

# ACCPM with a nonlinear constraint and an active set strategy to solve nonlinear multicommodity flow problems

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## Abstract

This paper proposes an implementation of a constrained analytic center cutting plane method to solve nonlinear multicommodity flow problems. The new approach exploits the property that the objective of the Lagrangian dual problem has a smooth component with second order derivatives readily available in closed form. The cutting planes issued from the nonsmooth component and the epigraph set of the smooth component form a localization set that is endowed with a self-concordant augmented barrier. Our implementation uses an approximate analytic center associated with that barrier to query the oracle of the nonsmooth component. The paper also proposes an approximation scheme for the original objective. An active set strategy can be applied to the transformed problem: it reduces the dimension of the dual space and accelerates computations. The new approach solves huge instances with high accuracy. The method is compared to alternative approaches proposed in the literature.

**Keywords.** Constrained ACCPM, approximation scheme, active set strategy.

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## 1 Introduction

The multicommodity network flow problem (MCF) consists of routing multiple commodities from a set of supply nodes to a set of demand nodes on a same underlying network. The cost associated with a routing is the sum of costs on the individual arcs. The cost on an individual arc is itself a function of the sum of the commodity flows on that arc. In the nonlinear multicommodity flow problem (NLMCF), the cost associated with an arc is a nonlinear (convex, increasing) function of the total flow on the arc, usually named *congestion* in transportation and *delay* in telecommunications. The main challenge of NLMCF is the size of problem instances. In this paper, we modify the Analytic Center Cutting Plane Method (ACCPM) in order to solve very large-scale instances.

NLMCF problems mainly arise in the areas of transportation and telecommunications. In the transportation sector, the concepts in use are “traffic assignment” and “Wardrop

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equilibrium". They are not equivalent, but both can be formulated as NLCMF, with arc costs that are convex polynomial functions of the flows. In the telecommunications sector, NLMCF models congestion on transmission networks. When several messages must be processed on a same link, the total processing time, hence the travel time of the messages, increases and tends to infinity when the flow approaches the arc capacity. This version of NLCMF is usually considered to be more difficult, due to the capacity constraint.

Two types of test problems can be found in the literature. In the first category, each commodity must be shipped from a single origin to a single destination. The potential number of commodities may be as large as the square of the number of nodes, a huge number on large networks, but finding the best route for a single commodity (independently of the other commodities) is a simple shortest path problem. In the second category, there are multiple supply nodes and demand nodes for each commodity. Finding the best route for a single commodity is then a transshipment problem, a more involved one, but the number of commodities is usually small to very small (with respect to the number of nodes). Papers in the literature deal with either one category, but not both. In this paper, we deal with problems in the first category.

The literature on NLMCF is abundant. We briefly review it. We first discuss methods that directly apply to the arc-flow formulation of the problem. The Frank-Wolfe method [10] works on a sequence of linearized versions of the problem. This approach is attractive because the direction finding subproblem is easy in the case of NLMCF problems with no capacity constraint. There, the direction finding subproblem turns out to be an unconstrained linear MCF that is separable in independent shortest path problems. The standard technique to cope with delay functions with a vertical asymptote consists in approximating the delay by a function with an unbounded domain. The main drawback of the Frank-Wolfe method is its slow convergence rate [26, 2]. The Frank-Wolfe algorithm has been applied to NLMCF, e.g. [20, 11]. The convergence of the method has been improved in [28]. More recently, the authors of [7] report dramatic acceleration using a conjugate gradient scheme.

Column generation is a standard tool to cope with large-scale optimization problems. In the case of NLMCF, this technique amounts to working on a sequence of restricted versions of the path-flow formulation of the problem. At each iteration, the method generates a shortest path for each commodity, with respect to arc lengths equal to the current marginal value of the delay or congestion function. The method then strives to allocate the flows on the generated paths in an optimal way. The authors in [3] use a scaled projected Newton method to find an improved allocation. In the comparative study carried in [25], the projected Newton method appeared to be one of the most efficient one. The method in [2] is an origin-based algorithm conceptually similar to [3]. Let us just mention that, as in the case of Frank-Wolfe, the method cannot handle arc capacity constraint directly. One must replace constraint violation by a fixed penalty function.

Most other methods can be classified as cutting plane methods (CPM). CPMs naturally arise in connection with a Lagrangian relaxation of the arc-flow formulation. It is easy to check that the relaxed master problem in a CPM is equivalent to the restricted primal in column generation. Lagrangian relaxation yields a Lagrangian dual problem of much smaller dimension. It is well-known that the Lagrangian dual is nondifferentiable. Kelley's cutting plane method [15] is appealing because the master problem is linear, but it often converges very slowly. The bundle method has been used in the context of linear MCF [9] to remedy this drawback. The method could be extended to NLMCF. Good numerical results have been obtained with the analytic center cutting plane method [14]. In that paper, ACCPM was used in a disaggregate mode, that is with as many objective

components as the number of commodities. An augmented Lagrangian technique has been used in [17] to solve non-linear traffic assignment problems with link capacity constraints. We conclude this brief review mentioning the proximal decomposition method of [21].

In this paper, we revisit ACCPM to improve its performance on transportation and telecommunications problems with nonlinear cost functions. The Lagrangian dual objective function of those problems has two main components: a piece-wise linear one (the same as in linear MCF) and one that is the negative Fenchel conjugate of the congestion function. The latter is smooth and can often be computed in closed form. In a traditional approach with ACCPM [14], the two components are approximated by cutting planes. The intersection of these half-spaces define a localization set whose analytic center becomes the point where to refine both approximations. In the present paper, we use in the definition of the localization set a direct representation of the epigraph of the smooth component as a fixed constraint. This approach is similar to [24] but our implementation does not use the embedding into a projective space.

The second contribution of this paper is an approximation scheme which replaces the congestion function near the origin by a linear function. This scheme is motivated by the fact that no Lagrangian dual variables need to be introduced in connection with arcs with a linear cost function. This results in a reduction of the dimension of the Lagrangian dual space and easier calculation of analytic center. This strategy has been implemented with success in [1] and can be described as an active set strategy aiming to find and eliminate unsaturated arcs. In the nonlinear case, the strategy is applied to arcs on which the optimal flow lies in a region where the cost function is well approximated by a linear function. The idea of active set has already been used in the context of linear multicommodity flow problems [1, 9, 22], but not within an approximation scheme for the nonlinear component in NLMCF.

The new method has been tested on standard problems that can be found in the open literature. We use four categories of problems. The first two categories, **planar** and **grid**, gather artificial problems that mimic telecommunication networks. Some of them are very large. The third category is made of four small to medium size telecommunication problems. One of them has been used in [25] to compare available specialized optimization codes. The last category includes six realistic traffic network problems; some of them are huge, with up to 13,000 nodes, 39,000 arcs and over 2,000,000 commodities.

The paper is organized as follows. In Section 2 we give formulation of the nonlinear multicommodity flow problem and in Section 3 we presents the associated Lagrangian relaxation. Section 4 provides a brief summary of the constrained ACCPM. Section 5 presents the most commonly used congestion functions and explicit their conjugate functions. Section 6 defines our approximation scheme. It discusses the active set and presents it as a partial Lagrangian relaxation. Section 7 details implementation issues and choices, while Section 8 is devoted to numerical results.

## 2 The nonlinear multicommodity flow problem

Let  $\mathcal{G}(\mathcal{N}, \mathcal{A})$  be an oriented graph, where  $\mathcal{N}$  is the set of nodes and  $\mathcal{A}$  the set of arcs. The graph represents a network on which multiple commodities must be shipped from specific origins to specific destinations. Given a shipment schedule, the total flow on an arc induces a congestion<sup>1</sup>. The objective is to find a shipment schedule that minimizes the total congestion on the network. In many applications, in particular in transportation

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<sup>1</sup>In the sequel, we shall not differentiate between "delay" and "congestion".

and telecommunications, the congestion function is nonlinear: the resulting problem is named the *nonlinear multicommodity flow problem*, in short NLMCF.

The arc-flow formulation of NLMCF is

$$\min g\left(\sum_{\tau \in \mathcal{T}} x^\tau\right) \equiv \sum_{a \in \mathcal{A}} g_a\left(\sum_{\tau \in \mathcal{T}} x_a^\tau\right) \quad (1a)$$

$$Nx^\tau = d_\tau \delta^\tau, \quad \forall \tau \in \mathcal{T}, \quad (1b)$$

$$x_a^\tau \geq 0, \quad \forall a \in \mathcal{A}, \forall \tau \in \mathcal{T}. \quad (1c)$$

Here,  $N$  is the network incidence matrix;  $\mathcal{T}$  is the set of commodities;  $d_\tau$  is the demand for the commodity  $\tau \in \mathcal{T}$ ; and  $\delta^\tau$  is vector of zeros except a “1” at the supply node and a “-1” at the demand node. The vector  $x^\tau = (x_a^\tau)_{a \in \mathcal{A}}$  represents the flows of commodity  $\tau$  on the arcs of the network. The number of arcs is  $m = |\mathcal{A}|$ ;  $|\mathcal{T}|$  is the number of commodities; and  $|\mathcal{N}|$  is the number of nodes.

The congestion function  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is assumed to be convex and twice continuously differentiable. The objective function is thus separable. Problem (1) has  $n = |\mathcal{A}| \times |\mathcal{T}|$  variables and  $|\mathcal{N}| \times |\mathcal{T}|$  constraints (plus possible upper bound constraints on variables). In general,  $|\mathcal{T}|$  is large and  $n \gg m > |\mathcal{N}|$ . (We have in mind instances for which  $m \geq 10^4$  and  $n \geq 10^{10}$ .)

The literature essentially deals with two types of congestion function: the *Kleinrock* function, used in telecommunications [4, 14, 16, 25] and the *BPR* (Bureau of Public Roads) function used in transportation [2, 6, 7, 17]. The Kleinrock function is given by

$$g_a(y_a) = \frac{y_a}{c_a - y_a}, \quad (2)$$

where  $c_a$  is the capacity on the arc. The function has a vertical asymptote at  $y_a = c_a$ .

The *BPR* function is

$$g_a(y_a) = r_a y_a \left(1 + \frac{\alpha}{\beta + 1} \left(\frac{y_a}{c_a}\right)^\beta\right). \quad (3)$$

In general, the parameter  $\alpha$  is very small and  $\beta > 1$  does not exceed 5. When the flow  $y_a$  is less than  $c_a$ , the second term under the parenthesis in (3) is negligible. Thus  $g_a(y_a) \approx r_a y_a$ : the parameter  $r_a$  is called *free-flow travel time* and it can be interpreted as a fixed travel time on a congestion-free arc. For larger values of  $y_a$  the nonlinear contribution to congestion increases. The threshold value  $c_a$  for the flow  $y_a$  is usually named the *practical capacity* of the arc, beyond which congestion becomes effective. In some applications, the parameters  $\alpha$  and  $\beta$  are arc-dependent.

### 3 Lagrangian relaxation

For the sake of simpler notation, let us consider the more general problem

$$\min\{g(Mx) \mid x \in X\}. \quad (4)$$

We assume that  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is convex,  $M$  is a  $m \times n$  matrix, while  $X \subset \mathbb{R}^n$  is convex. We easily identify problem (4) with (1). In nonlinear multicommodity flow problems on oriented graphs,  $X$  is defined as a set of network flow constraints (one per commodity). The matrix  $M$  collects the flows on the individual arcs; it is thus made of zeroes and ones. If the dimension of  $x$  is large, as it is the case in multicommodity flow problems, it becomes difficult to apply a direct method, even when the problem falls into the realm of

structural programming (see [23]), e.g.,  $g$  is self-concordant and  $X$  is endowed with a self-concordant barrier. An alternative to a direct method consists in applying a Lagrangian relaxation to a slightly transformed problem with an auxiliary variable  $y$  and an auxiliary constraint  $y = Mx$ . The new formulation

$$\min \quad g(y) \tag{5a}$$

$$Mx = y, \tag{5b}$$

$$x \in X, \tag{5c}$$

is equivalent to (4). Since the flows  $x$  on the oriented graph must be nonnegative, then  $Mx \geq 0$ . Therefore, the condition  $y \geq 0$  is implied by (5b) and (5c). We need not introduce it explicitly in formulation (5). Relaxing (5b) yields a concave programming problem in the dual variables  $u \in \mathbb{R}^m$  associated with the constraints  $y = Mx$ . The Lagrangian dual problem is

$$\max_u L(u), \tag{6}$$

where  $L$  is defined by

$$L(u) = \min_{y,x} \{g(y) + \langle u, Mx - y \rangle \mid x \in X\}. \tag{7}$$

By duality, this Lagrangian dual problem has the same optimal value as (4). Since  $m \ll n$ , the transformed problem has much smaller dimension: it is potentially solvable by a suitable convex optimization method.

A quick inspection shows that (7) is separable in the variables  $y$  and  $x$ . Then, (7) can be written as

$$L(u) = f_1(u) + f_2(u), \tag{8}$$

where

$$f_1(u) = \min_x \{\langle M^T u, x \rangle \mid x \in X\},$$

and

$$f_2(u) = \min_y \{g(y) - \langle u, y \rangle\}.$$

In view of our assumption on  $X$ ,  $f_1(u)$  can be routinely computed for arbitrary values of  $u$ . Since  $f_1$  is defined as the point-wise minimum of a collection of linear functions,  $f_1$  is concave but usually nondifferentiable. Besides, if

$$x(u) = \operatorname{argmin}\{\langle M^T u, x \rangle \mid x \in X\},$$

then

$$f_1(u') \geq \langle Mx(u), u' \rangle = f_1(u) + \langle Mx(u), u' - u \rangle, \quad \forall u' \in \mathbb{R}^m. \tag{9}$$

This shows that  $Mx(u)$  is an antsubgradient of  $f_1(u)$  at  $u$ , that is  $Mx(u) \in -\partial(-f_1(u))$ .

Akin, the function  $f_2(u)$  is the point-wise minimum of a collection of affine functions of  $u$ . It is thus concave and one may construct an inequality similar to (9). Actually, we can get more. From the definition, we observe that  $f_2(u)$  is the opposite of the Fenchel conjugate  $g_*(u)$  of  $g$ . In the cases under study,  $g_*(u)$  can be given in closed form and it also appears to be twice continuously differentiable. We certainly want to exploit this property when it is verified, and devise more efficient algorithms to solve the Lagrangian dual problem.

Let us introduce conditions that are relevant for our multicommodity network flow problem of interest. They are of considerable help in solving (6).

**Condition 1** *The linear programming problem*

$$\min\{\langle c, x \rangle \mid x \in X\},$$

*can be solved at low computational cost.*

In other words,  $f_1(u)$  can be computed routinely, without excessive burden on the overall algorithm.

**Condition 2** *The congestion function  $g(y)$  is separable, i.e.,  $g(y) = \sum_{i=1}^m g_i(y_i)$ . The functions  $g_i$  are nonnegative, convex, monotonically increasing and  $\text{dom } g_i \subset \mathbb{R}_+$ . Moreover, the convex conjugate  $(g_i)_*$  can be computed in closed form.*

Let us explore an immediate consequence of Condition 2. The first order optimality conditions for problem (5) are

$$0 \in \partial g(y) - u, \quad (10)$$

$$M^T u \in -\mathcal{N}_X(x), \quad (11)$$

where  $\mathcal{N}_X(x)$  is the normal cone of  $X$  at  $x$ . The right derivative  $g'_+(y)$  of  $g$  at  $y = 0$  is well-defined. Since  $g$  is monotone increasing, condition (10) implies that the constraint

$$u \geq g'_+(0), \quad (12)$$

is always met at the optimum. It is nevertheless convenient to introduce this redundant constraint in the formulation of problem (6).

## 4 Constrained ACCPM

We aim to solve (6) with a version of ACCPM in which the the smooth component  $f_2$  of the objective function is handled as fixed, explicit constraint on the localization set. This constraint can be viewed as a cutting surface. The general setting is a problem as (6) with the constraint (12)

$$\max\{f(u) = f_1(u) + f_2(u) \mid u \geq u_l\}, \quad (13)$$

in which  $u \in \mathbb{R}^m$ ,  $f_1 : \mathbb{R}^m \rightarrow \mathbb{R}$  is a concave function and  $f_2 : \mathbb{R}^m \rightarrow \mathbb{R}$  is a self-concordant function. Information on these functions is delivered by *oracles*.

**Definition 1** *A first order oracle for the concave function  $h : \mathbb{R}^m \rightarrow \mathbb{R}$  is a black-box procedure that returns a support to  $h$  at the query point  $\bar{u}$ . This support takes the form of the cutting plane, called optimality cut*

$$a^T(u - \bar{u}) + h(\bar{u}) \geq h(u), \quad \forall u \in \text{dom } h, \quad (14)$$

*where the vector  $a \in \mathbb{R}^m$  is an element of the anti-subgradient set,  $a \in -\partial(-h(\bar{u}))$ .*

**Definition 2** *A second-order oracle for the concave function  $h : \mathbb{R}^m \rightarrow \mathbb{R}$  is a black box procedure with the following property. When queried at  $\bar{u}$ , the oracle returns the function value and the first and second derivatives of  $h(u)$  at  $u = \bar{u}$ .*

We assume that  $f_1$  is revealed by a first order oracle, while  $f_2$  is accessed through a second-order oracle.

## 4.1 The algorithm

The hypograph set of the function  $f$  is the set defined by  $\{(z+\zeta, u) \mid z \leq f_1(u), \zeta \leq f_2(u)\}$ . The canonical problem (13) can be written with the help of the auxiliary variables  $z$  and  $\zeta$  as

$$\max\{z + \zeta \mid z \leq f_1(u), \zeta \leq f_2(u), u \geq u_l\}. \quad (15)$$

Optimality cuts (14) provide an outer polyhedral approximation of the hypograph set of  $f_1$ . Suppose that a certain number of query points  $u^k, k = 1, \dots, K$ , have been generated. The associated anti-subgradients  $a^k \in -\partial(-f_1(u^k))$  are collected in a matrix  $A$ . We further set  $\gamma_k = f_1(u^k) - \langle a^k, u^k \rangle$ . The polyhedral approximation of the hypograph set of  $f_1$  is  $\gamma \geq ze - A^T u$ , where  $e$  is the all-ones vector of appropriate dimension. Finally, let  $\underline{\theta}$  be the best recorded value:  $\underline{\theta} = \max_{k \leq K} \{f_1(u^k) + f_2(u^k)\}$ .

In view of the above definitions, we can define the so-called *localization set*, which is a subset of the hypograph of  $f$

$$\mathcal{F}_{\underline{\theta}} = \{(u, z, \zeta) \mid -A^T u + ze \leq \gamma, \zeta \leq f_2(u), z + \zeta \geq \underline{\theta}, u \geq u_l\}. \quad (16)$$

Clearly, the set contains all optimal pairs  $(u^*, f(u^*))$ . Thus, the search for a solution should be confined to the localization set.

In the proposed version of ACCPM, the query point is an approximate proximal analytic center of the localization set defined as the intersection of cutting planes and a fixed cutting surface. For the sake of clarity, we first sketch the basic step, or *outer iteration*, of a generic cutting plane method.

Outer iteration of constrained ACCPM

1. Select a query point in the localization set.
2. Send the query point to the first order oracle and get back an optimality cut to  $f_1$ .
3. Send the query point to the second order oracle to compute the objective function  $f_2$ .
4. Update the lower and upper bounds and the localization set.
5. Test termination.

## 4.2 Proximal analytic centers

The proximal analytic center is defined as the unique minimizer of a logarithmic barrier for the localization set, augmented with a proximal term

$$F(u, z, \zeta) = \frac{\rho}{2} \|u - \underline{u}\|^2 - \sum_{i=0}^K \log s_i - \log \sigma - \sum_{i=1}^m \log(u_i - u_{li}), \quad (17)$$

with  $s > 0$  and  $\sigma > 0$  defined by

$$\begin{aligned} s_0 &= z + \zeta - \underline{\theta}, \\ s_i &= \gamma_i - z + (a^i)^T u, \quad i = \{1, \dots, K\}, \\ \sigma &= f_2(u) - \zeta. \end{aligned}$$

Note that  $F(u, z, \zeta)$  is defined on the interior of the localization set  $\mathcal{F}_\theta$ . The proximal reference point  $\underline{u}$  and the proximal coefficient  $\rho$  are arbitrary. In practice,  $\underline{u}$  is chosen to be the query point  $u^k$  that achieves the best recorded value  $\underline{\theta}$ , i.e.,  $\underline{u} = \arg \max_{k \leq K} \{f_1(u^k) + f_2(u^k)\}$ .

The proximal analytic center method defines the next query point to be the  $u$  component of the solution  $(u, z, \zeta)$  to the minimization problem

$$\min \quad F(u, z, \zeta) = \frac{\rho}{2} \|u - \underline{u}\|^2 - \sum_{i=0}^K \log s_i - \log \sigma - \sum_{i=1}^m \log(u_i - u_{li})$$

$$s_0 = z + \zeta - \underline{\theta} \geq 0, \quad (18a)$$

$$s_i = \gamma_i - z + (a^i)^T u \geq 0, \quad i = \{1, \dots, K\}, \quad (18b)$$

$$\sigma = f_2(u) - \zeta \geq 0. \quad (18c)$$

**Theorem 1** *If the localization set has a non-empty interior, then  $F(u, z, \zeta)$  achieves its minimum value at a unique point.*

**Proof:** Let  $(\hat{u}, \hat{z}, \hat{\zeta})$  belong to the interior of  $\mathcal{F}_\theta$ ; thus  $\hat{s}_0 > 0$ ,  $\hat{s} > 0$  and  $\hat{\sigma} > 0$ . In view of the simple inequality  $-\log x \geq 1 - x$ , we have

$$F(u, z, \zeta) \geq \frac{\rho}{2} \|u - \underline{u}\|^2 - \sum_{i=0}^K s_i - \sigma - \sum_{i=1}^m (u_i - u_{li}) + K + m + 2.$$

Since  $s_0 \geq 0$  and  $\sigma \geq 0$  and in view of the concavity of  $f_2$ , we have

$$\begin{aligned} \sum_{i=0}^K s_i + \sigma &\leq \sum_{i=1}^K (s_i + s_0 + \sigma), \\ &\leq \sum_{i=1}^K (\gamma_i + (a^i)^T u) - K\underline{\theta} + Kf_2(u), \\ &\leq \sum_{i=1}^K (\gamma_i + (a^i)^T u) - K\underline{\theta} + K(f_2(\hat{u}) + f_2'(\hat{u})^T (u - \hat{u})). \end{aligned}$$

It follows that the objective  $F(u, z, \zeta)$  is bounded from below on the feasible set of (18) by a strongly quadratic convex function. This bounding quadratic function has bounded level sets. Thus, the projection of a level set of (18) on  $u$ , which is contained in some level set of the bounding function, is also bounded. We can easily show that (18a)–(18c) imply  $z$  and  $\zeta$  are also bounded from below and above on the level set of (18). Therefore, the function  $F(u, z, \zeta)$  achieves its minimum, and by strict convexity, this minimum is unique.  $\blacksquare$

**Remark 1** *The condition on the localization set is met if  $\text{dom } f_1 \cup \text{dom } f_2 \cup \{u \geq u_l\} \neq \emptyset$  and  $\underline{\theta} \leq \theta^*$ . The first condition holds in general. To enforce the second condition, we replace  $\underline{\theta}$  by  $\underline{\theta} - \nu$ , where  $\nu$  is a small fraction of the optimality tolerance  $\epsilon$ .*

### 4.3 Newton method

If  $f(x)$  is a self-concordant function with bounded level sets, then it is well-known [23] that Newton method converges quadratically to the neighborhood

$$\left\{ x \mid \langle -[f''(x)]^{-1}f'(x), f'(x) \rangle \leq \eta < \frac{3 - \sqrt{5}}{2} \right\}, \quad (19)$$

and that a damped Newton method converges to that neighborhood in a number of iterations that is polynomial. Thanks to a lemma in [13], it is easy to verify that the barrier function (17) with  $f_2$  equal to the Kleinrock or the BPR function is self-concordant.

In the rest of this subsection, we use the following notation. Given a vector  $s > 0$ ,  $S$  is the diagonal matrix whose main diagonal is  $s$ . We also use  $s^{-1} = S^{-1}e$  to denote the vector whose coordinates are the inverse of the coordinates of  $s$ . Similarly,  $s^{-2} = S^{-2}e$ . With this notation, the first order optimality conditions for Problem (18) are

$$\rho(u - \underline{u}) - As^{-1} - f'_2(u)\sigma^{-1} - (u - u_l)^{-1} = 0, \quad (20a)$$

$$e^T s^{-1} - s_0^{-1} = 0, \quad (20b)$$

$$\sigma^{-1} - s_0^{-1} = 0, \quad (20c)$$

$$s_0 - z - \zeta + \underline{\theta} = 0, \quad (20d)$$

$$s - \gamma + ze - A^T u = 0, \quad (20e)$$

$$\sigma - f_2(u) + \zeta = 0. \quad (20f)$$

The algorithm that computes the analytic center is a damped Newton method applied to (20). To write down the formulae, we introduce the residuals

$$\begin{aligned} r_u &= -(\rho(u - \underline{u}) - As^{-1} - f'_2(u)\sigma^{-1} - (u - u_l)^{-1}), \\ r_z &= -(e^T s^{-1} - s_0^{-1}), \\ r_\zeta &= -(\sigma^{-1} - s_0^{-1}), \\ r_{s_0} &= -(s_0 - z - \zeta + \underline{\theta}), \\ r_s &= -(s - \gamma + ze - A^T u), \\ r_\sigma &= -(\sigma - f_2(u) + \zeta). \end{aligned}$$

The Newton direction  $(du, dz, d\zeta, ds_0, ds, d\sigma)$  associated to (20) is given by

$$P \begin{pmatrix} du \\ dz \\ d\zeta \\ ds_0 \\ ds \\ d\sigma \end{pmatrix} = \begin{pmatrix} r_u \\ r_z \\ r_\zeta \\ r_{s_0} \\ r_s \\ r_\sigma \end{pmatrix}, \quad (21)$$

where

$$P = \begin{pmatrix} \rho I - f_2(u)''\sigma^{-1} + (U - U_l)^{-2} & 0 & 0 & 0 & AS^{-2} & f_2(u)'\sigma^{-2} \\ 0 & 0 & 0 & s_0^{-2} & -e^T S^{-2} & 0 \\ 0 & 0 & 0 & s_0^{-2} & 0 & -\sigma^{-2} \\ 0 & -1 & -1 & 1 & 0 & 0 \\ -A^T & e & 0 & 0 & I & 0 \\ -f'_2(u) & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Since (20f) is nonlinear, a full Newton step does not yield a feasible point with respect to (20f). Thus, we use the following empirical rule to compute the step length  $\alpha_{step}$ . Let  $0 < \gamma < 1$  be a fixed parameter and

$$\alpha_{max} = \max\{\alpha \mid s + \alpha ds > 0, \sigma + \alpha d\sigma > 0, u + \alpha du > u_l\},$$

the step length is

$$\alpha_{step} = \min(1, \gamma\alpha_{max}). \quad (22)$$

Since it is not essential in our solution method to compute an exact analytic center, we use the termination criteria  $\eta = 0.99$  that is looser than (19).

To summarize, a basic step of the Newton iteration, or *inner iteration*, is

**Inner iteration**

1. Send the current point to the second order oracle to compute the objective function  $f_2$  and its first and second derivatives.
2. Compute the Newton step  $(du, dz, d\zeta, ds_0, ds, d\sigma)$  by (21).
3. Compute a step length by (22) to update  $(u, z, \zeta, s_0, s, \sigma)$ .
4. Test termination.

**Remark 2** *The computation in the inner iteration uses second order derivatives of  $f_2$ . Since the derivatives change at each iteration, the inner iteration must have access to the second order oracle. In pure cutting plane methods, the inner iteration does not interact with the oracle.*

## 4.4 Upper and lower bounds

By duality, any feasible solution of (13) provides a lower bound for the original problem (1). Taking the values returned by the two oracles at the successive query points, we obtain the lower bound

$$\underline{\theta} = \max_{k \leq K} \{f_1(u^k) + f_2(u^k)\}. \quad (23)$$

An upper bound  $\bar{\theta}$  can be obtained from information collected in the computation of the analytic center. More precisely, assume that  $(u^c, z^c, \zeta^c)$  is an approximate analytic center, in the sense that  $(s_0^c, s^c, \sigma^c)$ , defined by (20d)-(20f), are strictly positive, and the equations (20a)-(20c) are approximately satisfied. Let

$$\lambda_c = (s_0^c)(s^c)^{-1} > 0. \quad (24)$$

If (20b) and (20c) are satisfied, then  $e^T \lambda_c = 1$  and  $(s_0^c)(\sigma^c)^{-1} = 1$ . Otherwise, we scale  $\lambda_c$  and  $(\sigma^c)^{-1}$  to meet the conditions. Using the cuts  $Mx^k$ , where  $x^k = x(u^k)$ ,  $k = 1, \dots, K$ , we define the vector

$$y^K = \sum_{k=1}^K \lambda_c^k Mx^k = Mx^K, \quad (25)$$

with  $x^K = \sum_{k=1}^K \lambda_c^k x^k$ . The vector  $x^K$  is a convex combination of vectors in  $X$ : it also belongs to  $X$ . If  $y^K \in \text{dom } g$  then,  $g(y^K)$  is a valid upper bound for the optimal value.

If we extend the definition of  $g$  to have  $g(y) = +\infty$  when  $y \notin \text{dom } g$ , we have the upper bound

$$\bar{\theta} = \min_{k \leq K} g(y^k). \quad (26)$$

Let us now argue that we may expect that  $y^k$  becomes feasible. It is easy to relate

$$r = -y^K - f'_2(u^c) - (s_0^c)(u^c - u_l)^{-1},$$

to the residual in (20a). If we are getting close to an optimal solution of the original problem, i.e.,  $\|u^c - u^*\|$  is small, we can reasonably hope that  $r$  is small with respect to  $(s_0^c)(u^c - u_l)^{-1}$ . Then, since  $(s_0^c)(u^c - u_l)^{-1} \geq 0$  we have

$$0 \leq y^K \leq -f'_2(u),$$

where  $-f'_2(u^c) = (g^*)'(u^c) \in \text{dom}(g)$ . Thus,  $y^K$  is a feasible solution for (1).

## 5 Congestion functions and their conjugates

The implementation of ACCPM with a constraint requires an explicit calculation of the conjugate function  $g_*$  of the congestion  $g$

$$g_*(u) = \max_y \{ \langle u, y \rangle - g(y) \}.$$

In the sequel we shall name  $-g_*$  the negative conjugate. Note that

$$g_*(u) = \sum_{a \in \mathcal{A}} g_{a*}(u_a) = \sum_{a \in \mathcal{A}} \max_{y_a} \{ u_a y_a - g_a(y_a) \}.$$

The computation for the two congestion functions, Kleinrock and BPR, is straightforward. Since the objective  $f_2(u)$  is the opposite of the conjugate function, we display the functional form of  $-g_{a*}$ , its domain and its first and second derivatives in Table 1. The functions and their conjugate are also plotted on Figures 1 and 2.

	Kleinrock delay function	BPR congestion function
Domain	$u_a \in [\frac{1}{c_a}, +\infty[$	$u_a \in [0, +\infty[$
Conjugate	$1 + u_a c_a - 2\sqrt{u_a c_a}$	$\frac{c_a(u_a - r_a)^{\frac{\beta+1}{\beta}}}{(\alpha r_a)^{\frac{1}{\beta}}} \frac{\beta}{\beta+1}$
Gradient	$c_a - \sqrt{\frac{c_a}{u_a}}$	$\frac{c_a}{(\alpha r_a)^{\frac{1}{\beta}}} (u_a - r_a)^{\frac{1}{\beta}}$
Hessian (diagonal)	$0.5\sqrt{c_a} u_a^{-1.5}$	$\frac{c_a}{\beta(\alpha r_a)^{\frac{1}{\beta}}} (u_a - r_a)^{\frac{1-\beta}{\beta}}$

Table 1: Conjugate functions.

In this paper, we shall also consider a more general class of congestion functions defined by

$$g_a(y_a) = \max_{y_a \in \text{dom} \tilde{g}_a} \{ t_a y_a, \tilde{g}_a(y_a) \}, \quad a \in \mathcal{A}, \quad (27)$$

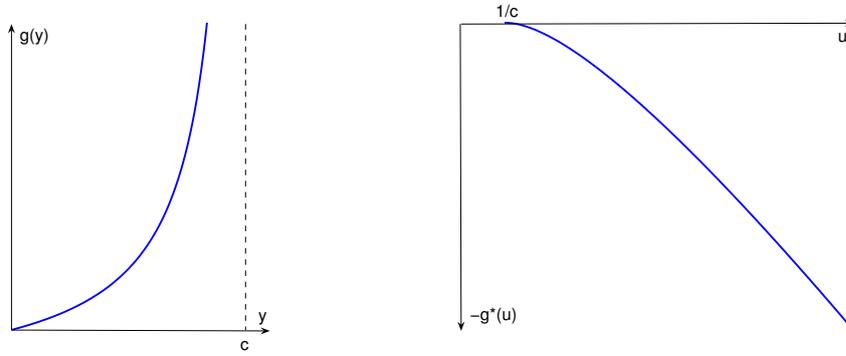


Figure 1: Kleinrock delay function and its negative conjugate

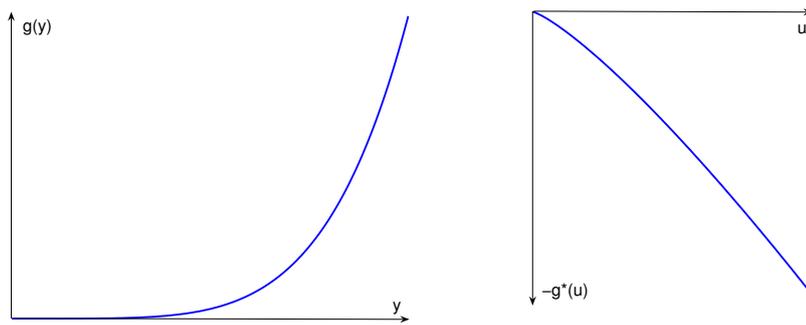


Figure 2: BPR congestion function and its negative conjugate

where  $t_a \geq 0$  and  $\tilde{g}_a(y_a)$  is a convex nondecreasing function whose domain is a closed or half-closed interval of  $\mathbb{R}_+$ :

$$\text{dom } \tilde{g}_a = [0, \bar{y}_a[ \text{ or } \text{dom } \tilde{g}_a = [0, \bar{y}_a].$$

The upper limit may be finite (e.g., the support function associated with the constraint  $y_a \leq \bar{y}_a$ ) or infinite. The meeting point between the linear and the nonlinear part is denoted  $y_a^c$  which is uniquely defined by  $t_a y_a^c = \tilde{g}_a(y_a^c)$ . In view of the convexity of  $g_a$ , we have  $\tilde{g}'_a(y_a^c) \geq t_a$ . We assume that  $\tilde{g}_a$  is continuously differentiable on the interior of its domain. Note that  $\text{dom } g_a = \text{dom } \tilde{g}_a$ . We name the function (27) a *compound congestion function*.

Let us compute the negative conjugate of the compound congestion function. Let

$$y(u) = \arg \min_{y \in \text{dom } g} \{g(y) - \langle u, y \rangle\},$$

whenever the minimum occurs in  $\text{dom } \tilde{g}$ . Simple calculation yields

$$-g_{a*}(u_a) = \begin{cases} (t_a - u_a)y_a^c, & \text{with } y_a(u_a) = y_a^c, & \text{if } t_a \leq u_a < \tilde{g}'_a(y_a^c), \\ -\tilde{g}_{a*}(u_a), & \text{with } y_a(u_a) = [g'_a]^{-1}(u_a), & \text{if } \tilde{g}'_a(y_a^c) \leq u_a \leq g'_a(\bar{y}_a). \end{cases}$$

It follows that  $\text{dom } g_* \subset \{u \geq t\}$ . In other words, we can add the constraint  $u \geq t$  in the maximization of the Lagrangian dual function.

Finally,  $g_*(u)$  is differentiable on the interior of  $\text{dom } g_* \subset \{t < u < g'(\bar{y})\}$  and

$$g'_*(u) = y(u).$$

Figure 3 and 4 display the plot of compound functions and their negative conjugate in the case of Kleinrock and BPR functions, respectively.

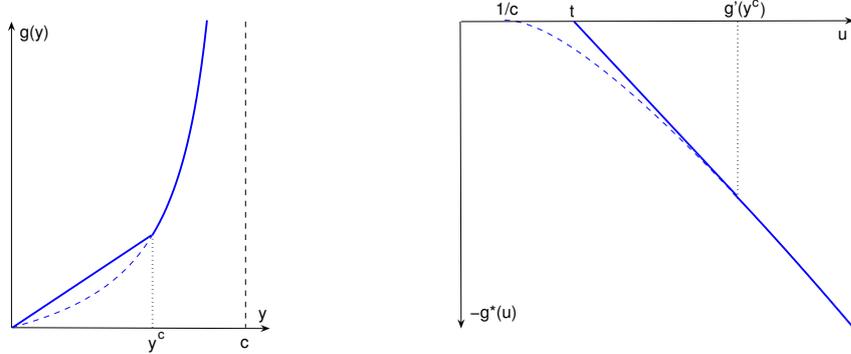


Figure 3: Compound Kleinrock function and its conjugate

## 6 Active set strategy on compound functions

Our interest for the class of compound congestion functions (27)

$$g_a(y_a) = \max_{y_a \in \text{dom } \tilde{g}_a} \{t_a y_a, \tilde{g}_a(y_a)\}, \quad a \in \mathcal{A},$$

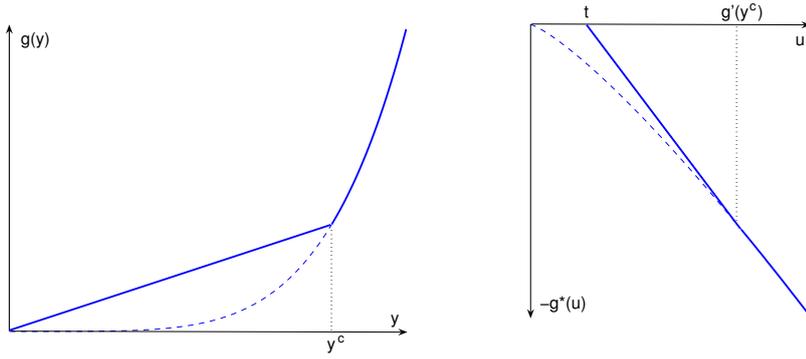


Figure 4: Compound BPR function and its conjugate

which were introduced in the previous section, has been triggered by the observation that multicommodity flow problems with linear congestion functions and no capacity constraint are easy to solve. Those problems are separable in the commodities, and for each commodity the minimization problem boils down to a shortest path calculation. In the case of capacity constraints on the arcs, the above property can still be exploited on those arcs where the capacity constraint is inactive at the optimum. A heuristic that dynamically estimates the sets of active and inactive arcs has proved successful for LMCF, [1]. This procedure can be extended to NLMCF problems with a compound congestion since, on those arcs where  $y_a < y_a^c$ , the function is linear. It can be further extended to standard NLMCF problems if one approximates the objective function by a compound one. Note that the approximation error is easily controlled by an appropriate choice of the meeting point  $y_a^c$  in (27). The approximation error tends to zero when  $t_a \downarrow (\tilde{g}_a)'_+(0)$ . In the previous section, we gave the formula for the negative conjugate of the compound congestion function. We can also use this expression to compute the maximal error on the dual side.

Consider the first order optimality conditions for the Lagrangian dual problem (6). In full generality, the condition stipulates that  $u \geq u_l$  is optimal if there exists a nonnegative vector  $\tau$  such that

$$\tau + \partial f_1(u) + \partial f_2(u) \ni 0, \quad (28)$$

$$\langle \tau, u - u_l \rangle = 0, \quad (29)$$

$$\tau \geq 0, \quad u \geq u_l, \quad (30)$$

where  $u_l = g'_+(0)$ . An anti-subgradient of  $f_1$  is of the form  $Mx$  with  $x \in X$ , while for  $f_2$  one can take  $-y(u)$ . Hence, (28) and (29) imply

$$Mx \leq y(u),$$

and

$$(Mx)_a < y_a(u_a) \Rightarrow u_a = u_{la} = (g_a)'_+(0). \quad (31)$$

A similar analysis can be performed on the primal side.

Our goal is to use condition (31) to find a set  $\mathcal{A}_1^* \subset \mathcal{A}$  with the property that for a optimal primal-dual pair  $(\tau^*, u^*)$

$$\tau_a^* > 0 \text{ and } u_a^* = u_{la}, \quad \forall a \in \mathcal{A}_1^*.$$

If the set  $\mathcal{A}_1^*$  were known in advance, the variables  $u_a$ ,  $a \in \mathcal{A}_1^*$ , could be fixed to their lower bound  $u_{la}$ . The original problem (6) would then boil down to a simpler problem in the variables  $u_a$ ,  $a \in \mathcal{A}_2^* = \mathcal{A} \setminus \mathcal{A}_1^*$ . Note that  $f_{2a}(u_{la}) = 0$ ; thus, the nonlinear term  $f_{2a}(u_a)$ ,  $a \in \mathcal{A}_1^*$ , can be removed from this equivalent formulation. The above reasoning applies to any subset  $\tilde{\mathcal{A}}_1 \subset \mathcal{A}_1^*$  and its complement  $\tilde{\mathcal{A}}_2 \supset \mathcal{A}_2^*$ .

In view of the above partition, we define (6) as the partial Lagrangian problem

$$\max\{L(u) \mid u_a = u_{la}, a \in \mathcal{A}_1^*; \quad u_a \geq u_{la}, a \in \mathcal{A}_2^*\}.$$

In practice, the partition is not known in advance and the proposed space reduction technique cannot be straightforwardly implemented. An active set strategy aims to guess the partition. Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be the current estimate of  $\mathcal{A}_1^*$  and  $\mathcal{A}_2^*$  with  $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ . This partition is used to work in a dual space of reduced dimension, a powerful gimmick if the cardinality of  $\mathcal{A}_2^*$  is small. How one should build these approximation sets? The danger is to have an arc that moves from a set to its complement back and forth. We propose heuristic rules to avoid this bad behavior.

Suppose that our iterative procedure has generated  $Mx^k$ ,  $k = 1, \dots, K$ , with  $x^k \in X$ . From (25), we know that it is always possible to construct flows  $y^K$  that meet all the demands. We also recall that  $y^c$  is the meeting point between the linear and the nonlinear part. Assuming we are given a current partition of  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$  into an active set and its complement, the rules that move elements between  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are:

- An arc  $a \in \mathcal{A}_1$  such that  $y_a^K > y_a^c$  is moved into the active set  $\mathcal{A}_2$ .
- An arc  $a \in \mathcal{A}_2$  such that  $y_a^K \leq \gamma y_a^c$  is moved into the non active set  $\mathcal{A}_1$ .

Note that we introduce the parameter  $\gamma < 1$  to increase the chances that an arc that is made inactive at some stage will not become active later in the process. In practice, we get  $\gamma = 0.9$ .

In the experiments, we approximate the Kleinrock delay function and the BPR congestion function by compound functions and use an active set strategy. The algorithm generates an  $\epsilon$ -optimal primal-dual solution  $(y^*, u^*)$  for the compound function. The primal-dual pair can also be used to measure the relative optimality gap with the original functions  $\tilde{g}$  and  $\tilde{g}_*$ . This gap depends on the quality of the approximation by the compound function and thus may be larger than  $\epsilon$ .

## 7 Implementation issues

In this section, we review the main items in the implementation of our solution method.

### 7.1 First order oracle

The first order oracle consists of  $|\mathcal{T}|$  shortest path computations, using Dijkstra's algorithm [8]. This algorithm computes shortest paths from a single node to all other nodes in a directed graph. To compute the shortest paths for all commodities, we partition the commodities according to the origin node of the demand. This defines a subset of nodes  $\mathcal{S} \subset \mathcal{N}$ . We apply  $|\mathcal{S}|$  times Dijkstra's algorithm, once for each  $s \in \mathcal{S}$ . For large graphs, most of the computational time is devoted to data handling. To speed-up computation, the algorithm is implemented with binary heap structures. This implementation is efficient enough, but probably not on par with the state-of-the-art. A better implementation would most likely improve the performance of the overall algorithm, but the focus of the paper is on the cutting plane method and not on shortest path computation.

## 7.2 Parameter settings in ACCPM

Few parameters have to be set in ACCPM. The important ones are the coefficient of the proximal term and the proximal reference point; the weight on the logarithmic barrier on the floor cut; and a heuristic to eliminate almost inactive cutting planes.

### 7.2.1 Proximal reference point and proximal coefficient

The initial proximal reference point is the first query point. Thereafter, the proximal reference point is updated to the current query point whenever the oracle returns an objective function value that improves upon the best lower bound.

The initial value for the proximal coefficient  $\rho$  is 1. The rule to update this parameter is the following. When the method do not improve the lower bound  $\underline{\theta}$  during few iterations, it may happen that the weight of the generated cuts is too large pushing the query point to far from the proximal reference point. To fix this behavior, we increase the impact of the proximal term multiplying  $\rho$  by 10 to ensure the new query point to remain closest from the best recorded value. It thus makes it easier for ACCPM to find a best dual solution, i.e., a best lower bound.

### 7.2.2 Weight on floor cut

The localization set is bounded below by the special constraint  $z + \zeta \geq \underline{\theta}$  in (16). We name it the *floor cut*. It is easily checked that the floor cut makes a negative angle with the cutting planes. When the number of cutting planes increases, their total weight dominates the weight of the floor cut in (18). Thus, the floor cut tends to become active at the analytic center, with the possible effect of slowing the global convergence. To counteract this negative effect, we put a weight to the floor cut that equals the total number of generated cuts.

### 7.2.3 Column elimination strategy

It is well-known that column generation techniques are adversely affected by the total number of generated columns. This is particularly true with ACCPM, since the Newton iterations in the computation of the analytic center have a complexity that is roughly proportional to the square of the number of generated columns. It is thus natural to try to reduce the total number of columns by eliminating irrelevant elements. Our criterion to eliminate columns is based on the contribution of a column to the primal flow solution. Let  $\lambda$  be defined by equation (24). (We assume without loss of generality that  $e^T \lambda = 1$ .) Then  $A\lambda$  is the total flow on the network. If  $\lambda_i$  is much smaller than the average of  $\lambda$ , then column  $i$  contributes little to the solution (dually, the distance  $s_i$  between the analytic center and the cut is high,) and is a good candidate for elimination. To make the elimination test robust, we compute the median of the  $\lambda$ 's and eliminate any column whose coefficient  $\lambda_i$  is less than  $1/\kappa$  times the median. In practice, we choose  $\kappa = 4$ . We also perform the test once every  $\tau = 20$  iterations. (For the largest problem, we took  $\tau = 100$ .)

## 7.3 Termination criterion

The standard termination criterion is a small enough relative optimality gap:

$$(\bar{\theta} - \underline{\theta})/\max(\underline{\theta}, 1) \leq \epsilon, \tag{32}$$

where  $\underline{\theta}$  is the best lower bound computed with (23) and  $\bar{\theta}$  is the best upper bound computed with (26). In our experiments we use  $\epsilon = 10^{-5}$ .

## 7.4 Approximation scheme

The goal of the approximation scheme is to replace the original nonlinear congestion function  $\tilde{g}_a$  by a compound one  $g_a$ . We have

$$g_a(y_a) = \max_{y_a \in \text{dom } \tilde{g}_a} \{t_a y_a, \tilde{g}_a(y_a)\} \quad (33)$$

where  $t_a \geq \tilde{g}'_{a+}(0)$ . The linear and the nonlinear parts meet at  $y_a^c$ . Note that  $\text{dom } g_a = \text{dom } \tilde{g}_a$ . The error function induced by the linear approximation

$$e_a(y_a) = t_a y_a - \tilde{g}_a(y_a), \quad y_a \in [0, y_a^c].$$

Let  $\hat{y}_a \in [0, y_a^c]$  be the point that maximizes the error  $e_a$ . Let  $\mathcal{A}_1^*$  be the set of inactive arcs at the optimum. The error due to the approximation is bounded by  $\sum_{a \in \mathcal{A}_1^*} e_a(\hat{y}_a)$ . We want that this error to be lower than  $\epsilon \tilde{g}^*$ , where  $\epsilon$  is the relative optimality gap and  $\tilde{g}^*$  is the optimal objective value. Furthermore we impose that  $e_a(\hat{y}_a) = \mu$  be the same for all  $a \in \mathcal{A}_1^*$ . We estimate  $\mu$  by

$$\mu = \frac{\hat{\epsilon} \tilde{g}^*}{|\mathcal{A}_1^*|}, \quad \text{with } \hat{\epsilon} < \epsilon. \quad (34)$$

Then we compute  $t_a$  such that  $e_a(\hat{y}_a) = \mu$ . Unfortunately  $\tilde{g}^*$  and  $|\mathcal{A}_1^*|$  must be estimated. In the experiments, we take the parameter  $\hat{\epsilon} = 10^{-6}$  and  $|\mathcal{A}_1^*| = n/2$ . The value of  $\tilde{g}^*$  is chosen empirically depending on the class of problems.

## 8 Numerical experiments

The main goal of our empirical study is to test the efficiency i) of using a nonlinear cutting surface, ii) of column elimination, iii) of active set strategy and iv) of column elimination and active set strategy jointly. We also use published results to benchmark the new algorithm.

### 8.1 Test problems

We used four sets of test problems. The first set, the **planar** problems, contains 10 instances that have been generated by Di Yuan to simulate telecommunication problems. Nodes are randomly chosen as points in the plane, and arcs link neighbor nodes in such a way that the resulting graph is planar. Commodities are pairs of origin and destination nodes, chosen at random. Demands and capacities are uniformly distributed in given intervals. The second set, the **grid** problems, contains 15 networks that have a grid structure such that each node has four incoming and four outgoing arcs. Note that the number of paths between two nodes in a grid network is usually large. Commodities, and demands are generated in a way similar to that of **planar** networks. These two sets of problems are used to solve the linear multicommodity flow problem in [1, 18]. The data include arc capacities and linear costs and can be downloaded from <http://www.di.unipi.it/di/groups/optimize/Data/MMCF.html>. We use directly these arc capacities in the definition of the Kleinrock function. To solve (1) with BPR function,

Problem ID	$ \mathcal{N} $	$ \mathcal{A} $	$ \mathcal{K} $	$z_{Kleinrock}^*$	$z_{BPR}^*$
planar problems					
planar30	30	150	92	40.5668	$4.44549 \times 10^7$
planar50	50	250	267	109.478	$1.21236 \times 10^8$
planar80	80	440	543	232.321	$1.81906 \times 10^8$
planar100	100	532	1085	226.299	$2.29114 \times 10^8$
planar300	300	1680	3584	329.120	$6.90748 \times 10^8$
planar500	500	2842	3525	196.394	$4.83309 \times 10^9$
planar800	800	4388	12756	354.008	$1.16952 \times 10^9$
planar1000	1000	5200	20026	1250.92	$3.41859 \times 10^9$
planar2500	2500	12990	81430	3289.05	$1.23827 \times 10^{10}$
grid problems					
grid1	25	80	50	66.4002	$8.33599 \times 10^5$
grid2	25	80	100	194.512	$1.72689 \times 10^6$
grid3	100	360	50	84.5618	$1.53241 \times 10^6$
grid4	100	360	100	171.331	$3.05543 \times 10^6$
grid5	225	840	100	236.699	$5.07921 \times 10^6$
grid6	225	840	200	652.877	$1.05075 \times 10^7$
grid7	400	1520	400	776.566	$2.60669 \times 10^7$
grid8	625	2400	500	1542.15	$4.21240 \times 10^7$
grid9	625	2400	1000	2199.83	$8.36394 \times 10^7$
grid10	625	2400	2000	2212.89	$1.66084 \times 10^8$
grid11	625	2400	3000	1502.75	$3.32475 \times 10^8$
grid12	900	3480	6000	1478.93	$5.81488 \times 10^8$
grid13	900	3480	12000	1760.53	$1.16933 \times 10^9$
grid14	1225	4760	16000	1414.39	$1.81297 \times 10^9$
grid15	1225	4760	32000	1544.15	$3.61568 \times 10^9$
Telecommunication-like problems					
ndo22	14	22	23	11.5631	$1.87110 \times 10^3$
ndo148	58	148	122	151.926	$1.40233 \times 10^5$
904	106	904	11130	33.4931	$1.29197 \times 10^7$
Transportation problems					
Sioux-Falls	24	76	528	600.679	$4.23133 \times 10^6$
Winnipeg	1067	2975	4345	1527.41	$8.25673 \times 10^5$
Barcelona	1020	2522	7922	845.872	$1.23277 \times 10^6$
Chicago-sketch	933	2950	93513	615.883	$1.67484 \times 10^7$
Chicago-region	12982	39018	2297945	3290.55	$2.58457 \times 10^7$
Philadelphia	13389	40003	1151166	2558.01	$1.27810 \times 10^8$

Table 2: Test problems.

we use the capacity as *practical capacity* and the linear cost as *free-flow travel time*. As suggested in [27], we use the parameter values  $\alpha = 0.15$  and  $\beta = 4$ .

The third collection of problems is composed of telecommunication problems of various sizes. The small problems `ndo22` and `ndo148` are two practical problems solved in [12, 14]. Problem 904 is based on a real telecommunication network and was used in the survey paper [25]. This problem set is adapted to solve (1) with Kleinrock function. To solve (1) with BPR function, we use the capacity as *practical capacity* and also use it as *free-flow travel time*. We choose the parameter values  $\alpha = 0.15$  and  $\beta = 4$ .

The last collection of problems is composed of six realistic transportation problems used in [1, 2, 7, 17]. Some of them are huge, with up to 13,000 nodes, 39,000 arcs and over 2,000,000 commodities. The data are adapted for the BPR function. They include *free-flow travel time*, *practical capacity* and the tuning parameters  $\alpha$  and  $\beta$ . These problems, can be downloaded from <http://www.bgu.ac.il/~bargera/tntp/>. To solve (1) with Kleinrock function we use *practical capacity* as capacity and to turn these problems feasible with respect to the capacity, which is handled by the objective function, the demands are reduced as in [1, 17].

Table 2 displays data on the four sets of problems. For each problem instance, we give the number of nodes  $|\mathcal{N}|$ , the number of arcs  $|\mathcal{A}|$ , the number of commodities  $|\mathcal{K}|$ , the optimal solution values to (1)  $z_{Kleinrock}^*$  for the Kleinrock function and  $z_{BPR}^*$  for the BPR function, with a relative optimality gap less than  $10^{-5}$ .

## 8.2 Numerical results

We carry the experiments with the two congestion functions: the Kleinrock delay function and the BPR congestion function. For each solution strategy, we attempt to solve all problem instances contained in Table 2 with a  $10^{-5}$  relative optimality gap. To benchmark the results with our best solution strategy, we use, for telecommunications problems (Kleinrock function), the results with the Projection Method reported in [25] and, for transportation problems (BPR function) several implementations of Frank-Wolfe algorithm reported in [7].

For all results using ACCPM, the tables give the number of outer iterations, denoted *Outer*, the number of Newton's iteration, or inner iterations, denoted *Inner*, the computational time in seconds *CPU* and the percentage of CPU time denoted *%Or* spent to compute the shortest path problems. When the active set strategy is activated, the working space of ACCPM is reduced to the active arcs only. Thus, we give the percentage of arcs in the active set,  $\%|\mathcal{A}_2|$ , at the end of the solution process. We display also the error, denoted *Error*, leaded from the approximation with respect to the optimal solution of the original problem. Finally, when the elimination column is activated we display the number of remaining cuts *Nb cuts* at the end of the process.

The ACCPM code we use has been developed in Matlab, while the shortest path algorithm is written in C. The tests were performed on a PC (Pentium IV, 2.8 GHz, 2 Gb of RAM) under Linux operating system.

### 8.2.1 Impact of using a nonlinear cutting surface

In this subsection, we experiment the impact of the second order information in the solution method solving all the instances. We compare ACCPM using a second order oracle and ACCPM in which the smooth function is handled implicitly by the first order oracle as in the traditional approach.

The results are reported in Table 3 for the Kleinrock function and in Table 4 for the BPR function. In the two cases, we observe that the new approach outperforms the classical ACCPM. The larger problems are not solved by the classical ACCPM, partly because too many cuts in the localization set jammed the memory space.

Problem ID	ACCPM with cutting surface				ACCPM without cutting surface			
	Outer	Inner	CPU	%Or	Outer	Inner	CPU	%Or
planar30	93	197	1.1	22	832	1664	59.2	17
planar50	134	279	2.8	20	1234	2468	253.5	14
planar80	182	383	8.1	16	1965	3930	1381.3	11
planar100	187	392	10.2	17	2342	4684	2593.5	10
planar300	175	367	29.5	24	-	-	-	-
planar500	127	324	32.2	37	-	-	-	-
planar800	182	429	110.5	40	-	-	-	-
planar1000	381	869	568.1	26	-	-	-	-
planar2500	543	1224	3471.7	45	-	-	-	-
grid1	52	118	0.4	24	462	924	11.5	21
grid2	93	212	1.0	25	456	912	11.4	21
grid3	138	341	4.1	15	1713	3426	798.4	12
grid4	167	344	5.7	17	1613	3226	710.2	12
grid5	204	474	18.5	17	3409	6818	11005.7	8
grid6	333	686	55.9	14	3326	6652	10457.4	8
grid7	410	811	155.4	15	-	-	-	-
grid8	845	1783	1416.8	10	-	-	-	-
grid9	582	1269	576.9	15	-	-	-	-
grid10	432	964	300.6	20	-	-	-	-
grid11	261	581	106.4	29	-	-	-	-
grid12	201	409	106.7	41	-	-	-	-
grid13	222	454	128.7	39	-	-	-	-
grid14	204	414	173.2	48	-	-	-	-
grid15	203	414	172.8	48	-	-	-	-
ndo22	12	86	0.2	7	173	346	1.5	27
ndo148	70	361	1.3	7	737	1474	45.7	17
904	135	294	10.4	27	-	-	-	-
Sioux-falls	140	345	1.7	24	533	1410	13.0	15
Winnipeg	338	988	215.0	14	-	-	-	-
Barcelona	253	678	101.1	15	-	-	-	-
Chicago-sketch	145	370	48.6	41	-	-	-	-
Chicago-region	190	500	8621.9	94	-	-	-	-
Philadelphia	279	822	13094.4	89	-	-	-	-

Table 3: Impact of the cutting surface (Kleinrock delay function).

### 8.2.2 Impact of column elimination

In this subsection, ACCPM solves the sets of problems using the column elimination. We report the results in Table 5 for the Kleinrock function and in Table 6 for the BPR function. The last column *CPU Ratio* displays the improvement ratio of the CPU time of ACCPM without column elimination (see Table 3 and 4), and with column elimination. We observe that column elimination speed-up the CPU time on all problems, with an average value 1.5. Since the number of outer iterations is about the same, the speed-up is due to a reduction of the computation time in ACCPM. It is apparent in comparing the proportion of time spent in the oracle.

### 8.2.3 Impact of the active set strategy

Problem ID	ACCPM with cutting surface				ACCPM without cutting surface			
	Outer	Inner	CPU	%Or	Outer	Inner	CPU	%Or
planar30	56	256	1.2	25	202	502	5.7	15
planar50	96	422	3.4	15	328	775	21.8	11
planar80	159	665	13.0	12	651	1400	165.0	7
planar100	101	423	6.5	16	629	1341	172.3	8
planar300	98	358	18.3	21	1101	2398	1980.2	6
planar500	42	164	10.0	37	986	2261	2548.9	7
planar800	88	299	51.1	43	2155	4513	32792.7	5
planar1000	192	552	209.6	37	-	-	-	-
planar2500	364	1744	3099.2	38	-	-	-	-
grid1	24	108	0.4	26	148	329	1.8	30
grid2	49	202	0.8	25	238	557	3.5	27
grid3	31	121	0.7	23	259	586	13.4	16
grid4	57	216	1.7	22	386	859	35.3	10
grid5	60	199	3.5	23	557	1230	174.2	7
grid6	125	385	11.5	16	918	1931	660.0	6
grid7	102	307	15.1	22	1111	2405	1872.9	5
grid8	158	422	49.5	23	1982	4119	13010.7	3
grid9	211	597	97.3	22	2379	4923	22202.1	3
grid10	207	586	94.8	24	2404	4965	22925.9	3
grid11	138	413	47.9	31	1966	4082	12873.5	4
grid12	107	323	52.7	46	2002	4369	20391.3	4
grid13	117	340	59.1	44	2300	4785	28038.6	4
grid14	84	274	61.9	56	1995	4179	26729.3	5
grid15	93	293	70.2	55	1588	4011	17449.6	4
ndo22	4	35	0.1	0	75	287	0.8	23
ndo148	6	45	0.2	0	171	390	3.6	24
904	93	316	8.4	19	802	1729	470.2	6
Sioux-falls	80	411	2.0	19	366	1057	10.1	20
Winnipeg	81	298	16.3	36	1307	2783	3352.8	8
Barcelona	56	245	10.4	29	925	2040	1493.7	8
Chicago-sketch	72	265	20.4	48	1828	4075	11891.0	5
Chicago-region	332	1502	10606.5	64	-	-	-	-
Philadelphia	287	1250	7469.9	63	-	-	-	-

Table 4: Impact of the cutting surface (BPR congestion function).

Problem ID	Nb cuts	Outer	Inner	CPU	%Or	ratio
planar30	54	114	293	1.1	26	1.0
planar50	67	164	385	2.2	27	1.3
planar80	126	239	544	6.5	24	1.2
planar100	95	208	481	6.0	24	1.7
planar300	104	205	484	22.2	32	1.3
planar500	75	129	319	24.1	47	1.3
planar800	99	179	447	77.1	54	1.4
planar1000	207	369	836	303.6	43	1.9
planar2500	339	567	1292	2398.5	64	1.5
grid1	45	59	194	0.5	27	0.8
grid2	55	108	242	0.8	46	1.0
grid3	71	121	307	2.3	22	1.8
grid4	71	188	394	3.1	31	1.8
grid5	137	203	472	12.0	21	1.5
grid6	113	389	828	24.3	26	2.3
grid7	244	471	1024	90.9	22	1.7
grid8	311	876	1845	384.0	21	3.7
grid9	344	646	1397	305.5	24	1.9
grid10	274	474	1039	199.8	29	1.5
grid11	228	270	599	96.6	32	1.1
grid12	189	212	431	107.2	43	1.0
grid13	207	234	478	125.8	41	1.0
grid14	183	212	430	166.5	51	1.0
grid15	181	205	418	161.5	52	1.1
ndo22	12	12	86	0.2	7	1.0
ndo148	47	73	176	0.8	25	1.6
904	85	138	300	7.6	27	1.4
Sioux-falls	73	144	353	1.6	25	1.1
Winnipeg	202	384	1155	135.3	19	1.6
Barcelona	182	261	732	71.2	19	1.4
Chicago-sketch	77	130	340	30.1	53	1.6
Chicago-region	92	194	597	8672.1	95	1.0
Philadelphia	109	290	826	13021.8	93	1.0

Table 5: Impact of column elimination (Kleinrock delay function).

Problem ID	Nb cuts	Outer	Inner	CPU	%Or	ratio
planar30	26	57	276	1.2	21	1.0
planar50	34	95	418	2.9	23	1.2
planar80	39	167	611	6.7	21	1.9
planar100	25	99	383	4.3	23	1.5
planar300	42	98	345	12.6	33	1.5
planar500	24	40	156	8.6	44	1.2
planar800	36	76	250	34.2	56	1.5
planar1000	72	177	519	135.1	51	1.5
planar2500	120	346	1350	1657.9	66	1.9
grid1	15	24	105	0.4	33	1.0
grid2	27	52	194	0.8	33	1.0
grid3	20	30	114	0.7	27	1.0
grid4	20	55	194	1.5	28	1.1
grid5	30	61	194	2.4	9	1.5
grid6	39	139	380	6.8	24	1.7
grid7	34	100	283	9.2	32	1.6
grid8	51	166	441	30.6	35	1.6
grid9	57	217	578	46.4	40	2.1
grid10	52	206	560	45.6	44	2.1
grid11	39	144	398	30.0	48	1.6
grid12	35	100	287	34.5	60	1.5
grid13	37	122	346	44.2	60	1.3
grid14	28	88	270	53.1	67	1.2
grid15	33	99	293	59.9	68	1.2
ndo22	4	4	35	0.1	0	1.0
ndo148	6	6	45	0.2	0	1.0
904	37	100	311	6.2	27	1.4
Sioux-falls	28	83	346	1.4	30	1.4
Winnipeg	36	76	259	10.6	47	1.5
Barcelona	20	55	231	7.9	38	1.3
Chicago-sketch	36	84	272	18.3	57	1.1
Chicago-region	124	342	1327	8224.8	85	1.2
Philadelphia	95	254	1031	4962.5	83	1.5

Table 6: Impact of column elimination (BPR congestion function).

In this subsection, we experiment the active strategy to solve (1) with the Kleinrock and BPR functions. This strategy turns out to be efficient only with the BPR function, but not with the Kleinrock function. The very steep slope of Kleinrock close its asymptote leads to a larger spread of the flows on the arcs. All arcs turn out to be moderately congested and the compound function does not provide a satisfactory approximation.

Table 7 gives the computational results using the active set strategy on the approximate BPR function. The last column, shows the improvement ratio of CPU time of ACCPM without active set strategy (see Table 4), and with the active set strategy. The value of  $g^*$  in (34) is empirical. We get  $g^* = 10^8$  for telecommunication instances (**planar**, **grid** and telecommunication-like problems) and  $g^* = 10^7$  for traffic networks.

Table 7 shows that active set strategy speed-up the CPU time on all problems (excepted

Problem ID	Error	%  $A_2$	Outer	Inner	CPU	%Or	ratio
planar30	$< 10^{-5}$	53	58	299	1.2	36	1.0
planar50	$< 10^{-5}$	63	99	473	2.9	33	1.2
planar80	$< 10^{-5}$	62	154	803	9.4	24	1.4
planar100	$< 10^{-5}$	60	97	536	4.9	29	1.3
planar300	$< 10^{-5}$	44	82	399	9.2	38	2.0
planar500	$< 10^{-5}$	26	38	216	5.8	62	1.7
planar800	$< 10^{-5}$	26	77	383	29.6	68	1.7
planar1000	$< 10^{-5}$	40	168	775	132.2	54	1.6
planar2500	$< 10^{-5}$	43	323	1996	2063.3	51	1.5
grid1	$< 10^{-5}$	86	25	118	0.4	32	1.0
grid2	$< 10^{-5}$	99	49	210	0.8	31	1.0
grid3	$< 10^{-5}$	40	25	122	0.6	33	1.1
grid4	$< 10^{-5}$	50	52	222	1.7	34	1.0
grid5	$< 10^{-5}$	47	50	224	2.3	37	1.5
grid6	$< 10^{-5}$	63	107	400	8.6	31	1.3
grid7	$< 10^{-5}$	52	76	312	8.8	39	1.7
grid8	$< 10^{-5}$	54	113	437	26.1	38	1.9
grid9	$< 10^{-5}$	64	178	612	60.8	34	1.6
grid10	$< 10^{-5}$	66	195	661	74.5	34	1.3
grid11	$< 10^{-5}$	61	139	475	41.9	43	1.1
grid12	$< 10^{-5}$	51	87	330	34.6	59	1.5
grid13	$< 10^{-5}$	57	113	448	53.4	51	1.1
grid14	$< 10^{-5}$	44	78	323	47.2	69	1.3
grid15	$< 10^{-5}$	49	88	346	56.9	66	1.2
ndo22	$< 10^{-5}$	50	5	46	0.1	-10	1.0
ndo148	$< 10^{-5}$	75	6	49	0.2	17	1.0
904	$< 10^{-5}$	32	93	358	5.1	42	1.6
Sioux-falls	$< 10^{-5}$	100	69	336	1.7	30	1.2
Winnipeg	$1.3 \times 10^{-5}$	33	48	246	6.8	49	2.4
Barcelona	$< 10^{-5}$	24	37	241	3.8	51	2.7
Chicago-sketch	$< 10^{-5}$	52	65	262	16.9	56	1.2
Chicago-region	$5 \times 10^{-4}$	34	55	698	1261.3	87	8.4
Philadelphia	$5 \times 10^{-5}$	54	97	1234	2157.9	74	3.5

Table 7: Impact of the active set strategy (BPR congestion function).

the smaller ones) until 8.4. This speed-up is partly due to the large number of inactive arcs in the optimal solution. The number of dual variables handled by ACCPM is usually lower than 60%. A second explanation is the reduction of the total number of outer iteration around 10%. Removing the inactive arcs from the Lagrangian relaxation seems to make ACCPM easier the converge. The point is important is that the quality of the optimal solution is not affected by the approximation, i.e., the computed optimal solution for the approximate problem is also a optimal solution with  $10^{-5}$  optimality gap for the original problem. For three instances, the approximation doesn't ensure a  $10^{-5}$  optimality

gap but it is also traduced by a larger decrease of number of outer iterations, of size of the active set, and obviously of CPU time. This observation shows the difficulties to guaranty a given optimality gap in a static approximation scheme.

### 8.2.4 Impact of active set strategy with column elimination

In this set of experiments, we combine both column elimination and the active set strategy. Obviously, since active set strategy is not efficient with Kleinrock function, we solve only (1) with the BPR congestion function. The settings of the active set strategy and the column elimination are those used in the previous subsections. The results are displayed on Table 8. In the last column of the table, we give the improvement ratio of the CPU time of ACCPM without using the two options (see Table 4), and with using them. As expected, column elimination reduces the computational time. As in subsection 8.2.2, it decreases the time spent in computing the analytic centers.

Problem ID	Error	$\% A_2 $	Nb cuts	Outer	Inner	CPU	%Or	ratio
planar30	$< 10^{-5}$	53	26	58	315	1.2	30	1.0
planar50	$< 10^{-5}$	63	26	92	475	2.9	31	1.2
planar80	$< 10^{-5}$	63	42	196	896	7.6	33	1.7
planar100	$< 10^{-5}$	60	36	98	506	4.2	32	1.5
planar300	$< 10^{-5}$	44	35	96	434	9.4	50	1.9
planar500	$< 10^{-5}$	26	16	34	196	5.1	70	2.0
planar800	$< 10^{-5}$	27	31	78	363	27.1	77	1.9
planar1000	$< 10^{-5}$	20	78	177	815	113.6	64	1.9
planar2500	$< 10^{-5}$	43	118	354	1898	1540.0	73	2.0
grid1	$< 10^{-5}$	86	16	25	117	0.4	34	1.0
grid2	$< 10^{-5}$	97	21	56	229	0.8	31	1.0
grid3	$< 10^{-5}$	40	18	26	125	0.7	33	1.0
grid4	$< 10^{-5}$	50	27	51	197	1.4	40	1.2
grid5	$< 10^{-5}$	47	22	45	207	1.9	39	1.8
grid6	$< 10^{-5}$	63	47	115	384	6.4	41	1.8
grid7	$< 10^{-5}$	52	37	73	290	6.7	47	2.3
grid8	$< 10^{-5}$	54	45	116	413	19.1	50	2.6
grid9	$< 10^{-5}$	64	45	177	680	39.0	48	2.5
grid10	$< 10^{-5}$	67	42	188	605	40.9	55	2.3
grid11	$< 10^{-5}$	61	44	139	455	28.4	59	1.7
grid12	$< 10^{-5}$	51	32	89	321	29.4	70	1.8
grid13	$< 10^{-5}$	57	35	112	418	38.9	64	1.5
grid14	$< 10^{-5}$	44	33	74	324	40.8	75	1.5
grid15	$< 10^{-5}$	49	35	84	328	47.4	75	1.5
ndo22	$< 10^{-5}$	50	5	5	46	0.1	-12	1.0
ndo148	$< 10^{-5}$	75	6	6	49	0.1	23	2.0
904	$< 10^{-5}$	32	37	114	358	3.5	55	2.4
Sioux-falls	$< 10^{-5}$	100	31	93	413	1.5	40	1.3
Winnipeg	$1.4 \times 10^{-5}$	33	21	47	251	5.7	51	2.9
Barcelona	$< 10^{-5}$	24	18	35	213	3.1	54	3.3
Chicago-sketch	$< 10^{-5}$	52	29	68	269	15.1	64	1.3
Chicago-region	$5 \times 10^{-4}$	34	22	55	681	1229.5	89	8.6
Philadelphia	$5 \times 10^{-5}$	54	41	99	1185	2004.0	81	3.7

Table 8: Active set strategy and column elimination (BPR congestion function).

### 8.2.5 Comparisons with other methods

In this subsection, we compare ACCPM with a Projection Method on telecommunication problems using the Kleinrock delay function. We also compare ACCPM with several implementations of Frank-Wolfe algorithm on transportation problems with the BPR

congestion function.

### ACCPM vs. the Projection Method

In this experiment, we compare the results of our solution method ACCPM using column elimination with the results of the Projection Method (PM) reported in [25]. As in [25], we solve problem 904 with a varying load factor to generate different demands. Table 9 gives the load factors we use and the corresponding optimal value with a  $10^{-5}$  relative optimality gap.

Problem ID	Load factor	$z_{Kleinrock}^*$
904	1	33.4931
904(1.5)	1.5	52.2678
904(2)	2	72.6437
904(2.5)	2.5	94.8839
904(3)	3	119.305

Table 9: Test problems.

In [25], the authors compare a previous version of ACCPM implemented in [14] with the Flow Deviation Method [19], the Projection Method [4], and the Proximal Decomposition Method [21]. In this comparative study, the Projection Method (PM) appears to be the most efficient method to solve the 904 instances. We use the figures reported in [25] for PM and ACCPM <sup>2</sup>.

The computational tests in [25] are performed on an IBM RISC/System 6000. We report the original computing times in Table 10. In order to compare these results with those we obtain with the new version of ACCPM on a Pentium IV, we have performed benchmark computations according to BYTEmark<sup>3</sup>. We found a ratio 14. We use this value to compare the speeds of the algorithms in the two columns entitled *ratio* in Table 10. These ratios are just indicative.

Problem ID	Nb cuts	ACCPM				previous ACCPM*			PM*		
		Outer	Inner	CPU	%Or	Outer	CPU	ratio	Outer	CPU	ratio
904	85	138	300	7.6	27	14	3233	30.4	579	380	3.6
904(1.5)	105	150	314	9.8	24	14	3441	25.0	663	434	3.2
904(2)	109	172	357	11.8	24	14	3186	19.3	688	471	2.8
904(2.5)	107	189	398	13.9	24	13	3276	16.8	741	558	2.9
904(3)	101	192	415	12.6	24	13	3544	20.1	691	501	2.8

\* Tests performed in [25] on an IBM RISC/System 6000 machine.

Table 10: ACCPM and PM.

Table 10 shows that the new ACCPM using column elimination outperforms the previous version of ACCPM and improves the computational time of the Projection Method with a ratio 3.

### ACCPM vs. Frank-Wolfe algorithm

<sup>2</sup>The ACCPM version [14] works on a disaggregated form of the objective function. It exploits the sparsity in the master problem to cope with the very large number of generated cuts. In the disaggregated approach, the oracle generates as many cuts as the number of commodities at each outer iteration. In the case of problem 904, this means 11130 cuts.

<sup>3</sup>BYTE Magazine's BYTEmark benchmark program (release 2) available at <http://www.byte.com/bmark/bmark.htm>.

In this experiment, we compare ACCPM with the results obtained in [7] with different versions of Frank-Wolfe algorithm: a classical Frank-Wolfe method (FW), a conjugate direction Frank-Wolfe method (CFW) and a bi-conjugate Frank-Wolfe method (BFW). These methods outperform the Frank-Wolfe method implemented in [2]. We solve the same set of transportation problems as in [7], with BPR function with a  $10^{-4}$  relative gap.

Since we have not at our disposal the machine used in [7], we cannot compare the computational times. To get an idea of performance, we focus on the number of iterations to solve the problems. In this experiment, we do not use the active set strategy to have perfect control on the precision of the optimal solution. The results displayed on Table

Problem ID	ACCPM	BFW	CFW	FW
Sioux-falls	44	124	357	1869
Winnipeg	39	163	243	838
Barcelona	31	41	34	51
Chicago-sketch	27	21	17	24
Chicago-region	115	43	53	126

Table 11: ACCPM vs. Frank-Wolfe.

11 show that ACCPM is competitive with the implementations of Frank-Wolfe algorithm in term of number of iterations, except for the last instance. ACCPM is more efficient on the smaller instances.

## 9 Conclusion

In this paper, we proposed two important modifications of the analytic center cutting plane method to solve nonlinear multicommodity flow problems: a cutting surface to handle the smooth component of the Lagrangian dual objective and an approximation scheme for the nonsmooth component of that objective. The approximation scheme is coupled with an active set strategy that leads to an expression of the Lagrangian dual in a space of smaller dimension. The new approach considerably improves the performance of the former implementation of ACCPM. It compares favorably with the known most efficient methods.

The computation in a Lagrangian relaxation approach breaks into two main components: computation of the query point (an approximate analytic center of the localization set, in our case) and the solving of the oracle subproblem (shortest paths computation, in our case). The overall computation depends on the computational efficiency of these two operations and on the number of times they are performed, i.e., on the number of calls to the oracle. In this paper, we focused on two items: the number of outer iterations and the computation of the query point. In contrast, our implementation of the shortest path algorithm is rather straightforward and could be improved, in particular if one wishes to exploit special network structures.

The present study suggests that possible further improvements could be achieved using the approximation/active set approach with a different linearization scheme for the cost function. Conceptually, this linearization could be performed around points that are dynamically chosen to lead more efficient approximations. This will be the object of further researches.

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