The Simplex Method - Computational Checks for the Simplex Calculation

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Abstract
The purpose of this paper is to derive computational checks for the simplex method of Linear Programming which can be applied at any iteration. The paper begins with a quick review of the simplex algorithm. It then goes through a theoretical development of the simplex method and its dual all the time focusing on the derivation of computational checks.

In this paper we describe the simplex method for the problem

$$\begin{cases} \text{maximize } cx \\ ax \le b \qquad\qquad x \ge 0, b \ge 0 \end{cases}.$$

The simplex algorithm is an iterative algorithm that iterates from one tableau to another until it reaches an optimal solution. There exists information about the linear program we are solving in the tableau of any iteration. We explain and prove a few interesting properties of these simplex tableaus. We show how to glean information about the problem from any given tableau. This paper will help give a deeper understanding of simplex method.

As a nice side benefit, we will derive some computational checks as we proceed from one step to the next. The computation of the simplex algorithm is fairly involved when done by hand even for moderately sized problems. It would thus be advantageous to develop some definite procedure for checking these computations at each step before one discovers at some later point that an error has been made at a previous point. Worse yet the error may not be discovered at all and the "solution" arrived at, is actually incorrect.

Problem Formulation

Consider the following linear programming problem:

Find $x \ge 0$ such that the objective function cx is maximized subject to the system of constraints $ax \le b$, $b \ge 0$ i.e.

(1) $$\begin{cases} \text{Maximize } cx \\ ax \le b \qquad x \ge 0,\ b \ge 0 \end{cases}$$

where a is an mxp matrix, x is a px1 column vector, b is an mx1 column vector and c is a 1xp row vector i.e.

$$a = \begin{pmatrix} a_{11} & . & . & . & a_{1P} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & . & . & . & a_{p} \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_p \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ . \\ . \\ . \\ b_m \end{pmatrix} \quad c = (c_1, \ . \ . \ ., \ c_p)$$

The system of inequalities $ax \le b$ is transformed into a system of equations by adding a slack variable to the left hand side of each inequality $ax \le b$. Thus

x_{p+1}

$$ax + I_m x_s = b \qquad x_s = \begin{pmatrix} x_{p+1} \\ . \\ . \\ . \\ x_{p+m} \end{pmatrix}$$

We now have an mxn system of equations (n=p+m)

$$\begin{pmatrix} a_{11} & . & . & . & a_{1p} & 1 & 0 & . & . & . & 0 \\ a_{21} & . & . & . & a_{2p} & 0 & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ a_{m1} & . & . & . & a_{mp} & 0 & 0 & . & . & . & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{pmatrix}$$

or

$$AX = b \qquad \text{where } A=(a\ I_m) \text{ and } X = \begin{pmatrix} x \\ x_s \end{pmatrix}$$

We note that this system has rank m. We can thus obtain a basic solution by assigning n-m variables the value 0 and solving for the remaining m variables. A basic solution will be called a basic feasible solution if all its variables are nonnegative. A basic feasible solution that also optimizes the objective function is called a basic optimal solution. It is clear that for our system x = 0 and a basic feasible solution is $x_s = b \geq 0$. We call any mxm matrix whose columns are chosen from A and are linearly independent a basic matrix, which we denote by B. The columns are called basic columns and the variables associated with these columns are called basic variables.

Our linear program (1) can be written as:

$$(2) \quad \begin{cases} \max CX \\ AX = b, \text{ where C is a 1xn row vector } (c_1, c_2, ..., c_p, 0, ..., 0) \end{cases}$$

It should be noted that (1) and (2) are equivalent. Firstly note that CX = cx. Moreover, if $x \geq 0$ satisfies $ax \leq b$ then AX=b is satisfied with $x_s \geq 0$. Conversely if X is a feasible solution to AX=b then the first p components of x yield a feasible solution to $ax \leq b$. Thus there is a one to one correspondence between feasible solutions to (1) and feasible solutions to (2).

It can be shown that if (2) has an optimal solution, one or more of the basic feasible solutions will be optimal (Hadley, p. 103). Thus our problem can be solved in a finite set of trials. Since there are m equations in n variables we would need to examine the C(n,m) basic solutions obtained by setting m of the variables equal to zero in all possible ways and each time solving for the remaining variables. We would then select the basic feasible solution which optimizes the objective function.

Such a method is not computationally practical even for linear programs of moderate size. For example, a problem consisting of m=10 inequalities and p=10 variables (n=p+m) would require C(20,10)=184,756 basic solutions to compute, examine, evaluate and compare.

The simplex method is an iterative procedure that begins with one basic feasible solution of AX=b and then by means of elementary row operations replaces the old basic feasible solution by a new one that will increase (or at least maintain) the value of the objective function. The procedure is repeated until an optimal solution is found or until it is determined that no optimal solution exists. Usually only a (small) subset of the entire set of basic feasible solutions need be examined.

The Simplex Method Described

Tableau Format of the Simplex Method

A useful tabular form displaying all the quantities of interest is given in figure 1. Such a format is called a tableau. A new tableau is constructed at each iteration i.e. each time a new column is introduced into the basis.

a_1	a_2	a_3	. . .	a_p	a_{p+1}	. .	a_n	b
y_{11}	y_{12}	y_{13}	. . .	y_{1p}	$y_{1,p+1}$	. .	y_{1n}	$y_{1,n+1}$
y_{21}	y_{22}	y_{23}	. . .	y_{2p}	$y_{2,p+1}$	. .	y_{2n}	$y_{2,n+1}$
.	.	.	. . .	.	.	. .	.	.
.	.	.	. . .	.	.	. .	.	.
.	.	.	. . .	.	.	. .	.	.
y_{m1}	y_{m2}	y_{m3}	. . .	y_{mp}	$y_{m,p+1}$	. .	y_{mn}	$y_{m,n+1}$
z_1-c_1	z_2-c_2	z_3-c_3	. . .	z_p-c_p	$z_{p+1}-c_{p+1}$	. .	z_n-c_n	z
$y_{m+1,1}$	$y_{m+1,2}$	$y_{m+1,3}$	. . .	$y_{m+1,p}$	$y_{m+1,p+1}$	. .	$y_{m+1,n}$	$y_{m+1,n+1}$

Figure 1

In figure 1 a_j is the j^{th} column of A, y_{ij} (i=1,...,m; j=1,...,n) is the element in the i^{th} row, j^{th} column of the tableau. The constant column b is column n+1 and its values are denoted by $y_{i,n+1}$. Notice also the extra row is denoted by $z_j-c_j=y_{m+1,j}$ for j=1,...,n and the value in column n+1 in the extra row is denoted by $z=y_{m+1,n+1}$. We will also use y_j (j=1,...,n+1) to represent the j^{th} column in the tableau. For the initial tableau, $y_j=a_j$ (j=1,...,n) and y_{n+1}=b.

Any basis matrix B determines a basic solution to AX=b. This basic solution defined by an m-component vector X_B is

$$X_B = B^{-1}b, \qquad X_B = (x_{B1}, x_{B2}, \ldots, x_{Bm}).$$

The subscript Bi means that x_{Bi} corresponds to the i^{th} column of B. Thus if a_5 is in column 2 of B then $x_{B2} = x_5$.

Let C_B be a 1xm vector whose coordinates are the coordinates of C corresponding to the basic variables i.e. $C_B=(c_{B1}, c_{B2}, \ldots, c_{Bm})$ where c_{Bi} is the coefficient of x_{Bi} in the objective function. Thus if a_5 is in column 3 of B then $c_{B3} = c_5$. The z_j are defined by

(3) $$z_j = C_B y_j = \sum_{i=1}^{m} c_{Bi} y_{ij}, \qquad j=1,2,\ldots,n$$

and z is given by

(4) $$z = C_B X_B = \sum_{i=1}^{m} c_{Bi} x_{Bi}.$$

For the initial tableau the basic matrix is $B=(a_{p+1}, \ldots, a_n)=I_m$ and the basic variables are $x_{p+1}, x_{p+2}, \ldots, x_n$; $y_j = a_j$, $j=1,2,\ldots,n$; $y_{n+1} = b$; the basic solution is $X_B = x_s = b$; C_B is the 1xm zero vector. Thus $z_j = C_B y_j = 0$; $z_j - c_j = -c_j$, $j=1,2,\ldots,n$ and $z=C_B X_B=0$.

Rules For Choosing the Pivot Element

The simplex method gives a criterion for finding another basic feasible solution with an improved (or at least as good) value of the objective function. At each stage of computation, a column of B is replaced by some a_j not in B. The following rules determine which basic column is to be replaced by which nonbasic column.

1. The column containing the most negative entry in the extra row is to be brought into the new basis i.e. column a_k such that

(5) $$z_k - c_k = \min(z_j - c_j) \quad \text{for all } j \text{ such that } z_j - c_j < 0 .$$

2. Once it has been determined that a_k is to be brought into the new basis, compute the ratios $y_{i,n+1}/y_{ik}$ for positive y_{ik}, and replace the i^{th} column of B for which this ratio is minimum, i.e. we determine the column r of B to be replaced by means of:

(6) $$y_{r,n+1}/y_{rk} = \min\{y_{i,n+1}/y_{ik} \, , \ y_{ik}>0\}, \ i=1,2,\ldots,m .$$

The element y_{rk} is commonly called the **pivot** element; the r^{th} row and k^{th} column are called the **pivot row** and **pivot column** respectively.
If the minimum in (5) or (6) is not unique, any one of the columns corresponding to the minimum value may be chosen. It can be shown that the new matrix formed by column ak replacing the r^{th} column of B forms a basis matrix.

Replacement Rules
The entries of the new tableau can be obtained from the entries of the current tableau by means of the following replacement rules:

$$(7)\quad \left.\begin{aligned} &(a)\ Y_{rj} = Y_{rj}/Y_{rk} \\ &(b)\ y_{ij} = y_{ij} - (y_{ij}/y_{rk})\, y_{rj}, i \neq r \end{aligned}\right\} \quad \begin{aligned} &j = 1,2,...,n+1 \\ &i = 1,2,...,m+1 \end{aligned}$$

These rules are equivalent to the elementary row operations:

(a') Divide the r^{th} row by y_{rk}.

(b') Add suitable multiples of the rth row to each of the other rows of the tableau so as to obtain zeros in the kth column other than in the (r,k) position.

The entries in column n+1 of each tableau are the x_{Bi} .i.e $y_{i,n+1}=x_{Bi}$ for i=1,2,...,m.

Termination Rules
The simplex method terminates when a tableau is arrived at in which either:
(a) No entry in the extra row is negative, in which case an optimal solution has been arrived at; the values of the basic optimal solution are in X_B, the values of the other variables are zero. One can simply read off the solution from the final tableau: If a column i is in 0-1 form (i.e., the column has a single 1 and the rest of its entries zeros) and there are no identical columns in 0-1 form, then $x_i = x_{Bk} = y_{k,n+1}$ where k is the row in which 1 appears. If two or more columns are identical and are in 0-1 form (i.e., they have their single 1 in the same row k) then choose any one of these columns and assign $y_{k,n+1}$ to its variable and 0 to the variables of the rest of these columns. For each column j not in 0-1 form, set $x_j = 0$. The optimal value is $z = C_BX_B = y_{m+1,n+1}$.

(b) There are no positive elements in the pivot column, in which case there is no optimal solution.

The reader may note that one could carry out the mechanical computation of the simplex method simply by following the rules for determining the pivot element and the replacement rules, without computing B, X_B, z_j, z, or C_B. However, these have theoretical importance in proving that the simplex method as described, works. Moreover we will make use of them in discussing the computational relationship between our maximizing problem and its dual. We will also see their importance when checking hand-computation. Before proceeding, however, we give an example to illustrate the method as well as to enhance our understanding of the notation.

Example: Computational Checks for the Maximizing Problem

$$\begin{cases} \max cx \\ ax \le b \end{cases}$$

where

$$a = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} \qquad c = (2\ 4\ 1\ 1)$$

or, in nonmatrix notation:

$$\max\ 2x_1 + 4x_2 + x_3 + x_4 \qquad x_1, x_2, x_3, x_4 \ge 0$$

subject to the constraints:

$$\begin{aligned} x_1 + 3x_2 + x_4 &\le 4 \\ 2x_1 + x_2 &\le 3 \\ x_2 + 4x_3 + x_4 &\le 3 \end{aligned}$$

adding slack variables, we have:

$$\begin{cases} \max CX \\ AX = b \end{cases}$$

where

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ x_s \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}, \quad C = (c_1,c_2,c_3,c_4,c_5,c_6,c_7)=(2,4,1,1,0,0,0)$$

and our initial tableau is:

1	(3)	0	1	1	0	0	4
2	1	0	0	0	1	0	3
0	1	4	1	0	0	1	3
−2	−4	−1	−1	0	0	0	0

$B = (a_5, a_6, a_7)$, $\qquad C_B = (0,0,0)$

$$X_B = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} = b \qquad z_j = C_B y_j = 0$$

$$z = C_B X_B = 0$$

Clearly this is not the final tableau since there exists negative values in the extra row. The most negative is -4; thus a_2 will enter the basis in the next iteration. We form the ratios $y_{i,n+1}/y_{i2}$, $(y_{i2} > 0)$ for i=1,2,3. The minimum value is $y_{1,n+1}/y_{12} = 4/3$. Thus a_2 replaces the first column in B, namely a_5. The pivot element, y_{12}, is usually circled as shown.

We now use the replacement rules to compute the tableau for the new basic feasible solution. We divide the pivot row (row 1) by y_{12} to obtain row 1 of the new tableau. For the new rows i=2,3,and 4 we compute:

$$y_{ij} = y_{ij} - (y_{i2}/y_{12})\, y_{1j}, \quad j=1,2,\ldots,8$$

The new tableau is given below:

1/3	1	0	1/3	1/3	0	0	4/3
5/3	0	0	−1/3	−1/3	1	0	5/3
−1/3	0	4	2/3	−1/3	0	1	5/3
−2/3	0	−1	1/3	4/3	0	0	16/3

$$B = (a_2, \; a_6, \; a_7)$$

$$X_B = \begin{pmatrix} 4/3 \\ 5/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} x_{B1} \\ x_{B2} \\ x_{B3} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_6 \\ x_7 \end{pmatrix}$$

$C_B = (c_{B1}, c_{B2}, c_{B3}) = (4,0,0)$

We note that by computing $z_j - c_j = C_B y_j - c_j$ and $z = C_B X_B$, we can check the extra row entries:

$$z_1 - c_1 = (4,0,0)\begin{pmatrix} 1/3 \\ 5/3 \\ -1/2 \end{pmatrix} - 2 = -2/3$$

$$z_2 - c_2 = (4,0,0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 4 = 0$$

The reader should carry out the remaining calculation checks.

If a matrix $[B|I_m]$ is reduced to $[I_m|E]$ by elementary row operations, then $E = B^{-1}$. We note that we have used elementary row operations to replace, at each stage, column i of the tableau (where a_i is a column of B) by the identity columns. Thus the columns of the tableau that originally contained the identity columns should now contain B^{-1}. In our case , the first column of B, namely a_2, was replaced in the tableau by the identity column (column 2 in the current tableau). Thus $B^{-1} = (y_5, y_6, y_7)$ where $B = (a_2, a_6, a_7)$. For any tableau, B^{-1} consists of the columns corresponding to the slack variables. This observation affords us another way to check our computation. For each tableau, multiply B and the matrix consisting of the slack variable columns and see whether I_m is obtained. In our case:

$$(a_2, a_6, a_7)(y_5, y_6, y_7) = I_m$$

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since at each tableau we obtain a basic feasible solution, we should check to see whether this solution does indeed satisfy $BX_B = b$. Also, for the current tableau

$$X = \begin{pmatrix} 0 \\ 4/3 \\ 0 \\ 0 \\ 0 \\ 5/3 \\ 5/3 \end{pmatrix}$$

The reader should verify that this satisfies AX=b.

Since there is a negative value in the extra row, we proceed to the next tableau. The pivot element is y_{33} = 4 since $z_3-c_3 = \min(z_j-c_j)$, $z_j-c_j < 0$ and $y_{38}/y_{33} = \min\{ y_{i,n+1}/y_{i3} , y_{i3}>0 \}$. a_3 is to replace the third column of B, i.e. a_7. Our new basis is B = (a_2,a_6,a_3). Using the replacement rules we obtain the next tableau:

$$\begin{array}{cccc|ccc|c} 1/3 & 1 & 0 & 1/3 & 1/3 & 0 & 0 & 4/3 \\ 5/3 & 0 & 0 & -1/3 & -1/3 & 1 & 0 & 5/3 \\ -1/12 & 0 & 1 & 1/6 & -1/12 & 0 & 1/4 & 5/12 \\ \hline -3/4 & 0 & 0 & 1/2 & 5/4 & 0 & 1/4 & 23/4 \end{array}$$

$$X_B = \begin{pmatrix} 4/3 \\ 5/3 \\ 5/12 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_6 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_{B1} \\ x_{B2} \\ x_{B3} \end{pmatrix}$$

$c_B = (c_2,c_6,c_3) = (4,0,1)$

The feasible solution is now:

$$X = \begin{pmatrix} 0 \\ 4/3 \\ 5/12 \\ 0 \\ 0 \\ 5/3 \\ 0 \end{pmatrix}$$

Once again the following checks can be made:

$z_j-c_j = C_By_j - c_j$ and $z = C_BX_B$,
AX = b and BX_B = b,
$(a_2,a_6,a_3)(y_5,y_6,y_7) = BB^{-1} = I_m$.

The negative entry in the extra row indicates that a new basic feasible solution must be obtained. Proceeding as before, we find that a_1 is to replace the second column of B i.e. a_1 replaces a_6. The pivot element is 5/3 and the next tableau is:

0	1	0	2/5	2/5	−1/5	0	1
1	0	0	−1/5	−1/5	3/5	0	1
0	0	1	3/20	−1/10	1/20	1/4	1/2
0	0	0	7/20	11/10	9/20	1/4	13/2

$B = (a_2, a_1, a_3)$

$$X_b = \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_{B1} \\ x_{B2} \\ x_{B3} \end{pmatrix}$$

$C_b = (c_2, c_1, c_3) = (4,2,1)$

and the feasible solution is:

$$X = \begin{pmatrix} 1 \\ 1 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can verify:

$z_j - c_j = C_B y_j - c_j$ and $z = C_B X_B$,

$(a_2, a_1, a_3)(y_5, y_6, y_7) = BB^{-1} = I_m$,

$AX = b$ and $BX_B = b$.

Since there are no negative entries in the extra row, the current feasible solution X is an optimal solution to:

$$\begin{cases} \max CX \\ AX = b \end{cases}$$

and an optimal solution to our original linear program

$$\begin{cases} \max cx \\ ax \le b \end{cases}$$

is:

$$x = \begin{pmatrix} 1 \\ 1 \\ ½ \\ 0 \end{pmatrix}$$

with the objective function having the optimal value of 13/2.

The Simplex Method - Theoretical Development

Forming a New Basis

Let $B=(\beta_1,\beta_2,...,\beta_m)$ be any basis matrix, where β_i is the i^{th} column of B. Any columns a_j and a_k of A can be written as a linear combination of the β_i:

(8) $$a_j = \sum_{i=1}^{m} \lambda_{ij}\beta_i = B\lambda j$$

where $\lambda_i = \begin{pmatrix} \lambda_{1i} \\ . \\ . \\ . \\ \lambda_{mi} \end{pmatrix}$

(9) $$a_k = \sum_{i=1}^{m} \lambda_{ik}\beta_i = B\lambda_k$$

From (9) with $\lambda_{rk} \neq 0$, we have:

(10) $$\beta_r = - \sum_{\substack{i=1 \\ i\neq r}}^{m} (\lambda_{ik}/\lambda_{rk})\beta_i + (1/\lambda_{rk})a_k$$

Substituting (10) into (8) , we get:

$$a_j = \sum_{\substack{i=1 \\ i\neq r}}^{m} (\lambda_{ij} - \lambda_{rj}\lambda_{ik}/\lambda_{rk})\beta_i + (\lambda_{rj}/\lambda_{rk})a_k = \sum_{i=1}^{m} \lambda_{ij}\beta_i = B\lambda_j$$

where $\beta_i=\beta_i$, $i\neq r$; $\beta_r=a_k$; the β_i are the columns of matrix B; and

(11) $$\begin{cases} \lambda_{ij} = \lambda_{ihj} - \lambda_{rj}\lambda_{ik}/\lambda_{rk} & i \neq r \\ \lambda_{rj} = \lambda_{rj}/\lambda_{rk} \end{cases}$$

Thus if we know how to express columns a_j and a_k as linear combinations of the $ß_i$, we express a_j as a linear combination of the $ß_i$.

To show that the $ß_i$ are independent, assume that

$$(12) \quad \sum_{\substack{i=1 \\ i\neq r}}^{m} t_i ß_i + s_k a_k = 0$$

since from (9), $a_k = \sum_{i=1}^{m} \lambda_{ik} ß_i$, (12) becomes:

$$(13) \quad \sum_{\substack{i=1 \\ i\neq r}}^{m} (t_i + s_k \lambda_{ik}) ß_i + s_k \lambda_{rk} ß_r = 0$$

Since all the $ß_i$ are independent (B is a basis matrix), all the coefficients in (13) are zero i.e. $t_i + s_k \lambda_{ik} = 0$ and $s_k \lambda_{rk} = 0$. But since $\lambda_{rk} \neq 0$ we must have $s_k = 0$ and so $t_i = 0$. Thus all the coefficients in (12) are zero, establishing the independence of the $ß_i$. B is, therefore, a basis matrix.

In the initial tableau of the simplex method $B = I_m$ and $a_j = \lambda_j = y_j$ for $j = 1, 2, ..., n$ and according to the rule for selecting the pivot element $\lambda_{rk} = y_{rk} \neq 0$. Thus if we replace the r^{th} column of B by a_k and use (11), which, in fact, is precisely the replacement rules (7), we see that B is a basis matrix and the elements of column j in the new tableau are the coefficients for expressing a_j as a linear combination of the $ß_i$. This argument can be applied to subsequent tableaus as well, i.e. for any tableau if B is the basis matrix for that tableau and y_j is the j^{th} column then

$$(14) \quad a_j = By_j = \sum_{i=1}^{m} y_{ij} ß_i$$

<u>Obtaining a New Basic Feasible Solution</u>

We now show that the new basic solution obtained from the new basis matrix B is feasible. Assume we have a basic feasible solution X_B corresponding to the basis matrix B .i.e $BX_B = b$ where $X_B \geq 0$. This can be written as

(15) $$\sum_{i=1}^{m} x_{Bi}\beta_i = b$$

From (9),(10) and (14), $a_k = \Sigma\, y_{ik}\beta_i$ can replace β_r ($y_{rk} \neq 0$) in B to form another basis matrix B, where

(16) $$\beta_r = -\sum_{\substack{i=1\\ i\neq r}}^{m} (y_{ik}/y_{rk})\beta_i + (1/y_{rk})a_k$$

Substituting (16) into (15) we obtain

(17) $$\sum_{\substack{i=1\\ i\neq r}}^{m} (x_{Bi} - x_{Br}y_{ik}/y_{rk})\beta_i + (x_{Br}/y_{rk})a_k = b \ .$$

Thus the new basic solution is

(18) $$\begin{cases} x_{Bi} - x_{Br} - x_{Br}y_{ik}/y_{rk} & i \neq r \\ x_{Br} = x_{Br}/y_{rk} & \end{cases}$$

We now show that $X_B \geq 0$. Recalling that $x_{Bi} = y_{i,n+1}$, we note that (18) is the replacement rule (7) for obtaining the entries in column n+1 of the new tableau from the entries of column n+1 of the current tableau. Furthermore, the rule (6) for determining that column r of B is to be replaced by column a_k implies that $y_{rk} > 0$ and for $i\neq r$

(19) $$(x_{Bi}/y_{ik}) - (x_{Br}/y_{rk}) \geq 0 \quad \text{when } y_{ik} > 0.$$

Thus, when $y_{ik} > 0$, we see from (18), that $X_B \geq 0$. We also note from (18) that if $y_{ik} \leq 0$, $X_B \geq 0$ is automatically satisfied. We have shown that if the pivot column has at least one $y_{ik} > 0$, the new basic solution will be feasible.

It can be seen from (18) and (19) that if the minimum in (6) is not unique, one or more variables in the new basic solution will be zero. A basic feasible solution that has one or more basic variables equal to zero is called **degenerate**. If a basic variable becomes zero in a given tableau, then one of the ratios in (6) used for finding the next pivot row may be zero. Since zero is the minimum possible value, the next pivot row would be that one with the zero basic variable (.i.e $x_{Br}=0$) and hence by (18), $x_{Br} = 0$ and $x_{Bi} = x_{Bi}$ $(i \neq r)$. This means that all basic variables in the new tableau have the same value that they had in the old tableau. However, it is also possible for the new solution not be degenerate although the previous solution was degenerate. This can occur if $y_{ik} \leq 0$ for every x_{Bi} which is zero; none of these variables enter into the computation in (6).

Improving The Value of The Objective Function

We now show that the value z is increased (or at least not decreased) with our new basic feasible solution. The new value of z is

(20) $$z = C_B X_B = \sum_{i=1}^{m} c_{Bi} x_{Bi}$$

where $c_{Bi} = c_{Bi}$ $(i \neq r)$ and $c_{Br} = c_k$ since only the component corresponding to the new variable entering the basis changes. If we substitute (18) into (20), we get

(21) $$z = \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{Bi}(x_{Bi} - x_{Br}y_{ik}/y_{rk}) + (x_{Br}/y_{rk})c_k$$

If we include the term $c_{Br}(x_{Bi} - x_{Br}y_{rk}/y_{rk}) = 0$ in the summation in (21), then from (3) and (4), we have

(22) $$z = z - (x_{Br}/y_{rk})(z_k - c_k)$$

and since $x_{Br}/y_{rk} \geq 0$, $z \geq z$ provided $z_k - c_k < 0$. We note that (22) is the replacement rule (7) where $z=y_{m+1,n+1}$, $z=y_{m+1,n+1}$, $x_{Br}=y_{r,n+1}$ and $z_k-c_k=y_{m+1,k}$.

Ideally we should select the column a_k to enter the basis which will provide the greatest increase in z. Accordingly, by (22), we should select a_k such that $(x_{Br}/y_{rk})(z_k-c_k)$ is most negative. However, we must compute x_{Br}/y_{rk} for each a_k having $z_k-c_k<0$. This involves additional computation and from a computational point of view there seems to be little or no advantage in using this more complicated criterion to determine the column to enter the basis (Hadley, p. 111). A simplified criterion (5) was thus suggested to make this determination.

It should be noted that in the presence of degeneracy, z may not increase in value when we move from one basic feasible solution to another (by (22) with x_{Br}=0). Moreover we may get into a situation where we repeat the same basis previously obtained. This cycling may occur indefinitely never reaching an optimal solution. Fortunately, in practice, degeneracy does not present a problem (Hadley, p. 113). Although degeneracy is a frequent phenomenon, it does not make it impossible to reach an optimal solution by means of the simplex method. The degeneracy problem and its resolution is discussed in detail in Hadley (ch. 6). In the absence of degeneracy, $x_{Br}/y_{rk} > 0$ and by (22), $z > z$ provided $z_k - c_k < 0$.

Computing The Extra Row

To compute the new values of $z_j - c_j$ for our new basis (i.e. to compute the extra row of our new tableau) we transform the values of $z_j - c_j$. We note that

$$(23) \quad z_j - c_j = C_B y_j - c_j = \sum_{i=1}^{m} c_{Bi} y_{ij} - c_j$$

Using (7) for y_{ij} in (23), we get

$$(24) \quad z_j - c_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{Bi}(y_{ij} - y_{rj} y_{ik}/y_{rk}) + (y_{rj}/y_{rk}) c_k - c_j$$

If we include the term $c_{Br}(y_{rj} - y_{rj} y_{rk}/y_{rk}) = 0$ in the summation, then by (3) we have

$$(25) \quad z_j - c_j = (z_j - c_j) - (y_{rj}/y_{rk})(z_k - c_k)$$

This is the replacement rule (7) where $y_{m+1,i} = z_i - c_i$ and $y_{m+1,i} = z_i - c_i$.

Termination Conditions

The simplex process can terminate in only one of two ways:

(a) There are no negative entries in the extra row.

(b) There are no positive entries in the pivot column.

We now show that if in any tableau all $z_j - c_j \geq 0$ (termination condition (a)), we have an optimal basic solution. Assume that $X_B = B^{-1}b$ is a basic feasible solution to (2) and the value of z for this solution is

(26) $$z_o = C_B X_B = \sum_{i=1}^{m} c_{Bi} x_{Bi}$$

Furthermore assume $z_j - c_j \geq 0$ for $j=1,2,...,n$. Let X be any feasible solution to AX = b, then

(27) $$\sum_{i=1}^{n} x_i a_i = b \quad \text{where } x_i \geq 0, \quad i=1,2,...,n$$

Let z^* be the corresponding value of the objective function

(28) $$z^* = \sum_{i=1}^{n} c_i x_i$$

Substituting (14) into (27), we obtain

(29) $$\sum_{j=1}^{n} x_j \sum_{i=1}^{m} y_{ij} \beta_i = b$$

which can be rearranged to yield

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_j y_{ij} \right) \beta_i = b$$

But $BX_B = b$ .i.e $\sum_{i=1}^{m} x_{Bi} \beta_i = b$ and using the fact that the representation of any vector in terms of the basis vectors is unique, we must conclude that

(30) $$x_{Bi} = \sum_{j=1}^{n} x_j y_{ij} \qquad i=1,2,...,m$$

If $z_j - c_j \geq 0$ for all j, then from (28)

(31) $$\sum_{i=1}^{n} z_i x_i \geq z^*$$

If we substitute (4) into (31) and rearrange the terms, we obtain

$$\sum_{i=1}^{m} c_{Bi}\left(\sum_{j=1}^{n} x_j y_{ij}\right) \geq z^*$$

However, by (26) and (30), this becomes

$$z_o = \sum_{i=1}^{m} c_{Bi} x_{Bi} \geq z^*$$

This tells us that when we have reached a solution X_B for which all $z_j - c_j \geq 0$, its value z_o is at least as large as that for any feasible solution z^*. Hence z_o is the maximum value of the objective function.

The existence of termination condition (b) indicates an unbounded solution .i.e if there are no positive entries in the pivot column, the objective function can be made arbitrarily large. We now show this. Let the k^{th} column be the pivot column .i.e a_k is the vector about to enter the basis B to form a new basis B. Let X_B be the current basic feasible solution, so that (15) is satisfied. If we add and subtract αa_k (α is any scalar) to the left hand side of (15), we have

$$(32) \qquad \sum_{i=1}^{m} x_{Bj} \beta_j - \alpha a_k + \alpha a_k = b$$

Substitute (14) into (32) to get

$$\sum_{i=1}^{m} x_{Bi} \beta_i - \alpha \sum_{i=1}^{m} y_{ik} \beta_i + \alpha a_k = b$$

This can be rearranged to yield

$$(33) \qquad \sum_{i=1}^{m} (x_{Bi} - \alpha y_{ik}) \beta_i + \alpha a_k = b$$

Since the pivot column contains no positive elements, $y_{ik} \le 0$ for $i=1,2,...,m$. If $\alpha > 0$, then

(34) $$\begin{cases} x_{Bi} - \alpha y_{ik} > 0 & i = 1,2,...,m \\ \alpha > 0 \end{cases}$$

Thus we have a feasible nonbasic solution since the m+1 variables in (34) are positive. The objective function for the solution under these conditions is

$$z^* = \sum_{i=1}^{m} c_{Bi}(x_{Bi} - \alpha y_{ik}) + \alpha c_k$$

and by (3) and (4) this becomes

(35) $$z^* = z - \alpha(z_k - c_k)$$

Hence, if $z_k - c_k < 0$ and $\alpha > 0$, z^* can be made arbitrarily large by making α sufficiently large and the linear programming problem has an unbounded solution. As indicated in Hadley (p. 14), no real world properly formulated linear programming problem has an unbounded solution. Normally, termination condition (b) is a sign that an error has been made in either data preparation or problem formulation.

The Mathematical Dual

The mathematical dual of

(36) $$\begin{cases} \max cx \\ \max \le b & x \ge 0, \quad b \ge 0 \end{cases}$$

is

(37) $$\begin{cases} \max b^t v \\ \max a^t v \ge c^t & v \ge 0 \end{cases}$$

where the superscript T denotes the transpose and v is an mx1 column vector.

Development of Computational Checks For The Minimizing Problem

Let $y_{p+1}, y_{p+2}, ..., y_{p+m}$ denote the slack variable columns in any tableau ($p+m=n$). We have seen that $B^{-1}=(y_{p+1}, y_{p+2}, ..., y_{p+m})$ is the inverse of the basis matrix B corresponding to the given tableau and $X_B=B^{-1}b$. From (14) $y_j=B^{-1}a_j$. Substituting this into (3), we obtain

(38) $z_j = C_B B^{-1} a_j \qquad j=1,2,...,n$

Since $a = (a_1,a_2,...,a_p)$ and $I_m = (a_{p+1},a_{p+2},...,a_{p+m})$, from (38) we see that

(39) $(z_1,z_2,...,z_p) = C_B B^{-1} a$

and

(40) $(z_{p+1},z_{p+2},...,z_{p+m}) = C_B B^{-1} I_m = C_B B^{-1}$

Combining (39) and (40), we get

(41) $(z_1,z_2,...,z_p) = (z_{p+1},z_{p+2},...,z_{p+m})a$

Let

(42) $v = \begin{pmatrix} z_{p+1} \\ . \\ . \\ z_{p+m} \end{pmatrix}$ and $u = \begin{pmatrix} z_1 \\ . \\ . \\ . \\ z_p \end{pmatrix}$

If we take the transpose of (41) we find that

(43) $a^T v = u$

Multiplying both sides of (40) on the right by b and then taking the transpose, we get

(44) $b^T v = (C_B B^{-1} b)^T = C_B X_B = z$.

From (44) we see that, in any tableau, the value of the objective function $b^T v$ of the dual is equal to the value of the objective function of the maximizing problem.

In the final tableau for the maximizing problem, $z_j - c_j \geq 0$ for $j=1,2,...,n$. Since $c_j = 0$ for $j=p+1,...,p+m$,

(45) $$\begin{cases} z_j \geq c_j & j = 1,2,\ldots,p \\ z_j \geq 0 & j = p+1,\ldots,p+m \end{cases}$$

Thus, in the final tableau, from (42) and (45), we see that

(46) $v \geq 0$ and $u \geq c^T$

so that by (43) and (46), v is a feasible solution to the dual (37). That this v is also an optimal solution to (37) can be seen as follows: Let U be a feasible solution to (36), .i.e

(47) $aU \leq b$

If we multiply both sides of (47) by w^T, where w is any feasible solution to (37), then we have the following sequence of steps:

$$\begin{aligned} w^T aU &\leq w^T b \\ (w^T aU)^T &\leq (w^T b)^T \\ U^T a^T w &\leq b^T w \\ U^T c^T &\leq b^T w \qquad \text{by (37)} \\ cU = (cU)^T &\leq b^T w \end{aligned}$$

This shows that for any feasible solution w to (37) and any feasible solution U to (36), the value of the objective function of the minimizing problem (37) is at least as large as that for the maximizing problem (36). Hence optimization occurs when the values of the two objective functions are equal. Thus, from (44), and using the fact that in the final tableau v is feasible, we can conclude that v is also optimal.

In any tableau, if there is one or more j such that $z_j - c_j < 0$, v is not a feasible solution to (37). However, since by (42)

$$\begin{pmatrix} z_1 - c_1 \\ \cdot \\ \cdot \\ \cdot \\ z_p - c_p \end{pmatrix} = u - c^T$$

so that from (43)

$$(48) \qquad a^T v = c^T + \begin{pmatrix} z_1 - c_1 \\ \cdot \\ \cdot \\ z_p - c_p \end{pmatrix}$$

These observations afford us another computational check. For each tableau, substitute v into the left hand side of $a^T v \geq c^T$. If $z_k - c_k < 0$, the k^{th} inequality of this system will not be satisfied; it will be "off" by the value $c_k - z_k$. If $z_k - c_k > 0$, the k^{th} inequality will be satisfied with the left hand side being greater than the right hand side by the value $z_k - c_k$. If $z_k - c_k = 0$, the k^{th} inequality will in fact be an equality.

Example: Computational Checks For The Minimizing Problem

The mathematical dual to the problem previously discussed is

$$\begin{cases} \min^T v \\ a^T v \geq c^T \end{cases}$$

where

$$a^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 4 \\ 1 & 0 & 1 \end{pmatrix} \quad c^T = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 1 \end{pmatrix} \quad b^T = (4,3,3) \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

or, in nonmatrix notation:

minimize $4v_1 + 3v_2 + 3v_3$ $\qquad v_1, v_2, v_3 \geq 0$

subject to the constraints

$$\begin{aligned} v_1 + v_2 \quad &\geq 2 \\ 3v_1 + v_2 + v_3 &\geq 4 \\ 4v_3 &\geq 1 \\ v_1 \quad + v_3 &\geq 1 \end{aligned}$$

For the initial tableau, we note that from the extra row

$$\begin{pmatrix} z_1 - c_1 \\ z_2 - c_2 \\ z_3 - c_3 \\ z_4 - c_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and since

$$a^T v = c^T + \begin{pmatrix} -2 \\ -4 \\ -1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad b^T v = 0 = z$$

we see that (44) and (48) is satisfied. Notice that none of the inequalities are satisfied; they are "off" by 2, 4, 1, and 1, respectively.

The extra row in the next tableau yields

$$\begin{pmatrix} z_1 - c_1 \\ z_2 - c_2 \\ z_3 - c_3 \\ z_4 - c_4 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 0 \\ -1 \\ 1/3 \end{pmatrix} \qquad \text{and} \quad v = \begin{pmatrix} z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 5/4 \\ 0 \\ 1/4 \end{pmatrix}$$

When v is substituted into the constraints, we see that the first and third inequalities are not satisfied; the first is "off" by 2/3 and the third is "off" by 1. The second is an equation and the fourth is satisfied with 1/3 to spare. Note also that $b^T v = 16/3 = z$. Thus (44) and (48) are satisfied.

For the next tableau

$$\begin{pmatrix} z_1 - c_1 \\ z_2 - c_2 \\ z_3 - c_3 \\ z_4 - c_4 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \qquad \text{and } v = \begin{pmatrix} z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 5/4 \\ 0 \\ 1/4 \end{pmatrix}$$

The reader should verify that (44) and (48) are satisfied.

In the next tableau, since $z_j - c_j \geq 0$ for $j=1,2,...,n$

$$V = \begin{pmatrix} z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 11/10 \\ 9/20 \\ 1/4 \end{pmatrix}$$

is an optimal solution with $b^T v = 13/2 = z$ and since

$$a^T v = c^T + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 7/20 \end{pmatrix}$$

we see that (48) is satisfied.

References

Adams, William J. (1981). Fundamentals of Mathematics for Business, Social, and Life Sciences. Englewood Cliffs: Prentice-Hall.

Cooper, Leon & Steinberg, David. (1974). Methods and Applications of Linear Programming. Philadelphia: W. B. Saunders.

Gale, David. (1960). The Theory of Linear Economic Models. New York: McGraw-Hill.

Hadley, G. (1962). Linear Programming. Reading, Mass: Addison-Wesley.