Constructing Risk Measures from Uncertainty Sets

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Abstract

We propose a unified theory that links uncertainty sets in robust optimization to risk measures in portfolio optimization. We illustrate the correspondence between uncertainty sets and some popular risk measures in finance, and show how robust optimization can be used to generalize the concepts of these measures. We also show that by using properly defined uncertainty sets in robust optimization models, one can in fact construct coherent risk measures. Our approach to creating coherent risk measures is easy to apply in practice, and computational experiments suggest that it may lead to superior portfolio performance. Our results have implications for efficient portfolio optimization under different measures of risk.

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1 Introduction

Markowitz [26] was the first to model the important tradeoff between risk and return in portfolio selection as an optimization problem. He suggested choosing an asset mix such that the portfolio variance is minimum for any target level of expected return. It is now known (Tobin [34], Chamberlain [10]) that the mean-variance framework is appropriate if the distribution of returns is elliptically symmetric (e.g., multivariate normal). In this case, the optimal mean-variance portfolio allocation is consistent with any set of preferences for market agents in the sense that given a fixed expected return, any investor will prefer the portfolio with the smallest standard deviation. However, when returns are not symmetrically distributed, or when a downside risk is more weighted than an upside risk, variance is not an accurate measure of investor risk preferences. Markowitz [27] acknowledges this shortcoming and discusses alternative risk measures in a more general mean-risk approach. Such considerations and the theory of stochastic dominance (Levy [25]) spurred interest in asymmetric or quantile-based portfolio risk measures such as expectation of loss, semi-variance, Value-at-Risk (VaR), and others (Jorion [21], Dowd [13], Konno and Yamazaki [22], Carino and Turner [9]). Generalizations of these approaches to worst-case risk measures when the distribution parameters are themselves unknown have been studied for the variance and the VaR risk measures (Halldorsson and Tutuncu [19], Goldfard and Iyengar [18], El Ghaoui et al. [15]). Artzner et al. [1] introduced an axiomatic methodology to characterize desirable properties in risk measures. Risk measures that satisfied their four axioms were termed *coherent*. A popular example of such a coherent risk measure is Conditional Value-at-Risk, or CVaR as discussed in Rockafellar and Uryasev [29, 30] and Rockafellar et al. [31].

If one thinks of future asset returns as unknown parameters, one can view the portfolio problem as an optimization problem with uncertain coefficients. It is then natural to approach it with tools for optimization under uncertainty, such as recently developed *robust optimization* techniques. The main idea in robust optimization is that the optimal solution must remain feasible for any realization of the uncertain parameters within a pre-specified uncertainty set. The “size” and “shape” of the uncertainty sets are usually based on probability estimates on the quality of the solution. It has been observed (Ben-Tal and Nemirovski [3]) that by stating the portfolio optimization problem as one of maximizing return subject to the constraint that future returns could vary in an ellipsoidal uncertainty set defined by the covariance structure of the uncertain returns, the robust counterpart of the portfolio optimization problem is reminiscent of the Markowitz formulation. This paper builds on this observation and presents a unified theory that relates portfolio risk measures to robust optimization uncertainty sets. Our contributions can be summarized as follows:

(a) We show explicitly how risk measures such as standard deviation, worst-case VaR, and CVaR can be mapped to robust optimization uncertainty sets. We also show how robust optimization can be used to generalize the concepts of these risk measures. For example, we formulate the problem of minimizing worst-case CVaR based on moment information when the exact distributions of uncertainties are unknown. This result extends the worst-case VaR results of El Ghaoui et al. [15].

(b) We show that by defining uncertainty sets appropriately, one can generate a set of coherent risk measures. Furthermore, we show how incoherent risk measures can be made coherent based on information about the support of the distribution of uncertainties, and explore the validity of probability bounds in doing so.

(c) We study the effect of modifying incoherent risk measures into coherent risk measures on the portfolio efficient frontier. Our computational results suggest that making non-coherent risk measures coherent by including support information results in better approximation and superior portfolio performance. These findings have implications for efficient portfolio optimization under different measures of risk.

While completing this paper, we became aware of work by Bertsimas and Brown [5] that relates robust linear optimization to coherent risk measures. While the spirit of the two papers are similar, the focus of our work is different. The results in [5] are based on a representation theorem that relates coherent risk measures to the supremum of the expected value function over a family of distributions. In particular, their focus in is on characterizing a family of coherent risk measures named *comonotone* risk measures that lead to robust linear optimization problems. By contrast, our focus is on using uncertainty sets to generate coherent risk measures, and relating these results to portfolio optimization techniques, both theoretically and numerically.

The structure of this paper is as follows: In Section 2, we review some popular financial measures of risk and the notion of coherent risk measures. In Section 3, we review the main concepts of robust optimization, and analyze financial risk measures in the context of robust counterpart risk measures. In Section 4, we link the notion of coherent risk measures to robust optimization uncertainty sets, propose a method for constructing coherent risk measures from uncertainty sets, and prove that the probability of constraint violation remains the same for the resulting coherent robust counterpart risk measures. We illustrate the technique with a numerical example that suggests that there is an advantage to constructing a coherent risk measure based on uncertainty sets that incorporate support information.

2 Risk Measures in Portfolio Optimization

In this section, we describe some commonly used risk measures in finance, and review the concept of coherence. Consider a generic portfolio risk minimization problem of the form:

$$\begin{aligned} \min \quad & \rho(f(\boldsymbol{x}, \tilde{\boldsymbol{z}})) \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X}, \end{aligned} \tag{1}$$

where ρ is a specified risk measure involving a function of the portfolio allocation weights $\boldsymbol{x}$ and the uncertain future returns $\tilde{\boldsymbol{z}}$. The set $\mathcal{X}$ may include constraints on the portfolio structure such as

(a) $\boldsymbol{x}'\boldsymbol{e} = 1$ (the weights of all assets in the portfolio add up to one where $\boldsymbol{e}$ is the vector of ones),

(b) $E(\tilde{\boldsymbol{z}}'\boldsymbol{x}) \geq r_{\text{target}}$ (the expected return of the portfolio should be at least as large as the target return of the portfolio manager), etc.

The random variables $\tilde{\boldsymbol{z}}$ defined on the sample space Ω are primitive uncertainties. We use $\tilde{v} = v(\tilde{\boldsymbol{z}}) \stackrel{\Delta}{=} f(\boldsymbol{x}, \tilde{\boldsymbol{z}})$ to denote random variables obtained from the primitive uncertainties $\tilde{\boldsymbol{z}}$. Let Ω_v denote the sample space of $\tilde{v}$ and $\mathcal{V}$ denote the set of random variables $\tilde{v} : \Omega_v \to \mathcal{R}$. Then, $\rho : \mathcal{V} \to \mathcal{R}$ is a risk functional that assigns a real value $\rho(\tilde{v})$ to an uncertain outcome $\tilde{v}$.

2.1 Examples of Risk Measures

The most commonly used portfolio risk formulations in finance include mean-standard deviation (or, equivalently, mean-variance), Value-at-Risk (VaR), and Conditional Value-at-Risk (CVaR). We describe these three risk measures in more detail.

Mean-Standard Deviation

For the classical mean-standard deviation portfolio optimization approach, we have:

$$\rho_\alpha(\tilde{v}) \stackrel{\Delta}{=} -E(\tilde{v}) + \alpha\sigma(\tilde{v}),$$

where $E(\tilde{v})$ is the expected value of $\tilde{v}$, $\sigma(\tilde{v})$ is the standard deviation of $\tilde{v}$, and α is a parameter associated with the level of the investor's risk aversion. The mean-standard deviation risk measure is an example of a moment-based portfolio risk measure - it incorporates information about the first and second moments of the distribution of returns. Higher moments of the distribution of returns have been suggested as well (Huang and Litzenberger [20]); however, such risk measures have not become as popular.

In contrast to moment-based risk measures, quantile-based risk measures are concerned with the probability or magnitude of losses. The advantage of the quantile-based approach to risk measurement is that asymmetry in the distribution of returns can be handled better.

Value-at-Risk (VaR)

The most popular quantile-based risk measure is Value-at-Risk. VaR measures the worst portfolio loss that can be expected with some small probability ϵ (ϵ typically equals 1% or 5%). Mathematically, the $(1-\epsilon)$-VaR is defined as follows:

$$VaR_{1-\epsilon}(\tilde{v}) \triangleq q_{1-\epsilon}(-\tilde{v}),$$

where $q_\epsilon(\tilde{v})$ denotes the ϵ-quantile of the random variable $\tilde{v}$,

$$q_\epsilon(\tilde{v}) = \min\{v \mid \mathrm{P}(\tilde{v} \leq v) \geq \epsilon\}. \tag{2}$$

Computationally, optimization of VaR is difficult to handle unless the distribution of returns is assumed to be normal or lognormal (Duffie and Pan [14], Jorion [21]). Heuristics for optimizing sample VaR have been proposed in Gaivoronski and Pflug [17] and Larsen et al. [23]. El Ghaoui et al. [15] suggest a probabilistic approach that optimizes the VaR for the worst-case distribution of the unknown asset returns. We will revisit their approach in Sections 3.2 and 4.3.

Conditional Value-at-Risk (CVaR)

In recent years, an alternative quantile-based measure of risk known as Conditional Value-at-Risk (CVaR) has been gaining ground due to its attractive computational properties (Rockafellar and Uryasev [29, 30]). CVaR measures the expected loss if the loss is above a specified quantile. Mathematically, the CVaR formulation can be written as:

$$CVaR_{1-\epsilon}(\tilde{v}) \triangleq \min_a \left(a + \frac{1}{\epsilon} E(-\tilde{v} - a)^+ \right).$$

It can be shown that $VaR_{1-\epsilon}(\tilde{v}) \leq CVaR_{1-\epsilon}(\tilde{v})$. Hence, CVaR is often used as a conservative approximation of VaR (Rockafellar and Uryasev [30]).

For the risk measures described above, the parameter α (in the case of mean-standard deviation) and ϵ (in the case of VaR and CVaR) determines the risk-averseness of the decision-maker.

2.2 Coherent Risk Measures

In formulation (1), the risk measure $\rho(\cdot)$ is a functional defined on a risky asset with uncertain returns, $\tilde{v}$. By convention, $\rho(\tilde{v}) \leq 0$ implies that the risk associated with an uncertain return $\tilde{v}$ is acceptable. A risk measure functional $\rho(\cdot)$ is *coherent* if it satisfies the following four axioms:

Axioms of coherent risk measures:

(i) Translation invariance: For all $\tilde{v} \in \mathcal{V}$ and $a \in \mathcal{R}$, $\rho(\tilde{v} + a) = \rho(\tilde{v}) - a$.

(ii) Subadditivity: For all random variables $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$, $\rho(\tilde{v}_1 + \tilde{v}_2) \leq \rho(\tilde{v}_1) + \rho(\tilde{v}_2)$.

(iii) Positive homogeneity: For all $\tilde{v} \in \mathcal{V}$ and $\lambda \geq 0$, $\rho(\lambda\tilde{v}) = \lambda\rho(\tilde{v})$.

(iv) Monotonicity: For all random variables $\tilde{v}_1$, $\tilde{v}_2 \in \mathcal{V}$ such that $\tilde{v}_1 \geq \tilde{v}_2$, $\rho(\tilde{v}_1) \leq \rho(\tilde{v}_2)$.

The four axioms were presented and justified in Artzner et al. [1]. The first axiom ensures that $\rho(\tilde{v} + \rho(\tilde{v})) = 0$, so that the risk of $\tilde{v}$ after compensation with $\rho(\tilde{v})$ is zero. It means that having a sure amount of a simply reduces the risk measure by a. The subadditivity axiom states that the risk associated with the sum of two financial instruments is not more than the sum of their individual risks. It appears naturally in finance - one can think equivalently of the fact that "a merger does not create extra risk," or of the "risk pooling effects" observed in the sum of random variables. The positive homogeneity axiom implies that the risk measure scales proportionally with its size. The final axiom is an obvious criterion, but it rules out the classical mean-standard deviation risk measure. Among the risk measures described previously, only CVaR is a coherent risk measure.

One important consequence of the coherent risk axioms is preservation of convexity, which is important for computational tractability (see also Ruszczynski and Shapiro [32]).

Theorem 1 *If $f(\boldsymbol{x}, \tilde{\boldsymbol{z}})$ is concave in $\boldsymbol{x}$ for all realizations of $\tilde{\boldsymbol{z}}$, then $\rho(f(\boldsymbol{x}, \boldsymbol{z}))$ is convex in $\boldsymbol{x}$ for any risk measure $\rho(\cdot)$ that satisfies the axioms of monotonicity, subadditivity and positive homogeneity.*

Proof : By concavity with respect to $\boldsymbol{x}$, we have

$$f(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \tilde{\boldsymbol{z}}) \geq \lambda f(\boldsymbol{x}_1, \tilde{\boldsymbol{z}}) + (1-\lambda) f(\boldsymbol{x}_2, \tilde{\boldsymbol{z}}) \text{ for all } \lambda \in [0,1].$$

Hence,

$$\begin{aligned}
\rho(f(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \tilde{\boldsymbol{z}})) &\leq \rho(\lambda f(\boldsymbol{x}_1, \tilde{\boldsymbol{z}}) + (1-\lambda) f(\boldsymbol{x}_2, \tilde{\boldsymbol{z}})) && \text{(Monotonicity)}\\
&\leq \rho(\lambda f(\boldsymbol{x}_1, \tilde{\boldsymbol{z}})) + \rho((1-\lambda) f(\boldsymbol{x}_2, \tilde{\boldsymbol{z}})) && \text{(Subadditivity)}\\
&= \lambda\rho(f(\boldsymbol{x}_1, \tilde{\boldsymbol{z}})) + (1-\lambda)\rho(f(\boldsymbol{x}_2, \tilde{\boldsymbol{z}})) && \text{(Positive homogeneity).}
\end{aligned}$$

3 Risk Measures and Optimization under Uncertainty

In this section, we point out parallels between the definition of risk measures in finance and optimization problems with chance or probabilistic constraints. We discuss how optimization problems with chance constraints can be handled with robust optimization techniques, and introduce the concept of robust counterpart risk measures.

The framework of risk measures in portfolio optimization can be extended to a fairly general optimization problem with parameter uncertainties. Consider the following family of optimization problems under parameter uncertainty:

$$\left\{ \begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & f_i(\boldsymbol{x}, \boldsymbol{z}_i) \geq 0, \quad i \in I \\ & \boldsymbol{x} \in X, \end{array} \right\}_{\boldsymbol{z}_i \in \{\tilde{\boldsymbol{z}}_i\},\ i \in I} \tag{3}$$

where $\boldsymbol{x}$ denotes the vector of decision variables, and $\tilde{\boldsymbol{z}}_i$, $i \in I$ are the primitive uncertainties that affect the optimization problem. Without loss of generality, we assume that $\boldsymbol{c}$ is known exactly and the uncertainty is present only in the constraints $f_i(\boldsymbol{x}, \tilde{\boldsymbol{z}}_i) \geq 0$, for all $i \in I$. Since $\tilde{\boldsymbol{z}}_i$ are random, for any fixed solution $\boldsymbol{x}$ the constraint $f_i(\boldsymbol{x}, \boldsymbol{z}_i) \geq 0$ may become infeasible for some realization of $\boldsymbol{z}_i \in \{\tilde{\boldsymbol{z}}_i\}$. In many applications of optimization, ensuring constraint feasibility for all realization of uncertainties can be overly conservative. In such problems, we can tolerate some risk of constraint violation for the benefit of improving the objective. Charnes and Cooper [11] introduce probabilistic constraints or chance constraints in optimization models as follows:

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \mathrm{P}(f_i(\boldsymbol{x}, \tilde{\boldsymbol{z}}_i) \geq 0) \geq 1 - \epsilon_i, \quad i \in I \\ & \boldsymbol{x} \in X, \end{array} \tag{4}$$

which is equivalent to a VaR formulation on the constraints,

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & VaR_{1-\epsilon_i}(f_i(\boldsymbol{x}, \tilde{\boldsymbol{z}}_i)) \leq 0, \quad i \in I \\ & \boldsymbol{x} \in X. \end{array} \tag{5}$$

Therefore, it is natural to extend the optimization framework to risk constraints as follows:

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \rho_i(f_i(\boldsymbol{x}, \tilde{\boldsymbol{z}}_i)) \leq 0, \quad i \in I \\ & \boldsymbol{x} \in X. \end{aligned} \tag{6}$$

We call model (6) the *risk counterpart* of (3). In line with model (3), a risk measure should satisfy the following *deterministic equivalence condition*:

$$\rho(c) = -c \tag{7}$$

for any constant, c, so that in the absence of uncertainty, the risk counterpart is the same as the nominal problem. Indeed, any risk measure $\rho(c)$ that satisfies the axiom of translation invariance and $\rho(0) = 0$ will satisfy the deterministic equivalence condition

$$\rho(c) = \rho(0) - c = -c.$$

For coherent risk measures, $\rho(0) = 0$ is implied by the axiom of positive homogeneity:

$$\rho(0) = \rho(0\tilde{v}) = 0\rho(\tilde{v}) = 0.$$

We also require risk measures to satisfy the translation invariance axiom of coherent risk measures, which will allow us to formulate the objective with the risk measure in the homogenized framework (6). For instance, under this condition, we can reformulate the portfolio risk minimization problem (1) as

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \rho(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) + t) \leq 0 \\ & \boldsymbol{x} \in \mathcal{X}. \end{aligned} \tag{8}$$

Even if the nominal problem without uncertainty is computationally tractable, the choice of risk measure can affect the tractability of the risk counterpart. Under the VaR risk measure, the risk counterpart can become non-convex and intractable. An important byproduct of using coherent risk measures, as illustrated in Theorem 1, is the preservation of convexity. Hence, risk counterparts with the CVaR measure are generally easier to optimize than VaR.

An important consideration with regards to tractability is also whether a risk measure can be computed with arbitrary accuracy. This is essential when an optimization problem contains constraints that need to be satisfied with high reliability, such as in the case of structural designs (see Ben-Tal and Nemirovski [4]). For example, even for affinely dependent functions $v(\tilde{\boldsymbol{z}})$, the computation of $CVaR_{1-\epsilon}(v(\tilde{\boldsymbol{z}}))$ involves multidimensional integration, which is computationally expensive. While the integrals can be approximated through Monte Carlo simulation, the number of trials in order to achieve high reliability can be prohibitive. At the same time, if first- and second-moment information about the distribution of the uncertainties is available, the mean-standard deviation risk measure has better computational characteristics despite the fact that it is not a coherent risk measure.

3.1 Robust Counterpart Risk Measures

In practice, the exact distributions of the uncertain coefficients in optimization models are rarely known, and exact solutions of optimization problems with chance constraints are virtually impossible to find. *Robust optimization* handles this issue by requiring the user to specify an uncertainty set for the coefficients based on some (possibly limited) information about their distributions. The key idea is then to find an optimal solution to the problem that remains feasible for any realization of the uncertain coefficients within the pre-specified deterministic uncertainty set. The *robust counterpart* of optimization problem (3) is therefore formulated as:

$$\begin{array}{lll} \min & \boldsymbol{c}'\boldsymbol{x} & \\ \text{s.t.} & f_i(\boldsymbol{x}, \boldsymbol{z}_i) \geq 0, & \forall \boldsymbol{z}_i \in \mathcal{U}_i, \quad i \in I \\ & \boldsymbol{x} \in X, & \end{array} \tag{9}$$

where $\mathcal{U}_i$ is an uncertainty set that is mapped out from the uncertain factors $\tilde{\boldsymbol{z}}_i$. Typically, the uncertainty set is convex. Its size is frequently related to some kind of guarantees on the probability that the constraint involving the uncertain data will not be violated (El Ghaoui et al. [16, 15], Ben-Tal and Nemirovski [4], Bertsimas and Sim [7], Bertsimas et al. [8], Chen et al. [12]).

The constraint containing uncertain data in (9) is equivalent to

$$-\min_{\boldsymbol{z}_i \in \mathcal{U}_i} f_i(\boldsymbol{x}, \boldsymbol{z}_i) \leq 0. \tag{10}$$

In view of (6) and (10), we define the concept of a *robust counterpart risk measure* as follows:

$$\eta_{\mathcal{U}}(\tilde{v}) \stackrel{\Delta}{=} -\min_{\boldsymbol{z} \in \mathcal{U}} v(\boldsymbol{z}), \tag{11}$$

where the definition of $\tilde{v} = v(\boldsymbol{z})$ is extended to real values over the uncertainty set $\mathcal{U}$. For our purposes, it will be sufficient to focus on the class $\mathcal{L}$ of functions $v(\tilde{\boldsymbol{z}})$ that are affinely dependent on the primitive uncertainties, i.e., $\tilde{v} = v(\tilde{\boldsymbol{z}}) = v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}$. $\mathcal{L}$ is undoubtedly the most widely considered class in optimization. In line with the convention for risk measures, $\eta_{\mathcal{U}}(\tilde{v}) \leq 0$ implies that the risk associated with the violation of the uncertainty constraint, $\{v \geq 0\}_{v \in \{\tilde{v}\}}$, is acceptable.

Hence, one can think of the definition of an uncertainty set as the definition of a risk measure on the uncertainties involved. This clearly indicates that the uncertainty set in robust optimization maps to the notion of a risk measure.

3.2 Examples of Uncertainty Sets and Corresponding Risk Measures

We illustrate the correspondence between risk measures and robust optimization uncertainty sets with several examples. In particular, we formulate the mean-standard deviation VaR, discrete CVaR, and worst-case CVaR problems. These examples show also that robust optimization uncertainty sets can be used to generalize the definitions of risk measures in finance.

Ellipsoidal uncertainty sets and the mean-standard deviation risk measure

One of the most commonly used uncertainty set in robust optimization is the ellipsoidal uncertainty set. It is well known (Ben-Tal and Nemirovski [4]) that the robust counterpart of

$$v_0 + \boldsymbol{v}'\boldsymbol{z} \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{E}_\alpha \stackrel{\Delta}{=} \{\boldsymbol{z} \mid \|\boldsymbol{z}\|_2 \leq \alpha\},$$

is equivalent to the second order conic constraint

$$v_0 - \alpha\|\boldsymbol{v}\|_2 \geq 0.$$

Clearly, the ellipsoidal uncertainty set maps to the mean-standard deviation risk measure.

If the means and covariance matrix of the primitive uncertainties are known, we can assume without loss of generality that $\mathrm{E}(\tilde{\boldsymbol{z}}) = \boldsymbol{0}$ and $\tilde{\boldsymbol{z}}$ are uncorrelated, that is, $\mathrm{E}(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}') = \boldsymbol{I}$. This can be achieved by a suitable transformation. El Ghaoui et al. [15] show that the worst-case $(1-\epsilon)$-VaR problem in this setting corresponds to the ellipsoidal uncertainty set formulation with $\alpha = \sqrt{\frac{1-\epsilon}{\epsilon}}$.

Discrete CVaR

We show the connection between robust optimization and CVaR for a given discrete distribution.

Theorem 2 *Consider discrete distributions of* $\tilde{\boldsymbol{z}}$ *such that* $\mathrm{P}(\tilde{\boldsymbol{z}} = \boldsymbol{z}_k) = p_k$, $k = 1, \ldots, M$. *For* $\tilde{v} \in \mathcal{L}$,

$$CVaR_{1-\epsilon}(\tilde{v}) = \eta_{\mathcal{U}_{1-\epsilon}}(\tilde{v}),$$

where the associated uncertainty set

$$\mathcal{U}_{1-\epsilon} = \left\{ \boldsymbol{z} : \begin{array}{l} \exists \boldsymbol{u} \in \mathcal{R}^M \\ \boldsymbol{z} = \sum_{k=1}^M u_k \boldsymbol{z}_k \\ \sum_{k=1}^M u_k = 1 \\ \boldsymbol{0} \le \boldsymbol{u} \le \frac{1}{\epsilon}\boldsymbol{p}. \end{array} \right.$$

Proof : The equivalent representation of the $(1-\epsilon)$-CVaR is

$$\begin{array}{rcll} CVaR_{1-\epsilon}(\tilde{v}) & = & \min_{a,\boldsymbol{y}} & a + \frac{1}{\epsilon}\sum_{k=1}^{M} p_k y_k \\ & & \text{s.t.} & a + y_k \ge -v(\boldsymbol{z}_k), \quad k = 1, \ldots, M \\ & & & \boldsymbol{y} \ge \boldsymbol{0}. \end{array}$$

Using strong LP duality,

$$\begin{array}{rcll} CVaR_{1-\epsilon}(\tilde{v}) & = & \max_{\boldsymbol{u}} & -\sum_{k=1}^{M} u_k v(\boldsymbol{z}_k) \\ & & \text{s.t.} & \sum_{k=1}^M u_k = 1 \\ & & & \boldsymbol{u} \le \frac{1}{\epsilon}\boldsymbol{p}. \end{array}$$

Since the function $v(\cdot)$ is affine and $\sum_{k=1}^M u_k = 1$, we have, equivalently,

$$\begin{array}{rcll} CVaR_{1-\epsilon}(\tilde{v}) & = & -\min_{\boldsymbol{u},\boldsymbol{z}} & v(\boldsymbol{z}) \\ & & \text{s.t.} & \boldsymbol{z} = \sum_{k=1}^M u_k \boldsymbol{z}_k \\ & & & \sum_{k=1}^M u_k = 1 \\ & & & \boldsymbol{0} \le \boldsymbol{u} \le \frac{1}{\epsilon}\boldsymbol{p}. \end{array}$$

This clearly yields the desired uncertainty set. ■

Worst-case CVaR

We now consider a worst-case CVaR formulation, where the distribution Q of the random variables $\tilde{\boldsymbol{z}}$ lies in a set of distributions $\mathcal{Q}$. The exact distribution is however unknown. It is natural in this setting to define the robust $(1-\epsilon)$-CVaR risk measure as:

$$\sup_{Q \in \mathcal{Q}} CVaR_{1-\epsilon}(\tilde{v}) = \sup_{Q \in \mathcal{Q}} \min_{a} \left(a + \frac{1}{\epsilon} \mathrm{E}_Q(-\tilde{v} - a)^+ \right). \tag{12}$$

Assume that $\mathcal{Q}$ is defined by a set of known moments on the random variables $\tilde{\boldsymbol{z}}$. Let $\mathcal{I}_d = \{\boldsymbol{\beta} \in \mathbb{N}^n : \beta_1 + \ldots + \beta_n \le d\}$ be an index set to define the set of monomials of degree less than or equal to

d. Suppose we are given a set of moments $\boldsymbol{m} \in \mathcal{R}^{|\mathcal{I}_d|}$. Let $\mathbb{M}(\Omega)$ denote the set of finite positive Borel measures supported by Ω. We can define the set of distributions $\mathcal{Q}$ as:

$$\mathcal{Q} = \left\{ Q \in \mathbb{M}(\Omega) \;\middle|\; E_Q[\tilde{\boldsymbol{z}}^{\boldsymbol{\beta}}] = m_{\boldsymbol{\beta}} \;\; \forall \boldsymbol{\beta} \in \mathcal{I}_d \right\}. \tag{13}$$

A simple example of such a moments model could include mean, variance and covariance information on $\tilde{\boldsymbol{z}}$. Note that no explicit assumptions on independence is made, thus naturally extending the multi-dimensional model of CVaR.

For $\Omega \subseteq \mathcal{R}^n$, let the cone of moments supported on Ω be defined as:

$$\mathbb{M}_{n,d}(\Omega) = \left\{ \boldsymbol{w} \in \mathcal{R}^{|\mathcal{I}_d|} \;\middle|\; w_{\boldsymbol{\beta}} = E_Q[\boldsymbol{z}^{\boldsymbol{\beta}}] \; \forall \beta \in \mathcal{I}_d \text{ for some Q } \in \mathbb{M}(\Omega) \right\},$$

and the cone of positive polynomials supported on Ω be defined as:

$$\mathbb{P}_{n,d}(\Omega) = \left\{ \boldsymbol{y} \in \mathcal{R}^{|\mathcal{I}_d|} \;\middle|\; y(\boldsymbol{z}) \geq 0 \; \forall \boldsymbol{z} \in \; \Omega \right\}.$$

It is well-known that $\mathbb{M}_{n,d}(\Omega)^* = \mathbb{P}_{n,d}(\Omega)$, and hence it follows that:

$$\overline{\mathbb{M}_{n,d}(\Omega)} = \mathbb{P}^*_{n,d}(\Omega),$$

i.e., the closure of the moment cone is precisely the dual cone of the set of non-negative polynomials on Ω. Furthermore, for a large class of Ω, membership in $\overline{\mathbb{M}_{n,d}(\Omega)}$ can either be exactly represented as semidefinite constraints, or else can be approximated by semidefinite constraints (Lasserre [24], Zuluaga and Pena [37]).

Theorem 3 *Consider a moments model for $\tilde{\boldsymbol{z}}$. For $\tilde{v} \in \mathcal{L}$,*

$$\sup_{Q \in \mathcal{Q}} CVaR_{1-\epsilon}(\tilde{v}) = \eta_{\mathcal{U}_{1-\epsilon}}(\tilde{v}),$$

where the associated uncertainty set

$$\mathcal{U}_{1-\epsilon} = \left\{ \boldsymbol{z} : \begin{array}{ll} \exists \boldsymbol{w}, \boldsymbol{s} \in \mathcal{R}^{|\mathcal{I}_d|} & \\ z_i = \boldsymbol{e_i}'\boldsymbol{w} & i = 1, \ldots, n \\ \boldsymbol{w} + \boldsymbol{s} = \frac{1}{\epsilon}\boldsymbol{m} & \\ \boldsymbol{w}, \boldsymbol{s} \in \overline{\mathbb{M}_{n,d}(\Omega)} & \\ w^{(0,\ldots,0)} = 1, & \end{array} \right.$$

where $\boldsymbol{e_i}$ has 1 for $(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) = (0, \ldots, 1, \ldots, 0)$, and 0 otherwise.

Proof : The worst-case $(1-\epsilon)$-CVaR for $\mathcal{Q}$ defined in (13) is:

$$\sup_{Q\in\mathcal{Q}} CVaR_{1-\epsilon}(\tilde{v}) = \sup_{Q\in\mathcal{Q}} \inf_a \left(a + \frac{1}{\epsilon}\mathrm{E}_Q(-v(\tilde{\boldsymbol{z}}) - a)^+ \right).$$

Changing the order of the supremum and minimum (Shapiro and Kleywegt [33]), we have the equivalent:

$$\sup_{Q\in\mathcal{Q}} CVaR_{1-\epsilon}(\tilde{v}) = \inf_a \left(a + \frac{1}{\epsilon} \sup_{Q\in\mathcal{Q}} \mathrm{E}_Q(-v(\tilde{\boldsymbol{z}}) - a)^+ \right).$$

For $\tilde{v} \in \mathcal{L}$, the inner problem can be expressed as (Zuluaga and Pena[37]):

$$\begin{aligned}
\sup_{Q\in\mathcal{Q}} \mathrm{E}_Q(-v(\tilde{\boldsymbol{z}}) - a)^+ = \inf \quad & \boldsymbol{m}'\boldsymbol{y} \\
\text{s.t.} \quad & \boldsymbol{y} + (a, v_1, \ldots, v_n, 0, \ldots, 0) \in \mathbb{P}_{n,d}(\Omega) \\
& \boldsymbol{y} \in \mathbb{P}_{n,d}(\Omega).
\end{aligned}$$

Substituting this inner formulation into the robust CVaR problem, we obtain

$$\begin{aligned}
\sup_{Q\in\mathcal{Q}} CVaR_{1-\epsilon}(\tilde{v}) = \inf \quad & \left(a + \frac{1}{\epsilon}\boldsymbol{m}'\boldsymbol{y} \right) \\
\text{s.t.} \quad & \boldsymbol{y} + (a, v_1, \ldots, v_n, 0, \ldots, 0) \in \mathbb{P}_{n,d}(\Omega) \\
& \boldsymbol{y} \in \mathbb{P}_{n,d}(\Omega).
\end{aligned}$$

Taking the conic dual, we obtain:

$$\begin{aligned}
\sup_{Q\in\mathcal{Q}} CVaR_{1-\epsilon}(\tilde{v}) = \sup \quad & -\sum_{i=1}^{n} v_i \boldsymbol{e_i}'\boldsymbol{w} \\
\text{s.t.} \quad & \boldsymbol{w} + \boldsymbol{s} = \frac{1}{\epsilon}\boldsymbol{m} \\
& \boldsymbol{w}, \boldsymbol{s} \in \overline{\mathbb{M}_{n,d}(\Omega)} \\
& w^{(0,\ldots,0)} = 1,
\end{aligned} \tag{14}$$

which yields the desired result. ■

This generalizes the idea of worst-case VaR introduced by El Ghaoui et al. [15] to worst-case CVaR. It should be noted that while extending the former notion to higher order moments is not easy (due to the non-convexity of the formulation), it is possible to obtain exactly or obtain stronger approximations for worst-case CVaR based on the description of Ω.

4 Coherent Risk Measures and Uncertainty Sets

In this section, we propose a method for constructing coherent risk measures based on robust optimization uncertainty sets with support information, and derive bounds on the probability of constraint violation under the so-constructed risk measures. We illustrate the method with a numerical example.

4.1 Creating Coherent Risk Measures

Our method is based on the following result:

Theorem 4 *Let Ω be the sample space of $\tilde{\boldsymbol{z}}$. For any uncertainty set $\mathcal{U}$ satisfying $\mathcal{U} \subseteq \Omega$, the robust counterpart risk measure $\eta_{\mathcal{U}}(\tilde{v})$ is a coherent risk measure.*

Proof : It is trivial to show translation invariance and positive homogeneity. With regard to subadditivity, observe that

$$\begin{aligned} \eta_{\mathcal{U}}(\tilde{v}_1 + \tilde{v}_2) &= -\min\nolimits_{\boldsymbol{z}\in\mathcal{U}}(v_1(\boldsymbol{z}) + v_2(\boldsymbol{z})) \\ &\leq (-\min\nolimits_{\boldsymbol{z}\in\mathcal{U}} v_1(\boldsymbol{z})) + (-\min\nolimits_{\boldsymbol{z}\in\mathcal{U}} v_2(\boldsymbol{z})) \\ &= \eta_{\mathcal{U}}(\tilde{v}_1) + \eta_{\mathcal{U}}(\tilde{v}_2). \end{aligned}$$

To show monotonicity, we note that if $\tilde{v} \geq 0$, then

$$\min_{\boldsymbol{z}\in\Omega} v(\boldsymbol{z}) \geq 0.$$

For $\mathcal{U} \subseteq \Omega$, we have

$$\eta_{\mathcal{U}}(\tilde{v}) = -\min_{\boldsymbol{z}\in\mathcal{U}} v(\boldsymbol{z}) \leq -\min_{\boldsymbol{z}\in\Omega} v(\boldsymbol{z}) \leq 0.$$

Without loss of generality, suppose $\tilde{v}_1 \geq \tilde{v}_2$. Therefore,

$$\begin{aligned} \eta_{\mathcal{U}}(\tilde{v}_1) &\leq \eta_{\mathcal{U}}(\underbrace{\tilde{v}_1 - \tilde{v}_2}_{\geq 0}) + \eta_{\mathcal{U}}(\tilde{v}_2) \quad \text{(Subadditivity)} \\ &\leq \eta_{\mathcal{U}}(\tilde{v}_2), \end{aligned}$$

which yields the desired result. ■

Robust counterparts in which $v(\boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z})$ is concave in $\boldsymbol{z}$ are well studied by Ben-Tal and Nemirovski [4]. It is easy to observe that

$$\eta_{\mathcal{CH}(\mathcal{U})}(\tilde{v}) \geq \eta_{\mathcal{U}}(\tilde{v}),$$

where $\mathcal{CH}(\mathcal{U})$ represents the convex hull of the set $\mathcal{U}$. However, if the function $v(\boldsymbol{z})$ is concave in $\boldsymbol{z}$, then

$$\eta_{\mathcal{CH}(\mathcal{U})}(\tilde{v}) = \eta_{\mathcal{U}}(\tilde{v}).$$

This holds for instance in the case of affine data uncertainty.

Based on Theorem 4, it is clear that given any uncertainty set $\mathcal{V}$ that is not necessarily a subset of Ω, we can make the associated risk measure a coherent one by modifying the uncertainty set to:

$$\mathcal{U} = \mathcal{V} \cap \bar{\Omega},$$

where $\bar{\Omega} \subseteq \mathcal{CH}(\Omega)$. Under mild assumptions, we assume that the uncertainty sets are conic representable,

$$\mathcal{U} = \{\boldsymbol{z} \mid \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} - \boldsymbol{b} \in \boldsymbol{K} \text{ for some } \boldsymbol{u}\}, \tag{15}$$

where the cone $\boldsymbol{K}$ is regular, i.e., it is closed, convex, pointed, and has a non-empty interior. Hence, the polar cone

$$\boldsymbol{K}^* = \left\{\boldsymbol{y} \mid \boldsymbol{y}'\boldsymbol{s} \geq 0 \ \forall \boldsymbol{s} \in \boldsymbol{K}\right\}$$

is also a regular cone (see the convex analysis in Rockafellar [28]). For technical reasons, we also assume that $\mathcal{U}$ is a compact set with nonempty interior.

Theorem 5 *For $\tilde{v} \in \mathcal{L}$, the risk constraint $\eta_{\mathcal{U}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) \leq 0$ is concisely representable as the conic constraints*

$$\begin{cases} v_0 + \boldsymbol{y}'\boldsymbol{b} \geq 0 \\ \boldsymbol{A}'\boldsymbol{y} = \boldsymbol{v} \\ \boldsymbol{B}'\boldsymbol{y} = \boldsymbol{0} \\ \boldsymbol{y} \in \boldsymbol{K}^*. \end{cases}$$

Proof : The application of duality theory in formulating robust counterparts is well known (see for instance Ben-Tal and Nemirovski [4]). Under the assumptions, the set $\mathcal{U}$ satisfies the necessary Slater condition for strong duality. Therefore

$$\begin{aligned} \eta_{\mathcal{U}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) \quad = \quad -\min_{\boldsymbol{z}} \quad & v_0 + \boldsymbol{v}'\boldsymbol{z} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} - \boldsymbol{b} \in \boldsymbol{K}, \end{aligned}$$

or equivalently,

$$\begin{aligned} \eta_{\mathcal{U}}(v_0 + \boldsymbol{v}'\tilde{\boldsymbol{z}}) \quad = \quad -\max_{\boldsymbol{y}} \quad & v_0 + \boldsymbol{y}'\boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{A}'\boldsymbol{y} = \boldsymbol{v} \\ & \boldsymbol{B}'\boldsymbol{y} = \boldsymbol{0} \\ & \boldsymbol{y} \in \boldsymbol{K}^*. \end{aligned}$$

This results in the conic constraint representation of the feasible region. ■

4.2 Probability Bounds on Risk Measures

In robust optimization, the conservativeness of the approach (equivalently, the tolerance to risk) is typically captured by the "size" of the uncertainty set. For example, one can think of $\mathcal{U}_\alpha$ as an uncertainty set of "size" α, where α is selected based on some probability estimate so that:

$$\eta_{\mathcal{U}_\alpha}(\tilde{v} - y) \leq 0 \Rightarrow \mathrm{P}(\tilde{v} \geq y) \geq 1 - g(\alpha). \tag{16}$$

Here $g(\alpha)$ provides an upper bound on the probability of constraint violation and typically decreases as α increases. Since this is true for all y, it implies that $\eta_{\mathcal{U}_\alpha}(\tilde{v}) \geq VaR_{1-g(\alpha)}(\tilde{v})$. Our concern is whether the following remains true:

$$\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(\tilde{v}) \geq VaR_{1-g(\alpha)}(\tilde{v}).$$

If it does, then making a risk measure coherent by using Theorem 4 does not require a tradeoff in terms of the probability of constraint violation.

More generally, suppose a robust counterpart risk measure $\eta_{\mathcal{U}_\alpha}(\tilde{v})$ is an upper bound of a risk measure $\rho(\tilde{v})$. We would like to know whether $\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(\tilde{v})$ remains an upper bound for $\rho(\tilde{v})$.

For this purpose, we assume that the set $\bar{\Omega}$ is compact with nonempty interior. We define the cone

$$\mathbf{\Pi} = \mathrm{cl}\{(\boldsymbol{z}, t) \ : \ \boldsymbol{z}/t \in \bar{\Omega}, t > 0\},$$

where $\mathrm{cl}(\cdot)$ denotes the closure of the cone. Therefore, the cone $\mathbf{\Pi}$ and its polar $\mathbf{\Pi}^*$ are regular cones. Again, for technical reasons, we assume that the Slater condition for $\mathcal{U}_\Omega \cap \bar{\Omega}$ is satisfied.

Theorem 6 *Let $\rho(\cdot)$ be a risk measure that satisfies the translation invariance and the monotonicity axioms. Suppose*

$$\eta_{\mathcal{U}_\alpha}(\tilde{v}) \geq \rho(\tilde{v}), \ \forall \tilde{v} \in \mathcal{L}.$$

Then

$$\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(\tilde{v}) \geq \rho(\tilde{v}), \ \forall \tilde{v} \in \mathcal{L} \quad \text{if} \quad \bar{\Omega} = \mathcal{CH}(\Omega).$$

Proof : Consider the following optimization problem:

$$\begin{aligned} -\eta_{\mathcal{U}_\alpha \cap \bar{\Omega}}(\boldsymbol{v}'\tilde{\boldsymbol{z}}) = \min_{\boldsymbol{z}} \quad & \boldsymbol{v}'\boldsymbol{z} \\ \text{s.t.} \quad & \boldsymbol{z} \in \mathcal{U}_\alpha \\ & (\boldsymbol{z}, 1) \in \mathbf{\Pi}, \end{aligned}$$

which is well defined in the compact set, and satisfies the Slater condition. Hence, the objective is the same as

$$\max_{(\boldsymbol{p},t)\in\boldsymbol{\Pi}^*}\left\{\min_{\boldsymbol{z}:\boldsymbol{z}\in\mathcal{U}_\alpha}\boldsymbol{v}'\boldsymbol{z}-\boldsymbol{z}'\boldsymbol{p}-t\right\}=\min_{\boldsymbol{z}:\boldsymbol{z}\in\mathcal{U}_\alpha}(\boldsymbol{v}-\boldsymbol{p}^*)'\boldsymbol{z}-t^*=-\eta_{\mathcal{U}_\alpha}((\boldsymbol{v}-\boldsymbol{p}^*)'\tilde{\boldsymbol{z}})-t^*$$

for some $(\boldsymbol{p}^*,t^*)\in\boldsymbol{\Pi}^*$. Therefore,

$$\begin{aligned}\eta_{\mathcal{U}_\alpha\cap\bar{\Omega}}(\boldsymbol{v}'\tilde{\boldsymbol{z}}) &= \eta_{\mathcal{U}_\alpha}((\boldsymbol{v}-\boldsymbol{p}^*)'\tilde{\boldsymbol{z}})+t^* \\ &\geq \rho((\boldsymbol{v}-\boldsymbol{p}^*)'\tilde{\boldsymbol{z}})+t^* \qquad (\text{Since } \rho(\tilde{v})\leq\eta_{\mathcal{U}_\alpha}(\tilde{v}) \text{ for all } \tilde{v}\in\mathcal{L}) \\ &= \rho((\boldsymbol{v}-\boldsymbol{p}^*)'\tilde{\boldsymbol{z}}-t^*). \qquad (\text{Translation invariance})\end{aligned}$$

Observe that $(\tilde{\boldsymbol{z}},1)\in\boldsymbol{\Pi}$. Therefore, $\boldsymbol{p}^{*\prime}\tilde{\boldsymbol{z}}+t^*\geq 0$. Hence, $(\boldsymbol{v}-\boldsymbol{p}^*)'\tilde{\boldsymbol{z}}-t^*\leq\boldsymbol{v}'\tilde{\boldsymbol{z}}$ and by the monotonicity axiom,

$$\eta_{\mathcal{U}_\alpha\cap\bar{\Omega}}(\boldsymbol{v}'\tilde{\boldsymbol{z}})\geq\rho((\boldsymbol{v}-\boldsymbol{p}^*)'\tilde{\boldsymbol{z}}-t^*)\geq\rho(\boldsymbol{v}'\tilde{\boldsymbol{z}}).$$

Finally, by the translation invariance axiom,

$$\eta_{\mathcal{U}_\alpha\cap\bar{\Omega}}(v_0+\boldsymbol{v}'\tilde{\boldsymbol{z}})\geq\rho(v_0+\boldsymbol{v}'\tilde{\boldsymbol{z}}).$$

■

The VaR measure satisfies the axioms of translation invariance and monotonicity. Hence, if the condition of (16) is true for all $\tilde{v}\in\mathcal{L}$, then

$$\eta_{\mathcal{U}_\alpha\cap\bar{\Omega}}(\tilde{v}-y)\leq 0\Rightarrow \mathrm{P}(\tilde{v}\geq y)\geq 1-g(\alpha).$$

4.3 A Numerical Example: Worst-Case VaR

In Section 3.2, we showed that the ellipsoidal uncertainty set maps to the mean-standard deviation risk measure. We also mentioned that El Ghaoui et al. [15] use this result to derive a formulation for the worst-case VaR based on first- and second-moment information about the distributions of uncertainties. However, formulating the problem using the ellipsoidal uncertainty set $\mathcal{E}_\alpha$ results in a non-coherent risk measure for general $\alpha>0$.

We now illustrate how one could make the resulting risk measure coherent. Suppose we have the additional information that

$$\bar{\Omega}=\{\boldsymbol{z}:-\underline{\boldsymbol{z}}\leq\boldsymbol{z}\leq\bar{\boldsymbol{z}}\}\subseteq\mathcal{CH}(\Omega).$$

Then, we can construct a coherent risk measure by intersecting the ellipsoidal uncertainty set with the set $\bar{\Omega}$. The robust counterpart of

$$v_0+\boldsymbol{v}'\boldsymbol{z}\geq 0 \qquad \forall\boldsymbol{z}\in\mathcal{E}_\alpha\cap\bar{\Omega}$$

then reduces to the set of constraints

$$\begin{cases} v_0 \geq \alpha \|\boldsymbol{v} + \boldsymbol{r} - \boldsymbol{s}\|_2 + \bar{\boldsymbol{z}}'\boldsymbol{r} + \underline{\boldsymbol{z}}'\boldsymbol{s} \\ \boldsymbol{r}, \boldsymbol{s} \geq \boldsymbol{0}. \end{cases} \tag{17}$$

We note that El Ghaoui et al. [15] discuss including support information in their worst-case VaR formulation, but they do not provide computational results on the performance of the modified formulation, and do not relate it to the idea of coherence.

We explore the potential of (17) with a numerical experiment.

We use daily historical returns from March 14, 1986 to December 31, 2003 for a portfolio of 30 Dow Jones stocks from different industry categories (Table 1). There are a total of 4493 observations for each stock. Our goal is to compare the VaR-efficient frontiers resulting from the worst-case VaR formulation and the coherent worst-case VaR formulation with support information.

Table 1: List of stocks and corresponding industries used in the computational experiment.

Industry	Company Name (Ticker)
Aerospace	AAR Corporation (AIR), Boeing Corporation (BA), Lockheed Martin (LMT), United Technologies (UTX)
Telecommunications	AT&T (T), Motorola (MOT)
Semiconductor	Applied Materials (AMD), Intel Corporation (INTC), Hitachi (HIT), Texas Instruments (TXN)
Computer Software	Microsoft (MSFT), Oracle (ORCL)
Computer Hardware	Hewlett Packard (HPQ), IBM Corporation (IBM), Sun Microsystems (SUNW)
Internet and Online	Northern Telecom (NT)
Biotech and Pharmaceutical	Bristol Myers Squibb (BMY), Chiron Co. (CHIR), Eli Lilly and Co. (LLY), Merck and Co. (MRK)
Utilities	Duke Energy Co. (DUK), Exelon Corporation (EXC), Pinnacle West (PNW)
Chemicals	Avery Denison Co. (AVY), Du Pont (DD), Dow Chemical (DOW)
Industrial Goods	FMC Corporation (FMC), General Electric (GE), Honeywell (HON), Ingersoll Rand (IR)

We assume that asset returns are generated by a factor model:

$$\tilde{\boldsymbol{r}} = \boldsymbol{\mu} + \boldsymbol{A}\tilde{\boldsymbol{z}}, \tag{18}$$

where $\boldsymbol{\mu}$ is the vector of expected returns. Here, the returns are an affine mapping of stochastically independent factors $\tilde{\boldsymbol{z}}$ that have zero means and support $\tilde{z}_j \in [-\underline{z}_j, \bar{z}_j]$, $j = 1, \ldots, N$. We create uncorrelated factors artificially for both the coherent and the noncoherent formulation by finding the covariance matrix of returns $\boldsymbol{\Sigma}$ from the data and choosing $\boldsymbol{A} = \boldsymbol{\Sigma}^{1/2}$. Hence, $\tilde{\boldsymbol{z}} = \boldsymbol{\Sigma}^{-1/2}(\tilde{\boldsymbol{r}} - \boldsymbol{\mu})$. In practice, portfolio managers could use more sophisticated factor models for asset returns.

The explicit formulations for the worst-case and the coherent worst-case VaR optimization problems are shown below:

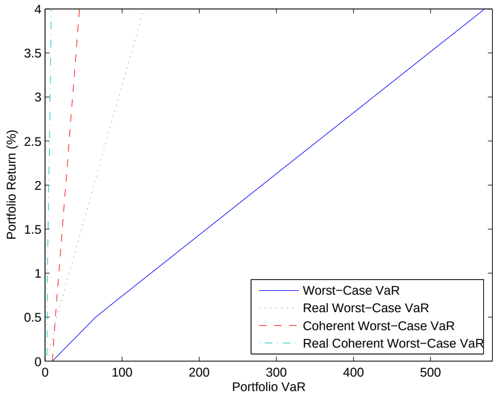
$$\begin{array}{lll} \text{(Worst Case VaR)} & \min & \gamma \\ & \text{s.t.} & -\boldsymbol{\mu}'\boldsymbol{x} + \alpha\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}} \leq \gamma \\ & & \boldsymbol{\mu}'\boldsymbol{x} \geq r_{\text{target}} \\ & & \boldsymbol{x}'\boldsymbol{e} = 1; \end{array} \tag{19}$$

$$\begin{array}{lll} \text{(Coherent Worst Case VaR)} & \min & \gamma \\ & \text{s.t.} & -\boldsymbol{\mu}'\boldsymbol{x} + \alpha\|\boldsymbol{\Sigma}^{1/2}\boldsymbol{x} + \boldsymbol{r} - \boldsymbol{s}\|_2 + \bar{\boldsymbol{z}}'\boldsymbol{r} + \underline{\boldsymbol{z}}'\boldsymbol{s} \leq \gamma \\ & & \boldsymbol{\mu}'\boldsymbol{x} \geq r_{\text{target}} \\ & & \boldsymbol{x}'\boldsymbol{e} = 1 \\ & & \boldsymbol{r}, \boldsymbol{s} \geq \boldsymbol{0}. \end{array} \tag{20}$$

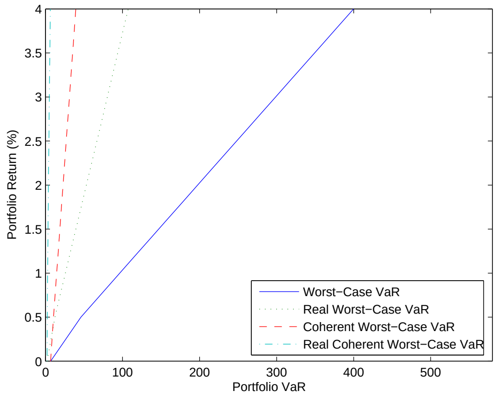
We use $\epsilon = 1\%, 2\%, 5\%$, and 10%, and compute the corresponding value for $\alpha = \sqrt{\frac{1-\epsilon}{\epsilon}}$. For each value of ϵ, we solve formulation (19) for different target portfolio returns to obtain the optimal objective function value ('Worst-Case VaR'). We then find the realized VaR using the optimal asset weights and the actual historical returns ('Real Worst-Case VaR'). Similarly, we solve formulation (20) to find the optimal objective function value for a coherent VaR measure ('Coherent Worst-Case VaR'), and use the optimal asset weights to find the realized portfolio VaR ('Real Coherent Worst-Case VaR'). All optimization problems are solved with SDPT3 [35].

The results are presented in Figure 1. Tables 2 and 3 show the computational output for $\epsilon = 1\%$ and $\epsilon = 10\%$ in more detail. The numbers in the computational output for VaR can be interpreted as the worst portfolio loss per dollar invested that can happen with probability ϵ when the expected portfolio return is the target return. It is therefore desirable to have low numbers for the VaR value. One can observe that the realized VaR is always lower than the objective function value in the optimization

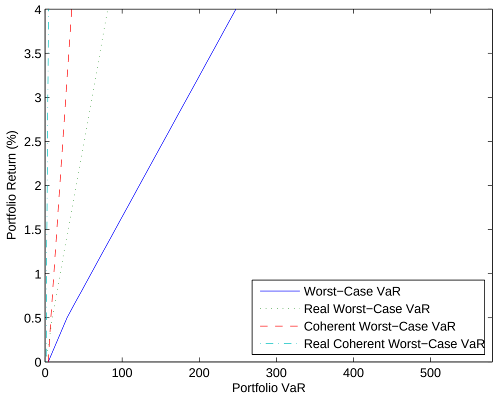
Figure 1: VaR efficient frontiers.



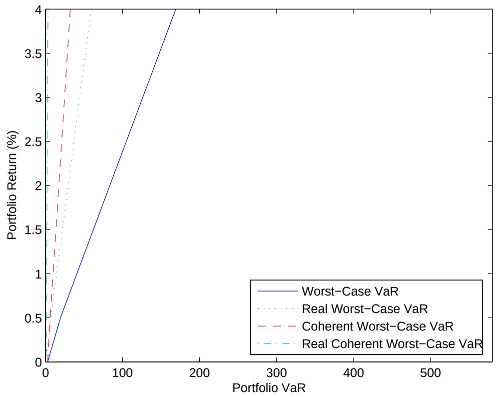
$\epsilon = 1\%$



$\epsilon = 2\%$



$\epsilon = 5\%$



$\epsilon = 10\%$

problems, i.e., a portfolio manager can be confident that the VaR estimate she gets from solving the optimization problem would be conservative. The computational results indicate also that using a coherent version of the worst-case VaR risk measure not only significantly improves the optimal objective function value in the VaR formulation, but also results in a substantial reduction in the actual realized VaR value. As Figure 1 illustrates, the efficient frontiers of the Coherent Worst-Case VaR and the Real Coherent Worst-Case VaR strongly dominate the efficient frontiers of their non-coherent counterparts. Furthermore, the relative improvement is higher for low values of ϵ, which is encouraging, considering the fact that portfolio managers are typically concerned about extreme events.

Table 2: Computational results for $\epsilon = 1\%$.

Target Return	VaR	Real VaR	Coherent VaR	Real Coherent VaR
0.500	65.541	15.152	13.402	2.757
0.600	79.850	18.395	14.295	2.815
0.700	94.197	21.691	15.188	2.886
0.800	108.566	24.916	16.081	2.984
0.900	122.949	28.150	16.974	3.043
1.000	137.343	31.402	17.867	3.154
1.200	166.151	37.851	19.653	3.495
1.400	194.975	44.230	21.440	3.771
1.600	223.809	50.608	23.225	4.146
1.800	252.650	56.986	25.011	4.526
2.000	281.495	63.365	26.797	4.872
2.200	310.344	69.743	28.583	5.226
2.400	339.195	76.122	30.370	5.559
2.600	368.049	82.500	32.156	5.935
2.800	396.903	89.001	33.942	6.262
3.000	425.759	95.596	35.728	6.592
3.200	454.616	102.191	37.514	6.952
3.400	483.474	108.745	39.300	7.271
3.600	512.332	115.143	41.086	7.483
3.800	541.191	121.542	42.872	7.763
4.000	570.050	127.940	44.658	8.063

Table 3: Computational results for $\epsilon = 10\%$.

Target Return	VaR	Real VaR	Coherent VaR	Real Coherent VaR
0.500	19.412	6.847	6.094	1.158
0.600	23.657	8.336	6.838	1.206
0.700	27.912	9.866	7.582	1.251
0.800	32.175	11.330	8.326	1.296
0.900	36.442	12.848	9.069	1.338
1.000	40.712	14.362	9.813	1.387
1.200	49.258	17.342	11.301	1.498
1.400	57.809	20.287	12.789	1.625
1.600	66.363	23.315	14.277	1.736
1.800	74.919	26.273	15.764	1.843
2.000	83.477	29.365	17.252	1.963
2.200	92.036	32.256	18.740	2.072
2.400	100.595	35.300	20.228	2.179
2.600	109.155	38.210	21.716	2.296
2.800	117.715	41.178	23.203	2.396
3.000	126.276	44.187	24.691	2.535
3.200	134.837	47.306	26.179	2.672
3.400	143.398	50.359	27.667	2.812
3.600	151.959	53.423	29.155	2.954
3.800	160.521	56.518	30.642	3.069
4.000	169.083	59.612	32.130	3.178

5 Concluding Remarks

We presented a unified view of risk measures in finance and uncertainty sets in robust optimization, and described how robust optimization can be used to enhance the concepts of some risk measures. We also proposed a practical approach to making existing risk measures coherent, and proved that the probability of constraint violation remains the same. Our computational experiments suggest that there may be practical benefits to using modified coherent risk measures with support information.

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