Complex Matrix Decomposition and Quadratic Programming

Yongwei Huang [*] Shuzhong Zhang [†]

June 2005

Abstract

This paper studies the possibilities of the Linear Matrix Inequality (LMI) characterization of the matrix cones formed by nonnegative complex Hermitian quadratic functions over specific domains in the complex space. In its real case analog, such studies were conducted in Sturm and Zhang [3]. In this paper it is shown that stronger results can be obtained for the complex Hermitian case. In particular, we show that the matrix rank-one decomposition result of Sturm and Zhang [3] can be strengthened for the complex Hermitian matrices. As a consequence, it is possible to characterize several new matrix co-positive cones (over specific domains) by means of LMI. We also present an upper bound on the minimum rank among optimal solutions for a standard complex SDP problem, as a byproduct of the new rank-one decomposition result.

Keywords: matrix rank-one decomposition, complex co-positivity cone, quadratic optimization, $\mathcal{S}$-procedure.

MSC subject classification: 90C20, 90C22.

---

[*] Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong (ywhuang@se.cuhk.edu.hk).

[†] Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong (zhang@se.cuhk.edu.hk). Research supported by Hong Kong RGC Earmarked Grants CUHK4174/03E and CUHK4242/04E.

1 Introduction

The aim of this paper is to extend the results on the cone of nonnegative quadratic functions, studied by Sturm and Zhang in [3], from the real-valued domains to the complex ones. Sturm and Zhang [3] developed a matrix rank-one decomposition technique, which is a key technique in their approach to establish the *Linear Matrix Inequality* representability of a class of matrix cones of nonnegative quadratic functions. It turns out that in the case of complex (Hermitian) quadratic forms, the rank-one decomposition result can actually be improved. In particular, we show in this paper that it is possible to find a rank-one decomposition for a positive semidefinite Hermitian matrix such that the inner-product between any of the rank-one matrices and *two* prescribed Hermitian matrices are constant respectively. As a comparison, in the real case, the inner-product of these rank-one matrices and only *one* prescribed matrix can be made constant in general.

The organization of this paper is as follows. Section 2 studies such matrix rank-one decompositions, and Section 3 is devoted to the description of the cone of nonnegative complex quadratic functions. The results of Sturm and Zhang [3] can be applied to solve quadratic optimization problem, as shown in Ye and Zhang [5]. Some of the results can be strengthened for the Hermitian forms, due to the results established in Sections 2 and 3. Section 4 is devoted to the complex quadratic programming problem. Finally, in Section 5 we study the rank of optimal solutions for a standard complex SDP, in light of the new rank-one decomposition result.

Notation. Throughout, we denote by $\bar{a}$ the conjugate of a complex number a, by $\mathbf{C}^n$ the space of n-dimensional complex vectors. For a given vector $z \in \mathbf{C}^n$, z^{H} denotes the conjugate transpose of z. The space of $n \times n$ real symmetric and complex Hermitian matrices are denoted by $\mathcal{S}^n$ and $\mathcal{H}^n$, respectively. For a matrix $Z \in \mathcal{H}^n$, we write $\mathrm{Re}\ Z$ and $\mathrm{Im}\ Z$ for the real and imaginary part of Z, respectively. Matrix Z being Hermitian implies that $\mathrm{Re}\ Z$ is symmetric and $\mathrm{Im}\ Z$ is skew-symmetric. We denote by $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$) and $\mathcal{H}^n_+$ ($\mathcal{H}^n_{++}$) the cones of real symmetric positive semidefinite (positive definite) and complex Hermitian positive semidefinite (positive definite) matrices, respectively. The notation $Z \succeq (\succ 0)$ means that Z is positive semidefinite (positive definite). For two complex matrices Y and Z, their inner product $Y \bullet Z = \mathrm{Re}\ (\mathrm{tr}\ Y^{\mathrm{H}} Z) = \mathrm{tr}\ \left[(\mathrm{Re}\ Y)^{\mathrm{T}}(\mathrm{Re}\ Z) + (\mathrm{Im}\ Y)^{\mathrm{T}}(\mathrm{Im}\ Z)\right]$, where tr denotes the trace of a matrix and $^{\mathrm{T}}$ denotes the transpose of a matrix. For a square matrix M, $\mathrm{diag}(M)$ stands for a column vector whose elements are diagonal components of M.

2 A rank-one decomposition of Hermitian PSD matrices

Let $X \in \mathcal{S}^n$ be a real symmetric positive semidefinite matrix, and $A \in \mathcal{S}^n$ be a real symmetric matrix. It follows by [3] that there is a rank-one decomposition of X:

$$X = \sum_{j=1}^{r} x_j x_j^{\mathrm{T}} \text{ such that } x_j^{\mathrm{T}} A x_j = \frac{A \bullet X}{r}, \text{ for } j = 1, \ldots, r,$$

where $r = \operatorname{rank} X$.

Now we shall show that in the Hermitian case, the decomposition result can be extended to two matrices.

Theorem 2.1 *Suppose that $Z \in \mathcal{H}^n$ is a complex Hermitian positive semidefinite matrix of rank r, and $A, B \in \mathcal{H}^n$ be two given Hermitian matrices. Then, there is a rank-one decomposition of Z,*

$$Z = \sum_{j=1}^{r} z_j z_j^H,$$

such that

$$z_j^H A z_j = \frac{A \bullet Z}{r}, \; z_j^H B z_j = \frac{B \bullet Z}{r}, \; j = 1, ..., r.$$

Proof. It follows from Corollary 4 of [3] that there is a decomposition of Z

$$Z = \sum_{j=1}^{r} u_j u_j^{\mathrm{H}} \text{ such that } u_j^{\mathrm{H}} A u_j = \frac{A \bullet Z}{r}, \text{ for } j = 1, \ldots, n.$$

If $u_j^{\mathrm{H}} B u_j = B \bullet Z / r$ for any $j = 1, \ldots, n$, then we are done. Otherwise, there should be two indices, say 1 and 2, such that

$$u_1^{\mathrm{H}} B u_1 > \frac{B \bullet Z}{r} \text{ and } u_2^{\mathrm{H}} B u_2 < \frac{B \bullet Z}{r}.$$

Denote $u_1^{\mathrm{H}} A u_2 = \gamma_1 e^{\boldsymbol{i}\alpha_1}$ and $u_1^{\mathrm{H}} B u_2 = \gamma_2 e^{\boldsymbol{i}\alpha_2}$. Let $w = \gamma e^{\boldsymbol{i}\alpha} \in \mathbf{C}$ with $\alpha = \alpha_1 + \frac{\pi}{2}$ and $\gamma > 0$ a root of the real quadratic equation in terms of x:

$$(u_1^{\mathrm{H}} B u_1 - \frac{B \bullet Z}{r}) x^2 + 2\left(\gamma_2 \sin(\alpha_2 - \alpha_1)\right) x + u_2^{\mathrm{H}} B u_2 - \frac{B \bullet Z}{r} = 0. \tag{1}$$

Remark that since $u_1^{\mathrm{H}} B u_1 - \frac{B \bullet Z}{r} > 0$ and $u_2^{\mathrm{H}} B u_2 - \frac{B \bullet Z}{r} < 0$, the above equation must have two real roots with opposite signs, and γ is taken to be the positive root.

Set

$$v_1 = (w u_1 + u_2)/\sqrt{1+\gamma^2}, \; v_2 = (-u_1 + \bar{w} u_2)/\sqrt{1+\gamma^2}.$$

It is easy to verify that

$$v_1 v_1^{\mathrm{H}} + v_2 v_2^{\mathrm{H}} = u_1 u_1^{\mathrm{H}} + u_2 u_2^{\mathrm{H}}. \tag{2}$$

Moreover

$$\begin{aligned}
(1+\gamma^2) v_1^{\mathrm{H}} A v_1 &= (\bar{w} u_1^{\mathrm{H}} + u_2^{\mathrm{H}}) A (w u_1 + u_2) \\
&= \gamma^2 u_1^{\mathrm{H}} A u_1 + u_2^{\mathrm{H}} A u_2 + \bar{w} u_1^{\mathrm{H}} A u_2 + w u_2^{\mathrm{H}} A u_1 \\
&= \gamma^2 u_1^{\mathrm{H}} A u_1 + 2\gamma\gamma_1 \mathrm{Re}\ e^{\boldsymbol{i}(\alpha_1 - \alpha)} + u_2^{\mathrm{H}} A u_2 \\
&= \gamma^2 u_1^{\mathrm{H}} A u_1 + u_2^{\mathrm{H}} A u_2 \\
&= (\gamma^2 + 1) \frac{A \bullet Z}{r},
\end{aligned}$$

which amounts to $v_1^{\mathrm{H}} A v_1 = A \bullet Z/r$. Therefore, $v_2^{\mathrm{H}} A v_2 = A \bullet Z/r$. At the same time, we have

$$\begin{aligned}
(1+\gamma^2) v_1^{\mathrm{H}} B v_1 &= (\bar{w} u_1^{\mathrm{H}} + u_2^{\mathrm{H}}) B (w u_1 + u_2) \\
&= \gamma^2 u_1^{\mathrm{H}} B u_1 + u_2^{\mathrm{H}} B u_2 + 2\mathrm{Re}\ (\bar{w} u_1^{\mathrm{H}} B u_2) \\
&= \gamma^2 u_1^{\mathrm{H}} B u_1 + 2\gamma\gamma_2 \sin(\alpha_2 - \alpha_1) + u_2^{\mathrm{H}} B u_2 \\
&= (1+\gamma^2) \frac{B \bullet Z}{r},
\end{aligned}$$

where in the last equality we use the fact that γ solves (1).

Due to (2), by letting $z_1 = v_1$, we get

$$Z - z_1 z_1^{\mathrm{H}} = Z - v_1 v_1^{\mathrm{H}} = \sum_{j=2}^{r} u_j u_j^{\mathrm{H}} \succeq 0.$$

We conclude that $z_1^{\mathrm{H}} A z_1 = A \bullet Z/r$ and $z_1^{\mathrm{H}} B z_1 = B \bullet Z/r$. Note that $\mathrm{rank}(Z - z_1 z_1^{\mathrm{H}}) = r - 1$ and $v_2^{\mathrm{H}} A v_2 = u_j^{\mathrm{H}} A u_j = A \bullet Z/r$ for $j = 3, \ldots, r$. Repeating this process, we obtain a rank-one decomposition for Z:

$$Z = \sum_{j=1}^{r} z_j z_j^{\mathrm{H}},$$

with the property that $z_j^{\mathrm{H}} A z_j = A \bullet Z/r$ and $z_j^{\mathrm{H}} B z_j = B \bullet Z/r$, $j = 2, \ldots, r$. $\square$

Denote '$\trianglerighteq$' to be '$=$', '$\geq$' or '$\leq$'. An immediate corollary follows.

Corollary 2.2 *Let $A, B \in \mathcal{H}^n$ be two arbitrary matrices. Let $Z \in \mathcal{H}^n$ be a positive semidefinite matrix of rank r. Suppose that $A \bullet Z \trianglerighteq_1 0$ and $B \bullet Z \trianglerighteq_2 0$. Then there is a rank-one decomposition for Z*

$$Z = \sum_{j=1}^{r} z_j z_j^{H}$$

such that $z_j^{H} A z_j \trianglerighteq_1 0$ and $z_j^{H} B z_j \trianglerighteq_2 0$ for all $j = 1, \ldots, r$.

3 Cone of complex nonnegative quadratic functions

Let $D \subset \mathbf{C}^n$ be a given set. Consider all Hermitian matrices which are co-positive over D, i.e.,

$$\mathcal{C}_+(D) = \{Z \in \mathcal{H}^n : z^{\mathrm{H}} Z z \geq 0, \forall z \in D\}. \tag{3}$$

Clearly, $\mathcal{C}_+(D)$ is a closed convex cone in $\mathcal{H}^n$. The cone of all complex quadratic functions that are non-negative over D, is defined by

$$\mathcal{FC}_+(D) = \left\{ \begin{bmatrix} c & b^{\mathrm{H}} \\ b & A \end{bmatrix} : c + 2\mathrm{Re}\ (b^{\mathrm{H}} z) + z^{\mathrm{H}} A z \geq 0, \forall z \in D \right\}. \tag{4}$$

For a quadratic function $q(z) = z^{\mathrm{H}} A z + 2\mathrm{Re}\ (b^{\mathrm{H}} z) + c$, we introduce its matrix representation as

$$q(z) = M(q(\cdot)) \bullet \begin{bmatrix} 1 & z^{\mathrm{H}} \\ z & z z^{\mathrm{H}} \end{bmatrix},$$

where

$$M(q(\cdot)) = \begin{bmatrix} c & b^{\mathrm{H}} \\ b & A \end{bmatrix}.$$

Apparently, $q(z) \geq 0$ for all $z \in D$ if and only if $M(q(\cdot)) \in \mathcal{FC}_+(D)$. The homogenization of a given set D is defined by

$$\mathcal{H}(D) = \mathrm{cl} \left\{ \begin{bmatrix} t \\ z \end{bmatrix} \in \Re_{++} \times \mathbf{C}^n : z/t \in D \right\},$$

where ‘cl’ stands for the closure operation.

For a given set D we denote by ‘conv (D)’ the convex hull of D, i.e., the intersection of all convex sets containing D, and by ‘cone (D)’ the convex cone hull of D, i.e., the intersection of all convex cones containing D. Following similar arguments as in Sturm and Zhang [3], the next two propositions are immediate.

Proposition 3.1 *It holds that*

$$\mathcal{C}_+(D) = (\mathrm{cone}\ \{z z^H : z \in D\})^*$$

and

$$\mathrm{cl\ cone}\ \{z z^H : z \in D\} = \mathrm{cone}\ \{z z^H : z \in \mathrm{cl}\ D\}.$$

Proposition 3.2 *For any nonempty set $D \subset \mathbf{C}^n$, there holds*

$$\mathcal{FC}_+(D) = \mathcal{C}_+ (\mathcal{H}(D)) = \mathcal{C}_+ \left(\bigcup_{u \in \mathbf{C}, |u|=1} (u \mathcal{H}(D)) \right).$$

In what follows, we shall give a characterization of $\mathcal{C}_+(D)$ where D is defined by

$$D = \{z \in \mathbf{C}^n : z^{\mathrm{H}} A z \geq 0, z^{\mathrm{H}} B z \geq 0\}.$$

Our next result follows directly from Theorem 2.1.

Theorem 3.3 *Suppose that $A, B \in \mathcal{H}^n$ and $D = \{z \in \mathbf{C}^n : z^H A z \geq 0, z^H B z \geq 0\}$. Then, we have*

$$\text{cone}\, \{zz^H : z \in D\} = \text{conv}\, \{zz^H : z \in D\} = \{Z \succeq 0 : A \bullet Z \geq 0, B \bullet Z \geq 0\}.$$

Proof. Obviously, conv $\{zz^{\mathrm{H}} : z \in D\} \subseteq$ cone $\{zz^{\mathrm{H}} : z \in D\}$. The equality follows from the observation that conv $\{zz^{\mathrm{H}} : z \in D\}$ is itself a convex cone. That conv $\{zz^{\mathrm{H}} : z \in D\} \subseteq \{Z \succeq 0 : A \bullet Z \geq 0, B \bullet Z \geq 0\}$ is clear, thanks to Carathéodory's theorem. Now, $\{Z \succeq 0 : A \bullet Z \geq 0, B \bullet Z \geq 0\} \subseteq$ conv $\{zz^{\mathrm{H}} : z \in D\}$ follows from Theorem 2.1 by construction. □

The dual form of Theorem 3.3 is also known as the $\mathcal{S}$-Lemma, which we shall present below. The result was first shown by Fradkov and Yakubovich [1], though their proof was totally different.

Theorem 3.4 *Suppose that $A, B \in \mathcal{H}^n$ and $D = \{z \in \mathbf{C}^n : z^H A z \geq 0, z^H B z \geq 0\}$. Furthermore, suppose that there is $z_0 \in \mathbf{C}^n$ such that $z_0^H A z_0 > 0, z_0^H B z_0 > 0$, then*

$$\{Z \in \mathcal{H}^n : z^H Z z \geq 0, \forall z \in D\} = \{Z : \exists \lambda_1 \geq 0, \lambda_2 \geq 0,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\}.$$

Proof. It follows from Theorem 3.3 and Proposition 3.1 that

$$\begin{aligned}
\mathcal{C}_+(D) &= \{Z : z^{\mathrm{H}} Z z \geq 0, \forall z \in D\} \\
&= (\text{conv}\, \{zz^{\mathrm{H}} : z \in D\})^* \\
&= (\{Z \succeq 0 : A \bullet Z \geq 0, B \bullet Z \geq 0\})^* \\
&= \text{cl}\, \{Z : \exists \lambda_1 \geq 0, \lambda_2 \geq 0,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\}.
\end{aligned}$$

It remains to show that $\{Z : \exists \lambda_1 \geq 0, \lambda_2 \geq 0,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\}$ is a closed set. To this end, take any sequence $Z_k, s_k \geq 0, t_k \geq 0$ such that

$$Z_k - s_k A - t_k B \succeq 0 \text{ for } k = 1, 2, \ldots \tag{5}$$

and $Z_k \to Z$. Then we have

$$z_0^{\mathrm{H}} Z_k z_0 \geq s_k z_0^{\mathrm{H}} A z_0 + t_k z_0^{\mathrm{H}} B z_0 \geq s_k z_0^{\mathrm{H}} A z_0 \geq 0,$$

which implies that $0 \le s_k \le z_0^{\rm H} Z_k z_0 / z_0^{\rm H} A z_0$ for each k. That is the sequence $\{s_k\}$ is bounded and has a cluster point, say $s_0 \ge 0$. In a similar way, one can prove that the sequence $\{t_k\}$ has a cluster point, say $t_0 \ge 0$. By (5) we have

$$Z - s_0 A - t_0 B \succeq 0.$$

Hence we conclude that

$$Z \in \{Z : \exists \lambda_1 \ge 0, \lambda_2 \ge 0,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\}.$$

That is,

$$\mathcal{C}_+(D) = \mathrm{cl}\ \{Z : \exists \lambda_1 \ge 0, \lambda_2 \ge 0,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\} = \{Z : \exists \lambda_1 \ge 0, \lambda_2 \ge 0,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\}.$$

The desired result is proven. $\square$

Theorem 3.4 can be further generalized. Consider $\trianglerighteq \in \{\ge, \le, =, \emptyset\}$, where $\emptyset$ means the relation to be 'unrelated', and denote

$$\trianglerighteq^* \text{ to be } \begin{cases} \ge, & \text{if } \trianglerighteq \text{ is } \ge; \\ \le, & \text{if } \trianglerighteq \text{ is } \le; \\ \emptyset, & \text{if } \trianglerighteq \text{ is } =; \\ =, & \text{if } \trianglerighteq \text{ is } \emptyset. \end{cases}$$

Theorem 3.5 *Suppose that $A, B \in \mathcal{H}^n$. Let $D = \{z \in \mathbf{C}^n : z^H A z \trianglerighteq_1 0, z^H B z \trianglerighteq_2 0\}$. Then*

$$\begin{aligned} \{Z \in \mathcal{H}^n : z^H Z z \ge 0, \forall z \in D\} &= (\{Z \succeq 0 : A \bullet Z \trianglerighteq_1 0, B \bullet Z \trianglerighteq_2 0\})^* \\ &= \mathrm{cl}\ \{Z : \exists \lambda_1 \trianglerighteq_1^* 0, \lambda_2 \trianglerighteq_2^* 0,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\}. \end{aligned}$$

Corollary 3.6 *Suppose that $A, B \in \mathcal{H}^n$. Let $D = \{z \in \mathbf{C}^n : z^H A z \ge 0, z^H B z = 0\}$. Suppose furthermore that there are $z_1, z_2, z_3 \in \mathbf{C}^n$ such that $z_1^H A z_1 > 0$, $z_1^H B z_1 = 0$, $z_2^H B z_2 > 0$ and $z_3^H B z_3 < 0$. Then*

$$\{Z \in \mathcal{H}^n : z^H Z z \ge 0, \forall z \in D\} = \{Z : \exists \lambda_1 \ge 0, \lambda_2 \in \Re,\ Z - \lambda_1 A - \lambda_2 B \succeq 0\}.$$

Proof. We need only to prove that the set

$$W = \{Z : \exists \lambda_1 \ge 0, \lambda_2 \in \Re, \text{ such that } Z - \lambda_1 A - \lambda_2 B \succeq 0\}$$

is closed. Suppose $Z \in \mathrm{cl}\ W$, i.e., there are Z_k, $s_k \ge 0$, $t_k \in \Re$ such that $Z_k \to Z$ and

$$Z_k - s_k A - t_k B \succeq 0,\ \forall k = 1, 2, \ldots. \tag{6}$$

Then we have

$$z_1^{\rm H} Z_k z_1 - s_k z_1^{\rm H} A z_1 - t_k z_1^{\rm H} B z_1 = z_1^{\rm H} Z_k z_1 - s_k z_1^{\rm H} A z_1 \geq 0, \, \forall k,$$

which implies that

$$0 \leq s_k \leq z_1^{\rm H} Z_k z_1 / (z_1^{\rm H} A z_1), \, \forall k.$$

Therefore the sequence $\{s_k\}$ is bounded and thus it has a cluster point, say $s_0 \geq 0$. It follows by the assumption that

$$z_2^{\rm H} Z_k z_2 - s_k z_2^{\rm H} A z_2 - t_k z_2^{\rm H} B z_2 \geq 0, z_3^{\rm H} Z_k z_3 - s_k z_3^{\rm H} A z_3 - t_k z_3^{\rm H} B z_3 \geq 0, \, \forall k.$$

These imply that

$$(z_3^{\rm H} Z_k z_3 - s_k z_3^{\rm H} A z_3) / z_3^{\rm H} B z_3 \leq t_k \leq (z_2^{\rm H} Z_k z_2 - s_k z_2^{\rm H} A z_2) / z_2^{\rm H} B z_2, \, \forall k.$$

Since $\{s_k\}$ is bounded, then $\{t_k\}$ is bounded as well. Let $t_0 \in \Re$ be a cluster point of $\{t_k\}$. By (6) we conclude that

$$Z - s_0 A - t_0 B \succeq 0,$$

i.e., $Z \in W$. Hence W is a closed set. ☐

Similarly, we have the following result.

Corollary 3.7 *Suppose that* $A, B \in \mathcal{H}^n$. *Let* $D = \{z \in \mathbf{C}^n : z^H A z = 0, z^H B z = 0\}$. *Suppose further that there are* $z_1, z_2, z_3, z_4 \in \mathbf{C}^n$ *such that* $z_1^H A z_1 > 0$, $z_1^H B z_1 = 0$, $z_2^H A z_2 < 0$, $z_2^H B z_2 = 0$, $z_3^H B z_3 > 0$ *and* $z_4^H B z_4 < 0$. *Then*

$$\{Z \in \mathcal{H}^n : z^H Z z \geq 0, \forall z \in D\} = \{Z : \exists \lambda_1 \in \Re, \lambda_2 \in \Re, \text{ such that } Z - \lambda_1 A - \lambda_2 B \succeq 0\}.$$

Let us now consider nonhomogeneous quadratic functions. Suppose that

$$D = \{z \in \mathbf{C}^n : q_1(z) \geq 0, q_2(z) \geq 0\},$$

where $q_j(z) = z^{\rm H} A_j z + 2{\rm Re}\ (b_j^{\rm H} z) + c_j$, $j = 1, 2$.

Lemma 3.8 *Suppose that there is* $z_0 \in \mathbf{C}^n$ *such that* $q_1(z_0) > 0$ *and* $q_2(z_0) > 0$, *and that there is no* $z \neq 0$ *such that* $z^H A_1 z = 0$ *and* $z^H A_2 z = 0$. *Then*

$$\bigcup_{u \in \mathbf{C}, |u|=1} (u \mathcal{H}(D)) = \left\{ \begin{bmatrix} t \\ z \end{bmatrix} \in \mathbf{C}^{n+1} : \begin{bmatrix} c_j & b_j^H \\ b_j & A_j \end{bmatrix} \bullet \begin{bmatrix} |t|^2 & t z^H \\ \bar{t} z & z z^H \end{bmatrix} \geq 0, j = 1, 2 \right\}.$$

Proof. That

$$\bigcup_{u\in\mathbf{C},|u|=1}(u\mathcal{H}(D))\subseteq\left\{\left[\begin{array}{c} t\\ z\end{array}\right]\in\mathbf{C}^{n+1}:\left[\begin{array}{cc} c_j & b_j^{\rm H}\\ b_j & A_j\end{array}\right]\bullet\left[\begin{array}{cc} |t|^2 & tz^{\rm H}\\ \bar{t}z & zz^{\rm H}\end{array}\right]\geq 0, j=1,2\right\}$$

is readily seen by definition. We need only show the other containing relationship. Take any arbitrary

$$\left[\begin{array}{c} t\\ z\end{array}\right]\in\left\{\left[\begin{array}{c} t\\ z\end{array}\right]\in\mathbf{C}^{n+1}:\left[\begin{array}{cc} c_j & b_j^{\rm H}\\ b_j & A_j\end{array}\right]\bullet\left[\begin{array}{cc} |t|^2 & tz^{\rm H}\\ \bar{t}z & zz^{\rm H}\end{array}\right]\geq 0, j=1,2\right\}.$$

If $t\neq 0$, then

$$\frac{z^{\rm H}}{\bar{t}}A_j\frac{z}{t}+2{\rm Re}\ (b_j^{\rm H}\frac{z}{t})+c_j\geq 0, j=1,2.$$

That is $z/t\in D$. Let $t=\gamma e^{\boldsymbol{i}\alpha}$. Then $\left[\begin{array}{c}\gamma\\ z/e^{\boldsymbol{i}\alpha}\end{array}\right]\in\mathcal{H}(D)$, which means $\left[\begin{array}{c}\gamma e^{\boldsymbol{i}\alpha}\\ z\end{array}\right]\in e^{\boldsymbol{i}\alpha}(\mathcal{H}(D))$, implying that $\left[\begin{array}{c} t\\ z\end{array}\right]\in\bigcup_{u\in\mathbf{C},|u|=1}(u\mathcal{H}(D))$.

If $t=0$, then $q_j(z)\geq 0$ amounts to $z^{\rm H}A_jz\geq 0$. In that case, if $z=0$, then $\left[\begin{array}{c} 0\\ 0\end{array}\right]=\left[\begin{array}{c} t\\ z\end{array}\right]\in\mathcal{H}(D)$ since $\mathcal{H}(D)$ is a closed cone.

Now we consider the case where $z\neq 0$. For $\epsilon\in\Re$, we consider

$$q_j(z_0+\epsilon z)=\epsilon^2z^{\rm H}A_jz+2\epsilon{\rm Re}\ (A_jz_0+b_j)^{\rm H}z+q_j(z_0), j=1,2.$$

For each j, we have four possibilities: (i) If $z^{\rm H}A_jz>0$, then $q_j(z_0+\epsilon z)>0$ when $|\epsilon|$ is sufficiently large (ϵ being negative or positive); (ii) If $z^{\rm H}A_jz=0$ and Re $(A_jz_0+b_j)^{\rm H}z=0$, then $q_j(z_0+\epsilon z)=q_j(z_0)>0$ for any real ϵ; (iii) If $z^{\rm H}A_jz=0$ and Re $(A_jz_0+b_j)^{\rm H}z>0$, then $q_j(z_0+\epsilon z)>0$ when $\epsilon>0$ is sufficiently large; (iv) If $z^{\rm H}A_jz=0$ and Re $(A_jz_0+b_j)^{\rm H}z<0$, then $q(z_0+\epsilon z)>0$ when $\epsilon<0$ and $|\epsilon|$ is sufficiently large.

Since $z^{\rm H}A_1z=0$ and $z^{\rm H}A_2z=0$ can not occur at the same time, we can find an ϵ (with $|\epsilon|$ sufficiently large), such that

$$q_1(z_0+\epsilon z)>0,\ \text{and}\ q_2(z_0+\epsilon z)>0.$$

That is, in this case we have $z_0+\epsilon z\in D$. Thus we claim $(z_0/\epsilon+z)/(1/\epsilon)\in D$ (for positive ϵ) or $(-z_0/\epsilon-z)/(-1/\epsilon)\in D$ (for negative ϵ). By letting $1/\epsilon\to 0$, we conclude that either $\left[\begin{array}{c} 0\\ z\end{array}\right]\in\mathcal{H}(D)$ or $\left[\begin{array}{c} 0\\ -z\end{array}\right]\in\mathcal{H}(D)$. In any of the two cases, we have

$$\left[\begin{array}{c} 0\\ z\end{array}\right]\in\bigcup_{u\in\mathbf{C},|u|=1}(u\mathcal{H}(D)).$$

The proof is completed. □

Theorem 3.9 *Let* $D = \{z \in \mathbf{C}^n : q_j(z) = z^H A_j z + 2\mathrm{Re}\ (b_j^H z) + c_j \geq 0, j = 1, 2\}$. *Suppose that there is* $z_0 \in \mathbf{C}^n$ *such that* $q_1(z_0) > 0$ *and* $q_2(z_0) > 0$, *and that there is no* $z \neq 0$ *such that* $z^H A_1 z = 0$ *and* $z^H A_2 z = 0$. *Then*

$$\mathcal{FC}_+(D) = \{Z : \exists \lambda_1 \geq 0, \lambda_2 \geq 0 \text{ such that } Z - \lambda_1 M(q_1(\cdot)) - \lambda_2 M(q_2(\cdot)) \succeq 0\}.$$

Proof. It follows that

$$\begin{aligned}
\mathcal{FC}_+(D) &= \mathcal{C}_+\left(\bigcup_{u\in\mathbf{C},|u|=1}(u\mathcal{H}(D))\right) \\
&= \left(\text{cone}\left\{zz^{\mathrm{H}} : z \in \bigcup_{u\in\mathbf{C},|u|=1}(u\mathcal{H}(D))\right\}\right)^* \\
&= \left(\text{conv}\left\{zz^{\mathrm{H}} : z \in \bigcup_{u\in\mathbf{C},|u|=1}(u\mathcal{H}(D))\right\}\right)^* \\
&= \left(\text{conv}\left\{zz^{\mathrm{H}} : \begin{bmatrix} c_j & b_j^{\mathrm{H}} \\ b_j & A_j \end{bmatrix} \bullet zz^{\mathrm{H}} \geq 0, j = 1, 2\right\}\right)^* \\
&= \left(\left\{Z \in \mathcal{H}_+^{n+1} : M(q_j(\cdot)) \bullet Z \geq 0, j = 1, 2\right\}\right)^* \\
&= \text{cl}\ \{Z : \exists \lambda_1 \geq 0, \lambda_2 \geq 0,\ Z - \lambda_1 M(q_1(\cdot)) - \lambda_2 M(q_2(\cdot)) \succeq 0\} \\
&= \{Z : \exists \lambda_1 \geq 0, \lambda_2 \geq 0,\ Z - \lambda_1 M(q_1(\cdot)) - \lambda_2 M(q_2(\cdot)) \succeq 0\}\,.
\end{aligned}$$

□

4 Complex quadratic programming and SDP relaxation

Consider the complex quadratically constrained quadratic programming:

$$\begin{array}{lll}
\text{(QCQP)} & \max & z^{\mathrm{H}} Q z + 2\mathrm{Re}\ z^{\mathrm{H}} q \\
 & \text{s.t.} & z^{\mathrm{H}} A_j z + 2\mathrm{Re}\ z^{\mathrm{H}} b_j + c_j \leq 0,\ j = 1, \ldots, m,
\end{array}$$

where $Q, A_j \in \mathcal{H}^n$, $b_j, q \in \mathbf{C}^n$, $c_j \in \Re$, $j = 1, \ldots, m$. Denote

$$B_0 = \begin{bmatrix} 0 & q^{\mathrm{H}} \\ q & Q \end{bmatrix},\ B_j = \begin{bmatrix} c_j & b_j^{\mathrm{H}} \\ b_j & A_j \end{bmatrix}, j = 1, \ldots, m,\ B_{m+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We rewrite (QCQP) equivalently as:

$$
\begin{array}{lll}
\text{(QCQP)} & \max & B_0 \bullet \begin{bmatrix} 1 & z^{\mathrm{H}} \\ z & zz^{\mathrm{H}} \end{bmatrix} \\
& \text{s.t.} & B_j \bullet \begin{bmatrix} 1 & z^{\mathrm{H}} \\ z & zz^{\mathrm{H}} \end{bmatrix} \le 0, j = 1, \ldots, m.
\end{array}
$$

A homogenized version of (QCQP) is

$$
\begin{array}{lll}
\text{(HQ)} & \max & B_0 \bullet \begin{bmatrix} |t|^2 & tz^{\mathrm{H}} \\ \bar{t}z & zz^{\mathrm{H}} \end{bmatrix} \\
& \text{s.t.} & B_j \bullet \begin{bmatrix} |t|^2 & tz^{\mathrm{H}} \\ \bar{t}z & zz^{\mathrm{H}} \end{bmatrix} \le 0, j = 1, \ldots, m, \\
& & B_{m+1} \bullet \begin{bmatrix} |t|^2 & tz^{\mathrm{H}} \\ \bar{t}z & zz^{\mathrm{H}} \end{bmatrix} = 1.
\end{array}
$$

It follows that if $\begin{bmatrix} t \\ z \end{bmatrix}$ solves (HQ), then z/t solves (QCQP). The SDP relaxation of (QCQP) is:

$$
\begin{array}{lll}
\text{(QCQPR)} & \max & B_0 \bullet Z \\
& \text{s.t.} & B_j \bullet Z \le 0, j = 1, \ldots, m, \\
& & B_{m+1} \bullet Z = 1, \\
& & Z \succeq 0,
\end{array}
$$

with the dual given by

$$
\begin{array}{lll}
\text{(DQCQPR)} & \min & y_0 \\
& \text{s.t.} & Y = \sum_{j=1}^{m} y_j B_j - B_0 + y_0 B_{m+1} \succeq 0, \\
& & y_j \ge 0, \ j = 1, \ldots, m, \ y_0 \text{ free}.
\end{array}
$$

Throughout this section we assume that (QCQP) satisfies the Slater condition, i.e., there exists $z_0 \in \mathbf{C}^n$ such that $q_j(z_0) < 0$, $j = 1, \ldots, m$. Accordingly, (QCQPR) satisfies the Slater condition as well.

It is well known that (QCQP) is NP-hard in general. In the remainder of this section, we shall study (QCQP) where m is small.

Theorem 4.1 *Suppose that (QCQPR) and (DQCQPR) have complementary optimal solutions. Moreover, suppose that $m = 2$. Then (QCQP) and (QCQPR) have the same optimal value. Moreover, an optimal solution for (QCQP) can be constructed from an optimal solution for (QCQPR) in polynomial time.*

Proof. This theorem is an extension of the result in Ye and Zhang [5] to the complex case.

Let $Z^* \succeq 0$ be an optimal solution of (QCQPR), and $(y_0^*, y_1^*, y_2^*, Y^*)$ be an optimal solution of (DQCQPR). Since the strong duality is satisfied, the respective primal and dual optimal solutions are complimentary, i.e., $Z^* \bullet Y^* = 0$.

Denote '$\trianglelefteq$' to be either '<' or '='. Then $B_j \bullet Z^* \trianglelefteq_j 0$, $j = 1, 2$. By Theorem 2.1 (or Corollary 2.2), there exist non-zero z_k, $k = 1, \ldots, r$, where r is the rank of Z^*, such that

$$Z^* = \sum_{k=1}^{r} z_k z_k^{\mathrm{H}},\ B_j \bullet z_k z_k^{\mathrm{H}} \trianglelefteq_j 0,\ j = 1, 2,\ k = 1, \ldots, r.$$

Since $Z_{11}^* = 1$, there is $l \in \{1, \ldots, r\}$ such that $t_l \neq 0$ where $z_l = \begin{bmatrix} t_l \\ \tilde{z}_l \end{bmatrix}$. Therefore, $q_j(\tilde{z}_l/t_l) \trianglelefteq_j 0$, $j = 1, 2$, which implies that $\tilde{z}_l/t_j$ is a feasible solution for (QCQP).

Due to $Y^* \succeq 0$, we have $Y^* \bullet z_l z_l^{\mathrm{H}} = 0$. If $B_j \bullet Z^* < 0$, then by complementarity, it follows that $y_j^* = 0$. If $B_j \bullet Z^* = 0$, then by the decomposition theorem we have $B_j \bullet z_l z_l^{\mathrm{H}} = 0$. Therefore we always have $y_j^*(B_j \bullet z_l z_l^{\mathrm{H}}) = 0$, $j = 1, 2$. This, combining with $Y^* \bullet z_l z_l^{\mathrm{H}} = 0$, leads to the conclusion that $\begin{bmatrix} 1 \\ \tilde{z}_l/t_l \end{bmatrix} [1\ \tilde{z}_l^{\mathrm{H}}/\bar{t}_l]$ is a solution of (QCQPR). Therefore $\tilde{z}_l/t_l$ is optimal for (QCQP). Note that the rank-one decomposition procedure runs in polynomial time. The theorem is proven. $\square$

Not all SDP problems are relaxations of quadratic programs. In the next section, we shall study the rank of optimal solutions for a general complex SDP problem.

5 The low rank optimal solutions for complex SDP

Consider a standard (complex) SDP problem

$$\begin{array}{lll} \text{(SDP)} & \min & C \bullet Z \\ & \text{s.t.} & A_j \bullet Z = b_j,\ j = 1, \ldots, m, \\ & & Z \succeq 0, \end{array}$$

and its dual

$$\begin{array}{lll} \text{(DSDP)} & \max & \sum_{j=1}^{m} y_j b_j \\ & \text{s.t.} & \sum_{j=1}^{m} y_j A_j \preceq C. \end{array}$$

Suppose that (SDP) and (DSDP) have a complementary optimal solution pair.

Consider now an optimal solution for (SDP) with minimum rank. In particular, let S_P be the set of all optimal solutions for (SDP), and introduce

$$r_P = \min\{\text{rank}(Z) : Z \in S_P\},$$

and let Z^* be such an optimal solution with $\text{rank}(Z^*) = r_P$.

For this given Z^*, let us introduce the following notion of minimum diagonal rank:

$$r_M = \min\left\{\text{rank}\left(\text{diag }(V^{\mathrm{H}}A_1V), ..., \text{diag }(V^{\mathrm{H}}A_mV)\right) : VV^{\mathrm{H}} = Z^*, \text{rank}(V) = \text{rank}(Z^*) = r_P\right\}. \quad (7)$$

The result below follows from Theorem 2.1.

Proposition 5.1 *Suppose that $S_P \neq \emptyset$ and $m \geq 3$. We have $r_M \leq m - 2$.*

Proof. Consider the case that at least three of the values $\{b_1, b_2, ..., b_m\}$ are nonzero. (In the other case, namely at most two of the b_j's are nonzero, the proof can be easily adapted). Without losing generality, let us assume that b_1, b_2 and b_3 are nonzero.

Take any $Z \in S_P$ with rank r. Since

$$(A_1/b_1 - A_2/b_2) \bullet Z = 0, \text{ and } (A_1/b_1 - A_3/b_3) \bullet Z = 0,$$

Theorem 2.1 asserts that there is a rank-one decomposition of Z,

$$Z = \sum_{k=1}^{r} v_k v_k^{\mathrm{H}}$$

such that

$$v_k^{\mathrm{H}}(A_1/b_1 - A_2/b_2)v_k = 0, \text{ and } v_k^{\mathrm{H}}(A_1/b_1 - A_3/b_3)v_k = 0$$

for all $k = 1, ..., r$. Letting $V = [v_1, ..., v_r]$, we thus have

$$\text{diag}(V^{\mathrm{H}}A_1V)/b_1 = \text{diag}(V^{\mathrm{H}}A_2V)/b_2 = \text{diag}(V^{\mathrm{H}}A_3V)/b_3.$$

Hence,

$$\text{rank}\left(\text{diag }(V^{\mathrm{H}}A_1V), ..., \text{diag }(V^{\mathrm{H}}A_mV)\right) \leq m - 2.$$

□

Let us denote d_D to be the dimension of the optimal solution set for (DSDP). Next theorem gives an upper bound on the minimum rank among optimal solutions.

Theorem 5.2 *Suppose that the complex SDP pair (SDP) and (DSDP) have a complementary optimal solution pair, and $m \geq 3$. Then, it holds that*

$$r_P \leq \min\{r_M, \lfloor \sqrt{m - d_D} \rfloor\}.$$

Proof. For $Z^* \in S_P$ with $\operatorname{rank}(Z^*) = r_P$, let $Z^* = VV^{\mathrm{H}}$ be the decomposition attaining the minimum diagonal rank as defined in (7). Consider the following system of linear equations

$$\begin{aligned}
\left(\operatorname{diag}\,(V^{\mathrm{H}} A_1 V)\right)^{\mathrm{T}} x &= A_1 \bullet \left(\sum_{k=1}^{r_P} x_k v_k v_k^{\mathrm{H}} \right) = 0 \\
\left(\operatorname{diag}\,(V^{\mathrm{H}} A_2 V)\right)^{\mathrm{T}} x &= A_2 \bullet \left(\sum_{k=1}^{r_P} x_k v_k v_k^{\mathrm{H}} \right) = 0 \\
&\vdots \\
\left(\operatorname{diag}\,(V^{\mathrm{H}} A_m V)\right)^{\mathrm{T}} x &= A_m \bullet \left(\sum_{k=1}^{r_P} x_k v_k v_k^{\mathrm{H}} \right) = 0.
\end{aligned}$$

The rank of the coefficient matrix of the above linear equation is r_M and the number of (real) variables is r_P ($x \in \Re^{r_P}$). If $r_P > r_M$, then the above equation must have a nonzero solution $x \in \Re^{r_M}$ with $x \not\leq 0$. In that case it would follow that

$$Z^*(t) = \sum_{k=1}^{r_P} (1 - t x_k) v_k v_k^{\mathrm{H}}$$

is also an optimal solution for (SDP) for all $t \leq \hat{t} := \min\{\frac{1}{x_k} : x_k > 0,\ k = 1, ..., r_P\}$. In particular, $\operatorname{rank}(Z^*(\hat{t})) \leq r_P - 1$, contradicting to the fact that Z^* is the minimum rank optimal solution. This shows that $r_P \leq r_M$.

We do not necessarily restrict the directions to be diagonal. As an alternative, consider the equation

$$\left.\begin{array}{c}
(V^{\mathrm{H}} A_1 V) \bullet \Delta = 0 \\
(V^{\mathrm{H}} A_2 V) \bullet \Delta = 0 \\
\vdots \\
(V^{\mathrm{H}} A_m V) \bullet \Delta = 0
\end{array}\right\} \tag{8}$$

where $\Delta \in \mathcal{H}^{r_P}$.

Due to the complementarity, we have

$$(C - \sum_{j=1}^{m} y_j A_j) VV^{\mathrm{H}} = 0$$

for any dual optimal solution $(y_1, ..., y_m)$. Thus, by the positive semidefiniteness of the matrices, we have

$$V^{\mathrm{H}} C V - \sum_{j=1}^{m} y_j V^{\mathrm{H}} A_j V = 0.$$

Therefore, the rank of the coefficient matrix in (8) is $m - d_D$. Since the dimension of Δ is r_P^2, this implies that as long as $r_P^2 > m - d_D$, the equation (8) would admit a nonzero solution $\Delta \in \mathcal{H}^{r_P}$, allowing $V(I - t\Delta)V^{\mathrm{H}}$ to be optimal for (SDP) with t satisfying $t\Delta \preceq I$, thereby enabling a possibility to further reduce the rank of Z^*. Again, this is in contradiction with the fact that Z^* has minimum rank among optimal solutions. This in turn shows that we must have $r_P^2 \le m - d_D$. Since r_P is integer, we thus have $r_P \le \lfloor \sqrt{m - d_D} \rfloor$. The theorem is proven. $\square$

Note that since the diagonal of a matrix is only a part of the whole matrix, naturally we have $r_M \le m - d_D$. In general, we may expect r_M to be much less than $m - d_D$ indeed.

As a consequence of Theorem 5.2, we conclude that (SDP) has a rank-one optimal solution if $m \le 3$, since in this case it follows from Proposition 5.1 and Theorem 5.2 that $r_P \le \min\{3-2, \lfloor \sqrt{3-0} \rfloor\} = 1$. It also follows from Theorem 5.2 that if $m = 4$ and the dual optimal solution is not unique ($d_D \ge 1$) then there is a rank-one optimal solution for (SDP).

Remark that in the real case, the discussion on the minimum rank optimal solutions for SDP can be found in Pataki [2], Ye and Zhang [5], and Ye [4].

References

[1] A.L. Fradkov and V.A. Yakubovich. The $\mathcal{S}$-procedure and the duality relation in convex quadratic programming problems. *Vestnik Leninggrad Univ.*, 155: 81 – 87, 1973.

[2] G. Pataki. On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues. *Mathematics of Operations Research*, 23: 339 – 358, 1998.

[3] J.F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Mathematics of Operations Research*, 28: 246 – 267, 2003.

[4] Y. Ye. *Linear Conic Programming.* Stanford University, 2004.

[5] Y. Ye and S. Zhang. New results on quadratic minimization. *SIAM Journal on Optimization*, 14: 245 – 267, 2003.