

# A Note on KKT Points of Homogeneous Programs <sup>1</sup>,

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**Abstract.** Homogeneous programming is an important class of optimization problems. The purpose of this note is to give a truly equivalent characterization of KKT-points of homogeneous programming problems, which corrects a result given by Lasserre and Hiriart-Urruty in Ref. 9.

**Key words.** Homogeneous programming, KKT-point, quadratic optimization

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# 1 Introduction

Denote by  $R^n$  the  $n$ -dimensional Euclidean space, and  $R_+^n$  the nonnegative orthant. Let  $\mathcal{D}$  be a subset in  $R^n$ . A function  $\phi : R^n \rightarrow R$  is said to be (positively) homogeneous with  $p$ -degree on the set  $\mathcal{D}$  if  $\phi(\lambda x) = \lambda^p \phi(x)$  for all  $\lambda > 0$  and  $x \in \mathcal{D}$ .

Given functions  $f : R^n \rightarrow R$  and  $g_i : R^n \rightarrow R$  ( $i = 1, \dots, m$ ), an optimization problem of the form

$$\min\{f(x) : g_i(x) = (\leq) b_i, i = 1, \dots, m\} \quad (1)$$

is said to be a homogeneous program if all  $f$  and  $g_i$ ,  $i = 1, \dots, m$ , are homogeneous on  $R^n$ . Homogeneous programs include linear programs and semi-definite programs (Ref. 1) as their special cases. Quadratic optimization (see for example, Refs. 2-3) and some other specific nonlinear programming problems can be also reformulated as homogeneous programs.

Properties of homogeneous optimization problems have been studied by many authors. See for example, Eisenberg (Ref. 4), Rubinov and Glover (Ref. 5), Gwinner (Refs. 6-7), Bagirov and Rubinov (Ref. 8). Recently, Lasserre and Hiriart-Urruty (Ref. 9) proved that under some suitable conditions, a homogeneous problem can be reformulated as a convex linear matrix inequality problem. Their result can be viewed as a new duality result for homogeneous programs. However, "Theorem 2.1" in Ref. 9, which states the property of KKT-points of homogeneous programming, is not correct. The purpose of this note is to correct such a result.

Assume that  $f$  and  $g_i$  ( $i = 1, \dots, m$ ) are continuous and homogeneous with  $p$ -degree and  $q_i$ -degree ( $i = 1, \dots, m$ ) on  $R^n$ , respectively. Lasserre and Hiriart-Urruty (Ref. 9) studied the relation between problem (1) in the case of inequalities and the following model:

$$\begin{aligned} \min \quad & u f(x) \\ \text{s.t.} \quad & u [g_i(x) - b_i(1 - q_i/p)] \leq (b_i q_i)/p, \quad i = 1, \dots, m, \\ & u \geq 0. \end{aligned} \quad (2)$$

However, there exist several weak points in their model and their "Theorem 2.1".

(i) As illustrated by Example 2.1 below, their result "Theorem 2.1" does not hold even for the case  $p = q_i$  for all  $i = 1, \dots, m$ .

(ii) Their result actually requires an assumption of  $\sum_{i=1}^m \lambda_i^* b_i^* \neq 0$ . They did not state this condition in their "Theorem 2.1", but they used this assumption at the end of the proof of their result. In fact, as the following example illustrates, their result fails to hold without this assumption.

(iii) If  $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})$  is a KKT-point of (2), there is no guarantee that  $\bar{u} > 0$ . Even when  $\bar{u} > 0$ , it cannot assure that  $\bar{u} = 1$ . See the following example.

**Example 1.1.** Let  $Q$  be a symmetric positive definite matrix,  $Q'$  be any symmetric matrix, and the scalar  $\gamma$  be a positive number. Consider the problem:

$$\min \left\{ (1/2)x^T Q x : (1/2)x^T Q' x \leq \gamma, x \in R^n \right\}.$$

Clearly,  $x = 0$  is the unique solution to the above problem. Consider the Lasserre and Hiriart-Urruty's model:

$$\min \left\{ u \left( (1/2)x^T Q x \right) : \left( (1/2)x^T Q' x \right) u \leq \gamma, x \in R^n, u \geq 0 \right\}.$$

Its optimality conditions are given as

$$\begin{aligned} \bar{u}(Q\bar{x} + \lambda Q'\bar{x}) &= 0, \quad (1/2)\bar{x}^T Q\bar{x} + (1/2)\lambda\bar{x}Q'\bar{x} - \bar{\mu} = 0, \quad \lambda \geq 0, \\ \lambda \left( (1/2)\bar{x}^T Q'\bar{x}\bar{u} - \gamma \right) &= 0, \quad ((1/2)\bar{x}Q'\bar{x})\bar{u} \leq \gamma, \quad \bar{u}\bar{\mu} = 0, \quad \bar{\mu} \geq 0, \quad \bar{u} \geq 0. \end{aligned}$$

Then all points  $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}) = (0, \bar{u} \in [0, \infty), 0, 0)$  are KKT-points of the problem. We note that  $\bar{u}$  can take any nonnegative number. In particular,  $\bar{u}$  can take zero. Even if  $\bar{u} > 0$ , it is not necessarily to be 1. This example shows that the result of "Theorem 2.1" of Lasserre and Hiriart-Urruty (Ref. 9) is incorrect. It is worth noting in this example  $p = q = 2$  and  $\sum_{i=1}^m \lambda_i^* b_i = 0$ . Therefore, Lasserre and Hiriart-Urruty's result "Theorem 2.1" does not really show the one-to-one correspondence of the KKT-points between problems (1) and (2).

In next section, we derive a truly equivalent characterization for KKT points of the homogeneous program.

## 2 Properties of KKT-points of homogeneous programs.

Note that the model (2) is linear with respect to  $u$ . In order to remove the above-mentioned weakness, we modify the model (2) and consider its quadratic version in  $u$ . As a result, we can give an equivalent characterization for KKT-points of homogeneous optimization problems.

We first focus our attention on the following homogeneous programming problem whose objective and constraint functions have the same homogeneous degrees:

$$\min \{ f(x) : g_i(x) = b_i, i = 1, \dots, m \}, \quad (3)$$

where  $f$  and  $g_i$  ( $i = 1, \dots, m$ ) are continuous and homogeneous with  $p$ -degree on  $R^n$ .

Let  $0 < \alpha \leq 1$  be a fixed scalar. Following an idea due to Ref. 9, we consider here a modification for their auxiliary problem to derive an equivalent characterization of KKT

points:

$$\begin{aligned}
\min \quad & F_\alpha(x, u) := (u + \alpha)f(x) + (1/2)(u + \alpha - 1)^2 \\
\text{s.t.} \quad & (u + \alpha)g_i(x) = b_i, \quad i = 1, \dots, m, \\
& u \geq 0.
\end{aligned} \tag{4}$$

In particular, setting  $\alpha = 1$ , problem (4) reduces to

$$\begin{aligned}
\min \quad & F(x, u) := (u + 1)f(x) + (1/2)u^2 \\
\text{s.t.} \quad & (u + 1)g_i(x) = b_i, \quad i = 1, \dots, m, \\
& u \geq 0.
\end{aligned} \tag{5}$$

For generality, we consider (4) instead of (5). All the results about (5) are special cases of those from (4). We are ready to prove that the KKT-points of problem (3) are of one-to-one correspondence to the KKT-points of problem (4).

**Theorem 2.1.** Let  $f$  and  $g_i$  ( $i = 1, \dots, m$ ) be continuously differentiable and be homogeneous with  $p$ -degree on  $R^n$ .

(i) If  $x^*$  is a KKT-point of problem (3) with associated Lagrange multiplier  $\lambda^* \in R^m$ , then  $(\hat{x}, \hat{u}) = (x^*, 1 - \alpha)$  is a KKT-point of problem (4) with Lagrange multiplier  $(\hat{\lambda}, \hat{\mu}) = (\lambda^*, 0) \in R^m \times R_+$ .

(ii) Conversely, if  $(\hat{x}, \hat{u})$  is a KKT-point of problem (4) with associated Lagrange multiplier  $(\hat{\lambda}, \hat{\mu}) \in R^m \times R_+$ , then it must have  $\hat{u} = 1 - \alpha$  and  $\hat{\mu} = 0$ , and hence  $\hat{x}$  is a KKT-point of (3) and  $\hat{\lambda}$  is just the Lagrange multiplier corresponding to  $\hat{x}$ .

**Proof.** We first note that  $x^*$  is a KKT-point of problem (3) with multiplier  $\lambda^*$  if and only if it satisfies the following conditions

$$\begin{aligned}
\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) &= 0, \\
g_i(x^*) &= b_i, \quad i = 1, \dots, m,
\end{aligned} \tag{6}$$

and that  $(\hat{x}, \hat{u})$  is a KKT-point of problem (4) with multipliers  $(\hat{\lambda}, \hat{\mu})$  if and only if it satisfies the following conditions

$$(\hat{u} + \alpha) \left( \nabla f(\hat{x}) + \sum_{i=1}^m \hat{\lambda}_i \nabla g_i(\hat{x}) \right) = 0, \tag{8}$$

$$f(\hat{x}) + (\hat{u} + \alpha - 1) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{x}) - \hat{\mu} = 0, \quad (9)$$

$$(\hat{u} + \alpha) g_i(\hat{x}) = b_i, \quad i = 1, \dots, m, \quad (10)$$

$$\hat{\mu} \geq 0, \quad \hat{u} \geq 0, \quad \hat{u}\hat{\mu} = 0. \quad (11)$$

(i) Assume that  $x^*$  is a KKT-point of (3) with multiplier  $\lambda^*$ . It is evident that  $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu}) = (x^*, 1 - \alpha, \lambda^*, 0)$  satisfies the conditions (8), (10) and (11). We now show that (9) is also satisfied. In fact, multiplying both sides of (6) by  $x^*$ , we have

$$\langle x^*, \nabla f(x^*) \rangle + \sum_{i=1}^m \lambda_i^* \langle x^*, \nabla g_i(x^*) \rangle = 0.$$

By Euler's property of homogeneous functions, i.e.,  $\langle x, \nabla f(x) \rangle = pf(x)$ , the above equality can be written as

$$p \left( f(x^*) + \sum_{i=1}^m \lambda_i^* g_i(x^*) \right) = 0,$$

which indicates that (9) holds with  $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu}) = (x^*, 1 - \alpha, \lambda^*, 0)$ . Result (i) is proved.

(ii) Assume that  $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$  satisfies conditions (8)-(11). We show that  $(x^*, \lambda^*) = (\hat{x}, \hat{\lambda})$  satisfies (6) and (7). Since  $\hat{u} + \alpha > 0$ , (8) reduces to

$$\nabla f(\hat{x}) + \sum_{i=1}^m \hat{\lambda}_i \nabla g_i(\hat{x}) = 0.$$

Thus (6) is satisfied. Furthermore, the above equality, by Euler's property, implies that

$$f(\hat{x}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{x}) = 0.$$

Hence, it follows from (9) that

$$\hat{\mu} = \hat{u} + \alpha - 1.$$

Since  $\hat{\mu} \geq 0$ , it follows that  $\hat{u} \geq 1 - \alpha \geq 0$ . Multiplying both sides of the above equality by  $\hat{u}$  and noting that  $\hat{u}\hat{\mu} = 0$ , we have

$$\hat{u}(\hat{u} + \alpha - 1) = 0,$$

which, together with the fact  $\hat{u} \geq 1 - \alpha \geq 0$ , implies that  $\hat{u} = 1 - \alpha$ . This in turn implies that  $\hat{\mu} = 0$ . Therefore, (9) reduces to  $g_i(\hat{x}) = b_i$  ( $i = 1, \dots, m$ ), i.e., condition (7) holds.  $\square$

In particular, setting  $\alpha = 1$ , we have the following result.

**Corollary 2.1.** Let  $f$  and  $g_i$  ( $i = 1, \dots, m$ ) be given as in Theorem 2.1.

(i) If  $x^*$  is a KKT-point of problem (3) with associated Lagrange multiplier  $\lambda^* \in R^m$ , then  $(\hat{x}, \hat{u}) = (x^*, 0)$  is a KKT-point of problem (4) with Lagrange multiplier  $(\hat{\lambda}, \hat{\mu}) = (\lambda^*, 0)$ .

(ii) Conversely, if  $(\hat{x}, \hat{u})$  is a KKT-point of problem (4) with associated Lagrange multiplier  $(\hat{\lambda}, \hat{\mu}) \in R^m \times R_+$ , then it must have  $\hat{u} = 0$  and  $\hat{\mu} = 0$ , and hence  $\hat{x}$  must be a KKT-point of problem (3) and  $\hat{\lambda}$  is just the Lagrange multiplier with respect to  $\hat{x}$ .

Furthermore, based on the above result, the next result shows the one-to-one correspondence of the global optimal solutions of (3) and (4) without requiring any assumption other than the homogeneity of  $f$  and  $g_i$ ,  $i = 1, 2, \dots, m$ .

**Theorem 2.2.** Let  $f$  and  $g_i$  ( $i = 1, \dots, m$ ) be homogeneous with  $p$ -degree on  $R^n$ . Then  $x^*$  is a global optimal solution to problem (3) if and only if  $(x^*, u^*)$ , where  $u^* = 1 - \alpha$ , is a global optimal solution to problem (4). Both problems have the same global optimal values.

**Proof.** Let  $S$  and  $D$  denote the feasible sets of problems (3) and (4), respectively. If  $x^*$  is a global optimal solution to (3), then  $(x^*, u^*) = (x^*, 1 - \alpha)$  is a feasible point of (4). Thus, we have

$$\min_{x \in S} f(x) = f(x^*) = F(x^*, u^*) \geq \min_{(x, u) \in D} F_\alpha(x, u). \quad (12)$$

It is sufficient to prove the converse. Let  $(x^*, u^*)$  be a global optimal solution to (4). We now show that  $x^*$  must be a global optimal solution to (3). Let  $y^* = (u^* + \alpha)^{1/p} x^*$ . We note that  $y^*$  is a feasible point to problem (3). Indeed, by the homogeneity of  $g_i$ , we have

$$g_i(y^*) = g_i((u^* + \alpha)^{1/p} x^*) = (u^* + \alpha) g_i(x^*) = b_i, \quad i = 1, \dots, m.$$

Therefore,

$$\begin{aligned} \min_{x \in S} f(x) \leq f(y^*) &= (u^* + \alpha) f(x^*) \\ &\leq (u^* + \alpha) f(x^*) + (1/2)(u^* + \alpha - 1)^2 \\ &= \min_{(x, u) \in D} F_\alpha(x, u). \end{aligned} \quad (13)$$

The first equality follows from the homogeneity of  $f$ . Combination of (12) and (13) implies that  $\min_{x \in S} f(x) = \min_{(x, u) \in D} F_\alpha(x, u)$ . This in turn implies that both inequalities in (13) are actually equalities. Thus  $u^* = 1 - \alpha$  and  $\min_{x \in S} f(x) = f(y^*)$  which indicates that  $y^*$  is a global optimal solution to the problem (3). The fact  $u^* = 1 - \alpha$  implies that  $y^* = (u^* + \alpha)^{1/p} x^* = x^*$ , and hence  $x^*$  is a global optimal solution to the problem (3).  $\square$

We now claim that the result in Theorem 2.1 can be extended to the case of inequality constraints, and to the case where the functions have different homogeneous degrees. Consider the following problem:

$$\min\{f(x) : g_i(x) \leq b_i, i = 1, \dots, m\}, \quad (14)$$

where  $f$  is homogeneous with  $p$ -degree and  $g_i$  is homogeneous with  $q_i$ -degree, where  $i = 1, \dots, m$ . Let  $0 < \alpha \leq 1$  be a positive scalar. Introducing a variable  $u$  to the above problem, we get an auxiliary problem below:

$$\begin{aligned} \min \quad & (u + \alpha)f(x) + (1/2)(u + \alpha - 1)^2 \\ \text{s.t.} \quad & (u + \alpha)[g_i(x) - b_i(1 - q_i/p)] \leq (b_i q_i)/p, i = 1, \dots, m, \\ & u \geq 0. \end{aligned} \quad (15)$$

The relation of KKT-points of problems (14) and (15) is disclosed by the following result.

**Theorem 2.3.** Assume that  $f$  and  $g_i$  ( $i = 1, \dots, m$ ) are differentiable and homogeneous with  $p$ -degree and  $q_i$ -degree on  $R^n$ , respectively.

(i) If  $x^*$  is a KKT-point of (14) with associated Lagrange multiplier  $\lambda^* \in R_+^m$ , then  $(x^*, u^*)$ , where  $u^* = 1 - \alpha$ , is a KKT-point of (15) with associated Lagrange multiplier  $(\lambda^*, \mu^* = 0)$ .

(ii) If  $(\bar{x}, \bar{u})$  is a KKT-point of (15) with associated Lagrange multiplier  $(\bar{\lambda}, \bar{\mu}) \in R_+^{m+1}$ , and

$$\sum_{i=1}^m \bar{\lambda}_i b_i (1 - q_i/p) (q_i/p) < \bar{u} + \alpha, \quad (16)$$

then  $\bar{u} = 1 - \alpha$ ,  $\bar{\mu} = 0$ , and hence  $\bar{x}$  is a KKT-point of (14) and  $\bar{\lambda}$  is a Lagrange multiplier associated with  $\bar{x}$ .

**Remark 2.1.** We note that inequality (16) holds trivially if one of the following holds:

- $f$  and  $g_i$  have the same homogeneous degree, i.e.,  $p = q_i$  for all  $i = 1, \dots, m$ .
- $b = 0$ .
- $b \leq 0$ , and  $p \geq \max_{1 \leq i \leq m} q_i$ .
- $b \geq 0$ , and  $p \leq \min_{1 \leq i \leq m} q_i$ .
- $b_i(p - q_i) \leq 0$ , for all  $i = 1, \dots, m$ .

Therefore, in each of the above-mentioned cases, the KKT-points of problem (14) are indeed of one-to-one correspondence to the KKT-points of (15). Theorem 2.3 can be proved by a way similar to that of Theorem 2.1. Its proof idea is due to Ref. 9. Our results (see, Theorems

2.1 and 2.3) establish a truly one-to-one correspondence of KKT-points between problem (3) (resp, (14)) and the auxiliary problem (4) (resp, (15)). It is worth stressing that our results include important special cases such as  $p = q_i$  for all  $i = 1, \dots, m$  and  $b = 0$ .

**Remark 2.2.** While the functions involved here are considered to be defined on  $R^n$ , it is not difficult to extend the discussion in this section to cases on a conic set which is actually a natural domain of a homogeneous function. It is also worth mentioning that the discussion in this section can be simplified for the nonnegative functions. In fact, if  $f$  is nonnegative and homogeneous of degree  $p$ , the minimization of  $f$  is equivalent to the minimization of the function  $f^{1/p}$ . If  $b$  is a nonnegative scalar and  $g$  is nonnegative and homogeneous of degree  $q > 0$ , the constraint  $g(x) \leq b$  is equivalent to the constraint  $g^{1/q}(x) \leq b^{1/q}$  which is homogeneous of degree 1.

**Remark 2.3.** It is well-known that some non-homogeneous nonlinear programs can be equivalently formulated as homogeneous optimization problems (see for example, Refs. 2-3). However, it should be pointed out that the resulting homogeneous problem might be non-differentiable somewhere.

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