

# Robust DWDM Routing and Provisioning under Polyhedral Demand Uncertainty \*

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## Abstract

We present mixed integer linear programming models that are robust in the face of uncertain traffic demands known to lie in a certain polyhedron for the problem of dense wavelength division multiplexing network routing and provisioning at minimal cost. We investigate the solution of the problem in a set of numerical experiments for two models of polyhedral uncertainty: (a) the hose model, (b) a restricted interval uncertainty model. We report the results of these numerical experiments in comparison with an alternative model of robustness due to Kennington *et al.*

**Keywords** Uncertainty modeling, robust DWDM routing and provisioning, hose uncertainty, interval uncertainty, valid inequalities.

## 1 Introduction

The purpose of this paper is to provide robust optimization models for the dense wavelength division multiplexing (DWDM) routing and provisioning problem under uncertain traffic demands. The main components of a dense wavelength division multiplexing network are fiber, terminal equipment (TE), optical amplifiers

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(As), and regenerators (Rs) [10]. The term TE is used as an acronym for the wavelength transponders performing optical-electrical-optical conversion. As in Kennington *et al.* [10], we assume that fiber becomes operational by installation of optical amplifiers and regenerators along links of the network, and of terminal equipment at incident nodes. Links containing these components constitute the main elements of a DWDM network. Each link may contain multiple fibers with multiple channels per fiber. The DWDM routing and provisioning problem is to determine the routing for each demand and the least cost DWDM equipment configuration needed to support the routing. The problem under known traffic demand can be modeled as an integer linear program and solved using off-the-shelf software for reasonable sizes. However, incorporating uncertainty into the problem renders its numerical solution more challenging.

The present study is inspired by the contribution from Kennington *et al.* [10] where a minimax regret based multicriteria framework is developed for the problem of DWDM routing and provisioning under demand uncertainty. The authors in [10] propose a scenario based robust optimization approach using a path based problem formulation whereby the number of paths for each demand pair is limited to four for tractability purposes. They incorporate a two-phase approach. In the first phase a minimax regret optimization model with a budget constraint is obtained over some demand scenarios. In the second phase, regret is fixed to its optimal value, and a minimum cost solution is sought.

The goal of the present paper is to show that other robustness measures are computationally viable in the context of DWDM routing and provisioning. We deviate from the aforementioned reference in two important aspects: 1. we adopt a flow based formulation for the problem, 2. we investigate uncertainty models based on polyhedral representations of the uncertain demands, and develop models leading to optimal designs which are feasible for all demand vectors lying in this polyhedral uncertainty set. Therefore, our models do not use a scenario based minimax regret approach doing away with the need to specify scenarios and weights (probabilities) for each scenario, but rather seek a minimum cost design that remains robust in the face of all potential demand scenarios within a certain polyhedral set. Furthermore, we unify both robustness and minimal cost design in single mixed integer linear programming models as opposed to the two-phase multicriteria approach of [10]. Studies related to the present paper, although substantially different, are [1, 3].

Numerical solutions of these uncertain problems are investigated using some test data from [10] and network design literature [6, 7, 14, 15]. We incorporate valid inequalities based on cuts and capacities. Comparisons with the robust model of [10] are also given and discussed.

The paper is organized as follows. In Section 2, we define and model the deterministic problem as in [10]. In Section 3, we present the hose model, derive a MIP formulation of the corresponding robust problem and present valid inequalities. The restricted interval uncertainty model is given in Section 4 along with the formulation of the robust problem and valid inequalities. Sections 5 and 6 are devoted to computational study for the hose and interval models respectively. We conclude in Section 7.

## 2 The model of Kennington *et al.*

Kennington *et al.* proposed a path-based formulation for the DWDM routing and provisioning problem when the demand matrix is known.

Let  $G = (N, E)$  be the underlying network with node set  $N$  and edge set  $E$ . Set  $A_n$  is the set of edges incident to node  $n \in N$ .

Let  $D$  denote the set of commodities. Commodity  $(o, d) \in D$  has demand  $R_{od}$ . The set of paths that can be used by commodity  $(o, d)$  is  $J_{od}$ .  $P_n$  denotes the set of paths that contain node  $n \in N$  and  $L_e$  denotes the set of paths that contain edge  $e \in E$ .

Let  $M^{TE}$ ,  $M^R$  and  $M^A$  be the number of DS3s (Digital Signal level 3) that each TE, regenerator and optical amplifier can accommodate, and  $C^{TE}$ ,  $C^R$  and  $C^A$  be the unit cost of a TE, a regenerator and an optical amplifier, respectively.  $F_e$  is the maximum number of fibers on link  $e \in E$ . Define  $G_e^A$  and  $G_e^R$  to be the number of amplifier sites and regenerator sites on link  $e \in E$ , respectively. In our experimentation we used these parameter values as reported in Kennington *et al.*

The model uses the following variables. Variable  $x_p$  is the amount of DS3s assigned to path  $p$  and  $l_n$  is the amount of TEs assigned to node  $n$ . Define also  $t_e$ ,  $a_e$ ,  $r_e$ , and  $z_e$  to be the amounts of TEs, optical amplifiers, regenerators, DS3s, and  $f_e$  and  $c_e$  to be the number of fibers and channels assigned to link  $e$ , respectively.

For given demand, the problem is:

$$\min \sum_{n \in N} C^{TE} l_n + \sum_{e \in E} (C^R r_e + C^A a_e) \quad (1)$$

$$\text{s.t.} \quad \sum_{p \in J_{od}} x_p = R_{od} \quad \forall (o, d) \in D \quad (2)$$

$$\sum_{p \in L_e} x_p = z_e \quad \forall e \in E \quad (3)$$

$$z_e \leq M^{TE} t_e \quad \forall e \in E \quad (4)$$

$$\sum_{e \in A_n} t_e = l_n \quad \forall n \in N \quad (5)$$

$$z_e \leq M^A f_e \quad \forall e \in E \quad (6)$$

$$z_e \leq M^R c_e \quad \forall e \in E \quad (7)$$

$$f_e \leq F_e \quad \forall e \in E \quad (8)$$

$$G_e^A f_e = a_e \quad \forall e \in E \quad (9)$$

$$G_e^R c_e = r_e \quad \forall e \in E \quad (10)$$

$$f_e, c_e \text{ integer} \quad \forall e \in E \quad (11)$$

$$\text{all variables are nonnegative.} \quad (12)$$

Constraints (2) imply that the demand of each commodity is satisfied over the union of paths reserved for this commodity. Constraints (3) compute the link capacities in terms of the path capacities. Similarly, constraints (4) convert link capacities to TEs. Note that these constraints hold as equality at optimality if costs are positive. The number of TEs at nodes are computed through constraints (5). Constraints (6), (7) and (8) imply the requirements of fibers and channels, and bound the number of fibers. Finally, constraints (9) and (10) compute the number of amplifiers and regenerators. The objective function (1) is the sum of cost of TEs, amplifiers and regenerators.

Some variables are defined only for the sake of clarity and can be projected out easily in order to reduce the model size. In particular, since constraints (4) will always be tight at optimality, we can eliminate the variables corresponding to the amounts of DS3s on each link, i.e.,  $z_e$ 's and eliminate constraints (4). Using equations (3) and (5) we can eliminate  $l_n$  variables along with constraints (3) and (5). In a similar fashion, constraints (9) and (10) are redundant and can be eliminated from the model along with the variables  $a_e$ 's and  $r_e$ 's. Thereby, the resulting model is as follows:

$$\begin{aligned}
\min \quad & \sum_{n \in N} \frac{C^{TE}}{M^{TE}} \sum_{e \in A_n} \sum_{p \in L_e} x_p + \sum_{e \in E} (C^R G_e^R c_e + C^A G_e^A f_e) \\
\text{s.t.} \quad & \sum_{p \in J_{od}} x_p = R_{od} \quad \forall (o, d) \in D \\
& \sum_{p \in L_e} x_p \leq M^A f_e \quad \forall e \in E \\
& \sum_{p \in L_e} x_p \leq M^R c_e \quad \forall e \in E \\
& f_e \leq F_e \quad \forall e \in E \\
& f_e, c_e \text{ integer} \quad \forall e \in E \\
& \text{all variables are nonnegative.}
\end{aligned}$$

In this research, we refrain from restricting our routing search space to a fixed set of paths for demand pairs. To this end, we provide the following flow-based formulation for the DWDM routing and provisioning problem under a fixed demand matrix:

$$\begin{aligned}
\min \quad & 2 \frac{C^{TE}}{M^{TE}} \sum_{\{i,j\} \in E} \sum_{(o,d) \in D} R_{od} (y_{ij}^{od} + y_{ji}^{od}) + \sum_{e \in E} (C^R G_e^R c_e + C^A G_e^A f_e) \\
\text{s.t.} \quad & \sum_{j: \{i,j\} \in E} (y_{ij}^{od} - y_{ji}^{od}) = \begin{cases} 1 & \text{if } i = o \\ -1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in N, (o, d) \in D \quad (13) \\
& \sum_{(o,d) \in D} R_{od} (y_{ij}^{od} + y_{ji}^{od}) \leq M^A f_e \quad \forall e = \{i, j\} \in E \quad (14) \\
& \sum_{(o,d) \in D} R_{od} (y_{ij}^{od} + y_{ji}^{od}) \leq M^R c_e \quad \forall e = \{i, j\} \in E \quad (15) \\
& f_e \leq F_e \quad \forall e \in E \quad (16) \\
& f_e, c_e \text{ integer} \quad \forall e \in E \quad (17) \\
& \text{all variables are nonnegative.} \quad (18)
\end{aligned}$$

The decision variable  $y_{ij}^{od}$  denotes the fraction of demand  $(o, d)$  being routed on edge  $\{i, j\}$  along the direction  $i \rightarrow j$ . Constraints (13) ensure that demand of each commodity  $(o, d)$  must be routed over a union of paths from  $o$  to  $d$ . Hence, the portion of commodity  $(o, d)$ 's demand carried through edge  $e = \{i, j\}$

is simply  $R_{od}(y_{ij}^{od} + y_{ji}^{od})$ . Consequently, constraints (14) and (15), respectively, decide on the number of fibers and channels to be used on each edge.

As expected, solving the flow-based formulation requires more effort than solving a restricted path-based formulation. However as will be evident with our experimentation, we managed to handle reasonably sized networks.

Finally, notice that for a fixed scenario, the DWDM routing and provisioning problem is a generalization of the single facility network loading problem (see e.g. [2, 6, 8, 11, 12, 13]).

### 3 The Robust Model under Hose Demand Uncertainty

This section is devoted to our first uncertainty model for the DWDM routing and provisioning problem. We adopt a polyhedral model of uncertainty known as the ‘‘hose’’ model [9] where the set of demand values are not known but there is a limit to the sum of demands originating and ending at a given node in the network. This is expressed as the set

$$\Theta = \{R \in \mathbb{R}_+^{|D|} : \sum_{d:(d,o) \in D} R_{do} + \sum_{d:(o,d) \in D} R_{od} \leq b_o \quad \forall o \in N\}.$$

Now, we consider the following robust model that guarantees an optimal design under all occurrences of demand vectors from the set  $\Theta$ . In particular, the resulting design should be flexible enough in supporting any, even the worst, demand vector from polyhedron  $\Theta$ . Consequently, we must solve the following nonlinear model.

$$\begin{aligned} \min \quad & 2 \frac{C^{TE}}{M^{TE}} \sum_{\{i,j\} \in E} \max_{R \in \Theta} \sum_{(o,d) \in D} R_{od}(y_{ij}^{od} + y_{ji}^{od}) + \sum_{e \in E} (C^R G_e^R c_e + C^A G_e^A f_e) \\ \text{s.t.} \quad & \sum_{j:\{i,j\} \in E} (y_{ij}^{od} - y_{ji}^{od}) = \begin{cases} 1 & \text{if } i = o \\ -1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in N, (o, d) \in D \\ & \max_{R \in \Theta} \sum_{(o,d) \in D} R_{od}(y_{ij}^{od} + y_{ji}^{od}) \leq M^A f_e \quad \forall e = \{i, j\} \in E \\ & \max_{R \in \Theta} \sum_{(o,d) \in D} R_{od}(y_{ij}^{od} + y_{ji}^{od}) \leq M^R c_e \quad \forall e = \{i, j\} \in E \\ & f_e \leq F_e \quad \forall e \in E \\ & f_e, c_e \text{ integer} \quad \forall e \in E \\ & \text{all variables are nonnegative.} \end{aligned}$$

Following the treatment of Altm *et al.*[1], we now proceed toward a compact MIP formulation. For a given edge  $e = \{i, j\} \in E$  and a fixed flow  $y$  satisfying flow balance constraints, the problem  $\max_{R \in \Theta} \sum_{(o,d) \in D} R_{od}(y_{ij}^{od} + y_{ji}^{od})$ , or equivalently,

$$\begin{aligned}
& \max \sum_{(o,d) \in D} R_{od}(y_{ij}^{od} + y_{ji}^{od}) \\
& \text{s.t.} \quad \sum_{d:(d,o) \in D} R_{do} + \sum_{d:(o,d) \in D} R_{od} \leq b_o \quad \forall o \in N \\
& \quad \quad R_{od} \geq 0 \quad \forall (o,d) \in D
\end{aligned} \tag{19}$$

is a linear programming problem. Now, assigning nonnegative dual variables  $w^o$ 's for each constraint in (19) we have the following dual LP. Note that since above LP is defined for each edge  $e = \{i, j\}$ , we indexed the dual variables with edges.

$$\begin{aligned}
& \min \sum_{o \in N} b_o w_e^o \\
& \text{s.t.} \quad w_e^o + w_e^d \geq y_{ij}^{od} + y_{ji}^{od} \quad \forall (o,d) \in D \\
& \quad \quad w_e^o \geq 0 \quad \forall o \in N.
\end{aligned}$$

Therefore, using the equivalence of the above primal-dual pair of LPs, our robust formulation can now be transformed into:

$$\min 2 \frac{C^{TE}}{M^{TE}} \sum_{e \in E} \sum_{o \in N} b_o w_e^o + \sum_{e \in E} (C^R G_e^R c_e + C^A G_e^A f_e) \tag{20}$$

$$\text{s.t.} \quad \sum_{j:\{i,j\} \in E} (y_{ij}^{od} - y_{ji}^{od}) = \begin{cases} 1 & \text{if } i = o \\ -1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in N, (o,d) \in D \tag{21}$$

$$w_e^o + w_e^d \geq y_{ij}^{od} + y_{ji}^{od} \quad \forall (o,d) \in D, e = \{i,j\} \in E \tag{22}$$

$$\sum_{o \in N} b_o w_e^o \leq M^A f_e \quad \forall e \in E \tag{23}$$

$$\sum_{o \in N} b_o w_e^o \leq M^R c_e \quad \forall e \in E \tag{24}$$

$$f_e \leq F_e \quad \forall e \in E \tag{25}$$

$$f_e, c_e \text{ integer} \quad \forall e \in E \tag{26}$$

$$\text{all variables are nonnegative.} \tag{27}$$

In the next subsection we shall attempt to strengthen the above formulation with the aid of valid inequalities.

### 3.1 Valid Inequalities for the Hose Uncertainty Model

Let  $P_H$  be the convex hull of the feasible set, i.e., the convex hull of the set defined by the constraints (21)–(27). We present valid inequalities for  $P_H$ . The first family of inequalities is a modification of the well known cutset inequalities (see e.g. [2, 11, 12]). For any set of nodes  $S \subset N$ , let  $[S : N \setminus S]$  be the set of edges with exactly one endpoint in  $S$ . Such a set  $[S : N \setminus S]$  is called a cut. Cutset inequalities imply that the capacity installed on a cut should be sufficient to route the demand across the cut. The difference in our problem is that we do not know the exact demand across a cut. However, we know the maximum value that this demand can attain.

**Proposition 1** *For any given cut  $[S : N \setminus S]$ , inequalities*

$$\sum_{e \in [S : N \setminus S]} f_e \geq \left\lceil \frac{\min\{\sum_{o \in S} b_o, \sum_{o \in N \setminus S} b_o\}}{M^A} \right\rceil$$

and

$$\sum_{e \in [S : N \setminus S]} c_e \geq \left\lceil \frac{\min\{\sum_{o \in S} b_o, \sum_{o \in N \setminus S} b_o\}}{M^R} \right\rceil$$

are valid for  $P_H$ .

*Proof.* The maximum amount of traffic between sets  $S$  and  $N \setminus S$  is

$$\min\left\{\sum_{o \in S} b_o, \sum_{o \in N \setminus S} b_o\right\}.$$

In the worst case, this cut should be able to support full demand/supply of set  $S$  or  $N \setminus S$ , whichever is minimum. The above inequalities simply state that the amount of capacities on edges in  $[S : N \setminus S]$  should be sufficient to support this traffic.  $\square$

For a given edge  $e \in E$ , consider the polyhedron

$$P_H^e = \text{conv}\left(\{(w_e, f_e) \in [0, 1]^{|N|} \times \mathbb{Z}_+ : \sum_{o \in N} \frac{b_o}{M^A} w_e^o \leq f_e\}\right).$$

Magnanti *et al.* [11] derive valid inequalities called “residual capacity inequalities” for  $P_H^e$ . For  $S \subseteq N$ , the residual capacity inequality is as follows:

$$\sum_{o \in S} \frac{b_o}{M^A} (1 - w_e^o) \geq \left(\frac{\sum_{o \in S} b_o}{M^A} - \left\lfloor \frac{\sum_{o \in S} b_o}{M^A} \right\rfloor\right) \left(\left\lceil \frac{\sum_{o \in S} b_o}{M^A} \right\rceil - f_e\right).$$

Magnanti *et al.* [11] show that linear constraints of  $P_H^e$  and residual capacity inequalities give the description of  $P_H^e$ .

In our formulation,  $w_e^o$  is not forced to be in  $[0, 1]$ . But at optimality, we know that  $y_{ij}^{od} + y_{ji}^{od} \leq 1$  for all  $(o, d) \in D$  and  $e = \{i, j\} \in E$ . Thus we can bound  $w_e^o$  by 1 from above for all  $o \in N$  and  $e \in E$  without losing optimality. As a result, any valid inequality for  $P_H^e$  can be used in solving the DWDM routing and provisioning problem.

Residual capacity inequalities for  $P_H$  are given in the following proposition.

**Proposition 2** *There exists an optimal solution to problem (20)–(27) satisfying*

$$\sum_{o \in S} \frac{b_o}{M^A} (1 - w_e^o) \geq \left( \frac{\sum_{o \in S} b_o}{M^A} - \left\lfloor \frac{\sum_{o \in S} b_o}{M^A} \right\rfloor \right) \left( \left\lceil \frac{\sum_{o \in S} b_o}{M^A} \right\rceil - f_e \right)$$

and

$$\sum_{o \in S} \frac{b_o}{M^R} (1 - w_e^o) \geq \left( \frac{\sum_{o \in S} b_o}{M^R} - \left\lfloor \frac{\sum_{o \in S} b_o}{M^R} \right\rfloor \right) \left( \left\lceil \frac{\sum_{o \in S} b_o}{M^R} \right\rceil - c_e \right)$$

for all  $S \subseteq N$  and  $e \in E$ .

In Section 5, we provide experimental results that demonstrate the efficiency of the valid inequalities studied.

## 4 The Robust Model under Restricted Interval Demand Uncertainty

We now turn our efforts to another model of polyhedral uncertainty. With interval demand uncertainty, we assure that the traffic demand of commodity  $(o, d) \in D$  takes a value in the interval  $[\hat{R}_{od}, \hat{R}_{od} + R'_{od}]$ . Here  $\hat{R}_{od}$  is the most likely demand value for commodity  $(o, d) \in D$ . One may ask to find a design that is feasible for any possible demand scenario in these intervals. Such a design can be computed by solving the deterministic problem for  $R_{od} = \hat{R}_{od} + R'_{od}$  for all  $(o, d) \in D$ . This is a very conservative approach and it is clear that it leads to high cost solutions. Bertsimas and Sim[4, 5] propose another robustness concept for interval uncertainty: instead of protecting ourselves against the scenario where all demand values are at their upper bounds, we limit ourselves to scenarios where at most  $0 \leq \Gamma \leq |D|$  commodities have demand at their upper bounds. If  $\Gamma = 0$ , we solve the deterministic problem for  $R_{od} = \hat{R}_{od}$  for all  $(o, d) \in D$ . This solution

may be infeasible even when one commodity has a demand larger than the most likely estimate. As  $\Gamma$  grows, both protection level and cost increase. For  $\Gamma = |D|$ , we obtain the most conservative solution. A similar approach is adopted for the Virtual Private Network design problem in [1].

For a given  $\Gamma$ , the robust DWDM routing and provisioning problem can be modeled as follows:

$$\begin{aligned}
\min & 2 \frac{C^{TE}}{M^{TE}} \sum_{\{i,j\} \in E} \left( \sum_{(o,d) \in D} \hat{R}_{od}(y_{ij}^{od} + y_{ji}^{od}) + \max_{D' \subseteq D: |D'| \leq \Gamma} \sum_{(o,d) \in D'} R'_{od}(y_{ij}^{od} + y_{ji}^{od}) \right) \\
& + \sum_{e \in E} (C^R G_e^R c_e + C^A G_e^A f_e) \\
\text{s.t.} & \sum_{j: \{i,j\} \in E} (y_{ij}^{od} - y_{ji}^{od}) = \begin{cases} 1 & \text{if } i = o \\ -1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in N, (o, d) \in D \\
& \sum_{(o,d) \in D} \hat{R}_{od}(y_{ij}^{od} + y_{ji}^{od}) + \max_{D' \subseteq D: |D'| \leq \Gamma} \sum_{(o,d) \in D'} R'_{od}(y_{ij}^{od} + y_{ji}^{od}) \leq M^A f_e \quad \forall e = \{i, j\} \in E \\
& \sum_{(o,d) \in D} \hat{R}_{od}(y_{ij}^{od} + y_{ji}^{od}) + \max_{D' \subseteq D: |D'| \leq \Gamma} \sum_{(o,d) \in D'} R'_{od}(y_{ij}^{od} + y_{ji}^{od}) \leq M^R c_e \quad \forall e = \{i, j\} \in E \\
& f_e \leq F_e \quad \forall e \in E \\
& f_e, c_e \text{ integer} \quad \forall e \in E \\
& \text{all variables are nonnegative.}
\end{aligned}$$

For  $e = \{i, j\} \in E$ , let  $\mu_e = \max_{D' \subseteq D: |D'| \leq \Gamma} \sum_{(o,d) \in D} R'_{od}(y_{ij}^{od} + y_{ji}^{od})$ . For a given  $y$  vector, this problem can be written as a linear integer programming problem:

$$\begin{aligned}
\mu_e &= \max \sum_{(o,d) \in D} R'_{od}(y_{ij}^{od} + y_{ji}^{od}) z_{od} \\
\text{s.t.} & \sum_{(o,d) \in D} z_{od} \leq \Gamma \\
& z_{od} \in \{0, 1\} \quad \forall (o, d) \in D.
\end{aligned}$$

It is easy to see that this problem can be solved by changing the binary requirements to  $0 \leq z_{od} \leq 1$  for all  $(o, d) \in D$ . So  $\mu_e$  can be computed by solving the corresponding dual which is the following minimization problem.

$$\begin{aligned}
\mu_e &= \min \Gamma \alpha_e + \sum_{(o,d) \in D} \beta_e^{od} \\
\text{s.t. } \alpha_e + \beta_e^{od} &\geq R'_{od}(y_{ij}^{od} + y_{ji}^{od}) \quad \forall (o,d) \in D \\
\alpha_e &\geq 0 \\
\beta_e^{od} &\geq 0 \quad \forall (o,d) \in D.
\end{aligned}$$

Now, appending the above problem in the original formulation, we obtain a mixed integer programming formulation for the robust DWDM routing and provisioning problem with restricted interval uncertainty:

$$\begin{aligned}
\min & 2 \frac{C^{TE}}{M^{TE}} \sum_{\{i,j\} \in E} \sum_{(o,d) \in D} \hat{R}_{od}(y_{ij}^{od} + y_{ji}^{od}) \\
& + \sum_{e \in E} (C^R G_e^R c_e + C^A G_e^A f_e + 2 \frac{C^{TE}}{M^{TE}} (\Gamma \alpha_e + \sum_{(o,d) \in D} \beta_e^{od})) \quad (28)
\end{aligned}$$

$$\text{s.t. } \sum_{j: \{i,j\} \in E} (y_{ij}^{od} - y_{ji}^{od}) = \begin{cases} 1 & \text{if } i = o \\ -1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in N, (o,d) \in D \quad (29)$$

$$\alpha_e + \beta_e^{od} \geq R'_{od}(y_{ij}^{od} + y_{ji}^{od}) \quad \forall e = \{i,j\} \in E, (o,d) \in D \quad (30)$$

$$\sum_{(o,d) \in D} \hat{R}_{od}(y_{ij}^{od} + y_{ji}^{od}) + \Gamma \alpha_e + \sum_{(o,d) \in D} \beta_e^{od} \leq M^A f_e \quad \forall e = \{i,j\} \in E \quad (31)$$

$$\sum_{(o,d) \in D} \hat{R}_{od}(y_{ij}^{od} + y_{ji}^{od}) + \Gamma \alpha_e + \sum_{(o,d) \in D} \beta_e^{od} \leq M^R c_e \quad \forall e = \{i,j\} \in E \quad (32)$$

$$f_e \leq F_e \quad \forall e \in E \quad (33)$$

$$f_e, c_e \text{ integer} \quad \forall e \in E \quad (34)$$

$$\text{all variables are nonnegative.} \quad (35)$$

## 4.1 Valid Inequalities for the Interval Uncertainty Model

As with our polyhedral uncertainty model, we shall adapt cutset inequalities to our model with interval uncertainty. Let  $P_I$  be the convex hull of feasible solutions to problem (28)-(35).

**Proposition 3** *For  $S \subset N$ , let  $[S : N \setminus S]$  be the set of edges with exactly one endpoint in  $S$ . Define  $D(S)$  to be the set of commodities with either the origin or the destination in  $S$  but not both.*

*Inequalities*

$$\sum_{e \in [S:N \setminus S]} f_e \geq \left\lceil \frac{\sum_{(o,d) \in D(S)} \hat{R}_{od} + \max_{D' \subseteq D(S): |D'| \leq \Gamma} \sum_{(o,d) \in D'} R'_{od}}{M^A} \right\rceil$$

and

$$\sum_{e \in [S:N \setminus S]} c_e \geq \left\lceil \frac{\sum_{(o,d) \in D(S)} \hat{R}_{od} + \max_{D' \subseteq D(S): |D'| \leq \Gamma} \sum_{(o,d) \in D'} R'_{od}}{M^R} \right\rceil$$

are valid for  $P_I$ .

*Proof.* The commodities in set  $D(S)$  should cross the cut  $[S : N \setminus S]$ . In the worst case  $\min\{|D(S)|, \Gamma\}$  commodities with maximum demand have their demand at upper bounds. Hence, the demand to cross the cut is

$$\sum_{(o,d) \in D(S)} \hat{R}_{od} + \max_{D' \subseteq D(S): |D'| \leq \Gamma} \sum_{(o,d) \in D'} R'_{od}.$$

Above inequalities imply that the capacities installed on the cut should support this traffic.  $\square$

We present our computational experience with these valid inequalities in Section 6.

## 5 Computational Study with the Hose Uncertainty Model

In our experimentation with the hose model, we borrowed the two datasets of Kennington *et al.* The DA data file is a US network composed of 68 nodes, 107 edges and 200 commodities. The KL data file represents a European network of 18 nodes, 35 edges and 100 commodities. Kennington *et al.* use five scenarios in testing their path-based multicriteria robust model.

All tests reported below are done on a 1.1 GHz 1256 MB RAM Pentium III using the MIP solver of CPLEX 9.0.

For our first set of computational tests, we evaluated the quality of our solution according to a regret criterion similar to the one used in [10]. The regret function is a piecewise linear function which computes a cost of under provisioning (depicted in Figure 1).

Let  $(\hat{y}, \hat{w}, \hat{f}, \hat{c})$  be an optimal solution to our model. Let  $S$  denote the set of scenarios,  $p_s$  denote the probability of scenario  $s \in S$ ,  $R_{od}^s$  denote the demand of

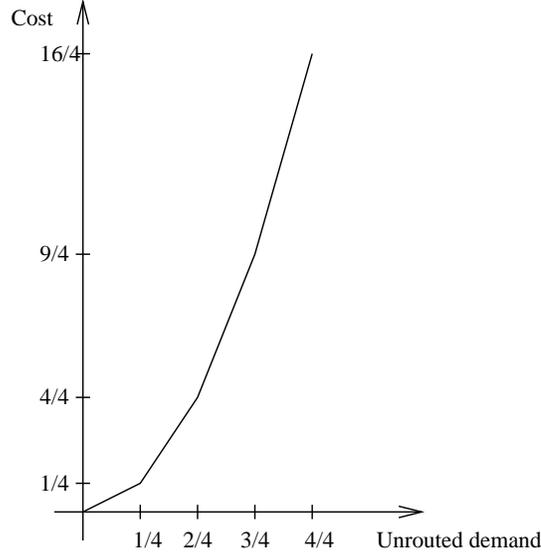


Figure 1: Under provisioning cost

commodity  $(o, d)$  in scenario  $s$  and  $R_{max} = \max_{s \in S} \max_{(o,d) \in D} R_{od}^s$ . For scenario  $s$ , we compute the piecewise linear regret function value  $r(s)$  by solving the following simple linear program:

$$r(s) = \min \sum_{(o,d) \in D} \left( \frac{R_{max}}{4} Z_{od}^1 + \frac{3R_{max}}{4} Z_{od}^2 + \frac{5R_{max}}{4} Z_{od}^3 + \frac{7R_{max}}{4} Z_{od}^4 \right) \quad (36)$$

$$\text{s.t.} \quad \sum_{(o,d) \in D} R_{od} (\hat{y}_{ij}^{od} + \hat{y}_{ji}^{od}) \leq \sum_{o \in N} b_o \hat{w}_e^o \quad \forall e = \{i, j\} \in E \quad (37)$$

$$R_{od}^s - R_{od} = Z_{od}^1 + Z_{od}^2 + Z_{od}^3 + Z_{od}^4 \quad \forall (o, d) \in D \quad (38)$$

$$0 \leq Z_{od}^k \leq \frac{R_{max}}{4} \quad \forall (o, d) \in D, k = 1, 2, 3, 4 \quad (39)$$

where  $R_{od}$  and  $Z_{od}^k$  for  $k = 1, 2, 3, 4$  and  $(o, d) \in D$  are the decision variables, representing the demand of commodity  $(o, d)$  routed and the auxiliary variables representing the level of usage of each branch of the piecewise linear regret function. Constraint (37) bounds the amount of traffic that can be routed on an edge by the amount of TE's on the edge (see the projection argument in Section 2).

The total regret is equal to  $\sum_{s \in S} p_s r(s)$ . Note that in our approaches, we do not have the over-provisioning cost present in [10] since our model decides on the fraction of flow on each arc for each commodity rather than the amount of demand routed along each path. Thus, in any scenario we do not route more than the actual demand.

We successfully solved our flow model for the KL network till optimality with

CPLEX 9.0 using its default settings. For the DA network which is considerably larger, we stopped the computation when the solver reached Kennington *et al.*'s 2% optimality gap tolerance.

To compare the regret of our solution with that of [10], we input the objective value of our solution as a budget constraint and solved the model of Kennington *et al.* using the AMPL files provided by the authors. For the KL network, Kennington *et al.*'s regret function value turned out to be 1.62 times the value of the regret function evaluated at our robust solution. As for the DA network, this ratio was 1.21.

Kennington *et al.* use three budget categories, low, medium and high budgets. The costs of our designs in KL and DA networks were \$ 1,838,068,672 and \$ 5,249,994,514 respectively. Both results fall between the medium and high budget categories of [10].

In our second phase of experimentation, we show that our robust design displays a stable behavior with respect to total design cost in the face of different demand scenarios. In this experiment, we use KL data for ease of computation. We randomly generate demand scenarios from our uncertainty polyhedron so as to make the inequalities  $\sum_{o:(o,i)\in D} R_{oi} + \sum_{d:(i,d)\in D} R_{id} \leq b_i$  tight for all  $i \in N$ . Then, for each demand scenario, we solve the deterministic optimization problem (1)–(12), and compute the optimal design cost for that scenario as well as its deviation from the robust total design cost. The difference between the robust design cost and the scenario optimal design cost varies in the range from \$  $2.84 \times 10^8$  to \$  $4.37 \times 10^8$  over 100 scenario runs. Its average value is \$  $3.73 \times 10^8$  which corresponds to 25 percent of the average of scenario optimal costs. Therefore we can say that the price to pay for robustness in our case is about 25 percent increase in design cost.

In our final experimental design, we test the usefulness of the valid inequalities presented in Section 4. To this end, we solved our model for KL and four other test problems from the literature with and without the inequalities. We borrowed two instances from Bienstock and Gunluk [7]. These are named as “Data1” and “Data2”. The other two instances are from Holmberg and Yuan [14, 15]. Their original names are “cexd3” and “cexd4” and we refer to them as “Data3” and “Data4” here.

In Data3 and Data4, the underlying graphs are directed. We make these graphs undirected by keeping one arc per edge arbitrarily. We take the data of

linear cost coefficients as distances.

Using these base data, we obtain the underlying graph, the distance matrix and the demand matrix. We use the cost and capacity data of KL to complete these into full data for the DWDM routing and provisioning problem. To make the distances and demands compatible with the cost and capacity data, we multiply each demand value by a coefficient  $\sigma_d$  and each distance by a coefficient  $\sigma_e$  (reported in Table 1). Then, we take  $b_o$  to be the sum of traffic of commodities incident at node  $o$ .

In Table 1 below, we summarize the properties of these problem instances. For each data, we report the numbers of nodes, edges and commodities and the parameters  $\sigma_d$  and  $\sigma_e$ .

Name	No. of Nodes	No. of Edges	No. of Commodities	$\sigma_d$	$\sigma_e$
KL	18	35	100	1	1
Data1	15	22	210	10000	5
Data2	16	49	240	10000	1
Data3	33	74	150	100	4
Data4	33	72	400	100	4

Table 1: Characteristics of problem instances

In Table 2, Model 1 is our original formulation and Model 2 contains cutset inequalities of size 1 and 2 and residual capacity inequalities for all subsets of size 1. For each model, we list the percentage gap of the LP relaxation from the optimal value (% gap column), the number of nodes in the branch and cut tree of CPLEX (nodes column), and the CPU seconds (cpu column). For the largest instance, Data4, we have an optimality tolerance of 0.05%.

As can be observed from the table, for KL data, Model 2 shows a better performance in all respects. There is an improvement of 63.27 % in the percentage gap, 31.94 % in the number of nodes and 66.18 % in the CPU time. Model 2 is also significantly better for Data1, Data3 and Data4. However, for Data2, the cpu time to solve Model 2 is almost two times the cpu to solve Model 1. For this instance, the inequalities we added could not result in a significant decrease in the size of the branch and cut tree. As they increase the size of the LP, the solution time increases. For Data4, Model 1 has a final gap of 0.05 % and Model 2 has a final gap of 0.0422%. The best MIP solution is found by Model 2 and is used to compute the percentage LP gaps. For this instance, our inequalities improved

significantly the computation time. On the average, there is an improvement of 70.5% in the percentage gap, 52.21% in the number of nodes and 30.57% in the cpu time.

		Model 1			Model 2		
Data	Optimal Value	% gap	nodes	cpu	% gap	nodes	cpu
KL	1838068671.8750	1.47	1988	2033.34	0.54	1353	687.72
Data1	8333169205.0965	0.55	1043	4126.89	0.46	691	2897.84
Data2	235730433.1771	8.68	488	7022.72	1.58	404	13876.11
Data3	1705616249.9990	1.76	43300	209716.03	0.07	8342	58073.42
Data4	4545062500.0000	0.86	1162	30035.52	0.05	30	5352.09

Table 2: Effects of Valid Inequalities - Hose Model

## 6 Computational Study with the Restricted Interval Uncertainty Model

We have done two experiments with the interval uncertainty model. In the first experiment, we want to observe how the cost of the robust solution changes as  $\Gamma$  changes. We let  $\Gamma$  take values 0, 5, 10,  $\dots$ , 50. For each value of  $\Gamma$ , we find a robust solution and report its cost.

We use data set KL for this experiment. We let the demand of a commodity vary between its values in scenarios 2 and 4 of [10]. The reason for this choice is to remove the most optimistic and pessimistic guesses and to work on the most probable ones.

We also compute the regret of the robust solution using the regret function given in Section 5. The linear program to compute the regret is adapted to the interval model as follows. Let  $(\hat{y}, \hat{\alpha}, \hat{\beta}, \hat{f}, \hat{c})$  be an optimal solution to our model. We solve the following linear program for each scenario  $s \in S$ .

$$\begin{aligned}
r(s) = \min & \sum_{(o,d) \in D} \left( \frac{R_{max}}{4} Z_{od}^1 + \frac{3R_{max}}{4} Z_{od}^2 + \frac{5R_{max}}{4} Z_{od}^3 + \frac{7R_{max}}{4} Z_{od}^4 \right) \\
\text{s.t.} & \sum_{(o,d) \in D} R_{od}(\hat{y}_{ij}^{od} + \hat{y}_{ji}^{od}) \leq \sum_{(o,d) \in D} R_{od}(\hat{y}_{ij}^{od} + \hat{y}_{ji}^{od}) + \Gamma \hat{\alpha}_e + \sum_{(o,d) \in D} \hat{\beta}_e^{od} \\
& \forall e = \{i, j\} \in E \\
& R_{od}^s - R_{od} = Z_{od}^1 + Z_{od}^2 + Z_{od}^3 + Z_{od}^4 \quad \forall (o, d) \in D \\
& 0 \leq Z_{od}^k \leq \frac{R_{max}}{4} \quad \forall (o, d) \in D, k = 1, 2, 3, 4.
\end{aligned}$$

Let  $Cost_\Gamma$  be the cost of the robust solution and  $Regret_\Gamma$  be the regret for given  $\Gamma$ . In Table 3, for each  $\Gamma$  value, we report the cost of the associated robust solution  $Cost_\Gamma$ , the percentage increase in the cost with respect to the cost of the solution for the previous  $\Gamma$  value, i.e.,  $\frac{Cost_\Gamma - Cost_{\Gamma-5}}{Cost_{\Gamma-5}} * 100$  (in column “% increase in cost”), the regret of the robust solution  $Regret_\Gamma$  and the percentage decrease in the regret with respect to the regret of the solution for the previous  $\Gamma$  value, i.e.,  $\frac{Regret_{\Gamma-5} - Regret_\Gamma}{Regret_{\Gamma-5}} * 100$  (in column “% decrease in regret”).

$\Gamma$	$Cost_\Gamma$	% increase in cost	regret	% decrease in regret
0	1254235313	-	442958800	-
5	1557527745	24.18	213227900	51.86
10	1710095259	9.80	134629000	36.86
15	1783628327	4.30	82258880	38.90
20	1806429175	1.28	58589290	28.77
25	1816218796	0.54	47637720	18.69
30	1816408549	0.01	47784790	-0.31
35	1816414064	0.00	47789400	-0.01
40	1816418254	0.00	47782290	0.01
45	1816421458	0.00	47225860	1.16
50	1816421458	0.00	47673520	-0.95

Table 3: Cost and regret behavior with respect to  $\Gamma$

The case with  $\Gamma = 0$  corresponds to the deterministic problem. Letting only 5 commodities have demand at their upper bounds results in an increase of 24.18% in the cost, but a decrease of 51.86% in the regret. When  $\Gamma$  goes from 5 to 10, both the percentage increase in cost and the percentage decrease in regret decrease. After  $\Gamma = 25$ , cost and regret do not change much. So by protecting ourselves against the bad performance of at most 25 demand values out of 100, we have almost full protection. The solution for  $\Gamma = 25$  is about 45% more expensive than the one for  $\Gamma = 0$ .

Notice that all these cost figures are lower than the cost of the robust optimal solution obtained for KL under hose model in Section 5. For  $\Gamma \geq 15$ , the regret of the corresponding robust solution is less than the one of the robust solution under hose model.

In the second experiment, we test the use of valid inequalities in solving our test instances. We solved our model for KL and the four data sets introduced

in the previous section with and without valid inequalities. For KL, we let the demand of a commodity vary between its demands in scenarios 2 and 4. For other instances, we take the demand of the commodity as the most likely estimate and set the upper bound to the smallest integer greater than or equal to 1.25 times this demand. We take  $\Gamma = \lceil 0.5|D| \rceil$ . For the largest instance, Data4, we have an optimality tolerance of 0.05%.

In Table 4, Model 1 refers to the initial formulation. In Model 2, we use all cutset inequalities of size 1 and 2. For each data set and model, we report the percentage LP gap, the number of nodes in the branch and cut tree and the CPU time in seconds.

		Model 1			Model 2		
Data	Optimal Value	% gap	nodes	cpu	% gap	nodes	cpu
KL	1816421458.3333	1.62	1328930	135418.22	0.99	31974	5591.98
Data1	5378401439.3229	0.87	7411	3786.13	0.56	1957	1764.63
Data2	229148941.3542	18.28	1696	4628.68	15.09	1436	6188.41
Data3	1663070885.4167	1.29	12265	2456.26	0.96	13144	2531.98
Data4	4692972916.6667	0.74	22341	23232.70	0.54	7330	7133.68

Table 4: Effects of Valid Inequalities - Interval Model

For the KL data set, we observe that there is an improvement of 38.94% in the percentage gap, 97.59% in the number of nodes and 95.87% in the CPU time. Model 2 performs worse for Data2 and Data3, but both of these instances are solved in less than 2 hours with Model 2. For Data 4, with Model 2, the optimality tolerance is reached within 30.7% of cpu seconds and 32.81% of number of nodes of Model 1.

The average improvements in percentage gap, number of nodes and CPU time are 29.02%, 49.31% and 36.36%, respectively.

## 7 Concluding Remarks

We proposed compact linear integer programming models for design of DWDM networks under traffic uncertainty that we represented using the so-called “hose model” and “restricted interval model”. We investigated the numerical solution of the two problems using two instances from [10] and other instances. Some valid inequalities that proved effective in reducing the initial duality gap were

derived and added to the LP relaxation. On the average, the cpu time is reduced by about its one third for both uncertainty models. As for the number of nodes, the improvement was about 50%.

For the KL instance, we compared our design costs to the robust design cost of [10], in particular, by calculating the regret values of our robust solution using a regret function similar to Kennington *et al.*'s regret function. It turned out that our robust optimal design cost in the hose model is in between the high and medium budgets of [10], while it resulted in lower regret values. In the interval model, design costs turned out to be in general lower compared to the hose model. The regret values also show the same behavior already for small  $\Gamma$ .

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