The Polar of a Simple Mixed-Integer Set

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Abstract

We study the convex hull P of the set $S = \{(x, y) \in \Re_+ \times Z^n : x + B_i y_{ij} \geq b_{ij}, j \in N_i, i \in M\}$, where $M = \{1, \ldots, m\}$, $N_i = \{1, \ldots, n_i\}$ $\forall i \in M$, $\sum_{i=1}^{m} n_i = n$, and $B_1 | \cdots | B_m$. The set P generalizes the mixed-integer rounding (MIR) set of Nemhauser and Wolsey and the mixing-MIR set of Günlük and Pochet. Our main result is a *compact* full inequality description of Ω, the polar of P. In the worst case, the number of inequalities $I(\Omega)$ in our description of Ω grows exponentially with n. However, when B_m/B_1 is bounded by a polynomial of n, $I(\Omega)$ grows polynomially with n. In particular, when B_m/B_1 is bounded by a constant, $I(\Omega) = O(n)$. Also, when m is fixed, $I(\Omega)$ grows polynomially with n, regardles of the values of the B_i's. This means that, in these cases, it is possible to find a most violated cut for $(x^*, y^*) \notin S$ in polynomial time. Finally, we indicate how our study may be extended to the case of nondivisible B_i's.

Keywords: mixed-integer programming, branch-and-cut, polyhedral combinatorics, simple mixed-integer set, lot-sizing

1 Introduction

The progress of computational mixed-integer programming (MIP) in the past 15 years has turned it into an indispensable managerial tool for today's economy, see for example [11]. This progress is the result of several factors, especially the use of cutting planes in LP-based branch-and-bound [3].

The importance of cutting planes to MIP has been recognized since the early days of operations research. They were instrumental, for example, in the study of the travelling salesman problem by Dantzig et al. [5]. Shortly after, Gomory introduced his mixed-integer cut [6], which recently was proven to be computationally effective [2], and is now part of all main academic and professional MIP software. Later, other related cuts were introduced, including the intersection cut [1], split cut [4], and mixed-integer rounding (MIR) cut [13]. For a review of the historical and recent progress on cutting planes for MIP we refer to the papers by Marchand et al. [9] and Wolsey [17].

An interesting framework to study the inequality description of MIP polyhedra is to consider *simple* MIPs and to use strong inequalities valid for them as cuts for more general MIPs. To the best of our knowledge, this point of view was introduced by Marchand and Wolsey [10]. They considered the set

$$X_{MIR} = \{(x, y) \in \Re_+ \times Z : x + y \geq b\}$$

and observed that adding the inequality

$$x + (b - \lfloor b \rfloor)y \geq (b - \lfloor b \rfloor)\lceil b \rceil \tag{1}$$

to the LP relaxation of X_{MIR} gives $\text{conv}(X_{MIR})$. Then, they showed that (1) implies several important cutting planes, such as the Gomory cut and MIR cut. In addition to obtaining remarkable computational results with their approach, by showing that the Gomory cut and MIR cut originate from a strong inequality of a relaxation, they shed light on the fact that these cuts are effective in practice.

Several authors extended the study in [10]. Günlük and Pochet gave the full inequality description of $\text{conv}(X^M_{MIR})$, the mixing-MIR polytope, where

$$X^M_{MIR} = \{(x, y) \in \Re_+ \times Z^n : x + y_i \geq b_i, i = 1, \ldots, n\}.$$

Miller and Wolsey [12] and van Vyve [15, 16] studied generalizations of X^M_{MIR}. The sets considered in all these studies are of the type $\{(x, y) \in \Re^m \times Z^n : Ax + By \geq b\}$, with A and B totaly unimodular (TU) matrices.

In this paper we study the convex hull P of the set

$$S = \{(x, y) \in \Re_+ \times Z^n : x + B_i y_{ij} \geq b_{ij}, j \in N_i, i \in M\},$$

where $M = \{1, \ldots, m\}$, $N_i = \{1, \ldots, n_i\}$ $\forall i \in M$, and $\sum_{i=1}^m n_i = n$. We assume that $B_i \in Q_+ - \{0\}$ $\forall i \in M$, $B_1 < \cdots < B_m$, and $B_1 | \cdots | B_m$, i.e.

$$\frac{B_r}{B_s} \in Z$$

whenever $r > s$, $r, s \in M$. The set P, which generalizes the previous studies to the case of a non-TU constraint coefficient matrix, appears for example in lot-sizing with production lower bounds [14].

To the best of our knowledge, P has not been considered in the literature, and the only result on it we are aware of is its full inequality description for $m = 2$, which is given in [15].

The inequality description of P seems to be hard, and it is not clear how to extend the results in [8, 15] to it. In fact, the description of P for $m = 2$ given in [15] is considerably more complicated than the one given in [8] for $m = 1$, and a description for $m = 3$ appears to be beyond reach. To overcome these difficulties, we consider the polar of P, i.e. the set Ω whose points are the coefficients of the inequalities valid for P. Our main result is a full inequality description of Ω that is *compact*.

Interestingly enough, the description of Ω is quite simple. In the worst case, the number of inequalities $I(\Omega)$ in our description of Ω grows exponentially with n. However, when B_m/B_1 is bounded by a polynomial of n, $I(\Omega)$ grows polynomially with n. In particular, when B_m/B_1 is bounded by a constant, $I(\Omega) = O(n)$. Also, when m is fixed, $I(\Omega)$ grows polynomially with n, regardles of the values of the B_i's. This means that, in these cases, it is possible to find a most violated cut for $(x^*, y^*) \notin S$ in polynomial time.

The remainder of the paper is organized as follows. In Section 2 we give a finite inequality description of Ω that grows as a polynomial of n when B_m/B_1 is bounded by a polynomial of n, but that grows exponentially with n otherwise. In Section 3 we show how to eliminate a substantial number of inequalities when the finite description is large. The resulting description still grows exponentially with n in the worst case. However, it grows polynomially with n when m is fixed, regardless of the values of the B_i's. In Section 4 we indicate how our study may be extended to the case of nondivisible B_i's and present directions for further research.

2 Finite Inequality Description of Ω

In this section we give a finite inequality description of Ω. We first show that it is possible to describe Ω with an infinite family of inequalities $\mathcal{F} = \{ine(i, j, k) : j \in N_i, i \in M, \text{ and } k \in Z_+\}$, which implies a particular inequality that we call *basic*. We then show that, even though *basic* is implied by $\mathcal{F}$, by adding *basic* to the description of Ω, the inequalities $ine(i, j, k)$ with $k \geq B_m/B_i$ become redundant $\forall j \in N_i, i \in M$, thus giving the finite description. This means that when B_m/B_1 is bounded by a polynomial of n, $I(\Omega)$ grows as a polynomial of n. In particular, if B_m/B_1 is bounded by a constant, $I(\Omega) = O(n)$. Finally, we give an inequality that, when added to the description of Ω gives a polytope. Even though the inequality is not valid for Ω, it only cuts off points that represent empty-face inequalities for P.

Let $MN = \cup_{i \in M} \{i\} \times N_j$. Following [8], we denote

$$\tau_{ij} = \left\lceil \frac{b_{ij}}{B_i} \right\rceil \text{ and } \gamma_{ij} = b_{ij} - (\tau_{ij} - 1)B_i \tag{2}$$

for every $ij \in MN$. It is also easy to adapt the proof of Lemma 2 in [8] to show that

Proposition 1 *Any valid inequality for P can be written as*

$$x + \alpha \geq \sum_{ij \in MN} \delta_{ij}(\tau_{ij} - y_{ij}), \tag{3}$$

where $(\alpha, \delta) \in R_+^{n+1}$. □

The *polar* of P is the set $\Omega \subseteq \Re^{n+1}$ whose points are the coefficients of the inequalities valid for P. From Proposition 1, it follows that

$$\Omega = \{(\alpha, \delta) \in R_{+}^{n+1} : (3) \text{ is valid for } P\}.$$

We now show that, for the description of Ω, we can restrict ourselves to the points $(x, y) \in P$ for which $\tau_{ij} - y_{ij} \geq 0 \ \forall ij \in MN$ and $\tau_{ij} - y_{ij} \geq 1$ for some $ij \in MN$.

Proposition 2 $\Omega = \{(\alpha, \delta) \in R_{+}^{n+1} : (\alpha, \delta)$ *satisfies* $(4)\ \forall l \in Z_{+}^{n} - \{0\}\}$, *where*

$$\max_{ij \in MN} \{\gamma_{ij} + B_i(l_{ij} - 1)\} + \alpha \geq \sum_{ij \in MN} \delta_{ij} l_{ij}. \tag{4}$$

Proof Note that $(\alpha, \delta) \in \Omega$ iff

$$\max_{ij \in MN} \{0, b_{ij} - B_i y_{ij}\} + \alpha \geq \sum_{ij \in MN} \delta_{ij}(\tau_{ij} - y_{ij})$$

for any $y \in Z^n$. Denoting $l_{ij} = \tau_{ij} - y_{ij} \ \forall ij \in MN$, and substituting (2), we have that $(\alpha, \delta) \in \Omega$ iff

$$\max_{ij \in MN} \{0, \gamma_{ij} + B_i(l_{ij} - 1)\} + \alpha \geq \sum_{ij \in MN} \delta_{ij} l_{ij} \tag{5}$$

for any $l \in Z^n$.

Since $\gamma_{ij} \leq B_i \ \forall ij \in MN$, if $l_{ij} \leq 0 \ \forall ij \in MN$, (5) becomes

$$\alpha \geq \sum_{ij \in MN} \delta_{ij} l_{ij},$$

which is redundant. So we may assume that $l_{ij} \geq 1$ for some $ij \in MN$. Now, let $\tilde{l} \in Z^n$ with $\tilde{l}_{rs} < 0$ for some $rs \in MN$, and $\hat{l} \in Z^n$ be such that $\hat{l}_{ij} = \tilde{l}_{ij} \ \forall ij \in MN - \{rs\}$ and $\hat{l}_{rs} = 0$. It follows that

$$\max_{ij \in MN} \{0, \gamma_{ij} + B_i(\hat{l}_{ij} - 1)\} = \max_{ij \in MN} \{0, \gamma_{ij} + B_i(\tilde{l}_{ij} - 1)\}$$

and

$$\sum_{ij \in MN} \delta_{ij} \hat{l}_{ij} \geq \sum_{ij \in MN} \delta_{ij} \tilde{l}_{ij}.$$

Therefore, (5) for $l = \hat{l}$ implies (5) for $l = \tilde{l}$, and we may assume that $l \geq 0$. □

We now sharpen the inequality description of Ω given in Proposition 2. In the resulting much simpler description, every inequality is indexed by only 3 indices, $ij \in MN$ and $k \in Z_{+}$.

Proposition 3 $\Omega = \{(\alpha, \delta) \in R_{+}^{n+1} : (\alpha, \delta)$ *satisfies* $(6)\ \forall rs \in MN$ *and* $k \in Z_{+}\}$, *where*

$$\gamma_{rs} + B_r k + \alpha \geq \sum_{ij \in MN} \delta_{ij} p_{ij}(r, s, k) \tag{6}$$

and

$$p_{ij}(r, s, k) = \left\lfloor \frac{\gamma_{rs} - \gamma_{ij} + B_r k + B_i}{B_i} \right\rfloor.$$

Proof Let $rs \in MN$. Because l is integral, we have that $\max\{\gamma_{ij} + B_i(l_{ij} - 1)\} = \gamma_{rs} + B_r(l_{rs} - 1)$ iff

$$l_{ij} \leq \left\lfloor \frac{\gamma_{rs} - \gamma_{ij} + B_r(l_{rs} - 1) + B_i}{B_i} \right\rfloor \tag{7}$$

for all $ij \in MN$. In this case, (4) becomes

$$\gamma_{rs} + B_r(l_{rs} - 1) + \alpha \geq \sum_{ij \in MN} \delta_{ij} l_{ij}. \tag{8}$$

Because of (7), (8) is implied by

$$\gamma_{rs} + B_r(l_{rs} - 1) + \alpha \geq \sum_{ij \in MN} \delta_{ij} \left\lfloor \frac{\gamma_{rs} - \gamma_{ij} + B_r(l_{rs} - 1) + B_i}{B_i} \right\rfloor.$$

Denoting $k = l_{rs} - 1$, (6) follows. □

We henceforth refer to (6) as $ine(r, s, k)$. The description of Ω given by Proposition 3 contains infinitely many inequalities. Our next goal is to obtain a finite description. To this end, we now introduce an inequality that is implied by the set $\mathcal{F} = \{ine(i, j, k) : ij \in MN, \text{ and } k \in Z_+\}$. However, by adding the inequality to the description of Ω, the inequalities $ine(i, j, k)$ with $k \geq B_m/B_i$ become redundant, thus giving the finite description.

Proposition 4 *The inequality*

$$\sum_{ij \in MN} \frac{\delta_{ij}}{B_i} \leq 1 \tag{9}$$

is valid for Ω*.*

Proof Let $rs \in MN$. We consider the inequalities $ine(r, s, k)$ with $k = \bar{k}\lambda$, where $\bar{k}$ and λ are positive integers and $B_r\bar{k}/B_i \in Z\ \forall i \in M$. Dividing $ine(r, s, k)$ by λ, we obtain

$$\frac{\gamma_{rs} + \alpha}{\lambda} + B_r\bar{k} \geq \frac{1}{\lambda} \sum_{ij \in MN} \delta_{ij} \left\lfloor \frac{\gamma_{rs} - \gamma_{ij} + B_r k + B_i}{B_i} \right\rfloor \geq \sum_{ij \in MN} \delta_{ij} \left\lfloor \frac{\gamma_{rs} - \gamma_{ij} + B_i}{B_i\lambda} + \frac{B_r\bar{k}}{B_i} \right\rfloor.$$

Because $ine(r, s, k)$ is valid for λ arbitrarily large, it must be valid in the limit $\lambda \to \infty$. Thus, (9) follows. □

We henceforth refer to (9) as the *basic* inequality. We also denote

$$B_{uv} = \frac{B_u}{B_v},$$

where $u, v \in M$. We now establish the finite description of Ω.

Theorem 1 *Let* $rs \in MN$ *and* $k \in Z_+$*. Inequality* $ine(r, s, k)$ *is implied by the set of inequalities* $\{ine(r, s, 0), \ldots, ine(r, s, B_{mr} - 1),\ basic\}$*.* □

Proof Let $\bar{k} \in \{0, \ldots, B_{mr} - 1\}$ be such that $\bar{k} \equiv k (mod\ B_{mr})$ and consider $ine(r, s, \bar{k})$:

$$\gamma_{rs} + B_r \bar{k} + \alpha \geq \sum_{ij \in MN} \delta_{ij} p_{ij}(r, s, \bar{k}).$$

Adding $B_r(k - \bar{k})$ to both sides of $ine(r, s, \bar{k})$ we obtain

$$\gamma_{rs} + B_r k + \alpha \geq \sum_{ij \in MN} \delta_{ij} p_{ij}(r, s, \bar{k}) + B_r(k - \bar{k}) \geq \sum_{ij \in MN} \delta_{ij} \left(p_{ij}(r, s, \bar{k}) + \frac{B_r(k - \bar{k})}{B_i} \right),$$

where the last inequality follows from *basic*. Since $B_{mr}|(k - \bar{k})$, $B_i | B_r(k - \bar{k})$ $\forall i \in M$, and $B_r(k - \bar{k})/B_i \in Z$. Thus, $\forall ij \in MN$,

$$p_{ij}(r, s, \bar{k}) + \frac{B_r(k - \bar{k})}{B_i} = \left\lfloor \frac{\gamma_{rs} - \gamma_{ij} + B_r \bar{k} + B_i}{B_i} \right\rfloor + \frac{B_r(k - \bar{k})}{B_i} =$$

$$\left\lfloor \frac{\gamma_{rs} - \gamma_{ij} + B_r k + B_i}{B_i} \right\rfloor = p_{ij}(r, s, k).$$

Therefore,

$$\sum_{ij \in MN} \delta_{ij} \left(p_{ij}(r, s, \bar{k}) + \frac{B_r(k - \bar{k})}{B_i} \right) = \sum_{ij \in MN} \delta_{ij} p_{ij}(r, s, k),$$

and $ine(r, s, k)$ is implied by $ine(r, s, \bar{k})$ and *basic*. □

Example 1 Consider the instance given by

$$\begin{aligned} x + y_{11} &\geq 1.6 \\ x + y_{12} &\geq 5.2 \\ x + 5y_{21} &\geq 9.4 \\ x + 15y_{31} &\geq 12.5. \end{aligned}$$

Here $m = 3$, $n_1 = 2$, $n_2 = n_3 = 1$, $B_1 = 1$, $B_2 = 5$, $B_3 = 15$, $\gamma_{11} = 0.6$, $\gamma_{12} = 0.2$, $\gamma_{21} = 4.4$, and $\gamma_{31} = 12.5$. The polar, in this case, is given by

$$\begin{aligned} basic: &\quad 1 \geq \delta_{11} + \delta_{12} + \frac{1}{5}\delta_{21} + \frac{1}{15}\delta_{31} \\ ine(1,1,0): &\quad 0.6 + \alpha \geq \delta_{11} + \delta_{12} \\ ine(1,1,1): &\quad 1.6 + \alpha \geq 2\delta_{11} + 2\delta_{12} \\ ine(1,1,2): &\quad 2.6 + \alpha \geq 3\delta_{11} + 3\delta_{12} \\ ine(1,1,3): &\quad 3.6 + \alpha \geq 4\delta_{11} + 4\delta_{12} \\ ine(1,1,4): &\quad 4.6 + \alpha \geq 5\delta_{11} + 5\delta_{12} + \delta_{21} \end{aligned}$$

$$
\begin{aligned}
ine(1,1,5): &\quad 5.6+\alpha \geq 6\delta_{11}+6\delta_{12}+\delta_{21} \\
ine(1,1,6): &\quad 6.6+\alpha \geq 7\delta_{11}+7\delta_{12}+\delta_{21} \\
ine(1,1,7): &\quad 7.6+\alpha \geq 8\delta_{11}+8\delta_{12}+\delta_{21} \\
ine(1,1,8): &\quad 8.6+\alpha \geq 9\delta_{11}+9\delta_{12}+\delta_{21} \\
ine(1,1,9): &\quad 9.6+\alpha \geq 10\delta_{11}+10\delta_{12}+2\delta_{21} \\
ine(1,1,10): &\quad 10.6+\alpha \geq 11\delta_{11}+11\delta_{12}+2\delta_{21} \\
ine(1,1,11): &\quad 11.6+\alpha \geq 12\delta_{11}+12\delta_{12}+2\delta_{21} \\
ine(1,1,12): &\quad 12.6+\alpha \geq 13\delta_{11}+13\delta_{12}+2\delta_{21}+\delta_{31} \\
ine(1,1,13): &\quad 13.6+\alpha \geq 14\delta_{11}+14\delta_{12}+2\delta_{21}+\delta_{31} \\
ine(1,1,14): &\quad 14.6+\alpha \geq 15\delta_{11}+15\delta_{12}+3\delta_{21}+\delta_{31}
\end{aligned}
$$

$$
\begin{aligned}
ine(1,2,0): &\quad 0.2+\alpha \geq \delta_{12} \\
ine(1,2,1): &\quad 1.2+\alpha \geq \delta_{11}+2\delta_{12} \\
ine(1,2,2): &\quad 2.2+\alpha \geq 2\delta_{11}+3\delta_{12} \\
ine(1,2,3): &\quad 3.2+\alpha \geq 3\delta_{11}+4\delta_{12} \\
ine(1,2,4): &\quad 4.2+\alpha \geq 4\delta_{11}+5\delta_{12} \\
ine(1,2,5): &\quad 5.2+\alpha \geq 5\delta_{11}+6\delta_{12}+\delta_{21} \\
ine(1,2,6): &\quad 6.2+\alpha \geq 6\delta_{11}+7\delta_{12}+\delta_{21} \\
ine(1,2,7): &\quad 7.2+\alpha \geq 7\delta_{11}+8\delta_{12}+\delta_{21} \\
ine(1,2,8): &\quad 8.2+\alpha \geq 8\delta_{11}+9\delta_{12}+\delta_{21} \\
ine(1,2,9): &\quad 9.2+\alpha \geq 9\delta_{11}+10\delta_{12}+\delta_{21} \\
ine(1,2,10): &\quad 10.2+\alpha \geq 10\delta_{11}+11\delta_{12}+2\delta_{21} \\
ine(1,2,11): &\quad 11.2+\alpha \geq 11\delta_{11}+12\delta_{12}+2\delta_{21} \\
ine(1,2,12): &\quad 12.2+\alpha \geq 12\delta_{11}+13\delta_{12}+2\delta_{21} \\
ine(1,2,13): &\quad 13.2+\alpha \geq 13\delta_{11}+14\delta_{12}+2\delta_{21}+\delta_{31} \\
ine(1,2,14): &\quad 14.2+\alpha \geq 14\delta_{11}+15\delta_{12}+2\delta_{21}+\delta_{31}
\end{aligned}
$$

$$
\begin{aligned}
ine(2,1,0): &\quad 4.4+\alpha \geq 4\delta_{11}+5\delta_{12}+\delta_{21} \\
ine(2,1,1): &\quad 9.4+\alpha \geq 9\delta_{11}+10\delta_{12}+2\delta_{21} \\
ine(2,1,2): &\quad 14.4+\alpha \geq 14\delta_{11}+15\delta_{12}+3\delta_{21}+\delta_{31} \\
ine(2,1,3): &\quad 19.4+\alpha \geq 19\delta_{11}+20\delta_{12}+4\delta_{21}+\delta_{31} \\
ine(2,1,4): &\quad 24.4+\alpha \geq 24\delta_{11}+25\delta_{12}+5\delta_{21}+\delta_{31}
\end{aligned}
$$

$$
ine(3,1,0): \quad 12.5+\alpha \geq 12\delta_{11}+13\delta_{12}+2\delta_{21}+\delta_{31}.
$$

□

An immediate consequence of Proposition 1 is that when B_{m1} is bounded by a polynomial of

n, $I(\Omega)$ grows as a polynomial of n. In particular, if B_{m1} is bounded by a constant, $I(\Omega) = O(n)$. Note that when $m = 1$, Ω is given by the $n+1$ inequalities *basic* and $ine(1, 1, 0)$, ..., $ine(1, n, 0)$. From this result, the inequality description of $\text{conv}(X^M_{MIR})$ given in [8] follows.

We now give inequality (10), which, when added to description of Ω, gives a polytope. Inequality (10) is not valid for Ω. However, the only points of Ω it cuts off correspond to inequalities (3) that are not satisfied at equality by any point of S.

Proposition 5 *Let $(\alpha, \delta) \in \Omega$. Suppose that the corresponding inequality (3) defines a nonempty face of P. Then,*

$$\alpha \leq \sum_{ij \in MN} \frac{B_i - \gamma_{ij}}{B_i} \delta_{ij}. \tag{10}$$

Proof Let $(\bar{x}, \bar{y})$ be a point of the face of P defined by (3), i.e.

$$\bar{x} + \alpha = \sum_{ij \in MN} \delta_{ij}(\tau_{ij} - \bar{y}_{ij}).$$

Suppose that

$$\alpha > \sum_{ij \in MN} \frac{B_i - \gamma_{ij}}{B_i} \delta_{ij}.$$

Then,

$$\sum_{ij \in MN} \delta_{ij}(\tau_{ij} - \bar{y}_{ij}) > \bar{x} + \sum_{ij \in <N} \frac{B_i - \gamma_{ij}}{B_i} \delta_{ij}. \tag{11}$$

Since $(\bar{x}, \bar{y}) \in S$, $\bar{x} + (B_i - \gamma_{ij}) \geq B_i(\tau_{ij} - \bar{y}_{ij})$, i.e.

$$\frac{\bar{x}}{B_i} \geq (\tau_{ij} - \bar{y}_{ij}) - \frac{b_i - \gamma_{ij}}{B_i}. \tag{12}$$

But *basic* and (12) imply that

$$\sum_{ij \in MN} \delta_{ij} \left(\tau_{ij} - \bar{y}_{ij} - \frac{B_i - \gamma_{ij}}{B_i} \right) \leq \bar{x},$$

which contradicts (11). Thus, (10) holds. □

3 Compact Inequality Description of Ω

When B_{m1} grows exponentially with n, so does $I(\Omega)$ for the finite inequality description of Section 2, even for $m = 2$. In this section we show how to reduce $I(\Omega)$ substantially when B_{m1} is large. In the worst case, $I(\Omega)$ still grows exponentially with n. However, now, $I(\Omega)$ grows as a polynomial of n when m is fixed, regardless of the values of the B_i's. We first present Lemma 1, which gives a sufficient condition for some of the inequalities in the description of Ω to be redundant. Then we present Lemma 2, which, after the redundant inequalities have been eliminated, shows how to apply Lemma 1 again to possibly eliminate even more inequalities. Combining Lemmas 1 and 2, we obtain Algorithm 1, which gives, for $rs \in MN$, the values of all k's for $ine(r,s,k)$ in the compact inequality description. Finally, we summarise the main results of the section in Theorem 2.

Let $rs \in MN$. From Theorem 1, it follows that the inequalities $ine(r,s,k)$ with $k \geq B_{mr}$ are redundant in the description of Ω. We now present Lemma 1, where we establish the following result. Let $u \in \{r,\ldots,m-1\}$, $a, b \in Z_+$ with $b \geq a + B_{ur}$, and suppose that the inequalities $ine(r,s,k)$ with $k \in \{a, a+1, \ldots, b\}$ are included in the description of Ω. If u satisfies a certain condition (condition (13) in the lemma), the inequalities $ine(r,s,k)$ with $k \geq a+B_{ur}$ are redundant in the description of Ω. This means that, in this case, we can eliminate some of the inequalities from the current description of Ω.

Lemma 1 *Let $rs \in MN$, $u \in M$ with $r \leq u \leq m-1$, and $a, b \in Z_+$ with $a < b$. Suppose that for each $ij \in MN$ with $i \in \{u+1,\ldots,m\}$,*

$$p_{ij}(r,s,k') = p_{ij}(r,s,k'') \tag{13}$$

$\forall k', k'' \in \{a,\ldots,b\}$. Then, $\forall k \in \{a,\ldots,b\}$, $ine(r,s,k)$ is implied by the inequalities $ine(r,s,a)$, $\ldots$, $ine(r,s,min\{a+B_{ur}-1,b\})$ and basic. □

Before proving Lemma 1, we illustrate it with an example.

Example 1 (Continued) Let $u = 2$. Consider $(r,s) = (1,1)$, $a = 0$, and $b = 11$. Note that $p_{31}(1,1,0) = \cdots = p_{31}(1,1,11)$. Therefore, $ine(1,1,5)$, $\ldots$, $ine(1,1,11)$ are redundant. Consider next $(r,s) = (1,2)$, $a = 0$, and $b = 12$. Note that $p_{31}(1,2,0) = \cdots = p_{31}(1,2,12)$. Therefore, $ine(1,2,5)$, $\ldots$, $ine(1,2,12)$ are redundant. Consider $(r,s) = (2,1)$, $a = 0$, and $b = 1$. Note that $p_{31}(2,1,0) = p_{31}(2,1,1)$. Therefore, $ine(2,1,1)$ is redundant. Finally, consider $(r,s) = (2,1)$, $a = 2$, and $b = 4$. Note that $p_{31}(2,1,2) = \cdots = p_{31}(2,1,4)$. Therefore, $ine(2,1,3)$ and $ine(2,1,3)$ are redundant. This way, the inequality description of Ω reduces to

$$\begin{aligned}
basic: &\quad 1 \geq \delta_{11} + \delta_{12} + \frac{1}{5}\delta_{21} + \frac{1}{15}\delta_{31} \\
ine(1,1,0): &\quad 0.6 + \alpha \geq \delta_{11} + \delta_{12} \\
ine(1,1,1): &\quad 1.6 + \alpha \geq 2\delta_{11} + 2\delta_{12} \\
ine(1,1,2): &\quad 2.6 + \alpha \geq 3\delta_{11} + 3\delta_{12} \\
ine(1,1,3): &\quad 3.6 + \alpha \geq 4\delta_{11} + 4\delta_{12} \\
ine(1,1,4): &\quad 4.6 + \alpha \geq 5\delta_{11} + 5\delta_{12} + \delta_{21} \\
ine(1,1,12): &\quad 12.6 + \alpha \geq 13\delta_{11} + 13\delta_{12} + 2\delta_{21} + \delta_{31} \\
ine(1,1,13): &\quad 13.6 + \alpha \geq 14\delta_{11} + 14\delta_{12} + 2\delta_{21} + \delta_{31} \\
ine(1,1,14): &\quad 14.6 + \alpha \geq 15\delta_{11} + 15\delta_{12} + 3\delta_{21} + \delta_{31} \\
\\
ine(1,2,0): &\quad 0.2 + \alpha \geq \delta_{12} \\
ine(1,2,1): &\quad 1.2 + \alpha \geq \delta_{11} + 2\delta_{12}
\end{aligned}$$

$$\begin{aligned}
ine(1,2,2): &\quad 2.2+\alpha \geq 2\delta_{11}+3\delta_{12} \\
ine(1,2,3): &\quad 3.2+\alpha \geq 3\delta_{11}+4\delta_{12} \\
ine(1,2,4): &\quad 4.2+\alpha \geq 4\delta_{11}+5\delta_{12} \\
ine(1,2,13): &\quad 13.2+\alpha \geq 13\delta_{11}+14\delta_{12}+2\delta_{21}+\delta_{31} \\
ine(1,2,14): &\quad 14.2+\alpha \geq 14\delta_{11}+15\delta_{12}+2\delta_{21}+\delta_{31} \\
\\
ine(2,1,0): &\quad 4.4+\alpha \geq 4\delta_{11}+5\delta_{12}+\delta_{21} \\
ine(2,1,2): &\quad 14.4+\alpha \geq 14\delta_{11}+15\delta_{12}+3\delta_{21}+\delta_{31} \\
\\
ine(3,1,0): &\quad 12.5+\alpha \geq 12\delta_{11}+13\delta_{12}+2\delta_{21}+\delta_{31}.
\end{aligned}$$

□

Proof of Lemma 1 It is sufficient to consider the case $b \geq a + B_{ur}$ and to prove the lemma for $k \in \{a + B_{ur}, \ldots, b\}$. Let $\bar{k} \in \{a, \ldots, a + B_{ur} - 1\}$ be such that $\bar{k} \equiv k(mod\ B_{ur})$. As in the proof of Proposition 1, $ine(r, s, \bar{k})$ and *basic* imply that

$$\gamma_{rs} + B_r k + \alpha \geq \sum_{ij \in MN} \delta_{ij} \left(p_{ij}(r, s, \bar{k}) + \frac{B_r(k - \bar{k})}{B_i} \right).$$

Because $B_{ri} \in Z$ when $i \leq r$, $B_r(k - \bar{k})/B_i \in Z\ \forall i \leq r$. In addition, because $\bar{k} \equiv k(mod\ B_{ur})$, $B_r(k - \bar{k})/B_i \in Z\ \forall i \in \{r + 1, \ldots, u\}$. In both cases, $p_{ij}(r, s, \bar{k}) + B_r(k - \bar{k})/B_i = p_{ij}(r, s, k)$. Finally, because of (13), $p_{ij}(r, s, \bar{k}) = p_{ij}(r, s, k)\ \forall i \in \{u + 1, \ldots, m\}$. Thus, $ine(r, s, k)$ follows. □

Let rs, u, a, and b be as in Lemma 1, and $\bar{b} = min\{a + B_{ur} - 1, b\}$. If $b > \bar{b}$, then, through Lemma 1, the inequalities $ine(r, s, k)$ with $\bar{b} + 1 \leq k \leq b$ can be eliminated from the description of Ω. We now present Lemma 2, where we establish the following result. It is possible to partition $\{a, \ldots, \bar{b}\}$ into at most $n_u + 1$ subsets in such a way that (13) holds $\forall k', k''$ in the same subset and $ij \in \{u\} \times N_u$. This means that if $\{a_t, a_t + 1, \ldots, b_t\}$ is one of the subsets and $b_t \geq a_t + B_{u-1,r}$, we can apply Lemma 1 again to eliminate the inequalities $ine(r, s, k)$ with $a_t + B_{u-1,r} \leq k \leq b_t$ from the description of Ω.

Lemma 2 *Let rs, u, a, and b be as in Lemma 1. Suppose that the hypotheses of Lemma 1 hold. Then, it is possible to partition the set $\{a, \ldots, \bar{b}\}$ into the subsets $\{a_0, \ldots, b_0\}$, …, $\{a_l, \ldots, b_l\}$, with $a_0 = a$, $b_l = \bar{b}$, $a_i \leq b_i = a_{i+1} - 1 \leq \bar{b}\ \forall i \in \{0, \ldots, l - 1\}$, and $0 \leq l \leq n_u$, in such a way that the following property is satisfied. For each $t \in \{0, \ldots, l\}$, condition (13) holds $\forall k', k'' \in \{a_t, \ldots, b_t\}$ and $ij \in \{u\} \times N_u$.* □

Proof For each $j \in N_u$, let

$$k_{uj} = \left\lceil \frac{\gamma_{uj} - \gamma_{rs}}{B_r} \right\rceil$$

and $c_{uj} \in \{a, \ldots, a + B_{ur} - 1\}$ be such that $c_{uj} \equiv k_{uj}\ (mod\ B_{ur})$. (Note that $k_{uj} \in \{0, \ldots, B_{ur}\}$.)

Because the hypotheses of Lemma 1 hold, for each $vw \in MN$ with $v \in \{u+1, \ldots, m\}$, $p_{vw}(r,s,k') = p_{vw}(r,s,k'')\ \forall k', k'' \in \{a, \ldots, \bar{b}\}$. So it suffices to show that for each $j \in N_u$, $p_{uj}(r,s,k') = p_{uj}(r,s,a)\ \forall k' \in \{a, \ldots, c_{uj}-1\}$ and $p_{uj}(r,s,k') = p_{uj}(r,s,c_{uj})\ \forall k' \in \{c_{uj}, \ldots, a+B_{ur}-1\}$. (If $c_{uj} = a$ for some $j \in N_u$, the set $\{a, \ldots, c_{uj}-1\}$ is empty, and the assertion holds trivially for it.)

Let then $k' \in \{a, \ldots, a+B_{ur}-1\}$. We have that

$$p_{uj}(r,s,k') = \left\lfloor \frac{\gamma_{rs} - \gamma_{uj} + B_r k' + B_u}{B_u} \right\rfloor = \left\lfloor \frac{B_r(k' - k_{uj} + f) + B_u}{B_u} \right\rfloor,$$

where $f \in [0,1)$. Since $c_{uj} = k_{uj} + B_{ur}q$ for some integer q, it follows that

$$p_{uj}(r,s,k') = \left\lfloor \frac{B_r(k' + B_{ur}q - c_{uj} + f) + B_u}{B_u} \right\rfloor = \left\lfloor \frac{B_r(k' - c_{uj} + f)}{B_u} \right\rfloor + q + 1.$$

However, since $|k' - c_{uj}| < B_{ur}$,

$$\left| \frac{B_r(k' - c_{uj} + f)}{B_u} \right| < 1,$$

and therefore $p_{uj}(r,s,k') = q\ \forall k' \in \{a, \ldots, c_{uj}-1\}$, and $p_{uj}(r,s,k') = q+1\ \forall k' \in \{c_{uj}, \ldots, a+B_{ur}-1\}$.

We now consider the l integers $a_1, \ldots, a_l$, where $a_i \in \{c_{u1}, \ldots, c_{un_u}\} \cap \{a+1, \ldots, \bar{b}\}\ \forall i \in \{1, \ldots, l\}$ and $a_1 < \cdots < a_l$. If $l = 0$, the lemma holds for the entire set $\{a, \ldots, \bar{b}\}$. If $l \geq 1$, the lemma holds for the sets $\{a, \ldots, a_1 - 1\}$, $\{a_1, \ldots, a_2 - 1\}$, ..., and $\{a_l, \ldots, \bar{b}\}$, which partition $\{a, \ldots, \bar{b}\}$. In any case, it is clear that $l \leq n_u$. □

Example 1 (Continued) We have that $\{u\} \times N_u = \{(2,1)\}$. Consider first $(r,s) = (1,1)$, $a = 0$, and $\bar{b} = 4$. Then, $k_{21} = \lceil (4.4 - 0.6)/1 \rceil = 4$, $c_{21} = 4$, and $l = 1$. So we partition $\{0, \ldots, 4\}$ into $\{0,1,2,3\}$ and $\{4\}$. Note that $p_{21}(1,1,0) = \cdots = p_{21}(1,1,3)$ and $p_{31}(1,1,0) = \cdots = p_{31}(1,1,3)$. Therefore, we can use Lemma 1 to show that $ine(1,1,1)$, $ine(1,1,2)$, and $ine(1,1,3)$ are redundant.

We now consider $(r,s) = (1,1)$, $a = 12$, and $\bar{b} = 14$. Again, $k_{21} = \lceil (4.4 - 0.6)/1 \rceil = 4$. But this time, $c_{21} = 14 \equiv 4\ (mod\ 5)$, and once more $l = 1$. So we partition $\{12, 13, 14\}$ into $\{12, 13\}$ and $\{14\}$. Note that $p_{21}(1,1,12) = p_{21}(1,1,13)$ and $p_{31}(1,1,12) = p_{31}(1,1,13)$. Therefore, we can use Lemma 1 to show that $ine(1,1,13)$ is redundant.

Next, we consider $(r,s) = (1,2)$, $a = 0$, and $\bar{b} = 4$. Then, $k_{21} = \lceil (4.4 - 0.2)/1 \rceil = 5$, $c_{21} = 0$, and $l = 0$, i.e. we do not partition $\{0, \ldots, 4\}$. Note that $p_{21}(1,2,0) = \cdots = p_{21}(1,2,4)$ and $p_{31}(1,2,0) = \cdots = p_{31}(1,2,4)$. Therefore, we can use Lemma 1 to show that $ine(1,2,1)$, ..., $ine(1,2,4)$ are redundant.

Finally, we consider $(r,s) = (1,2)$, $a = 13$, and $\bar{b} = 14$. Again, $k_{21} = \lceil (4.4 - 0.2)/1 \rceil = 5$. But this time, $c_{21} = 15 \equiv 5\ (mod\ 5)$, and once more $l = 0$. So we do not partition $\{13, 14\}$. Note that $p_{21}(1,2,13) = p_{21}(1,1,14)$ and $p_{31}(1,2,13) = p_{12}(1,1,14)$. Therefore, we can use Lemma 1 to show that $ine(1,2,13)$, is redundant. The inequality description of Ω reduces then to

$$\begin{aligned} basic: &\quad 1 \geq \delta_{11} + \delta_{12} + \frac{1}{5}\delta_{21} + \frac{1}{15}\delta_{31} \\ ine(1,1,0): &\quad 0.6 + \alpha \geq \delta_{11} + \delta_{12} \end{aligned}$$

$$
\begin{aligned}
ine(1,1,4): &\quad 4.6+\alpha \geq 5\delta_{11}+5\delta_{12}+\delta_{21}\\
ine(1,1,12): &\quad 12.6+\alpha \geq 13\delta_{11}+13\delta_{12}+2\delta_{21}+\delta_{31}\\
ine(1,1,14): &\quad 14.6+\alpha \geq 15\delta_{11}+15\delta_{12}+3\delta_{21}+\delta_{31}\\
\\
ine(1,2,0): &\quad 0.2+\alpha \geq \delta_{12}\\
ine(1,2,13): &\quad 13.2+\alpha \geq 13\delta_{11}+14\delta_{12}+2\delta_{21}+\delta_{31}\\
\\
ine(2,1,0): &\quad 4.4+\alpha \geq 4\delta_{11}+5\delta_{12}+\delta_{21}\\
ine(2,1,2): &\quad 14.4+\alpha \geq 14\delta_{11}+15\delta_{12}+3\delta_{21}+\delta_{31}\\
\\
ine(3,1,0): &\quad 12.5+\alpha \geq 12\delta_{11}+13\delta_{12}+2\delta_{21}+\delta_{31}.
\end{aligned}
$$

The polyhedron P, in this case, has 44 facets, which are the 44 vertices of Ω. □

By combining Lemmas 1 and 2, it is possible to obtain the compact description of Ω. We give the details in Algorithm 1. Given $rs \in MN$, The algorithm gives the set of all values k for the inequalities $ine(r,s,k)$ of the compact description. The algorithm mantains a collection L of sets of values of k. At each iteration, we reduce the sizes of the sets in L according to Lemma 1. Then, each set in L is divided into l subsets according to Lemma 2. If $l \geq 1$, we replace the set with the subsets in L. We iterate the algorithm from $u = m$ through $u = r$. At the end, L is a set of singletons, and the set of all values of k is $\bigcup L$.

Algorithm 1

$L \leftarrow \{Z_+\}$ and $u \leftarrow m$

While $r \leq u$

For each set $I = \{a, \ldots, b\} \in L$, $I \leftarrow \{a, \ldots, \min\{b, a + B_{ur} - 1\}\}$

If $r < u$

1. For each $j \in N_u$, $k_{uj} \leftarrow \left\lceil \frac{\gamma_{uj} - \gamma_{rs}}{B_r} \right\rceil$
2. For each $j \in N_u$, let $c_{uj} \in \{a, \ldots, a + B_{ur} - 1\}$ be such that $c_{uj} \equiv k_{uj}\ (mod\ B_{ur})$
3. Collect the l distinct integers $a_1, \ldots, a_l \in \{c_{uj} > a : j \in N_u\}$ such that $a_1 < \cdots < a_l$
4. If $l > 0$
   (a) Form the sets $I_0 = \{a, \ldots, a_1 - 1\}$, $I_1 = \{a_1, \ldots, a_2 - 1\}$ …, $I_l = \{a_l, \ldots, \bar{b}\}$
   (b) $L \leftarrow L \cup \{I_0, \ldots I_l\}$
   (c) $L \leftarrow L - \{I\}$
5. end if
6. $u \leftarrow u - 1$

End if, end while

Return $\bigcup L$. □

Example 1 (Continued) Consider $(r,s)=(1,1)$. Initially, $L=\{Z_+\}$ and $u=3$. At the begining of the first iteration, L becomes $\{\{0,\ldots,14\}\}$. Now, $r=1<3=u$ and $N_3=\{1\}$. Then, $k_{31}=\lceil(12.5-0.6)/1\rceil=12$. So, $c_{31}=12$, $l=1$, and L becomes $\{\{0,\ldots,11\},\{12,13,14\}\}$. At the end, we obtain the compact description given after Lemma 2 by proceeding as in the parts of the example that follow Lemmas 1 and 2. □

Note that the number of subsets generated by Algorithm 1, and consequently the number of inequalities $ine(r,s,k)$ in the compact description of Ω, is $O(\prod_{i=r+1}^{m}(n_i+1))$. Therefore, the compact description has $O(n(n/m)^{m-1})$ inequalities. The main results of this section can be summarized in Theorem 2.

Theorem 2 *For each $rs\in MN$, Algorithm 1 gives all values of k for the inequalities $ine(r,s,k)$ of the compact description of Ω. The compact description of Ω contains*

$$I(\Omega)=O\left(\frac{n}{m}\left(\frac{n}{m}+2\right)^{m-1}\right)$$

inequalities. □

Note that, in the worst case, $I(\Omega)$ grows exponentially with n. However, for m fixed, $I(\Omega)$ grows as a polynomial of n, regardless of the values of the B_i's.

4 Nondivisible Coefficients and Further Research

In this section we indicate how our study may be extended to the case of nondivisible B_i's and present directions for further research. We first note that Propositions 3 and 4 ramain valid when the B_i's are not divisible. Also, it is easy to extend Theorem 1 to the case of nondivisible B_i's, which we do in Theorem 3. Finally, as a direct consequence of Theorem 3, we obtain Corollary 1. There, we denote $B_{21}=C_2/C_1$, where $C_1,C_2>0$ are relative prime integers. Corollary 1 gives a full inequality description of Ω for the case $m=2$ for which $I(\Omega)=O(C_2)$.

We now present Theorem 3. There, we denote $B_i=p_i/q_i$, where $p_i,q_i>0$ are integers $\forall i\in M$. We also denote, for each $r\in M$, $d_r=lcm(p_1,\ldots,p_{r-1},p_{r+1},\ldots,p_m,q_1,\ldots,q_m)$, where *lcm* stands for *least common multiple*.

Theorem 3 *Let $rs\in MN$ and $k\in Z_+$. Inequality $ine(r,s,k)$ is implied by the set of inequalities $\{ine(r,s,0),\ldots,ine(r,s,d_r-1),basic\}$.*

Proof Let $k\in Z_+$ and $\bar{k}\in\{0,\ldots,d_r-1\}$ be such that $\bar{k}\equiv k(mod\ d_r)$. Then, for all $j\in M$, $B_r(k-\bar{k})/B_j\in Z$. Following the proof of Theorem 1, we conclude that that $ine(r,s,k)$ is implied by $ine(r,s,\bar{k})$ and *basic*. □

From Theorem 3, it follows immediately that

Corollary 1 *Suppose that $m = 2$. Let $B_{21} = C_2/C_1$, where C_1 and C_2 are relative primes. Then, Ω can be described by basic and the $n_1C_2 + n_2C_1$ inequalities $ine(i,j,k)$ with $ij \in MN$, $k \in \{0, \ldots, C_2 - 1\}$ for $i = 1$, and $k \in \{0, \ldots, C_1 - 1\}$ for $i = 2$.* □

Example 2 Consider the instance given by

$$\begin{aligned}
&x + y_{11} \geq 1.6\\
&x + y_{12} \geq 2.8\\
&x + \frac{5}{3}y_{21} \geq \frac{14}{3}\\
&x + \frac{5}{3}y_{22} \geq 2.
\end{aligned}$$

Here, $m = 2$, $n_1 = n_2 = 2$, $C_1 = 3$, $C_2 = 5$, $\gamma_{11} = 0.6$, $\gamma_{12} = 0.8$, $\gamma_{21} = 4/3$, and $\gamma_{22} = 1/3$. The polar, in this case, is given by

$$\begin{aligned}
basic: &\quad 1 \geq \delta_{11} + \delta_{12} + \frac{3}{5}\delta_{21} + \frac{3}{5}\delta_{22}\\
ine(1,1,0): &\quad 0.6 + \alpha \geq \delta_{11} + \delta_{22}\\
ine(1,1,1): &\quad 1.6 + \alpha \geq 2\delta_{11} + \delta_{12} + \delta_{21} + \delta_{22}\\
ine(1,1,2): &\quad 2.6 + \alpha \geq 3\delta_{11} + 2\delta_{12} + \delta_{21} + 2\delta_{22}\\
ine(1,1,3): &\quad 3.6 + \alpha \geq 4\delta_{11} + 3\delta_{12} + 2\delta_{21} + 2\delta_{22}\\
ine(1,1,4): &\quad 4.6 + \alpha \geq 5\delta_{11} + 4\delta_{12} + 2\delta_{21} + 3\delta_{22}\\
ine(1,2,0): &\quad 0.8 + \alpha \geq \delta_{11} + \delta_{12} + \delta_{22}\\
ine(1,2,1): &\quad 1.8 + \alpha \geq 2\delta_{11} + 2\delta_{12} + \delta_{21} + \delta_{22}\\
ine(1,2,2): &\quad 2.8 + \alpha \geq 3\delta_{11} + 3\delta_{12} + \delta_{21} + 2\delta_{22}\\
ine(1,2,3): &\quad 3.8 + \alpha \geq 4\delta_{11} + 4\delta_{12} + 2\delta_{21} + 3\delta_{22}\\
ine(1,2,4): &\quad 4.8 + \alpha \geq 5\delta_{11} + 5\delta_{12} + 3\delta_{21} + 3\delta_{22}\\
ine(2,1,0): &\quad \frac{4}{3} + \alpha \geq \delta_{11} + \delta_{12} + \delta_{21} + \delta_{22}\\
ine(2,1,1): &\quad 3 + \alpha \geq 3\delta_{11} + 3\delta_{12} + 2\delta_{21} + 2\delta_{22}\\
ine(2,1,2): &\quad \frac{14}{3} + \alpha \geq 5\delta_{11} + 4\delta_{12} + 3\delta_{21} + 3\delta_{22}\\
ine(2,2,0): &\quad \frac{1}{3} + \alpha \geq \delta_{22}\\
ine(2,2,1): &\quad 2 + \alpha \geq 2\delta_{11} + 2\delta_{12} + \delta_{21} + 2\delta_{22}\\
ine(2,2,2): &\quad \frac{11}{3} + \alpha \geq 4\delta_{11} + 3\delta_{12} + 2\delta_{21} + 3\delta_{22}.
\end{aligned}$$

The polyhedron P, in this case, has 44 facets, which are the 44 vertices of Ω. □

When C_2 is bounded by a polynomial of n, the description of Ω given in Corollary 1 grows polynomially with n. Note, however, that this descrition may be improved. For instance, in Example 2, $ine(1,1,4)$ is equal to 4 time *basic* plus $ine(1,1,0)$. So an important question that we leave open for future research is how to obtain a compact inequality description of Ω for the case of nondivisible B_i's. For that matter, another important direction for further research is whether we can reduce the number of inequalities of the compact description given in Theorem 2.

A question that seems to be related with the polyhedral description of P is the complexity of the optimization problem $Opt(c, S)$: $\min\{cx : x \in S\}$. To the best of our knowledge, this question is open. Previously to this study, this question was settled for $m = 1$ in [8] and for $m = 2$ with $B_1|B_2$ in [15] with polynomial time algorithms. Here, as a result of Theorems 1 and 2, Corollary 1, and the equivalence of separation and optimization [7], we proved the polynomiality of $Opt(c, S)$ for the case of divisible B_i's when B_{m1} is bounded by a polynomial of n or m is fixed, and for the case of nondivisible B_i's when $m = 2$ and C_2 is bounded by a polynomial of n. But, for the most general case, the question remains unanswered. We suggest, however, that because we have a closed form inequality description of Ω, $Opt(c, S)$ can be solved in polynomial time, even for nondivisible B_i's.

The model studied here can be generalized in several ways, and it would be interesting to investigate the applicability of our polar approach to these models (see [14] for possible model extensions.) One particular set of interest is $SC = \{(x, y) \in \Re \times Z^n : xe + Cy \geq b\}$, where e is the n-dimensional column vector with all components equal to 1, and $C_{n\times n}$ is the lower triangular matrix with $C_{ij} = C_j > 0$ for $j \leq i$, $i, j \in \{1, \ldots, n\}$. We have that $C = D\bar{C}$, where $D_{n\times n}$ is the diagonal matrix with $D_{ii} = C_i$, $i \in \{1, \ldots, n\}$, and $\bar{C}_{n\times n}$ is the lower triangular matrix with $C_{ij} = C_j/C_i$ for $j \leq i$, $i, j \in \{1, \ldots, n\}$. If $C_n|\cdots|C_1$, then by making the change of variables $z = \bar{C}y$, SC becomes a special case of S. However, for the more interesting case where $C_1|\cdots|C_n$, it is not obvious how to apply our approach. We are currently investigating this issue.

Finally, motivated by the spetacular computational success of MIR inequalities, we are currently investigating the computational use of our theory to MIP problems in general.

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