Non-Linear Stochastic Fractional Programming Model of Financial Derivatives

V.Charles
Department of Management Science
SDM Institute for Management Development, Mysore
Karanataka, India – 570 011. Ph: 9885430036.

D.Dutta
Department of Mathematics & Humanities
National Institute of Technology Warangal
A.P.,India – 506 004. Ph: 91-870-2468820.

Email : v_chals@yahoo.com & ddutta@nitw.ernet.in

Abstract

Non-Linear Stochastic Fractional programming models provide numerous insights into a wide variety of areas such as in financial derivatives. Portfolio optimization has been one of the important research fields in modern finance. The most important character within this optimization problem is the uncertainty of the future returns on assets. The objective of this study is to achieve a maximum profit with minimum investment in the share market. In this paper, we have discussed about linear and nonlinear stochastic fractional programming problems with mixed constraints, which is the key aspect of this model. The application of the model is discussed with an example.

Key words: Stochastic Programming, Fractional Programming, Financial Derivatives.

Introduction

Let us define what are financial derivatives. A derivative is a financial instrument whose value is derived from the price of a more basic asset called the underlying. The underlying may not necessarily be a tradable product. Examples of underlying are stock market indices, shares, commodities, currencies, credits, weather temperatures, sunshine, results of sport matches, wind speed and so on. Basically, anything that may have a certain degree of an unpredictable effect on any business activity can be considered as an underlying of a certain derivative. The most popular derivative is stock option.

Before we discuss about stock option, one should know what is an investment? An investment is a sacrifice of current money for future benefits, Prasanna (2002). Now a days number of avenue of investment are available. One can have chances of investing the money in the following form, deposit money in a bank account or purchase a long-term government bond or invest in the equity shares of a company or contribute to provident fund account or buy a stock option or acquire a plot of land.

Very important attributes of any investment are time and risk. The sacrifice takes place now and is certain. The benefit is expected in the future and tends to be uncertain and here the stochastic nature is peeping in to picture. One can find the stochastic models in Robert A. Strong (2003) and Sen et al. (1999). In some investment, the time element is the vital attribute and in some investment the risk factor is the vital attribute. For example, in government bonds, time place a vital role whereas in stock option, risk matters.

In this paper we have designed a model, which helps us to place minimum investment on the stock market to achieve maximum profit in single period. The model has been derived from the concepts of stochastic fractional programming (SFP) Charles and Dutta (2004a, 2004b) and Sen et al. (1999). The SFP is having wide spread of applications, some of them can be found in Jeeva et al. (2002,2004).

This paper has been classified into seven sections. The first and second section deals with preliminary financial concepts and optimal portfolio. In Section three and four we have discussed about linear and non-linear stochastic fractional programming with mixed constraints. The model description has been given in section five and section six deals with portfolio optimization. The application of the model and the required algorithm with numerical example are given in section seven.

1. Financial Concepts

In this section we briefly recall a number of fundamental financial concepts， A financial derivative is a security whose value is derived by an underlying asset. The most popular financial derivative is the “stock option” very well known as “contingent claim”. An option is a right negotiated between two parties to buy or sell an asset at a later date or a price agreed now or certain price, called the exercise or strike price.

A “ call” option give the buyer the right to buy a specified number of shares of a certain stock at any time at or before the expiration date. Once the option has been “exercised”, the buyer of the call option profits and the writer loses if the market value of the stock rises above the exercise price by an amount exceeding the original amount paid for the option. e.g., if the price of the call is Rs.10 and the exercise price is Rs.80, the buyer would gain if the price of the stock were above Rs.90 at expiration.

A “put” option provides the buyer with the right to sell shares at any time within the option period at the exercise price. The buyer of the put profits if the market value of the stock when the put is exercised is below the exercise price by more than the original cost of the put. e.g., if the price of the put is Rs.10 and the exercise price is Rs.80, the put buyer would gain if the stock price were below Rs.70 at expiration.

An option that can only be exercised at the expiration date is called a “ European option” whereas an option that can be exercised at or before expiration is referred to as an “American option”. “in-the-money” options - where the exercise price is below the stock price, i.e., provides a positive payoff for the owner of the contract. “out-of-the-money” or “underwater” options - where the exercise price is above the stock price, i.e., does not provide a positive payoff. “at-the-money” options - where the exercise price equal the stock price, i.e. no payoff. “Bermudan option” can be exercised prior to maturity but on certain pre-determined days. Options are also used to manage the risk investment portfolios, so that wild upward or downward swings are minimized.

2. Optimal Portfolio

Portfolio theory deals with the problem of how to allocate wealth among several assets (stocks, bounds), many of which have an unknown outcome. Some of the model portfolios can be found in Sudhir Malik (2003). The optimization problem on portfolio selection, portfolio optimization problem, has been one of the important research fields in modern finance. The most important character within this optimization problem is the uncertainty of the future returns on assets. Generally speaking, an investor always prefers to have the return on their portfolio as large as possible. At the same time, he also wants to make the risk as small as possible. However, some investors pursue a high return even though it is accompanied with a higher risk. Markowitz presented the basic theory of portfolio optimization in his pioneering article Markowitz (1952). By employing the standard deviation and expected value of the asset as the representation of return, Markowitz (1952) introduced the famous mean-variance model, which has been regarded as a quadratic programming problem.

3. Linear Stochastic Fractional Programming Problem (LSFPP)

Linear Stochastic Fractional programming supports decision making under uncertainty. It is a methodology for bringing uncertain future scenarios into the traditional decision making framework of linear programming. Stochastic fractional programming model is an optimization problem in which the allocation of today's resources will meet tomorrow's unknown returns in such a way that the user can explore the trade offs with respect to expected risks and rewards and make informed decisions.

Let us consider LSFPP as defined in Charles and Dutta (2001a, 2001b, 2002),

$$\text{Min } Z(X) = \frac{N(X)+\alpha}{D(X)+\beta} \quad (3.1)$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} t_{ij}\, x_j \geq b_i\right) \geq 1-p_i \quad (3.2)$$

Where $T_{mxn} = \| t_{ij} \|$, $X_{nx1} = \| x_j \|$, $b_{mx1} = \| b_i \|$, $i = 1,2,..,m$; $j = 1,2,..,n$. α and β are scalars. In this model, atleast one of T, N(X)= $\sum_{j=1}^{n} c_j x_j$, D(X)= $\sum_{j=1}^{n} d_j x_j$ or b is random in nature and X is deterministic decision variable. Let $S = \{ X \mid \Pr(\sum_{j=1}^{n} t_{ij} x_j \geq b_i) \geq 1\text{-}p_i, X \geq 0, X \in R^n \}$ be the non-empty convex set.

4. Non-linear Stochastic Fractional Programming Problem (NLSFPP)

Non-Linear Stochastic fractional programming deals with a class of optimization models and algorithms in which some of the data may be subject to significant uncertainty with non-linear fractional objective. Such models are appropriate when data evolve over time and decisions need to be made prior to observing the entire data stream. For instance, investment decisions in portfolio planning problems must be implemented before stock performance can be observed.

Let us consider NLSFPP of the first form

$$\text{Min } Z(X) = \frac{N(X) + \alpha}{D(X) + \beta} \qquad (4.1)$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} t_{ij} x_j \geq b_i \right) \geq 1\text{-}p_i \qquad (4.2)$$

Where $T_{mxn} = \| t_{ij} \|$, $X_{nx1} = \| x_j \|$, $b_{mx1} = \| b_i \|$, $i = 1,2,..,m$; $j = 1,2,..,n$. α and β are scalars. In this model, atleast one of T, N(X)= $\sum_{j=1}^{n} c_j x_j^2$, D(X)= $\sum_{j=1}^{n} d_j x_j$ or b is random in nature and X is deterministic decision variable. Let $S = \{ X \mid \Pr(\sum_{j=1}^{n} t_{ij} x_j \geq b_i) \geq 1\text{-}p_i, X \geq 0, X \in R^n \}$ be the non-empty convex set.

4.1. NLSFPP with Mixed Constraints

Let us consider NLSFPP of the form

$$\text{Min } Z(X) = \frac{N(X)+\alpha}{D(X)+\beta} \tag{4.3}$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} t_{ij}\, x_j \geq b_i \right) \geq 1\text{-}p_i \tag{4.4}$$

along with the deterministic constraints as defined in Charles and Dutta (2002, 2003, 2004b)

$$\sum_{j=1}^{n} r_{kj}\, x_j \leq b_{rk} \tag{4.5}$$

where $T_{mxn} = \| t_{ij} \|$, $X_{nx1} = \| x_j \|$, $b_{mx1} = \| b_i \|$, $b_{r\,hx1} = \| b_{rk} \|$, i =1,2,..,m ; j =1,2,..,n; k=1,2,..,h. In this model T and b are random in nature and X is deterministic decision variable. Let S = $\{X \mid \Pr(\sum_{j=1}^{n} t_{ij}\, x_j \geq b_i) \geq 1\text{-}p_i,\ X \geq 0,\ X \in R^n\}$ be the non-empty convex set.

5. Model Description

The model described in Markowitz (1952), assumes a series of steps between a writer and buyer of an options contract:

1. At the beginning of the period, a contract writer sells an option to the buyer and receives F_0.
2. With this F_0, the writer invests the money into the market which consists of securities X={x_1, x_2, x_3,…,x_n}, until the end of the period.

3. At the end of the period, if the options are "in the money", the writer has to pay out to the buyer.

5.1. Multi Period Model

In multi period model, F_0 is the value of the stock option and $x_1, x_2, x_3,\ldots,x_n$ are the only securities in the market. Among the available securities some of them are "numeraire" i.e riskless and the other securities are having some sorts of risk. The mathematical formulation of the prescribed model as follows

$$\text{Maximize} \quad E[C_t X_t] \tag{5.1}$$

$$\text{Subject to} \quad C_o X_0 \leq F_0 \tag{5.2}$$

$$C_t [X_0 - X_t] \geq F_t \tag{5.3}$$

$$C_t X_t \geq 0 \tag{5.4}$$

where F_0 be the cost of the stock option, F_t : payouts from buyer because of options, C_0 : the initial value of the securities, C_t : the value of each stock at the end of the period and X_0 : the amount held of each security.

The model depicts that C_t, F_t are random in nature. One can view the constraint (5.3) as stochastic constraint with appropriate probability. By eliminating X_t from the above multi period model we can get the following one period model.

5.2. One Period Model

The one period model is of the form

Maximize	$E[C_t X_t]$	(5.5)
Subject to	$C_o X_0 \leq F_0$	(5.6)
	$C_t X_0 \geq F_t$	(5.7)

Constraint (5.7) has been obtained by adding constraints (5.3) with (5.4). Note that in this model X_t is totally eradicated. Still the stochastic nature is retained in this model. Here constraint (5.6) is deterministic where as constraint (5.7) is probabilistic. The objective function can be viewed as the terminal portfolio minus the expected payout. Notational form: $C_t X_0 - E[F_t]$.

5.3. Option Valuation

The cost or value of the stock option, F_0 is an unknown variable. One can get the feasible region in the linear programming by attempting to convert the one period model into linear programming from and minimize $F_{0.}$

Minimize F_0		(5.8)
Subject to	$C_o X_0 - F_0 \leq 0$	(5.6)
	$C_t X_0 \geq F_t$	(5.7)

Once F_0 is determined then one can obtain the optimal portfolio by substituting F_0 in the one period model and solving it. This leads us to solve more than one problem.

6. Portfolio Optimization

In this section an attempt has been made to minimize the value of the stock option and maximize the net profit with help of nonlinear fractional objective function subject to mixed constraints. Let the model be of the form

$$\text{Minimize } \frac{V[C_t X_0] + F_0}{E[C_t X_0] + \beta} \qquad (6.1)$$

$$\text{Subject to} \quad C_o X_0 - F_0 \leq 0 \qquad (6.2)$$

$$P[C_t X_0 \geq F_t] \geq 1\text{-}p \qquad (6.3)$$

where $\beta \in (0,1]$ and here the stochastic constraint (6.3) is designed with help of constraint (5.7) and the concepts of Charles and Dutta (2001a, 2001b, 2004a). The technical meaning of the constraint (6.3) is that it should atleast hold with probability not less than 1-p. Where $p \in (0,1]$. We should note that C_t and F_t are random variables, which are assumed to follow normal distribution. The above stochastic constraint can be converted into deterministic form Charles and Dutta (2001a, 2003, 2004a), which is as follows:

$$E[C_t X_0 - F_t] - k_{1-p}\sqrt{V[C_t X_0 - F_t]} \geq 0 \qquad (6.4)$$

Where k_{1-p} is the table value of normal distribution at 1-p level. Once we obtain the solution for the above problem, we can optimize the portfolios by substituting F_0 in the following portfolio optimization model.

Maximize $E[C_t X_0]$ (6.5)

Subject to $C_o X_0 - F_0 \leq 0$ (6.6)

$E[C_t X_0 - F_t] - k_{1-p}\sqrt{V}[C_t X_0 - F_t] \geq 0$ (6.7)

The profit of the portfolio model is $E[C_t X_0] - E[F_t]$.

7. Application of the Model

Let us consider the application of this model for three securities, by creating example with some hypothetical values for the stock prices.

Let $C_0 = [c_0^{(1)}, c_0^{(2)}, c_0^{(3)}]$ be the initial value of the stock price. Let us assume that the first security is riskless whereas the second and third securities have high and medium risk. Let $C_t = [c_0^{(1)}, c_t^{(2)}, c_t^{(3)}]$ be the stock price at the end of the first period i.e. at time t, which is usually liable to change with respect to time period. One can note that the first stock price of C_0 and C_t are same since it is assumed to be risk free security.

Let us also assume that for the second security, the value can go from $c_0^{(2)}$ at the start of the period to $c_1^{(2)}$, $c_2^{(2)}$…,or $c_{n1}^{(2)}$ each value having the probability $1/n_1$. For the third security, the value can go from $c_0^{(3)}$ to $c_1^{(3)}$, $c_2^{(3)}$…,or $c_{n2}^{(3)}$, each with probability $1/n_2$. At the end of the period the option allows the owner to buy second security at strike price φ_1 and third security at strike price φ_2. One can observe that at the end of the period, there will be $n_1 * n_2$ possible outcomes with each out come having probability $1/n_1 n_2$.

7.1. Algorithm to determine C_t and F_t

Step 1: Read φ_1, φ_2
Step 2: K = 0
Step 3: For i = { $c_1^{(2)}, c_2^{(2)} \ldots, c_{n1}^{(2)}$ }
For j = { $c_1^{(3)}, c_2^{(3)} \ldots, c_{n2}^{(3)}$ }
$c_t^{(1)}$ [K] = i
$c_t^{(2)}$ [K] = j
If i < φ_1
F_{t1}= 0
Else
$F_{t1} = \varphi_1 - i$
End If.
If j < φ_2
F_{t2}= 0
Else
$F_{t2} = \varphi_2 - j$
End If.
F_t [k++] = $F_{t1} + F_{t2}$
End For.
End For.
Step 4: Stop.

7.2. Numerical Example

Now we will solve the proposed model numerically by considering the market data for three securities, which is given in table 1. The initial values of the security in rupees be C_0 = [591.8,218,25.8]. The values of φ_1, φ_2 and φ_3 are Rs. 478, Rs.224 and Rs. 25 respectively. Here we should note that the first, second and third securities have high, medium and low risks. Using the proposed algorithm in section 7.1 the following expressions has been obtained.

<Place for Table 1 – Which is given separately at the end>

$$E(C_tX_0) = 549.7375\, x_1 + 222.6375\, x_2 + 26.7535\, x_3 \quad (7.1)$$

$V(C_tX_0) = 1356.9290\, x_1^2 + 111.8748\, x_2^2 + 1.6895\, x_3^2$ (7.2)

$E(F_t) = 77.2536$ (7.3)

$V(F_t) = 1371.3870$ (7.4)

$V(CtX_0 - F_t) = 1356.9290\, x_1^2 + 111.8748\, x_2^2 + 1.6895\, x_3^2 + 1371.3870$ (7.5)

$$\text{Minimize } \frac{V[C_t X_0] + F_0}{E[C_t X_0] + \beta} \quad (7.6)$$

Subject to $591.8\, x_1 + 218\, x_2 + 25.8\, x_3 - F_0 \leq 0$ (7.7)

$549.7375\, x_1 + 222.6375\, x_2 + 26.7535\, x_3 – 1.28 \sqrt{(1356.9290\, x_1^2 + 111.8748\, x_2^2 + 1.6895\, x_3^2 + 1371.3870)} \geq 77.2536$ (7.8)

$0 < \beta \leq 1$ (7.9)

Solving the above non-linear stochastic fractional programming problem we obtain the following result. $\{ x_1, x_2, x_3, F_0, \beta \} = \{0.02963, 0.2422, 2.0442, 123.0843, 1\}$.

Let us substitute F_0 value in (6.6) and solve (6.5) to (6.7). We find the optimal solution to be $\{x_1, x_2, x_3\} = \{0.02963, 0.2422, 2.0442\}$ or 14.2512 % in first security, 42.8971% in second security and 42.8511% in third security. This yields an objective value of Rs.124.9092, which gives a net profit of Rs.124.9092 - Rs.123.0843 = Rs. 1.8249.

Conclusion

Stochastic fractional programming is a promising technology for handling portfolio problems in uncertain environments. Unfortunately, due to modeling difficulties this technology has not yet reached a wide range of audience it deserves. In this paper, non-linear stochastic fractional programming of financial derivatives has been discussed with a numerical example to illustrate the proposed model. With the help of this model one can get a commendable profit with a least amount of investment in the market.

References

1. Charles, V. and Dutta, D., Non-linear stochastic fractional programming models of financial derivatives- II, Proceedings of International Conference on Business and Finance, Vol. III, pp 253-263, 2004a.
2. Charles, V. and Dutta, D., A method for solving linear stochastic fractional programming problem with mixed constraints, Acta Ciencia Indica, Vol. XXX M, No. 3, pp 497-506, 2004b.
3. Charles, V. and Dutta, D., Bi-weighted multi-objective stochastic fractional programming problem with mixed constraints, Proceedings of the Second National Conference on Mathematical and Computational Methods, pp. 29-36, 2003.
4. Charles, V. and Dutta, D., Two level linear stochastic fractional programming problem with discrepancy vector, Journal of Indian Society of Statistics and Operations Research, Vol. XXIII, No. 1-4, pp. 59-67, 2002.
5. Charles, V., Dutta, D. and Appal Raju, K., Linear stochastic fractional programming problem, Proceedings of International Conference on Mathematical Modelling, IIT Roorkee, 211-217, 2001a.

6. Charles, V. and Dutta, D., Linear stochastic fractional programming with branch-and-bound technique, Proceedings of National Conference on Mathematical and Computational Models, PSG College of Technology Coimbator, 131-139, 2001b.
7. Jeeva, M., Rajalakshmi Rajagopal, Charles, V. and Yadavalli, V.S.S., An application of stochastic programming with weibull distribution-cluster based optimum allocation of recruitment in manpower planning, International Journal of Stochastic Analysis and Applications, Vol. 22, No.3, pp 801-812, 2004.
8. Jeeva, M., Rajalakshmi Rajagopal, and Charles, V., Stochastic Programming in Manpower Planning- Cluster Based Optimum Allocation of Recruitments, Advances in Stochastic Modelling. Notable Publication, Inc., New Jersey, USA, 147-156, 2002.
9. Markowitz, H., Portfolio selection, Journal of Finance, 7, 77-91, 1952.
10. Prasanna Chandra, Investment analysis and portfolio management, Indian Edition, CFM-TMH Professional Series in Finance, Tata McGraw-Hill Publication Co., 2002.
11. Robert A. Strong, Porfolio construction, management, & protection, International Student Edition, Taxmann Publications Pvt Ltd, 2003.
12. Sen, S. and Higle, J.L., An introductory tutorial on stochastic linear programming models, Interfaces, 29, 33-61, 1999.
13. Sudhir Malik, plan your investments with investment ready reckoner, India Edition, Taxmann Allied Service Pvt. Ltd, 2003.

Table 1: Data for example 7.2

Date	CMC	Bajaj Tempo	Amarjothi Special
16/07/2004	474.3	236.9	27.75
17/07/2004	478.9	249.9	29
20/07/2004	514.85	225.75	28.8
21/07/2004	562	223.3	26.85
22/07/2004	565.65	211.05	27.9
23/07/2004	570	224.9	26.95
24/07/2004	564.5	215.7	26.1
27/07/2004	567.1	220.75	25.7
28/07/2004	561.2	214.1	26.1
29/07/2004	571.6	214.7	25.1
30/07/2004	574.95	216.6	25
31/07/2004	591.8	218	25.8