose that the set

$$C_D := \begin{pmatrix} A^T & 0 \\ b^T & 1 \end{pmatrix} [g^c] + (-[f^c])$$

is closed.

Theorem 4.2. *Suppose that* f, g *are closed, the dual closedness assum*
tion is satisfied, and $\mathbf{P} \cup \mathbf{D} \neq \emptyset$. *Then for every* $\delta \in \mathcal{R}$ *the following statemen*
are equivalent:
a) there exists a vector $y \in \mathcal{R}^m$ *such that* $g^c(y) - f^c(A^T y - c) + b^T y \geq \delta$ *hol*
b) for every $x \in \mathcal{R}^n$ *the inequality* $f(x) - g(Ax - b) + c^T x \geq \delta$ *holds.* □

Now the strong duality theorem described below follows from Theorems 4.1 and 4.2 the same way as in the theory of linear programming.

Theorem 4.3.
a) Under the assumptions of Theorem 4.1, the optimal values of programs (P) *and* (D) *are equal. Furthermore, the primal optimal value* v_P *is attained if it is finite.*
b) Under the assumptions of Theorem 4.2, the optimal values of programs (P) *and* (D) *are equal. Furthermore, the dual optimal value* v_D *is attained if it is finite.* □

Theorem 4.3 is similar to Rockafellar's duality theorem in [7], only the stability conditions in the latter theorem are replaced with closedness conditions. Using our closedness theorems (Theorems 1.3, 3.1 and 3.2) we show that Rockafellar's duality theorem and Theorem 4.3 have important common special cases. We will need the following lemma, a consequence of Corollary 13.5.1 in [5].

Lemma 4.1. *Let* f *be a convex function. Then the barrier cone of the epigraph of this function and the relative interior of this cone can be described as follows:*
a) $\mathrm{bar}\,[f] = \hat{K}_1 \cup \hat{K}_2$ *where*

$$\hat{K}_1 = (\mathrm{bar}\, F(f)) \times \{0\},\ \hat{K}_2 = \mathrm{cone}\,(F(f^c) \times \{1\});$$

b) $\mathrm{ri}\,(\mathrm{bar}[f]) = \mathrm{ri}\,\hat{K}_2 = \{(\lambda a, \lambda) : \lambda > 0,\, a \in \mathrm{ri}\, F(f^c)\}.$

Proof. We omit the simple calculation that proves part a). To prove part b), note that

$$\mathrm{ri}\,(\hat{K}_1 \cup \hat{K}_2) \subseteq \hat{K}_2 \subseteq \hat{K}_1 \cup \hat{K}_2.$$

Thus the convex cones $\hat{K}_1 \cup \hat{K}_2$ and $\hat{K}_2$ has the same closure and therefore the same relative interior. The remaining equality in b) is proved by Lemma 2.1 then. □

Similar statement can be derived in the case of the hypograph of g as well. Using Lemma 4.1 and the closedness theorems 1.3, 3.1 and 3.2, we can deduce sufficient conditions for the primal resp. the dual closedness assumption to hold. These sufficient conditions are described in the following two propositions.

Proposition 4.1. *The primal closedness assumption is satisfied if any of the following statements holds:*

a) f, $-g$ *are closed convex functions, and there exists a vector* $y_0 \in \mathcal{R}^m$ *su that* $y_0 \in\mathrm{ri}\, F(g^c)$, $A^T y_0 - c \in\mathrm{ri}\, F(f^c)$.
b) f *is a closed convex function,* $-g$ *is a polyhedral convex function, and the exists a vector* $y_0 \in \mathcal{R}^m$ *such that* $y_0 \in F(g^c)$, $A^T y_0 - c \in\mathrm{ri}\, F(f^c)$.
c) f *is a polyhedral convex function,* $-g$ *is a closed convex function, and the exists a vector* $y_0 \in \mathcal{R}^m$ *such that* $y_0 \in\mathrm{ri}\, F(g^c)$, $A^T y_0 - c \in F(f^c)$.
d) f, $-g$ *are polyhedral convex functions, and* $\mathbf{D} \neq \emptyset$.

Proof. First we prove that statement a) implies the closedness of the s C_P. Let

$$\hat{A} := \begin{pmatrix} A & 0 \\ c^T & 1 \end{pmatrix},\ C_1 := [f],\ C_2 := -[g].$$

Then, as f, $-g$ are closed convex functions, C_1 and C_2 are closed conv sets. By Theorem 3.1 $C_P = \hat{A}C_1 + C_2$ is closed if there exists a vect $\hat{y} \in\mathrm{ri}\,\mathrm{bar}\, C_2$ such that $\hat{A}^T \hat{y} \in\mathrm{ri}\,\mathrm{bar}\, C_1$. Let y_0 be a vector with the properti described in statement a). Using Lemma 4.1 it can be easily seen that t vector $\hat{y} := (-y_0, 1)$ then meets the requirements. Thus C_P is closed, and t primal closedness assumption is satisfied.

The statement that b) resp. c) implies the closedness of the set C_P c be dealt similarly, only now using Theorem 3.2 and Theorem 1.3 instead Theorem 3.1.

Finally, if statement d) holds then $[f]$, $[g]$ are polyhedrons, therefore C is also a polyhedron (a consequence of Motzkin's theorem). Specially C_P closed, and the primal closedness assumption is satisfied. □

As in the case of Theorem 4.1, Proposition 4.1 can also be dualized. Th way we obtain the following result:

Proposition 4.2. *The dual closedness assumption is satisfied if any the following statements holds:*
a) f, $-g$ *are closed convex functions, and there exists a vector* $x_0 \in \mathcal{R}^n$ *su that* $x_0 \in\mathrm{ri}\, F(f)$, $Ax_0 - b \in\mathrm{ri}\, F(g)$.
b) f *is a closed convex function,* $-g$ *is a polyhedral convex function, and the exists a vector* $x_0 \in \mathcal{R}^n$ *such that* $x_0 \in\mathrm{ri}\, F(f)$, $Ax_0 - b \in F(g)$.
c) f *is a polyhedral convex function,* $-g$ *is a closed convex function, and the exists a vector* $x_0 \in \mathcal{R}^n$ *such that* $x_0 \in F(f)$, $Ax_0 - b \in\mathrm{ri}\, F(g)$.
d) f, $-g$ *are polyhedral convex functions, and* $\mathbf{P} \neq \emptyset$. □

Combining Theorem 4.3 with Propositions 4.1 and 4.2, we arrive at t following well-known theorem, a common special case of Rockafellar's duali theorem and Theorem 4.3.

Theorem 4.4.
a) Suppose that from Proposition 4.1 statement a), b), c) or d) holds. Then the optimal values of programs (P) and (D) are equal. Furthermore, the primal optimal value v_P is attained if it is finite.
b) Suppose that from Proposition 4.2 statement a), b), c) or d) holds. Then the optimal values of programs (P) and (D) are equal. Furthermore, the dual optimal value v_D is attained if it is finite. □

Finally we mention an open problem: It would be interesting to see further connections (if any) between the stability conditions in Rockafellar's duality theorem in [7] and the closedness conditions in Theorem 4.3.

Conclusion. In this paper we described a common generalization of several closedness theorems from [5]. We presented two proofs for the main theorem; the first is based on the Bolzano-Weierstrass theorem, the second can be carried through without using this theorem. We give proofs of the separation lemmas as well, therefore the paper is almost self-contained. Also we presented a duality theorem for convex programs which is similar to Rockafellar's duality theorem in [7]. As a consequence of our general closedness theorem we showed that these duality theorems have important common special cases.

Acknowledgements. I am indebted to András Frank, Margit Kovács and Károly Böröczky for the several consultations.

References

[1] A. Berman, *Cones, matrices and mathematical programming*, Springer-Verlag, Berlin, 1973.

[2] B. D. Craven and J. J. Koliha, *Generalizations of Farkas' theorem*, SIAM J. Math. Anal., Vol. 8 No. 6 (1977), 983–997.

[3] J. Hirriart-Urruty and C. Lemarèchal, *Convex analysis and minimization algorithms I*, Springer-Verlag, Berlin, 1993.

[4] M. Kovács, *Theory of nonlinear programming*, Typotex Kiadó, Budapest, 1997 (in Hungarian).

[5] R. T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

[6] A. Schrijver, *Theory of linear and integer programming*, John Wiley
Sons, New York, 1986.

[7] J. Stoer and C. Witzgall, *Convexity and optimization in finite*
mensions I., Springer-Verlag, Berlin, 1970.

[8] G. Strang, *Linear algebra and its applications*, Academic Press, N
York, 1980.

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