

Approximation Algorithms for Indefinite Complex Quadratic Maximization Problems

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July 2005

Abstract

In this paper we consider the following two types of complex quadratic maximization problems: (i) maximize $z^H Q z$, subject to $z_k^m = 1$, $k = 1, \dots, n$, where Q is a Hermitian matrix with $\text{tr } Q = 0$ and $z \in \mathbf{C}^n$ is the decision vector; (ii) maximize $\text{Re } y^H A z$, subject to $y_k^m = 1$, $k = 1, \dots, p$, and $z_l^m = 1$, $l = 1, \dots, q$, where $A \in \mathbf{C}^{p \times q}$ and $y \in \mathbf{C}^p$ and $z \in \mathbf{C}^q$ are the decision vectors. In the real cases (namely $m = 2$ and the matrices are all real-valued), such problems were recently considered by Charikar and Wirth [6] and Alon and Naor [1] respectively. In particular, Charikar and Wirth [6] presented an $\Omega(1/\log n)$ approximation algorithm for Problem (i) in the real case, and Alon and Naor [1] presented a 0.56-approximation algorithm for Problem (ii) in the real case. In this paper, we propose an $\Omega(1/\log n)$ approximation algorithm for the general version of complex Problem (i), and a $\left(\frac{m^2(1-\cos \frac{2\pi}{m})}{4\pi} - 1\right)$ -approximation algorithm for the general version of complex Problem (ii). For the limit and continuous version of complex Problem (ii) ($m \rightarrow \infty$), we further show that a 0.7118 approximation ratio can be achieved.

Keywords: indefinite Hermitian matrix, randomized algorithms, approximation ratio, semidefinite programming relaxation.

MSC subject classification: 90C20, 90C22.

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1 Introduction

Polynomial-time approximation algorithms for NP-hard problems via semidefinite programming (SDP) have received much attention in the last decade, since, in several important cases, this approach leads to significant improvements on the worst-case approximation ratios. The pioneering work along this direction is the famous 0.87856-approximation algorithm of Goemans and Williamson [7] for the Max-Cut problem. Nesterov [11] and Ye [14] obtained $\frac{2}{\pi}$ -approximation algorithms for quadratic maximization problem with a positive semidefinite objective matrix. Actually, the $\frac{2}{\pi}$ approximation result can also be derived from the so-called real matrix cube theorem developed by Ben-Tal and Nemirovski in [3]. Alternatively, Alon and Naor [1] showed that the $\frac{2}{\pi}$ approximation result can also be obtained via the so-called Grothendieck's Inequality. Moreover, Ben-Tal and Nemirovski [3] and Alon and Naor [1] showed that the $\frac{2}{\pi}$ bound is essentially tight.

Recently, Goemans and Williamson [8] proposed a randomized approximation algorithm, via complex SDP relaxation, for solving the Max-3-Cut problem, which is formulated as a quadratic maximization problem with complex-valued decision variables. In particular, they consider the following model: maximize $z^H Q z$ subject to $z_k^3 = 1$, $k = 1, \dots, n$, where Q is the Laplacian of the graph (hence positive semidefinite for a nonnegatively weighted graph). By a SDP relaxation and random hyperplane method, Goemans and Williamson showed that the algorithm achieves a 0.836 approximation ratio. Recently, Zhang and Huang [16] extended Goemans and Williamson's model [8], where they first developed a closed-form formula for computing the probability of a complex-valued normally distributed bivariate random vector to be in a given angular region, and then applying this formula they obtained an approximation ratio $\frac{\pi}{4}$ for the problem: maximize $z^H Q z$ subject to $|z_k| = 1$, $k = 1, \dots, n$, where Q is Hermitian positive semidefinite. Similar as with its real-case counter-part, in fact this $\frac{\pi}{4}$ approximation ratio can also be obtained in two other ways: either by the so-called complex matrix cube theorem developed by Ben-Tal, Nemirovski and Roos [4], or by the complex Grothendieck's Inequality approach developed by Haagerup [9]. However, these approaches shed lights on the problem from very different angles. For the discrete version of the model: maximize $z^H Q z$ subject to $z_k^m = 1$, $k = 1, \dots, n$, So, Zhang and Ye [12] obtained the $m^2(1 - \cos \frac{2\pi}{m})/8\pi$ approximation ratio based on Grothendieck's Inequality. We obtained the same approximation ratio in a later version of [16], by using the probability formula that we developed earlier.

Charikar and Wirth [6] considered the following problem: maximize $x^T Q x$ subject to $x_k^2 = 1$, $k = 1, \dots, n$, where Q is a symmetric matrix with zero diagonal elements. They presented an $\Omega(1/\log n)$ -approximation algorithm for such problems. The novelty of their approach lies in a rounding procedure, taking into account the size of the projections of a random vector onto the solution vectors.

A related paper was due to Alon and Naor [1], where they considered the so-called cut-norm of a real

matrix. In fact, they first considered another norm, denoted as $\|A\|_{\infty \mapsto 1}$, of a real matrix $A \in \Re^{m \times n}$, with $\|A\|_{\infty \mapsto 1} := \max\{x^T A y \mid x_i^2 = 1, y_j^2 = 1, i = 1, \dots, m, j = 1, \dots, n\}$. Alon and Naor [1] proposed a 0.56-approximation algorithm for computing the norm $\|A\|_{\infty \mapsto 1}$ of a real matrix A , where the ratio 0.56 is the reciprocal of an upper bound $\frac{\pi}{2 \ln(1+\sqrt{2})}$ of the real Grothendieck's constant. Clearly, $\|A\|_{\infty \mapsto 1}$ can also be viewed as the optimal value of a binary quadratic programming, which in fact is a subclass of the problem considered by Charikar and Wirth [6].

In this paper, we consider approximation algorithms for indefinite complex quadratic programming with m -point constellation constraint. In particular, we consider the following two problems. Problem (i): maximize $z^H Q z$, subject to $z_k^m = 1, k = 1, \dots, n$, where Q is a Hermitian matrix with $\text{tr } Q = 0$; and Problem (ii): maximize $\text{Re } y^H A z$, subject to $y_k^m = 1, k = 1, \dots, p$, and $z_l^m = 1, l = 1, \dots, q$, where $A \in \mathbf{C}^{p \times q}$. These models are extensions of Charikar and Wirth [6], and Alon and Naor [1] respectively. We propose an $\Omega(1/\log n)$ -approximation algorithm for Problem (i), and a $\left(\frac{m^2(1-\cos \frac{2\pi}{m})}{4\pi} - 1\right)$ -approximation algorithm for Problem (ii). Some further extensions of the model in Problem (i) are also considered. With regard to Problem (ii), we propose a 0.7118-approximation algorithm in the limiting continuous case ($m \rightarrow \infty$), where we make use of several results established in Zhang and Huang [16], and Haagerup [9].

This paper is organized as follows. In Section 2 we discuss an approximation algorithm for the indefinite complex quadratic maximization problem: Problem (i), and show that the approximation results can be obtained for quadratic programming with convex constraint on squared modulus and continuous complex quadratic programming. In Section 3, we study Problem (ii), and study approximation algorithms for both the discrete and continuous cases.

Notation. We denote by \bar{a} the conjugate of a complex number a , $\text{Arg } z$ the argument of z , $|z|$ the modulus of z , and by \mathbf{C}^n the space of n -dimensional complex vectors. As a convention we assume $\text{Arg } z = 0$ if $z = 0$. For a given vector $z \in \mathbf{C}^n$, we denote z^H the conjugate transpose of z , and $\text{Diag}(z)$ the $n \times n$ diagonal matrix with diagonal entries taken from z , and if Z is an $n \times n$ matrix, then $\text{diag}(Z)$ denotes an n -dimensional vector formed by the diagonal elements of Z . The space of $n \times n$ real symmetric and the space of complex Hermitian matrices are denoted by \mathcal{S}^n and \mathcal{H}^n , respectively. For a matrix $Z \in \mathcal{H}^n$, we write $\text{Re } Z$ and $\text{Im } Z$ for the real part and imaginary part of Z , respectively. Matrix Z being Hermitian implies that $\text{Re } Z$ is symmetric and $\text{Im } Z$ is skew-symmetric. We denote by \mathcal{S}_+^n (\mathcal{S}_{++}^n) and \mathcal{H}_+^n (\mathcal{H}_{++}^n) the cones of real symmetric positive semidefinite (positive definite) and complex Hermitian positive semidefinite (positive definite) matrices, respectively. The notation $Z \succeq 0$ ($\succ 0$) means that Z is positive semidefinite (positive definite). For two complex matrices Y and Z , their inner product $Y \bullet Z$ is defined to be $\text{Re}(\text{tr } Y^H Z) = \text{tr}[(\text{Re } Y)^T (\text{Re } Z) + (\text{Im } Y)^T (\text{Im } Z)]$, where tr denotes the trace of a matrix and T denotes the transpose of a matrix.

2 Indefinite complex quadratic maximization

In this section, we consider the following (indefinite) complex quadratic maximization problem

$$\begin{aligned} \text{(DQP)} \quad & \max \quad z^H Q z \\ \text{s.t.} \quad & z_k \in \mathbf{C} \text{ and } z_k^m = 1, \quad k = 1, \dots, n, \end{aligned}$$

where $Q \neq 0$ is an indefinite Hermitian matrix with $\text{diag}(Q) = 0$, and $m \geq 2$ is an integer which is a part of the input parameter of the problem. Clearly, the problem can be more explicitly written as

$$\begin{aligned} \text{(DQP)} \quad & \max \quad z^H Q z \\ \text{s.t.} \quad & z_k \in \{1, \omega, \dots, \omega^{m-1}\}, \quad k = 1, \dots, n, \end{aligned}$$

where $\omega = e^{i\frac{2\pi}{m}} = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m}$.

In case Q is real symmetric and $m = 2$, the above model coincide with the one considered in Charikar and Wirth [6]. We remark that, as $|z_k| = 1$, the objective function value remains unchanged if we replace the condition $\text{diag}(Q) = 0$ by $\text{tr } Q = 0$. Zhang and Huang [16] considered approximation algorithms for such problem when Q is Hermitian positive semidefinite. In that case, applications of such models arise from solving the Max-3-Cut problem (Goemans and Williamson [8]) and signal processing (Luo, Luo, and Kisialiou [10]).

We solve the following semidefinite programming as a relaxation of (DQP):

$$\begin{aligned} \text{(SDP)} \quad & \max \quad Q \bullet Z \\ \text{s.t.} \quad & Z_{kk} = 1, \quad k = 1, \dots, n, \\ & Z \succeq 0. \end{aligned}$$

This relaxed problem (complex SDP) can be solved in polynomial time up to any prescribed precision. For practical solution methods, see e.g. Sturm [13].

Next we shall discuss what to do with the optimal solution of the relaxed problem, and our method is inspired by Charikar and Wirth [6].

Suppose that Z^* is an optimal solution of (SDP). We draw a random complex vector $\xi \in \mathcal{N}_c(0, Z^*)$, where $\mathcal{N}_c(0, Z^*)$ stands for the n -dimensional complex-valued normal distribution with mean vector zero and covariance matrix Z^* . (More discussions on complex-valued normal distribution can be found in [2] and [16]). For $k = 1, 2, \dots, n$, let

$$x_k := \begin{cases} \xi_k / |\xi_k|, & \text{if } |\xi_k| > T; \\ \xi_k / T, & \text{if } |\xi_k| \leq T, \end{cases} \quad (1)$$

where $T > 0$ is a number to be specified later.

This process naturally produces a complex vector $x \in \mathbf{C}^n$ with $|x_k| \leq 1$, $k = 1, \dots, n$. We then use x to further randomly generate z_k independently from each other as follows:

$$z_k = \begin{cases} 1, & \text{with probability } (1 + \operatorname{Re} x_k)/m \\ \vdots \\ \omega^j, & \text{with probability } (1 + \operatorname{Re} (\omega^{-j} x_k))/m \\ \vdots \\ \omega^{m-1}, & \text{with probability } (1 + \operatorname{Re} (\omega^{-(m-1)} x_k))/m \end{cases} \quad (2)$$

for $k = 1, \dots, n$. Indeed we note that $(1 + \operatorname{Re} (\omega^{-j} x_k))/m \geq 0$ for all $j = 0, 1, \dots, m-1$, and that

$$\sum_{j=0}^{m-1} (1 + \operatorname{Re} (\omega^{-j} x_k))/m = 1 + \frac{1}{m} \operatorname{Re} \left(\left(\sum_{j=0}^{m-1} \omega^{-j} \right) x_k \right) = 1.$$

With regard to this second randomization process (from x to z), we have the following general result.

Lemma 2.1 *For $k \neq l$, it holds that*

$$\mathbb{E}[z_k \bar{z}_l] = \begin{cases} \mathbb{E}[\operatorname{Re} x_k \operatorname{Re} \bar{x}_l], & \text{for } m = 2, \\ \frac{1}{4} \mathbb{E}[x_k \bar{x}_l], & \text{for } m \geq 3. \end{cases}$$

The case $m = 2$ is easy to see, and is actually used in Charikar and Wirth [6]. For the case $m \geq 3$, the proof of Lemma 2.1 involves some tedious calculations, and we postpone the proof to the appendix.

We remark that there is an immediate consequence of Lemma 2.1 regarding the relationship between the optimal Max-2-Cut value and the optimal Max-3-Cut value for the same weighted graph. To be specific, consider a weighted graph (undirected) with n nodes, and the weight on the edge (k, l) is w_{kl} ($k \neq l$). Let Q be the Laplacian matrix of a weighted graph, i.e., $Q_{kl} = -w_{kl}$ for $k \neq l$, and $Q_{kk} = \sum_{l \neq k} w_{kl}$, $k = 1, \dots, n$. Let $x_k \in \{1, -1\}$, $k = 1, \dots, n$, and $z_k \in \{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$. Let $X = xx^T$ and $Z = zz^H$. It is easy to verify that the corresponding 2-Cut value associated with x is $\frac{1}{4}Q \bullet X$, and the corresponding 3-Cut value associated with z is $\frac{1}{3}Q \bullet Z$. Let us denote the sum of all weights be $W^* := \sum_{k < l} w_{kl}$. Now let $x \in \{1, -1\}^n$ be corresponding to the optimal Max-2-Cut solution. Based on x we again generate $z \in \{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$ as described in the above procedure. Then we have

$$\begin{aligned} v(\text{M3C}) &\geq \frac{1}{3} \mathbb{E}[Q \bullet Z] \\ &= \frac{1}{3} \left(\sum_{k=1}^n Q_{kk} + 2 \sum_{k < l} Q_{kl} \operatorname{Re} \mathbb{E}[z_k \bar{z}_l] \right) \\ &= \frac{1}{3} \left(2W^* + \frac{1}{2} \sum_{k < l} Q_{kl} x_k x_l \right) \\ &= \frac{1}{2} W^* + \frac{1}{3} v(\text{M2C}), \end{aligned} \quad (3)$$

where $v(\text{M3C})$ is the optimal Max-3-Cut value and $v(\text{M2C})$ is the optimal Max-2-Cut value. Note that in relation (3), no assumption is made regarding the signs of the weights.

In the construction of vector x , its components differ from that of ξ/T only when $|\xi_k| > T$. Our next target is to show that the difference is small if T is large.

For $k \neq l$, define $\Delta_{kl} = \xi_k \bar{\xi}_l / T^2 - x_k \bar{x}_l$.

Lemma 2.2 *For $k \neq l$ and $T > 1$ it holds that $\mathbf{E}[|\Delta_{kl}|] < e^{-T^2}(\frac{5}{T} + 4)$.*

Proof. Let us divide the complex space \mathbf{C}^2 (with overlaps) into

$$A = \{(\xi_k, \xi_l) : |\xi_k| \leq T, |\xi_l| \leq T\}, B = \{(\xi_k, \xi_l) : |\xi_l| > T\} \text{ and } C = \{(\xi_k, \xi_l) : |\xi_k| > T\}.$$

Suppose, without loss of generality due to symmetricity, that $\mathbf{E}_C[|\Delta_{kl}|] \leq \mathbf{E}_B[|\Delta_{kl}|]$. We have

$$\mathbf{E}[|\Delta_{kl}|] \leq \mathbf{E}_A[|\Delta_{kl}|] + \mathbf{E}_B[|\Delta_{kl}|] + \mathbf{E}_C[|\Delta_{kl}|] \leq \mathbf{E}_A[|\Delta_{kl}|] + 2\mathbf{E}_B[|\Delta_{kl}|]. \quad (4)$$

Obviously, $\mathbf{E}_A[|\Delta_{kl}|] = 0$, and we need only to treat the term $\mathbf{E}_B[|\Delta_{kl}|]$.

Suppose that $\mathbf{E}[\xi_k \bar{\xi}_l] = Z_{kl}^* = \gamma e^{i\alpha}$. Since

$$\begin{pmatrix} \xi_k \\ \xi_l \end{pmatrix} \in \mathcal{N}_c \left(0, \begin{pmatrix} 1 & \gamma e^{i\alpha} \\ \gamma e^{-i\alpha} & 1 \end{pmatrix} \right),$$

we can express (ξ_k, ξ_l) in terms of (η, λ) as

$$\xi_k = \gamma e^{i\alpha} \eta + \sqrt{1 - \gamma^2} \lambda, \quad \xi_l = \eta,$$

where $\begin{pmatrix} \eta \\ \lambda \end{pmatrix} \in \mathcal{N}_c(0, I_2)$. Thus, we have

$$\text{Prob}\{|\xi_l| > T\} = \text{Prob}\{|\eta| > T\} = \int_T^\infty \int_0^{2\pi} \frac{r}{\pi} e^{-r^2} d\theta dr = 2 \int_T^\infty r e^{-r^2} dr = e^{-T^2}.$$

Moreover,

$$\mathbf{E}_B[|x_k \bar{x}_l|] \leq \mathbf{E}_B[1] = \text{Prob}\{(\xi_k, \xi_l) \in B\} = \text{Prob}\{|\eta| > T\} = e^{-T^2}.$$

On the other hand, since $\gamma \leq 1$, we have

$$\begin{aligned} \mathbf{E}_B[|\xi_k \xi_l|] &= \mathbf{E}_B[\gamma e^{i\alpha} \eta \bar{\eta} + \sqrt{1 - \gamma^2} \lambda \bar{\eta}] \\ &\leq \mathbf{E}_B[|\eta|^2 + |\eta| \cdot |\lambda|] \\ &= \int_T^\infty \int_0^{2\pi} \frac{r^3}{\pi} e^{-r^2} d\theta dr + \int_T^\infty \int_0^{2\pi} \frac{r^2}{\pi} d\theta_1 dr \int_0^\infty \int_0^{2\pi} \frac{s^2}{\pi} e^{-s^2} d\theta_2 ds \\ &= (T^2 + \frac{\sqrt{\pi}}{2} T + 1) e^{-T^2} + \frac{\pi}{2} (1 - \Phi(\sqrt{2} T)), \end{aligned}$$

where in the last equality we use the facts that

$$\int_T^\infty r^3 e^{-r^2} dr = \frac{1}{2}(T^2 + 1)e^{-T^2}, \text{ and } \int_T^\infty r^2 e^{-r^2} dr = \frac{T}{2}e^{-T^2} + \frac{\sqrt{\pi}}{2}(1 - \Phi(\sqrt{2}T)),$$

and

$$4 \int_0^\infty s^2 e^{-s^2} ds = \sqrt{\pi},$$

where $\Phi(\cdot)$ is the cumulative distribution function of the real-valued standard normal distribution $\mathcal{N}(0, 1)$.

Therefore,

$$\begin{aligned} \mathbb{E}_B[|\Delta_{kl}|] &\leq \mathbb{E}_B[|\xi_k \bar{\xi}_l|/T^2 + |x_k \bar{x}_l|] \\ &\leq e^{-T^2} \left(\frac{1}{T^2} + \frac{\sqrt{\pi}}{2T} + 2 \right) + \frac{\pi}{2T^2} (1 - \Phi(\sqrt{2}T)), \end{aligned}$$

and so from (4) we have

$$\mathbb{E}[|\Delta_{kl}|] \leq e^{-T^2} \left(\frac{2}{T^2} + \frac{\sqrt{\pi}}{T} + 4 \right) + \frac{\pi}{T^2} (1 - \Phi(\sqrt{2}T)). \quad (5)$$

Further notice that

$$\frac{\pi}{T^2} (1 - \Phi(\sqrt{2}T)) = \frac{\sqrt{\pi}}{T^2} \int_T^\infty e^{-s^2} ds \leq \frac{\sqrt{\pi}}{T^2} \int_T^\infty s e^{-s^2} ds = \frac{\sqrt{\pi}}{2T^2} e^{-T^2},$$

and so it follows from (5) that

$$\mathbb{E}[|\Delta_{kl}|] \leq e^{-T^2} \left(\frac{4 + \sqrt{\pi}}{2T^2} + \frac{\sqrt{\pi}}{T} + 4 \right) < e^{-T^2} \left(\frac{5}{T} + 4 \right).$$

Note that in the above derivations we used the fact that $T > 1$. □

Lemma 4 of [6] asserts that

$$v(\text{DRQP}) \geq \frac{1}{n} \sum_{k,l} |a_{kl}|, \quad (6)$$

where $v(\text{DRQP})$ is the optimal value of

$$\begin{aligned} (\text{DRQP}) \quad &\max \quad x^T A x \\ &\text{s.t.} \quad x_k \in \{\pm 1\}, \quad k = 1, \dots, n, \end{aligned}$$

where A is a real matrix with $\text{diag}(A) = 0$.

Since (SDP) is a relaxation of the following continuous complex quadratic programming (CQP)

$$\begin{aligned} (\text{CQP}) \quad &\max \quad z^H Q z \\ &\text{s.t.} \quad |z_k| = 1, \quad k = 1, \dots, n, \end{aligned}$$

it follows that the optimal value of (SDP), $v(\text{SDP})$, is no less than the optimal value of (CQP), $v(\text{CQP})$. By denoting $\text{Re } Q = A$, $\text{Im } Q = B$, $\text{Re } z = x$ and $\text{Im } z = y$, we may write (CQP) as an equivalent real quadratic program (CRQP):

$$\begin{aligned} (\text{CRQP}) \quad & \max \quad (x^\text{T}, y^\text{T}) \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ & \text{s.t.} \quad x_k^2 + y_k^2 = 1, \quad k = 1, \dots, n. \end{aligned}$$

That is, $v(\text{CQP}) = v(\text{CRQP})$. Obviously, the optimal value of (CRQP), $v(\text{CRQP})$, is no less than $v(\text{SRQP})$, the optimal value of

$$\begin{aligned} (\text{SRQP}) \quad & \max \quad (x^\text{T}, y^\text{T}) \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ & \text{s.t.} \quad x_k^2 = y_k^2 = 1/2, \quad k = 1, \dots, n, \end{aligned}$$

which is in fact equivalent to the following problem, by substitution of variables $u := \sqrt{2}x$ and $v := \sqrt{2}y$,

$$\begin{aligned} & \max \quad \frac{1}{2}(u^\text{T}, v^\text{T}) \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ & \text{s.t.} \quad u_k^2 = v_k^2 = 1, \quad k = 1, \dots, n. \end{aligned}$$

Hence, using (6) one obtains

$$v(\text{SRQP}) \geq \frac{1}{2n} \sum_{k,l} (|A_{kl}| + |B_{kl}|) \geq \frac{1}{2n} \sum_{k,l} |Q_{kl}|.$$

Finally, by noting $v(\text{SDP}) \geq v(\text{CQP}) = v(\text{CRQP}) \geq v(\text{SRQP})$, we arrive at the following result:

Lemma 2.3 *It holds that $v(\text{SDP}) \geq \frac{1}{2n} \sum_{k,l} |Q_{kl}|$.*

Theorem 2.4 *Suppose that $n \geq 3$. By setting $T = \sqrt{9 \log n}$, we have $\mathbb{E}[z^H Q z] \geq \frac{1}{40 \log n} v(\text{SDP})$.*

Proof. It follows that

$$\begin{aligned}
\mathbb{E}[z^H Q z] &= \sum_{k \neq l} Q_{lk} \mathbb{E}[z_k \bar{z}_l] \\
&= \frac{1}{4} \sum_{k \neq l} Q_{lk} \mathbb{E}[x_k \bar{x}_l] \\
&= \frac{1}{4} \sum_{k \neq l} Q_{lk} \mathbb{E}[-\Delta_{kl} + \xi_k \bar{\xi}_l / T^2] \\
&= \frac{1}{4} \left(\sum_{k \neq l} Q_{lk} \mathbb{E}[\xi_k \bar{\xi}_l / T^2] - \sum_{k \neq l} Q_{lk} \mathbb{E}[\Delta_{kl}] \right) \\
&= \frac{1}{4} \left(v(\text{SDP}) / T^2 - \sum_{k \neq l} Q_{lk} \mathbb{E}[\Delta_{kl}] \right) \\
&\geq \frac{1}{4} \left(v(\text{SDP}) / T^2 - \sum_{k \neq l} |Q_{lk}| \times |\mathbb{E}[\Delta_{kl}]| \right) \\
&\geq \frac{1}{4} \left(v(\text{SDP}) / T^2 - e^{-T^2} \left(\frac{5}{T} + 4 \right) \times 2n \times v(\text{SDP}) \right) \\
&= \frac{1 - e^{-T^2} (4T^2 + 5T) 2n}{4T^2} v(\text{SDP}),
\end{aligned} \tag{7}$$

where, equality (7) is due to Lemma 2.1, and inequality (8) is due to Lemmas 2.2 and 2.3.

If $T = \sqrt{9 \log n}$ then $\frac{1 - e^{-T^2} (4T^2 + 5T) 2n}{T^2} > \frac{1}{10 \log n}$, whenever $n \geq 3$. Thus we have

$$\mathbb{E}[z^H Q z] \geq \frac{1}{40 \log n} v(\text{SDP}).$$

□

Theorem 2.4 implies that the approximation ratio of our randomized algorithm for (DQP) is at least $\Omega(1/\log n)$, extending the approximation result of Charikar and Wirth [6] to the complex discrete quadratic maximization model (DQP).

In the next two subsections, we shall see that the result can be further extended.

2.1 Constrained squared modulus

Consider the following constrained model

$$\begin{aligned}
(\text{DQPC}) \quad &\max \quad z^H Q z \\
&\text{s.t.} \quad \text{Arg } z_k \in \{0, \frac{1}{m} 2\pi, \dots, \frac{m-1}{m} 2\pi\}, \quad k = 1, \dots, n, \\
&\quad (|z_1|^2, \dots, |z_n|^2)^T \in \mathcal{F},
\end{aligned}$$

where $Q \in \mathcal{H}^n$ with $\text{diag}(Q) = 0$, and \mathcal{F} is a closed convex set in \mathfrak{R}^n . In particular, if \mathcal{F} is the singleton e (the all-one vector), then (DQPC) reduces to (DQP).

Consider the following convex SDP relaxation for (DQPC)

$$\begin{aligned} \text{(CSDP)} \quad & \max \quad Q \bullet Z \\ & \text{s.t.} \quad \text{diag}(Z) \in \mathcal{F}, \\ & \quad \quad Z \succeq 0. \end{aligned}$$

For $0 \leq d \in \mathfrak{R}^n$, define its generalized inverse vector d^{-1} to be

$$(d^{-1})_k = \begin{cases} 1/d_k, & \text{if } d_k > 0; \\ 1, & \text{if } d_k = 0, \end{cases}$$

where $k = 1, \dots, n$. Thus the diagonal matrix $D^{-1} := \text{Diag}(d^{-1}) \succ 0$. Thus,

$$(D^{-1}D)_{kk} = \begin{cases} 1, & \text{if } d_k > 0; \\ 0, & \text{if } d_k = 0. \end{cases}$$

For a Hermitian positive semidefinite matrix Z , let $d := \sqrt{\text{diag}(Z)}$ and $D := \text{Diag}(d)$, and

$$\tilde{Z} := Z + (I - D^{-1}D). \quad (9)$$

Then $\tilde{Z}_{kl} = Z_{kl}$ for any $k \neq l$, $\tilde{Z}_{kk} = Z_{kk}$ if $Z_{kk} > 0$, and $\tilde{Z}_{kk} = 1$ if $Z_{kk} = 0$.

Lemma 2.5 *Let $Z \in \mathcal{H}_+^n$, and \tilde{Z} be defined as in (9). Then the matrix $D^{-1}\tilde{Z}D^{-1}$ has the properties:*

- (i) $D^{-1}\tilde{Z}D^{-1} \succeq 0$;
- (ii) $\text{diag}(D^{-1}\tilde{Z}D^{-1})$ is the n -dimensional all-one vector;
- (iii) $DD^{-1}\tilde{Z}D^{-1}D = Z$.

Proof. The first two assertions follow from the definition, and the last assertion can be seen from the fact that

$$DD^{-1}\tilde{Z}D^{-1}D = DD^{-1}(Z + (I - D^{-1}D))D^{-1}D = DD^{-1}ZD^{-1}D = Z.$$

This completes the proof of (iii). □

Now we consider the following randomization method for solving (DQPC).

Let Z^* be an optimal solution of (CSDP). Let $d := \sqrt{\text{diag}(Z^*)}$, $D := \text{Diag}(d)$, $D^{-1} := \text{Diag}(d^{-1})$, and \tilde{Z}^* is defined according to (9).

Draw a random complex vector $\xi \in \mathcal{N}_c(0, D^{-1}\tilde{Z}^*D^{-1})$, and set x according to (1), and generate z according to (2). Finally, set $z := Dz$.

Next we shall analyze the performance of this algorithm. Consider the following complex quadratic program

$$\begin{aligned} \max \quad & DQD \bullet zz^H \\ \text{s.t.} \quad & z_k \in \{1, \omega, \dots, \omega^{m-1}\}, k = 1, \dots, n, \end{aligned}$$

and its SDP relaxation

$$\begin{aligned} \text{(S)} \quad \max \quad & DQD \bullet Z \\ \text{s.t.} \quad & Z_{kk} = 1, k = 1, \dots, n, \\ & Z \succeq 0. \end{aligned}$$

By Lemma 2.3 we have $v(\text{S}) \geq \frac{1}{2n} \sum_{k \neq l} |Q_{kl}| \sqrt{Z_{kk}^* Z_{ll}^*}$.

Let Y^* be optimal for (S). Since DY^*D is feasible for (CSDP),

$$v(\text{CSDP}) \geq Q \bullet DY^*D = DQD \bullet Y^* = v(\text{S}) \geq \frac{1}{2n} \sum_{k \neq l} |Q_{kl}| \sqrt{Z_{kk}^* Z_{ll}^*}.$$

That is,

$$-\sum_{k \neq l} |Q_{kl}| \sqrt{Z_{kk}^* Z_{ll}^*} \geq -2n \times v(\text{CSDP}).$$

Therefore,

$$\begin{aligned} \mathbb{E}[Q \bullet zz^H] &= \frac{1}{4} \sum_{k \neq l} Q_{kl} \sqrt{Z_{kk}^* Z_{ll}^*} \mathbb{E}[x_k \bar{x}_l] \\ &= \frac{1}{4} \sum_{k \neq l} Q_{kl} \sqrt{Z_{kk}^* Z_{ll}^*} (\mathbb{E}[\xi_k \bar{\xi}_l] / T^2 - \mathbb{E}[\Delta_{kl}]) \\ &\geq \frac{1}{4} \left(\frac{DQD \bullet D^{-1}\tilde{Z}^*D^{-1}}{T^2} - \sum_{k \neq l} Q_{kl} \sqrt{Z_{kk}^* Z_{ll}^*} \mathbb{E}[|\Delta_{kl}|] \right) \\ &\geq \frac{1}{4} \left(\frac{DQD \bullet D^{-1}\tilde{Z}^*D^{-1}}{T^2} - e^{-T^2} \left(\frac{5}{T} + 4 \right) \sum_{k \neq l} |Q_{kl}| \sqrt{Z_{kk}^* Z_{ll}^*} \right) \\ &\geq \frac{1}{4} \left(\frac{v(\text{CSDP})}{T^2} - e^{-T^2} \left(\frac{5}{T} + 4 \right) 2n \times v(\text{CSDP}) \right) \\ &= \frac{1 - e^{-T^2} (4T^2 + 5T) 2n}{4T^2} v(\text{CSDP}) \\ &\geq \frac{1}{40 \log n} v(\text{CSDP}), \end{aligned}$$

where in the last step we take $T = \sqrt{9 \log n}$ and $n \geq 3$. This concludes the theorem below.

Theorem 2.6 *Let $v(\text{DQPC})$ be the optimal value of (DQPC) and $v(\text{CSDP})$ be the optimal value of (CSDP). Then $v(\text{DQPC}) \geq \frac{1}{40 \log n} v(\text{CSDP})$ for $n \geq 3$. This implies the approximation ratio for the algorithm of (DQPC) is at least $\Omega(1/\log n)$.*

2.2 Continuous indefinite complex quadratic maximization

Consider the following model

$$\begin{aligned} (\text{CQP}) \quad & \max \quad z^H Q z \\ & \text{s.t.} \quad |z_k| = 1, k = 1, \dots, n, \end{aligned}$$

where $Q \in \mathcal{H}^n$ and $\text{diag}(Q) = 0$. We use (SDP) a relaxation for the above problem (CQP). Suppose that Z^* is an optimal solution of (SDP).

Draw a random complex vector $\xi \in \mathcal{N}_c(0, Z^*)$, and set x according to (1), and then independently generate $z_k \in \mathbf{C}^n$ as follows:

$$z_k = \begin{cases} e^{i \text{Arg } x_k}, & \text{with probability } (1 + |x_k|)/2, \\ -e^{i \text{Arg } x_k}, & \text{with probability } (1 - |x_k|)/2, \end{cases}$$

for $k = 1, \dots, n$.

In that case, we have $\mathbf{E}[z_k \bar{z}_l] = \mathbf{E}[x_k \bar{x}_l]$ (see also Charikar and Wirth [6]). This leads to

$$\begin{aligned} \mathbf{E}[z^H Q z] &= \sum_{k \neq l} Q_{kl} \mathbf{E}[z_k \bar{z}_l] = \sum_{k \neq l} Q_{kl} \mathbf{E}[x_k \bar{x}_l] \\ &\geq \frac{1 - e^{-T^2} (4T^2 + 5T) 2n}{T^2} v(\text{SDP}) > \frac{1}{10 \log n} v(\text{SDP}), \end{aligned}$$

if T is taken to be $\sqrt{9 \log n}$ (with $n \geq 3$). Hence the approximation ratio of the above approximation algorithm is at least $\Omega(1/\log n)$. So, Zhang, and Ye [12] obtained the same result for this model. However, the basic techniques used in [12] are quite different from ours.

3 Discrete complex bilinear maximization

Alon and Naor [1] studied the problem of approximating the cut-norm $\|A\|_C$ of a real matrix $A \in \mathbb{R}^{p \times q}$. They in fact worked on approximation method for computing the norm $\|A\|_{\infty \rightarrow 1}$, which turned out to provide a scheme to approximate the cut-norm of A . By definition, the value of $\|A\|_{\infty \rightarrow 1}$ is given by the following bilinear binary optimization problem

$$\begin{aligned} \max \quad & y^T A z \\ \text{s.t.} \quad & y_k, z_l \in \{\pm 1\}, k = 1, \dots, p, l = 1, \dots, q, \end{aligned}$$

which can be equivalently cast as a binary quadratic maximization problem

$$\begin{aligned} \max \quad & \frac{1}{2}(y^T, z^T) \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \\ \text{s.t.} \quad & y_k, z_l \in \{\pm 1\}, \quad k = 1, \dots, p, \quad l = 1, \dots, q. \end{aligned}$$

At this point, we remark that computing $\|A\|_{\infty \rightarrow 1}$ is clearly a special case of Charikar and Wirth [6]'s model (DRQP). Based on an identity due to Grothendieck, Alon and Naor [1] managed to derive a 0.56-approximation algorithm for computing $\|A\|_{\infty \rightarrow 1}$.

In this section, we shall consider the discrete complex bilinear maximization problem

$$\begin{aligned} \max \quad & \text{Re } y^H Q z \\ \text{s.t.} \quad & y_k, z_l \in \{1, \omega, \dots, \omega^{m-1}\}, \quad k = 1, \dots, p, \quad l = 1, \dots, q, \end{aligned}$$

or, equivalently, the following complex discrete quadratic program (DBLP):

$$\begin{aligned} \text{(DBLP)} \quad \max \quad & \frac{1}{2}(y^H, z^H) \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \\ \text{s.t.} \quad & y_k, z_l \in \{1, \omega, \dots, \omega^{m-1}\}, \quad k = 1, \dots, p, \quad l = 1, \dots, q, \end{aligned}$$

where Q is an $p \times q$ complex matrix, and $\omega = \cos(2\pi/m) + i \sin(2\pi/m)$, $m \geq 3$. We also consider the following continuous version of (DBLP),

$$\begin{aligned} \text{(CBLP)} \quad \max \quad & \frac{1}{2}(y^H, z^H) \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \\ \text{s.t.} \quad & |y_k| = |z_l| = 1, \quad k = 1, \dots, p, \quad l = 1, \dots, q. \end{aligned}$$

Clearly, the objective matrix of (DBLP) is Hermitian with zero diagonals. As its real counter-part, (DBLP) is a subclass of (DQP) and (CBLP) is a subclass of (CQP). The SDP relaxation for (DBLP) and (CBLP) is

$$\begin{aligned} \text{(SDBLP)} \quad \max \quad & \frac{1}{2} \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \bullet W \\ \text{s.t.} \quad & W_{kk} = 1, \quad k = 1, \dots, p+q, \\ & W \succeq 0. \end{aligned}$$

Lemma 2.3 asserts that if $Q \neq 0$ then $v(\text{SDBLP}) > 0$. Also, we note that in the summation form, the objective function of (SDBLP) is $\sum_{k=1}^p \sum_{l=1}^q \text{Re}(\bar{Q}_{kl} W_{k,p+l})$, and the objective function of (DBLP) and (CBLP) is $\sum_{k=1}^p \sum_{l=1}^q \text{Re}(\bar{Q}_{kl} y_k \bar{z}_l)$.

3.1 An approximation algorithm for the discrete complex bilinear program

Let W^* be an optimal solution for (SDBLP). We draw a random complex vector as follows:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathcal{N}_c(0, W^*),$$

and generate complex vectors $y \in \mathbf{C}^p$ and $z \in \mathbf{C}^q$ as follows:

For $k = 1, \dots, p$, assign $y_k := \omega^j$ if $\text{Arg } \xi_k \in [\frac{j}{m}2\pi, \frac{j+1}{m}2\pi)$ with $j \in \{0, 1, \dots, m-1\}$;

and

For $l = 1, \dots, q$, assign $z_l := \omega^j$ if $\text{Arg } \eta_l \in [\frac{j}{m}2\pi, \frac{j+1}{m}2\pi)$ with $j \in \{0, 1, \dots, m-1\}$.

In Zhang and Huang [16] we have shown that

$$\mathbb{E}[y_k \bar{z}_l] = \frac{m(2 - \omega - \omega^{-1})}{8\pi^2} \sum_{j=0}^{m-1} \omega^j (\arccos(-\text{Re}(\omega^{-j} W_{k,p+l}^*)))^2 =: F_m(W_{k,p+l}^*), \forall k, l,$$

and $F_m(z) = z$ for $z \in \{1, \omega, \dots, \omega^{m-1}\}$. Furthermore, in Appendix of [16] we established that

$$F_m(z) = \frac{m^2(1 - \cos \frac{2\pi}{m})}{8\pi} z + \phi_1(z) + \phi_2(z),$$

with

$$\phi_1(z) = \frac{m^2(1 - \cos \frac{2\pi}{m})}{4\pi} \sum_{r=1}^{\infty} a_r \sum_{i=0}^{2r+1} b_{2r+2-2i} z^i (\bar{z})^{2r+1-i},$$

and

$$\phi_2(z) = \frac{m^2(1 - \cos \frac{2\pi}{m})}{4\pi^2} \sum_{s=0, t=0}^{\infty} a_s a_t \sum_{i=0}^{2s+2t+2} b_{2s+2t+3-2i} z^i (\bar{z})^{2s+2t+2-i},$$

where

$$a_r = \frac{(2r)!}{2^{4r+1}(r!)^2(2r+1)}, \quad b_{k+1-2i} = \binom{k}{i} \sum_{j=0}^{m-1} e^{\mathbf{i}(\frac{2\pi}{m}ji)}.$$

Note that $\sum_{j=0}^{m-1} e^{\mathbf{i}(\frac{2\pi}{m}ji)}$ is either 0 or m .

Let $\phi(z) := \phi_1(z) + \phi_2(z)$.

If $Z \succeq 0$ then $\bar{Z} \succeq 0$. Moreover, the Hadamard product of Hermitian positive semidefinite matrices remain positive semidefinite. This implies that if $Z \succeq 0$ then $\phi(Z) \succeq 0$, where $\phi(Z) := (\phi(Z_{kl}))_{n \times n}$.

On the other hand, since $F_m(1) = 1$ we have $1 = \frac{m^2(1 - \cos \frac{2\pi}{m})}{8\pi} + \phi(1)$. Let $\beta_m := \frac{m^2(1 - \cos \frac{2\pi}{m})}{8\pi}$. We conclude that $(\phi(W^*))_{kk}/(1 - \beta_m) = 1$, for $k = 1, \dots, p+q$, and so, $\phi(W^*)/(1 - \beta_m)$ is itself a feasible

solution for (SDBLP). Now observe that for any feasible solution of (SDBLP), say W , it necessarily follows that

$$-v(\text{SDBLP}) \leq \frac{1}{2} \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \bullet W \leq v(\text{SDBLP}). \quad (10)$$

The second inequality is obvious, by definition of the feasibility. To argue that the first inequality also holds, we consider a decomposition of $W \succeq 0$, i.e.,

$$W = \begin{pmatrix} U^H \\ V^H \end{pmatrix} \cdot (U, V),$$

where the number of rows in U^H is p , and the number of rows in V^H is q . Let us now consider another solution,

$$\tilde{W} := \begin{pmatrix} U^H \\ -V^H \end{pmatrix} \cdot (U, -V) \succeq 0.$$

Since the diagonal of \tilde{W} remains the all-one vector, it is also a feasible solution for (SDBLP). Therefore

$$v(\text{SDBLP}) \geq \frac{1}{2} \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \bullet \tilde{W} = -\frac{1}{2} \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \bullet W,$$

and so the first inequality in (10) follows. Therefore,

$$\frac{1}{2} \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \bullet \phi(W^*) / (1 - \beta_m) \geq -v(\text{SDBLP}).$$

Now we are in a position to calculate the expected value of the randomized solution

$$\begin{aligned} \mathbb{E} \left[\sum_{k=1}^p \sum_{l=1}^q \text{Re} (\bar{Q}_{kl} y_k \bar{z}_l) \right] &= \sum_{k=1}^p \sum_{l=1}^q \text{Re} (\bar{Q}_{kl} F_m(W_{k,p+l}^*)) \\ &= \sum_{k=1}^p \sum_{l=1}^q \text{Re} (\bar{Q}_{kl} (\beta_m W_{k,p+l}^* + \phi(W_{k,p+l}^*))) \\ &= \beta_m \times v(\text{SDBLP}) + (1 - \beta_m) \times \frac{1}{2} \begin{pmatrix} 0 & Q \\ Q^H & 0 \end{pmatrix} \bullet \phi(W^*) / (1 - \beta_m) \\ &\geq \beta_m \times v(\text{SDBLP}) + (\beta_m - 1) \times v(\text{SDBLP}) \\ &= (2\beta_m - 1) \times v(\text{SDBLP}) \\ &= \left(\frac{m^2(1 - \cos \frac{2\pi}{m})}{4\pi} - 1 \right) \times v(\text{SDBLP}). \end{aligned}$$

This leads to the following result.

Theorem 3.1 *There is an approximation algorithm for (DBLP) with the ratio $\alpha_m := \frac{m^2(1 - \cos \frac{2\pi}{m})}{4\pi} - 1$ for $m \geq 3$. In particular, $\alpha_3 \geq 0.0742$, $\alpha_4 \geq 0.2732$, $\alpha_5 \geq 0.3746$, $\alpha_{10} \geq 0.5198$, and $\alpha_{100} \geq 0.5702$.*

3.2 An improved approximation algorithm for the continuous bilinear problem

Since (CBLP) is the limit of (DBLP) with $m \rightarrow \infty$, it is clear that if we let $y_k = e^{i \text{Arg } \xi_k}$ and $z_l = e^{i \text{Arg } \eta_l}$ where $(\xi^H, \eta^H)^H$ is generated from $\mathcal{N}_c(0, W^*)$ and W^* is an optimal solution of (SDBLP), then we will get an approximation algorithm with an approximation ratio of $\lim_{m \rightarrow \infty} \frac{m^2(1 - \cos \frac{2\pi}{m})}{4\pi} - 1 = \frac{\pi}{2} - 1 \approx 0.5708$.

In this subsection, we shall show that this ratio can be improved if we further exploit particular structures of Problem (CBLP). Our analysis below makes use of some of the results in our previous paper [16], and also important insights presented in a paper by Haagerup, [9].

According to the analysis in Sections 3.1 and 3.3 of Zhang and Huang [16], if we generate $y_k = e^{i \text{Arg } \xi_k}$ and $z_l = e^{i \text{Arg } \eta_l}$ with $(\xi^H, \eta^H)^H$, then we have

$$\begin{aligned} \mathbb{E}[y_k \bar{z}_l] &= \lim_{m \rightarrow \infty} F_m(W_{k,p+l}^*) (= F(W_{k,p+l}^*)) \\ &= \frac{1}{4\pi} \int_0^{2\pi} e^{i\theta} (\arccos(-\gamma \cos(\theta - \alpha)))^2 d\theta \\ &= e^{i\alpha} \int_0^{\pi/2} \arcsin(\gamma \sin \theta) \sin \theta d\theta \\ &= \frac{\pi}{4} e^{i\alpha} \sum_{r=0}^{\infty} c_r \gamma^{2r+1}, \end{aligned}$$

where we denoted $W_{k,p+l}^*$ as $\gamma e^{i\alpha}$ and $c_r = \frac{((2r)!)^2}{2^{4r}(r!)^2(r+1)}$. For $\gamma \in [-1, 1]$, let

$$\psi(\gamma) := \int_0^{\pi/2} \arcsin(\gamma \sin \theta) \sin \theta d\theta \left(= \gamma \int_0^{\pi/2} \frac{(\cos \theta)^2}{\sqrt{1 - (\gamma \sin \theta)^2}} d\theta \right).$$

In that notation, the transformation function F can be rewritten as

$$F(z) = e^{i \text{Arg } z} \psi(|z|).$$

We remark here that equation (3.9) in Zhang and Huang [16] coincides with Lemma 3.2 in Haagerup [9], although we were not aware of [9] at the time when we derived that equation.

Important properties of the function $\psi(\gamma)$ are discussed in [9]. In particular, Theorem 2.1 of [9] states that the inverse function $\psi^{-1} : [-1, 1] \rightarrow [-1, 1]$ of ψ exists and it can be expanded into an absolutely convergent power series:

$$\psi^{-1}(s) = \sum_{r=0}^{\infty} b_{2r+1} s^{2r+1}, \quad s \in [-1, 1],$$

with $b_1 = \frac{4}{\pi}$ and $b_{2r+1} \leq 0$ for all $r \geq 1$. Specifically, $b_3 = -8/\pi^3$, $b_5 = 0$, $b_7 = -16/\pi^7$, $b_9 = -80/\pi^9$, $b_{11} = -480/\pi^{11}$, $b_{13} = -3136/\pi^{13}$ and $b_{2r+1} \sim -4/((2r+1) \log(2r+1))^2$ for $r \rightarrow \infty$. Moreover, the following result was shown in [9], which was used to bound the complex Grothendieck constant.

Lemma 3.2 *There is a unique $\beta \in (0, 1)$ for which $\sum_{r=0}^{\infty} |b_{2r+1}| \beta^{2r+1} = 1$ (i.e., $\psi^{-1}(\beta) = \frac{8}{\pi} \beta - 1$), with $\beta \approx 0.7118$.*

Now the inverse function of $F(z)$ can be written as

$$F^{-1}(z) = e^{i \operatorname{Arg} z} \psi^{-1}(|z|) = \sum_{r=0}^{\infty} b_{2r+1} z^{r+1} \bar{z}^r.$$

For a given $W \in \mathcal{H}_+^{p+q}$ with all-one diagonal elements, let us construct another Hermitian matrix $G(W) \in \mathcal{H}^{p+q}$ as follows:

$$\begin{aligned} G_{k,p+l}(W) &:= \frac{4}{\pi} \beta W_{k,p+l} - \sum_{r=1}^{\infty} |b_{2r+1}| \beta^{2r+1} (W_{k,p+l})^{r+1} (\bar{W}_{k,p+l})^r, \quad k = 1, \dots, p, \quad l = 1, \dots, q, \\ G_{k_1, k_2}(W) &:= \frac{4}{\pi} \beta W_{k_1, k_2} + \sum_{r=1}^{\infty} |b_{2r+1}| \beta^{2r+1} (W_{k_1, k_2})^{r+1} (\bar{W}_{k_1, k_2})^r, \quad k_1, k_2 = 1, \dots, p, \\ G_{p+l_1, p+l_2}(W) &:= \frac{4}{\pi} \beta W_{p+l_1, p+l_2} + \sum_{r=1}^{\infty} |b_{2r+1}| \beta^{2r+1} (W_{p+l_1, p+l_2})^{r+1} (\bar{W}_{p+l_1, p+l_2})^r, \quad l_1, l_2 = 1, \dots, q. \end{aligned}$$

That is,

$$\begin{aligned} G_{k,p+l}(W) &= F^{-1}(\beta W_{k,p+l}), \quad k = 1, \dots, p, \quad l = 1, \dots, q, \\ G_{k_1, k_2}(W) &= \frac{8}{\pi} \beta W_{k_1, k_2} - F^{-1}(\beta W_{k_1, k_2}), \quad k_1, k_2 = 1, \dots, p, \\ G_{p+l_1, p+l_2}(W) &= \frac{8}{\pi} \beta W_{p+l_1, p+l_2} - F^{-1}(\beta W_{p+l_1, p+l_2}), \quad l_1, l_2 = 1, \dots, q. \end{aligned}$$

By the choice of β (Lemma 3.2), we see that if W has all-one diagonal elements, then so is true for the Hermitian matrix $G(W)$. Denote $E := (e_p^T, -e_q^T)^T (e_p^T, -e_q^T)$ ($\succeq 0$), where e_p and e_q are the all-one vectors in \mathfrak{R}^p and \mathfrak{R}^q respectively. We can now write $G(W)$ in a uniform fashion as follows

$$G(W) = \frac{4}{\pi} \beta W + \sum_{r=1}^{\infty} |b_{2r+1}| \beta^{2r+1} E \circ (W)^{(r+1)} \circ (W^T)^{(r)}.$$

where ' $A \circ B$ ' stands for the Hadamard product between A and B , and $A^{(r)}$ is the r th power in the Hadamard sense, i.e., $A^{(r)} = \overbrace{A \circ A \circ \dots \circ A}^r$. If $W \succeq 0$, then, by the fact that the Hadamard product of positive semidefinite matrices remains positive semidefinite, we have $G(W) \succeq 0$. As a remark, we note here that combining this and the previous observation (regarding the diagonals of $G(W)$) leads to the conclusion that if W is a feasible solution for (SDBLP) then so is true for $G(W)$.

Suppose that W^* is an optimal solution of (SDBLP). Let us take $y_k = e^{i \text{Arg } \xi_k}$ and $z_l = e^{i \text{Arg } \eta_l}$ with $(\xi^H, \eta^H)^H$ randomly generated from $\mathcal{N}_c(0, G(W^*))$. In that case, the expected objective value is

$$\begin{aligned}
\mathbb{E} \left[\sum_{k,l} \text{Re} (\bar{Q}_{kl} y_k \bar{z}_l) \right] &= \sum_{k,l} \text{Re} (\bar{Q}_{kl} \mathbb{E}[y_k \bar{z}_l]) \\
&= \sum_{k,l} \text{Re} (\bar{Q}_{kl} F(G(W_{k,p+l}^*))) \\
&= \sum_{k,l} \text{Re} (\bar{Q}_{kl} F(F^{-1}(\beta W_{k,p+l}^*))) \\
&= \sum_{k,l} \text{Re} (\bar{Q}_{kl} \beta W_{k,p+l}^*) \\
&= \beta \sum_{k,l} \text{Re} (\bar{Q}_{kl} W_{k,p+l}^*) \\
&= \beta \times v(\text{SDBLP}) \\
&\approx 0.7118 \times v(\text{SDBLP}).
\end{aligned}$$

This proves the following theorem.

Theorem 3.3 *The above randomized algorithm has an approximation ratio 0.7118 for the continuous complex bilinear maximization problem (CBLP).*

A Proof of Lemma 2.1

We only consider the case $m \geq 3$ here. Since for the random variables z and x it holds that

$$\mathbb{E}[z_k \bar{z}_l] = \mathbb{E}[\mathbb{E}[z_k \bar{z}_l \mid (x_k, x_l)]], \text{ for } k \neq l,$$

we shall first compute $\mathbb{E}[z_k \bar{z}_l \mid (x_k = x_k^0, x_l = x_l^0)]$. For simplicity, we drop the superscript naughts of x_k^0 and x_l^0 , and denote the expectation $\mathbb{E}[z_k \bar{z}_l \mid (x_k = x_k^0, x_l = x_l^0)]$ simply by $\mathbb{E}[z_k \bar{z}_l \mid (x_k, x_l)]$, and $\text{Prob}\{z_k = \omega^j, z_l = \omega^{j-i} \mid (x_k = x_k^0, x_l = x_l^0)\}$ by $\text{Prob}\{z_k = \omega^j, z_l = \omega^{j-i} \mid (x_k, x_l)\}$. We have

$$\begin{aligned}
\mathbb{E}[z_k \bar{z}_l \mid (x_k, x_l)] &= 1 \times \sum_{j=0}^{m-1} \text{Prob}\{z_k = \omega^j, z_l = \omega^j \mid (x_k, x_l)\} + \cdots \\
&\quad + \omega^i \times \sum_{j=0}^{m-1} \text{Prob}\{z_k = \omega^j, z_l = \omega^{j-i} \mid (x_k, x_l)\} + \cdots \\
&\quad + \omega^{m-1} \times \sum_{j=0}^{m-1} \text{Prob}\{z_k = \omega^j, z_l = \omega^{j-m+1} \mid (x_k, x_l)\}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
& \sum_{j=0}^{m-1} \text{Prob} \{z_k = \omega^j, z_l = \omega^{j-i} \mid (x_k, x_l)\} \\
&= \sum_{j=0}^{m-1} \frac{1 + \text{Re}(\omega^{-j}x_k)}{m} \times \frac{1 + \text{Re}(\omega^{-j+i}x_l)}{m} \\
&= \frac{1}{m} + \frac{1}{m^2} \sum_{j=0}^{m-1} \text{Re}(\omega^{-j}x_k) \text{Re}(\omega^{-j+i}x_l).
\end{aligned}$$

Thus we further have

$$\begin{aligned}
\mathbb{E}[z_k \bar{z}_l \mid (x_k, x_l)] &= \frac{1}{m} \sum_{j=0}^{m-1} \omega^j + \frac{1}{m^2} \sum_{i=0}^{m-1} \omega^i \sum_{j=0}^{m-1} \text{Re}(\omega^{-j}x_k) \text{Re}(\omega^{-j+i}x_l) \\
&= \frac{1}{m^2} \sum_{j=0}^{m-1} \text{Re}(\omega^{-j}x_k) \sum_{i=0}^{m-1} \omega^i \text{Re}(\omega^{-j+i}x_l).
\end{aligned}$$

To simplify the expression, let us write $x_k = a + ib$ and $x_l = c + id$. Then we have

$$\begin{aligned}
& \sum_{j=0}^{m-1} \text{Re}(\omega^{-j}x_k) \sum_{i=0}^{m-1} \omega^i \text{Re}(\omega^{-j+i}x_l) \\
&= ac \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] + bd \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] \\
&+ ad \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] + bc \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] \\
&+ i \left(ad \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] + bc \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] \right. \\
&\left. + ac \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] - bd \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] \right).
\end{aligned}$$

Observing $\sum_{j=0}^{m-1} \omega^{2j} = 0$ (note here that $m \geq 3$), one has

$$\sum_{j=0}^{m-1} \cos\left(\frac{2j}{m} 2\pi\right) = \sum_{j=0}^{m-1} \sin\left(\frac{2j}{m} 2\pi\right) = 0.$$

This in turn leads to

$$\sum_{j=0}^{m-1} \sin \frac{j}{m} 2\pi \cos \frac{j}{m} 2\pi = 0, \text{ and } \sum_{j=0}^{m-1} (\sin \frac{j}{m} 2\pi)^2 = \sum_{j=0}^{m-1} (\cos \frac{j}{m} 2\pi)^2 = m/2.$$

It follows that

$$\begin{aligned}
& \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] \\
&= \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \left(\cos \frac{j}{m} 2\pi (\cos \frac{i}{m} 2\pi)^2 + \sin \frac{j}{m} 2\pi \sin \frac{i}{m} 2\pi \cos \frac{i}{m} 2\pi \right) \right] \\
&= \sum_{j=0}^{m-1} \left[(\cos \frac{j}{m} 2\pi)^2 \sum_{i=0}^{m-1} (\cos \frac{i}{m} 2\pi)^2 + \sin \frac{j}{m} 2\pi \cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \cos \frac{i}{m} 2\pi \right] \\
&= \sum_{j=0}^{m-1} \left[(\cos \frac{j}{m} 2\pi)^2 \sum_{i=0}^{m-1} (\cos \frac{i}{m} 2\pi)^2 \right] \\
&= \frac{m^2}{4}.
\end{aligned}$$

In a similar way, one further calculates that

$$\begin{aligned}
& \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] = \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] \\
&= - \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] = \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] \\
&= \frac{m^2}{4}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] = \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \cos \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] \\
&= \sum_{j=0}^{m-1} \left[\cos \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \cos \frac{j-i}{m} 2\pi \right] = \sum_{j=0}^{m-1} \left[\sin \frac{j}{m} 2\pi \sum_{i=0}^{m-1} \sin \frac{i}{m} 2\pi \sin \frac{j-i}{m} 2\pi \right] \\
&= 0.
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{j=0}^{m-1} (\operatorname{Re}(\omega^j x_k) \sum_{i=0}^{m-1} \omega^i \operatorname{Re}(\omega^{j-i} x_l)) \\
&= \frac{m^2}{4} (ac + bd + \mathbf{i}(-ad + bc)) \\
&= \frac{m^2}{4} (a + \mathbf{i}b)(c - \mathbf{i}d) \\
&= \frac{m^2}{4} x_k \bar{x}_l.
\end{aligned}$$

This means that

$$\mathbf{E} [z_k \bar{z}_l \mid (x_k, x_l)] = \frac{1}{m^2} \times \frac{m^2}{4} x_k \bar{x}_l = \frac{1}{4} x_k \bar{x}_l,$$

i.e., $\mathbf{E} [z_k \bar{z}_l \mid (x_k = x_k^0, x_l = x_l^0)] = \frac{1}{4} x_k^0 \bar{x}_l^0$. Therefore

$$\mathbf{E} [z_k \bar{z}_l] = \mathbf{E} [\mathbf{E} [z_k \bar{z}_l \mid (x_k, x_l)]] = \frac{1}{4} \mathbf{E} [x_k \bar{x}_l].$$

□

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