

# Linear Stochastic Fractional Programming with Sum-of-Probabilistic-Fractional Objective

V.Charles<sup>1</sup> & D.Dutta<sup>2</sup>

<sup>1</sup>Department of Management Science  
SDM Institute for Management Development  
Mysore, Karnataka, INDIA – 570 011.

<sup>2</sup>Department of Mathematics & Humanities  
National Institute of Technology Warangal  
Andhra Pradesh, INDIA-506 004.  
Email: v\_chals@yahoo.com & ddutta@nitw.ernet.in

## Abstract

Fractional programming deals with the optimization of one or several ratios of functions subject to constraints. Most of these optimization problems are not convex while some of them are still generalised convex. After about forty years of research, well over one thousand articles have appeared on applications, theory and solution methods for various types of fractional programs. The present article focuses on the stochastic sum-of-probabilistic-fractional program, which was one of the least researched fractional program until nineteenth century. We propose an interactive conversion technique with the help of deterministic parameter, which converts the sum-of-probabilistic-fractional objective into stochastic constraint. Then the problem reduces to stochastic programming with linear objective of sum-of-deterministic parameters. The reduced problem has been solved and illustrated with numerical examples.

Keywords: Stochastic Programming, Fractional Programming, Sum-of-Probabilistic, Quadratic Fractional Programming.

**AMS Classification:** 90C15, 90C32.

## **1. Introduction**

Over the past four decades, fractional programming has become one of the planning tools. It is routinely applied in engineering, business, finance, economics and other disciplines. This acceptance may be due to (1) good algorithms, (2) practitioners' understanding of the power and scope of fractional programming. Furthermore, research on specialized problems, such as max-min fractional programming, fractional transportation programming, quadratic fractional programming and bi-level fractional programming problems has made fractional programming methodology indispensable in many industries including airlines, energy, manufacturing and telecommunications. Notwithstanding its successes, however, the assumption that all model parameters are known with certainty limits its usefulness in planning under uncertainty. When one or more of the data elements in a fractional program is represented by a random variable, a stochastic program results, which is defined as stochastic fractional programming problem [7-11].

Stochastic programming is a powerful analytical approach that permits the solving of models under uncertainty. Yet, it is not widely used, mainly because practitioners still find it difficult to handle and implement it. The application of stochastic programming in manpower planning can be seen in [15,16] and the basic concepts and stochastic modeling is given in Sen [27]. In this paper we consider a particular type of stochastic fractional programming problem where the objective function is given as a sum-of-probabilistic-fractional random function, which can be converted into stochastic constraint and then to deterministic constraint. We also focus on obtaining solutions for

problems with continuous random variables influencing not only the fractional objective functions but also the constraints. That is, we consider that the feasible set of problem is deterministic or that it has been transformed into its equivalent deterministic, requiring that the constraints hold with, at least, a given probability. Moreover, we focus on problems in which the random variables appearing in the problem are continuous and it follows a well-known distribution, in our case we have considered it as normal distribution.

The rest of the paper proceeds as follows. Section 2 deals with stochastic sum-of-fractional programming problem. Section 3 deals with conversion of stochastic constraints into deterministic constraints. Section 4 provides the conversion technique that helps us to convert stochastic sum-of-fractional objective functions into stochastic constraints and section 5 gives deterministic version. Section 6 numerical examples are provided for the sake of illustration and conclusion has been drawn at the end.

## **2. Stochastic Sum-of-Fractional Programming Problem**

The problem of optimizing sum of more than one ratios of function is called stochastic sum-of-probabilistic-fractional programming problem when the data under study are random in nature.

A Stochastic Sum-of-probabilistic-fractional programming problem in a so-called criterion space can be defined as follows

$$\text{Max}_{X \in S} R(X) = \sum_{y=1}^k R_y(X) \text{ where } R_y(X) = \frac{N_y(X) + \alpha_y}{D_y(X) + \beta_y}, \quad y = 1, 2, \dots, k \quad (2.1)$$

Subject to

$$\text{Pr}\left[\sum_{j=1}^n t_{ij}x_j \leq b_i^{(1)}\right] \geq 1 - p_i^{(1)} \quad i = 1, 2, \dots, m \quad (2.2)$$

$$\sum_{j=1}^n t_{ij}x_j \leq b_i^{(2)} \quad i = m+1, \dots, h \quad (2.3)$$

where  $0 \leq X_{n \times 1} = \|x_j\| \subset \mathbb{R}^n$  is a feasible set and  $R : \mathbb{R}^n \rightarrow \mathbb{R}^k$ ,  $T_{m \times n} = \|t_{ij}^{(1)}\|$ ,  $b^{(1)}_{m \times 1} = \|b_i^{(1)}\|$ ,  $i = 1, 2, \dots, m$ ;  $j = 1, 2, \dots, n$ ;  $b^{(2)}_{h-(m+1) \times 1} = \|b_i^{(2)}\|$ ,  $i = m+1..h$ ;  $\alpha_y, \beta_y$  are

scalars.  $N_y(X) = \sum_{j=1}^n c_{yj}x_j$  and  $D_y(X) = \sum_{j=1}^n d_{yj}x_j$ .

In this model, out of  $N_y(X)$ ,  $D_y(X)$ ,  $T$  and  $b^{(1)}$  atleast one may be a random variable.  $S = \{X \mid \text{Equation (2.2-2.3)}, X \geq 0, X \subset \mathbb{R}^n\}$  is non-empty, convex and compact set in  $\mathbb{R}^n$ .

**Assumptions:** 1.  $N_y(X) + \alpha_y : \mathbb{R}^n \rightarrow \mathbb{R}$  are nonnegative concave and

$D_y(X) + \beta_y : \mathbb{R}^n \rightarrow \mathbb{R}$  are positive, convex for each  $y$ .

2. Underlying random variables follow normal distribution.

Model (2.1) arises naturally in decision making when several rates are to be optimized simultaneously. In light of the applications of single-ratio fractional programming [22-25] numerators and denominators may be representing output, input, profit, cost, capital, risk or time.

Almogly and Levin [1] analyze a multistage stochastic shipping problem. A deterministic equivalent of this stochastic problem is formulated which turns out to be a sum-of-ratios

problem. Another presentation of this application can be seen in [14]. Benson's algorithms in [3-5] are based on branch-and-bound approaches. Kuno [17] also proposes a branch-and-bound method for deterministic case of problem (2.1) and Kuno's algorithms works well for nonlinear case too. Bykadorov [6] has studied generalized concavity properties of sum-of-linear-ratios and, more generally, of sum-of-ratios of polynomials.

Rao [21] discusses various models in cluster analysis. The problem of optimal partitioning of a given set of entities into a number of mutually exclusive and exhaustive groups (clusters) gives rise to various mathematical programming problems depending on which optimality criterion is used. If the objective is to minimize the sum of the average squared distances within groups, then a minimum of a sum-of-ratios is to be determined.

More recently other applications of the sum-of-ratios problem have been identified. Mathis and Mathis [20] formulated a hospital fee optimization problem in this way. The model is used by hospital administrators in the State of Texas to decide on relative increases of charges for different medical procedures in various departments. According to [12] a number of geometric optimization problems give rise to the sum-of-ratios problem. These often occur in layered manufacturing [18,19,26], for instance in material layout and clothe manufacturing. For various examples we refer to the survey by Chen et al. [12] and the references therein. Quite in contrast to other applications of the sum-of-ratios problems mentioned before, the number of variables is very small (one, two or

three), but the number of ratios is large; often there are hundreds or even thousands of ratios involved.

### 3. Deterministic Equivalents of Probabilistic Constraints

Let  $T$  be a random variable in (2.2) and it follows  $N(u_{ij}, s_{ij}^2)$ ,  $i = 1, 2, \dots, m$ ;  $j = 1, 2, \dots, n$ ,

where  $u_{ij}$  and  $s_{ij}^2$  are the mean and variance respectively. Let  $l_i = \sum_{j=1}^n t_{ij} x_j$ ,  $i = 1, 2, \dots, m$ .

$E(l_i) = \sum_{j=1}^n u_{ij} x_j$ ;  $V(l_i) = X^T V_i X = \sum_{j=1}^n s_{ij}^2 x_j^2$ , where  $V_i$  is the  $i^{\text{th}}$  covariance matrix. When

$T$  is an independent random variable then covariance terms becomes zero in covariance matrix. The  $i^{\text{th}}$  deterministic constraint for (2.2) is obtained from [7-11] as

$$\sum_{j=1}^n u_{ij} x_j + Kq_i \sqrt{\left( \sum_{j=1}^n s_{ij}^2 x_j^2 \right)} \leq b_i \quad (3.1)$$

where  $Kq_i$  is the cumulative distribution function of standard normal distribution.

If  $b$  is a random variable in (2.2) i.e.  $b_i \sim N(u_{bi}, s_{bi}^2)$ ,  $i = 1, 2, \dots, m$ , where  $u_{bi}$  and  $s_{bi}^2$  are the mean and variance respectively. The  $i^{\text{th}}$  deterministic constraint for (2.2) is obtained from [7-11] as

$$\sum_{j=1}^n t_{ij} x_j \leq u_{bi} + Kp_i s_{bi} \quad (3.2)$$

where  $Kp_i$  is the cumulative distribution function of standard normal distribution.

Suppose  $T$  &  $b$  are random variables in (2.2) i.e.  $T \sim N(u_{ij}, s_{ij}^2)$  and  $b_i \sim N(u_{bi}, s_{bi}^2)$ ,  
 $i = 1, 2, \dots, m ; j = 1, 2, \dots, n$ , where  $u_{ij}, u_{bi}$  are means and  $s_{ij}^2, s_{bi}^2$  are variances respectively.

The  $i^{\text{th}}$  deterministic constraint for (2.2) is obtained from [7-11] as

$$\sum_{j=1}^n u_{ij}x_j - \text{Kp}_i \sqrt{\sum_{j=1}^{n+1} s_{ij}^2 x_j^2} \leq u_{bi} \quad (3.3)$$

where  $x_{n+1} = -1$  and  $\text{Kp}_i$  is the cumulative distribution function of standard normal distribution.

#### 4. Conversion of Objective Function into Constraint

The aim of this section is to consider sum of fractional objective function in the form of constraint. The feature of our model is that it takes into account the probability distribution of the sum-of-ratio (objective) function by maximizing the lower allowable limit of the objective function under chance constraints with three cases discussed here. In [2], Almogly and Levin try to extend Dinkelbach's method [13] to sum-of-ratios problem. The algorithm is based on decoupling numerators and denominators. In this paper, we have extended Almogly and Levin method [2] to stochastic sum-of- fractional problem.

Let us define the deterministic unknown parameter  $\lambda_y$ , which is less than or equal to  $R_y(X)$ . That is

$$R_y(X) \geq \lambda_y \text{ i.e. } \frac{N_y(X) + \alpha_y}{D_y(X) + \beta_y} \geq \lambda_y$$

$$\Rightarrow 0 \leq N_y(X) + \alpha_y - \lambda_y [D_y(X) + \beta_y] \quad (4.1)$$

This can be extended to sum of ratios case as follows:

$$\sum_{y=1}^k R_y(X) \geq \sum_{y=1}^k \lambda_y \quad \text{i.e.} \quad \sum_{y=1}^k \frac{N_y(X) + \alpha_y}{D_y(X) + \beta_y} \geq \sum_{y=1}^k \lambda_y \quad (4.2)$$

$$\Rightarrow 0 \leq \sum_{y=1}^k N_y + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y]$$

where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  is a vector of  $k$  deterministic parameters.

**Case 1:**  $N_y(X)$  consists of random variables.

**Assumption:**  $N_y(X) \sim N(\sum_{j=1}^n u_{cyj} X_j, \sum_{j=1}^n s_{cyj}^2 X_j^2)$ , where  $\sum_{j=1}^n u_{cyj} X_j$  and  $\sum_{j=1}^n s_{cyj}^2 X_j^2$  are mean and variance respectively.

There are two sub-cases in this problem

**Case 1.1:** When  $\sum_{y=1}^k \alpha_y > 0$

$$\text{Let } f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) = \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y(X) \leq \sum_{y=1}^k \alpha_y$$

$$\text{and } E[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)] = F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)$$

Then

$$\sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y^E(X) = \sum_{y=1}^k \lambda_y [\sum_{j=1}^n d_{yj} X_j + \beta_y] - \sum_{y=1}^k \sum_{j=1}^n u_{cyj} X_j \quad (4.3)$$

$$V[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)] = F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) = \sum_{y=1}^k N_y^V(X) = \sum_{y=1}^k \sum_{j=1}^n s_{cyj}^2 X_j^2 \quad (4.4)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) \leq \sum_{y=1}^k \alpha_y] \geq 1 - p^{(2)} \quad (4.5)$$

$$\Pr \left[ \frac{f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}} \leq \frac{\sum_{y=1}^k \alpha_y - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}} \right] \geq q^{(2)}$$

$$\Pr \left[ \frac{\sum_{y=1}^k N_y^E(X) - \sum_{y=1}^k N_y(X)}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \leq \frac{\sum_{y=1}^k N_y^E(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y]}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \right] \geq q^{(2)}$$

$$\phi \left[ \frac{\sum_{y=1}^k N_y^E(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y]}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \right] \geq q^{(2)}$$

$$\left[ \frac{\sum_{y=1}^k N_y^E(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y]}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \right] \geq \phi^{-1}(q^{(2)})$$

$$\sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y^E(X) + \phi^{-1}(q^{(2)}) \sqrt{\sum_{y=1}^k N_y^V(X)} \leq \sum_{y=1}^k \alpha_y$$

$$\sum_{y=1}^k \lambda_y [\sum_{j=1}^n d_{yj} x_j + \beta_y] - \sum_{y=1}^k \sum_{j=1}^n u_{cyj} x_j + \phi^{-1}(q^{(2)}) \sqrt{\sum_{y=1}^k \sum_{j=1}^n s_{cyj}^2 x_j^2} \leq \sum_{y=1}^k \alpha_y \quad (4.6)$$

**Case 1.2:** When  $\sum_{y=1}^k \alpha_y \leq 0$

$$\text{Let } f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) = \sum_{y=1}^k N_y(X) - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] \geq \sum_{y=1}^k \alpha_y$$

$$\text{and } E[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)] = F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)$$

Then

$$\sum_{y=1}^k N_y^E(X) - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] = \sum_{y=1}^k \sum_{j=1}^n u_{cyj} x_j - \sum_{y=1}^k \lambda_y [\sum_{j=1}^n d_{yj} x_j + \beta_y] \quad (4.7)$$

$$V[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)] = F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) = \sum_{y=1}^k N_y^V(X) = \sum_{y=1}^k \sum_{j=1}^n s_{cyj}^2 x_j^2 \quad (4.8)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) \geq \sum_{y=1}^k \alpha_y] \geq 1 - p^{(2)} \quad (4.9)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) \leq \sum_{y=1}^k \alpha_y] \leq p^{(2)}$$

$$\Pr \left[ \frac{f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}} \leq \frac{\sum_{y=1}^k \alpha_y - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}} \right] \leq p^{(2)}$$

$$\Pr \left[ \frac{\sum_{y=1}^k N_y(X) - \sum_{y=1}^k N_y^E(X)}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \leq \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y^E(X)}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \right] \leq p^{(2)}$$

$$\Phi \left[ \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y^E(X)}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \right] \leq p^{(2)}$$

$$\left[ \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y^E(X)}{\sqrt{\sum_{y=1}^k N_y^V(X)}} \right] \leq \Phi^{-1}(p^{(2)})$$

$$\sum_{y=1}^k N_y^E(X) - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] + \Phi^{-1}(p^{(2)}) \sqrt{\sum_{y=1}^k N_y^V(X)} \geq \sum_{y=1}^k \alpha_y$$

$$\sum_{y=1}^k \sum_{j=1}^n u_{cyj} x_j - \sum_{y=1}^k \lambda_y \left[ \sum_{j=1}^n d_{yj} x_j + \beta_y \right] + \Phi^{-1}(p^{(2)}) \sqrt{\sum_{y=1}^k \sum_{j=1}^n s_{cyj}^2 x_j^2} \geq \sum_{y=1}^k \alpha_y \quad (4.10)$$

Note that in both the cases variance part is common

**i.e.**  $F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) = F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)$ . Similarly we see for the Case1 & Case 2

also variance part will be common.

**Case 2:  $D_y(X)$  consists of random variables**

**Assumption:**  $D_y(X) \sim N(\sum_{j=1}^n u_{dyj} x_j, \sum_{j=1}^n s_{dyj}^2 x_j^2)$ , where  $\sum_{j=1}^n u_{dyj} x_j$  and  $\sum_{j=1}^n s_{dyj}^2 x_j^2$  are mean and variance respectively.

There are two sub-cases in this problem

**Case 2.1:** When  $\sum_{y=1}^k \alpha_y > 0$

Let  $f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) = \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y(X) \leq \sum_{y=1}^k \alpha_y$

and  $E[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)] = F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)$

Then

$$\sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y(X) = \sum_{y=1}^k \lambda_y [\sum_{j=1}^n \sum_{j=1}^n u_{dyj} x_j + \beta_y] - \sum_{y=1}^k \sum_{j=1}^n c_{yj} x_j \quad (4.11)$$

$$V[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)] = F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) = \sum_{y=1}^k \lambda_y^2 D_y^V(X) = \sum_{y=1}^k \lambda_y^2 \sum_{j=1}^n s_{dyj}^2 x_j^2 \quad (4.12)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) \leq \sum_{y=1}^k \alpha_y] \geq 1 - p^{(2)} \quad (4.13)$$

$$\Pr \left[ \frac{f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}} \leq \frac{\sum_{y=1}^k \alpha_y - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}} \right] \geq q^{(2)}$$

$$\Pr \left[ \frac{\sum_{y=1}^k \lambda_y [D_y(X) - D_y^E(X)]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \leq \frac{\sum_{y=1}^k N_y(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \right] \geq q^{(2)}$$

$$\phi \left[ \frac{\sum_{y=1}^k N_y(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \right] \geq q^{(2)}$$

$$\left[ \frac{\sum_{y=1}^k N_y(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \right] \geq \phi^{-1}(q^{(2)})$$

$$\sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y(X) + \phi^{-1}(q^{(2)}) \sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)} \leq \sum_{y=1}^k \alpha_y$$

$$\sum_{y=1}^k \lambda_y \left[ \sum_{j=1}^n u_{dyj} x_j + \beta_y \right] - \sum_{y=1}^k \sum_{j=1}^n c_{yj} x_j + \phi^{-1}(q^{(2)}) \sqrt{\sum_{y=1}^k \lambda_y^2 \sum_{j=1}^n s_{dyj}^2 x_j^2} \leq \sum_{y=1}^k \alpha_y \quad (4.14)$$

**Case 2.2:** When  $\sum_{y=1}^k \alpha_y \leq 0$

$$\text{Let } f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) = \sum_{y=1}^k N_y(X) - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] \geq \sum_{y=1}^k \alpha_y$$

$$\text{and } E[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)] = F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)$$

$$\sum_{y=1}^k N_y(X) - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] = \sum_{y=1}^k \sum_{j=1}^n c_{yj} x_j - \sum_{y=1}^k \lambda_y \left[ \sum_{j=1}^n u_{dyj} x_j + \beta_y \right] \quad (4.15)$$

Then

$$V[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)] = F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) = \sum_{y=1}^k \lambda_y^2 D_y^V(X) = \sum_{y=1}^k \lambda_y^2 \sum_{j=1}^n s_{dyj}^2 x_j^2 \quad (4.16)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) \geq \sum_{y=1}^k \alpha_y] \geq 1 - p^{(2)} \quad (4.17)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) \leq \sum_{y=1}^k \alpha_y] \leq p^{(2)}$$

$$\Pr \left[ \frac{f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}} \leq \frac{\sum_{y=1}^k \alpha_y - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}} \right] \leq p^{(2)}$$

$$\Pr \left[ \frac{\sum_{y=1}^k \lambda_y [D_y(X) - D_y^E(X)]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \leq \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y(X)}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \right] \leq p^{(2)}$$

$$\phi \left[ \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y(X)}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \right] \leq p^{(2)}$$

$$\left[ \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y(X)}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)}} \right] \leq \phi^{-1}(p^{(2)})$$

$$\sum_{y=1}^k N_y(X) - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] + \phi^{-1}(p^{(2)}) \sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X)} \geq \sum_{y=1}^k \alpha_y$$

$$\sum_{j=1}^n \sum_{y=1}^k c_{yj} x_j - \sum_{y=1}^k \lambda_y \left[ \sum_{j=1}^n u_{dyj} x_j + \beta_y \right] + \phi^{-1}(p^{(2)}) \sqrt{\sum_{y=1}^k \lambda_y^2 \sum_{j=1}^n s_{dyj}^2 x_j^2} \geq \sum_{y=1}^k \alpha_y \quad (4.18)$$

**Case 3:**  $N_y(X)$  and  $D_y(X)$  consist of random variables

**Assumption:**  $N_y(X) \sim N(\sum_{j=1}^n u_{cyj} x_j, \sum_{j=1}^n s_{cyj}^2 x_j^2)$  and  $D_y(X) \sim N(\sum_{j=1}^n u_{dyj} x_j, \sum_{j=1}^n s_{dyj}^2 x_j^2)$ , where

$\sum_{j=1}^n u_{cyj} x_j, \sum_{j=1}^n u_{dyj} x_j$  are means and  $\sum_{j=1}^n s_{cyj}^2 x_j^2, \sum_{j=1}^n s_{dyj}^2 x_j^2$  are variances.

There are two sub-cases in this problem

**Case 3.1:** When  $\sum_{y=1}^k \alpha_y > 0$

$$\text{Let } f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) = \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] - \sum_{y=1}^k N_y(X) \leq \sum_{y=1}^k \alpha_y$$

$$\text{and } E[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)] = F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)$$

$$\sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y^E(X) = \sum_{y=1}^k \lambda_y [\sum_{j=1}^n \sum_{j=1}^n u_{dyj} x_j + \beta_y] - \sum_{y=1}^k \sum_{j=1}^n u_{cyj} x_j \quad (4.19)$$

Then

$$\begin{aligned} V[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)] &= F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) = \sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X) \\ &= \sum_{y=1}^k \lambda_y^2 \sum_{j=1}^n s_{dyj}^2 x_j^2 + \sum_{y=1}^k \sum_{j=1}^n s_{cyj}^2 x_j^2 = \sum_{y=1}^k \sum_{j=1}^n (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2 \end{aligned} \quad (4.20)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) \leq \sum_{y=1}^k \alpha_y] \geq 1 - p^{(2)} \quad (4.21)$$

$$\begin{aligned} \Pr \left[ \frac{f(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0) - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}} \leq \frac{\sum_{y=1}^k \alpha_y - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y > 0)}} \right] &\geq q^{(2)} \\ \Pr \left[ \frac{\sum_{y=1}^k N_y^E(X) - \sum_{y=1}^k N_y(X) + \sum_{y=1}^k \lambda_y [D_y(X) - D_y^E(X)]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \leq \frac{\sum_{y=1}^k N_y^E(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \right] &\geq q^{(2)} \\ \phi \left[ \frac{\sum_{y=1}^k N_y^E(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \right] &\geq q^{(2)} \\ \left[ \frac{\sum_{y=1}^k N_y^E(X) + \sum_{y=1}^k \alpha_y - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \right] &\geq \phi^{-1}(q^{(2)}) \end{aligned}$$

$$\sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y^E(X) + \phi^{-1}(q^{(2)}) \sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)} \leq \sum_{y=1}^k \alpha_y$$

$$\sum_{y=1}^k \lambda_y \left[ \sum_{j=1}^n u_{dyj} x_j + \beta_y \right] - \sum_{y=1}^k \sum_{j=1}^n u_{cyj} x_j + \phi^{-1}(q^{(2)}) \sqrt{\sum_{y=1}^k \sum_{j=1}^n (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \leq \sum_{y=1}^k \alpha_y \quad (4.22)$$

**Case 3.2:** When  $\sum_{y=1}^k \alpha_y \leq 0$

$$\text{Let } f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) = \sum_{y=1}^k N_y(X) - \sum_{y=1}^k \lambda_y [D_y(X) + \beta_y] \geq \sum_{y=1}^k \alpha_y$$

$$\text{and } E[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)] = F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)$$

$$\sum_{y=1}^k N_y^E(X) - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] = \sum_{y=1}^k \sum_{j=1}^n u_{cyj} x_j - \sum_{y=1}^k \lambda_y \left[ \sum_{j=1}^n u_{dyj} x_j + \beta_y \right] \quad (4.23)$$

Then

$$V[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)] = F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) = \sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X) \quad (4.24)$$

$$= \sum_{y=1}^k \lambda_y^2 \sum_{j=1}^n s_{dyj}^2 x_j^2 + \sum_{y=1}^k \sum_{j=1}^n s_{cyj}^2 x_j^2 = \sum_{y=1}^k \sum_{j=1}^n (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) \geq \sum_{y=1}^k \alpha_y] \geq 1 - p^{(2)} \quad (4.25)$$

$$\Pr[f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) \leq \sum_{y=1}^k \alpha_y] \leq p^{(2)}$$

$$\Pr \left[ \frac{f(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0) - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}} \leq \frac{\sum_{y=1}^k \alpha_y - F^E(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}{\sqrt{F^V(X, \lambda_y; \sum_{y=1}^k \alpha_y \leq 0)}} \right] \leq p^{(2)}$$

$$\Pr \left[ \frac{\sum_{y=1}^k N_y(X) - \sum_{y=1}^k N_y^E(X) - \sum_{y=1}^k \lambda_y [D_y(X) - D_y^E(X)]}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \leq \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y^E(X)}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \right] \leq p^{(2)}$$

$$\phi \left[ \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y^E(X)}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \right] \leq p^{(2)}$$

$$\left[ \frac{\sum_{y=1}^k \alpha_y + \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] - \sum_{y=1}^k N_y^E(X)}{\sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)}} \right] \leq \phi^{-1}(p^{(2)})$$

$$\sum_{y=1}^k N_y^E(X) - \sum_{y=1}^k \lambda_y [D_y^E(X) + \beta_y] + \phi^{-1}(p^{(2)}) \sqrt{\sum_{y=1}^k \lambda_y^2 D_y^V(X) + \sum_{y=1}^k N_y^V(X)} \geq \sum_{y=1}^k \alpha_y$$

$$\sum_{y=1}^k \sum_{j=1}^n u_{cyj} x_j - \sum_{y=1}^k \lambda_y [\sum_{j=1}^n u_{dyj} x_j + \beta_y] + \phi^{-1}(p^{(2)}) \sqrt{\sum_{y=1}^k \sum_{j=1}^n (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \geq \sum_{y=1}^k \alpha_y \quad (4.26)$$

Note that once the objective function is converted into deterministic constraint then the new objective function is sum of deterministic parameters subject to converted objective function along with other constraints.

$$\text{i.e. Max } \sum_{y=1}^k \lambda_y \quad (4.17)$$

## 5. Deterministic Version

Suppose when  $\sum_{y=1}^k \alpha_y > 0$  and T is a random variable and numerator of the objective

function is random then we have the following optimization problem

Objective function (4.17)

Subject to (4.6), (2.3) and (3.1).

Similarly one can form the possible optimization problem with different combinations of randomness.

## 6. Numerical Examples

### Example 1:

$$\text{Max } R(X) = \left[ \frac{c_{11}x_1 + c_{12}x_2 + \alpha_1}{d_{11}x_1 + d_{12}x_2 + \beta_1} + \frac{c_{21}x_1 + c_{22}x_2 + \alpha_2}{d_{21}x_1 + d_{22}x_2 + \beta_2} \right] \quad (6.1)$$

subject to

$$a_{11}x_1 + a_{12}x_2 \leq 1$$

$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

$$16x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Let the second and third constraint satisfy atleast 99 and 80 percentage respectively.

Table 1 provides deterministic costs and  $\alpha$  and  $\beta$  values.

**Table 1:**

$d_{11}$	$d_{12}$	$d_{21}$	$d_{22}$	$\alpha_1$	$\alpha_2$	$\beta_1$	$\beta_2$
1	1	2	3	1	2	2	4

The means and variances of the normal random variables are given in table 2.

**Table 2:**

	$c_{11}$	$c_{12}$	$c_{21}$	$c_{22}$	$a_{11}$	$a_{12}$	$a_{21}$	$a_{22}$	$b_2$
Mean	4	3	1	1	2	1	3	4	3
Variance	1	0.5	0	0.5	1	1	2	3	2

The deterministic equivalent of (6.1) is given below

$$\text{Max } \lambda_1 + \lambda_2 \quad (6.2)$$

Subject to

$$(\lambda_1 + 2\lambda_2 - 5)x_1 + (\lambda_1 + 3\lambda_2 - 4)x_2 + 2\lambda_1 + 4\lambda_2 + 1.28\sqrt{x_1^2 + x_2^2} \leq 3$$

$$(2x_1 + x_2) + 1.645\sqrt{x_1^2 + x_2^2} \leq 1$$

$$(3x_1 + 4x_2) + 0.84\sqrt{2x_1^2 + 3x_2^2 + 2} \leq 3$$

$$16x_1 + x_2 \leq 4$$

$$x_1, x_2, \lambda_1, \lambda_2 \geq 0$$

The solution is obtained as  $x_1 = 0.0602$ ,  $x_2 = 0.3292$ ,  $\lambda_1 = 1.7533$ ,  $\lambda_2 = 0.0000$ . The corresponding objective function value of (6.1) is 1.40035.

### Example 2:

$$\text{Max } R(X) = \sum_{y=1}^3 \frac{c_{y1}x_1 + c_{y2}x_2 + \alpha_y}{d_{y1}x_1 + d_{y2}x_2 + \beta_y} \quad (6.3)$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leq 20$$

$$x_1 + x_2 + x_3 \leq b_3$$

$$5x_1 + 3x_2 + 4x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Let the first, second and third constraint satisfies at least 99, 90 and 80 percentage respectively. Table 3 provides deterministic costs and  $\alpha$  and  $\beta$  values.

**Table 3:**

$y$	$c_{y1}$	$c_{y2}$	$c_{y3}$	$\alpha_y$	$\beta_y$
1	2	5	10	10	2
2	1	3	15	22	10
3	5	10	8	-20	5

The means and variances of the normal random variables are given in table 4 and table 5.

**Table 4:**

$Y$	$d_{y1}$		$d_{y2}$		$d_{y3}$	
	Mean	Variance	Mean	Variance	Mean	Variance
1	1	0	2	1	4	0.5
2	1	0	1	0.5	3	2
3	1	0	4	2	7	3

**Table 5:**

	$a_{11}$	$a_{12}$	$a_{13}$	$a_{31}$	$a_{32}$	$a_{33}$	$b_1$	$b_3$
Mean	4	2	4	6	4	6	12	4
Variance	0.5	0.25	0.5	1	0.5	0.75	0.25	1

The deterministic equivalent of (6.3) is given below

$$\text{Max } \lambda_1 + \lambda_2 + \lambda_3 \quad (6.4)$$

Subject to

$$(\lambda_1 + 2\lambda_2 + 4\lambda_3 - 17)x_1 + (\lambda_1 + \lambda_2 + 3\lambda_3 - 19)x_2 + (\lambda_1 + 4\lambda_2 + 7\lambda_3 - 23)x_3 + 2\lambda_1 + 10\lambda_2 + 5\lambda_3 + 1.645\sqrt{(\lambda_2^2 + 0.5\lambda_3^2)x_1^2 + (0.5\lambda_2^2 + 2\lambda_3^2)x_2^2 + (2\lambda_2^2 + 3\lambda_3^2)x_3^2} \leq 12$$

$$4x_1 + 2x_2 + 4x_3 + 1.645\sqrt{0.5x_1^2 + 0.25x_2^2 + 0.5x_3^2 + 0.25} \leq 12$$

$$6x_1 + 4x_2 + 6x_3 + 1.28\sqrt{x_1^2 + 0.5x_2^2 + 0.75x_3^2} \leq 20$$

$$x_1 + x_2 + x_3 \leq 3.16$$

$$5x_1 + 3x_2 + 4x_3 \leq 15$$

$$x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3 \geq 0$$

The solution is obtained as  $x_1 = 0.0000$ ,  $x_2 = 1.4248$ ,  $x_3 = 1.6816$ ,  $\lambda_1 = 15.1931$ ,  $\lambda_2 = \lambda_3 = 0.0000$ . The corresponding objective function value of (6.3) is 6.9623.

**Example 3:**

$$\text{Max } R(X) = \sum_{y=1}^2 \frac{c_{y1}x_1 + c_{y2}x_2 + \alpha_y}{d_{y1}x_1 + d_{y2}x_2 + \beta_y} \quad (6.5)$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq 27$$

$$5x_1 + 3x_2 + x_3 \leq 12$$

$$x_1, x_2 \geq 0$$

Where  $\alpha_1 = -8, \alpha_2 = 5, \beta_1 = 10$  and  $\beta_2 = 12$ ;  $a_{11} \sim N(3, 2), a_{12} \sim N(4, 1)$  and  $a_{13} \sim N(8, 1)$ .

Let the first constraint satisfy atleast 99 percent. The means and variances of the normal random variables are given in table 6.

**Table 6:**

y	C <sub>y1</sub>		C <sub>y2</sub>		C <sub>y3</sub>	
	Mean	Variance	Mean	Variance	Mean	Variance
1	10	2	8	1	5	3
2	20	8	8	3	7	2

y	$d_{y1}$		$d_{y2}$		$d_{y3}$	
	Mean	Variance	Mean	Variance	Mean	Variance
1	2	1	3	2	5	3
2	4	1	2	1	2	1

The deterministic equivalent of (6.5) is given below

$$\text{Max } \lambda_1 + \lambda_2 \quad (6.6)$$

Subject to

$$(20 - 2\lambda_1 - 4\lambda_2)x_1 + (16 - 3\lambda_1 - 2\lambda_2)x_2 + (12 - 5\lambda_1 - 2\lambda_2)x_3 - 10\lambda_1 - 12\lambda_2 - 1.28\sqrt{(\lambda_1^2 + \lambda_2^2 + 10)x_1^2 + (2\lambda_1^2 + \lambda_2^2 + 4)x_2^2 + (3\lambda_1^2 + \lambda_2^2 + 5)x_3^2} \geq 3$$

$$3x_1 + 4x_2 + 8x_3 + 1.645\sqrt{2x_1^2 + x_2^2 + x_3^2} \leq 27$$

$$5x_1 + 3x_2 + x_3 \leq 12$$

$$x_1, x_2, \lambda_1, \lambda_2 \geq 0$$

The solution is obtained as  $x_1 = 2.4000$ ,  $x_2 = x_3 = 0.0000$ ,  $\lambda_1 = 3.6584$ ,  $\lambda_2 = 0.0000$ . The corresponding objective function value of (5.5) is 3.5348.

## 7. Conclusion

The stochastic sum-of-probabilistic-fractional program has important real world applications in several areas such as production, transportation, finance, etc. In this paper we have dealt with modeling and converting objective function of the form, sum-of-probabilistic-fractional function into constraint and solved along with other mixed constraints i.e. stochastic and non-stochastic constraints. In this paper we have discussed

sum-of-probabilistic-fractional objective, when stochastic nature involved in numerator or denominator or both functions. This model can be easily extended to mixed stochastic nature concepts that is stochastic nature can be involved in numerator or denominator or both functions in single sum itself.

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