A Special Ordered Set Approach to Discontinuous Piecewise Linear Optimization

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October 17, 2005

Abstract

Piecewise linear functions (PLFs) are commonly used to approximate nonlinear functions. They are also of interest in their own, arising for example in problems with economies of scale. Early approaches to piecewise linear optimization (PLO) assumed continuous PLFs. They include the incremental cost MIP model of Markowitz and Manne and the convex combination MIP model of Dantzig. Later, Beale and Tomlin gave an approach alternative to MIP for continuous PLO based on the concept of special ordered set (SOS). It is well established today that the SOS approach is considerably more efficient than MIP for continuous PLO. Recently, Croxton, Gendron, and Magnanti studied three MIP models for discontinuous PLO that are correct when the PLFs are lower semi-continuous, and showed that they give the same LP relaxation bound. Here we present a SOS approach for discontinuous PLO that gives the same LP relaxation bound as their MIP models. In addition to the usual advantages of SOS over MIP for PLO, our SOS approach is more robust than MIP in the sense that it solves PLO even when some of the PLFs are not lower semi-continuous.

Keywords: piecewise linear optimization, discontinuous piecewise linear function, special ordered set, mixed-integer programming

1 Introduction

Any nonlinear function can be approximated to an arbitrary degree of accuracy as a piecewise linear function (PLF). PLF approximation abounds in optimization. It is used in electronic circuit design [3, 13], portfolio selection [16, 19], and optimization of gas network [18]. In addition, piecewise linear optimization (PLO) is of interest in its own. It arises, for example, in problems with economies of scale [1].

PLO can be solved as a linear programming problem (LP) when it is convex [11]. Otherwise it is NP-hard [12], in which case it can be tackled as a mixed-integer programming problem (MIP). To the best of our knowledge, the first MIP models introduced for PLO were the *incremental cost* model of Markowitz and Manne [17] and the *convex combination* model of Dantzig [5]. In both models PLO is assumed to be continuous.

Beale and Tomlin [2] gave an approach alternative to MIP for continuous PLO based on the concept of *special ordered set of type 2* (SOS2). A set of variables $\{\lambda_1, \dots, \lambda_n\}$ is SOS2 when:

at most two variables can be nonzero, and two nonzero variables must be adjacent. (1)

In the SOS2 approach, one keeps in the model only $\lambda_1, \dots, \lambda_n$ and dispenses with the introduction of auxiliary binary variables to enforce (1). Rather, constraint (1) is enforced by branching on the SOS2.

Keha et al. [14] showed that the LP relaxation bound of the incremental cost, convex combination, and SOS2 models is the same. However, the SOS2 approach has much advantage over MIP. The SOS2 approach:

- branches on sets of variables rather than on individual variables
- has a considerably smaller number of variables and constraints and no big-M constants
- introduces less degeneracy in the LP relaxation
- is more stable numerically,

see [7, 8, 10]. As a result, it is well established that using SOS2 is more efficient computationally than MIP for continuous PLO, see for example Keha et al. [15].

Given a function $g : [0, u] \to \Re$, we say that g is *lower semi-continuous* if:

$$g(x) \leq \liminf_{x' \to x} g(x') \ \forall x \in [0, u] \,,$$

where $\{x'\}$ is a sequence of points in the interval $[0, u]$. Recently, Croxton et al. [4] studied three MIP models for discontinuous PLO that are correct when the PLFs are lower semi-continuous, and showed that they give the same LP relaxation bound. Here we introduce a SOS approach for discontinuous PLO, which we call *SOS of type D (SOSD)*. We show that when the PLFs are lower semi-continuous, the LP relaxation bound of SOSD is the same as the one given by the LP relaxation of the MIP models in [4]. However, in addition to the usual advantages of SOS over MIP, the SOSD approach is robust in the sense that it solves PLO even when some of the PLFs are not lower semi-continuous.

2 SOSD for Discontinuous PLO

Let $g : [0, u] \to \Re$ be a PLF with breakpoints $a_0 = 0$, a_1, …, $a_T = u$, slopes c_1, …, c_T, and fixed-charges f_1, …, f_T for each line segment, see Figure 1. Croxton et al. [4] gave the following MIP formulation (CGM) for g:

$$g(x) = \sum_{i=1}^{T} \left[\lambda_i \left(c_i a_{i-1} + f_i\right) + \mu_i \left(c_i a_i + f_i\right)\right] \tag{2}$$
$$x = \sum_{i=1}^{T} \left(\lambda_i a_{i-1} + \mu_i a_i\right) \tag{3}$$
$$\lambda_i + \mu_i = y_i, \qquad i = 1, \ldots, T \tag{4}$$
$$\sum_{i=1}^{T} y_i = 1 \tag{5}$$
$$\lambda_i, \mu_i \geq 0, \qquad i = 1, \ldots, T \tag{6}$$
$$y_i \in \{0, 1\}, \qquad i = 1, \ldots, T, \tag{7}$$

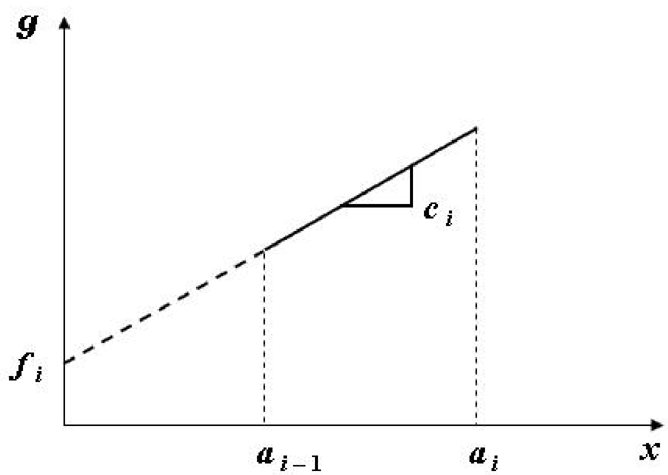

Figure 1: PLF line segment

and showed that it is possible to solve the PLO:

$$\begin{aligned} \textit{minimize} \quad & \sum_{i=1}^{n} g_i(x_i) \\ s.t. \quad & \\ & Ax = b \\ & x_i \in [0, u_i], \quad i = 1, \ldots, n \end{aligned}$$

as a MIP by formulating the PLFs $g_1(x_1)$, ..., $g_n(x_n)$ as (CGM), as long as they are lower semi-continuous. As we show in Example 1, however, (CGM) may fail to give a correct MIP model for PLO when some of the PLFs are not lower semi-continuous.

Example 1 Consider the PLO:

$$\begin{aligned} \textit{minimize} \quad & g_1(x_1) + g_2(x_2) \\ s.t. \quad & \\ & x_1 + x_2 = 2 \\ & x_1, x_2 \geq 0, \end{aligned}$$

where g_1 and g_2 are given in Figure 2. The (CGM) MIP for the PLO is:

$$\begin{aligned} \textit{minimize} \quad & 2\lambda_1^1 + \mu_1^1 + 6\lambda_2^1 + 6\mu_2^1 + 2\lambda_1^2 + \mu_1^2 + 5\lambda_2^2 + 4\mu_2^2 + 4\lambda_3^2 + 4\mu_3^2 \\ s.t. \quad & \\ & \mu_1^1 + \lambda_2^1 + 2\mu_2^1 + \mu_1^2 + \lambda_2^2 + \tfrac{3}{2}\mu_2^2 + \tfrac{3}{2}\lambda_3^2 + 2\mu_3^2 = 2 \\ & \lambda_i^j + \mu_i^j = y_i^j \qquad \forall i, j \\ & y_1^1 + y_2^1 = 1 \\ & y_1^2 + y_2^2 + y_3^2 = 1 \\ & \lambda_i^j, \mu_i^j \geq 0 \qquad \forall i, j \\ & y_i^j \in \{0, 1\} \qquad \forall i, j, \end{aligned}$$

where λ_i^1 and μ_i^1 correspond to x_1 $\forall i \in \{1, 2\}$, and λ_i^2 and μ_i^2 correspond to x_2 $\forall i \in \{1, 2, 3\}$.

The optimal solution to the LP relaxation of the MIP is $\mu_1^1 = y_1^1 = \mu_1^2 = y_1^2 = 1$ and $\lambda_i^j = \mu_r^s = y_2^1 = y_2^2 = y_3^2 = 0$ otherwise, with objective function value 2. It satisfies integrality, so it is an optimal solution to the MIP. However, it gives the incorrect values $g_1(1) = g_2(1) = 1$ to the PLFs. □

A set of variables $\{\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n\}$ is SOSD if at most two variables can be positive, and in case two variables are positive they must be a pair λ_i, μ_i for some $i = 1, \ldots, T$. We now give a SOSD formulation and approach for PLO that is correct even when some of the PLFs are not lower semi-continuous. Let $L = \{i \in \{1, \ldots, T\} : g(a_{i-1}) \neq c_i a_{i-1} + f_i\}$ and $M = \{i \in \{1, \ldots, T\} : g(a_i) \neq c_i a_i + f_i\}$. The SOSD formulation for $g(x)$ consists of (2), (3), (6), and:

$$\sum_{i=1}^{T} (\lambda_i + \mu_i) = 1 \tag{8}$$

$$\lambda_i \neq 1 \qquad \forall i \in L \tag{9}$$

$$\mu_i \neq 1 \qquad \forall i \in M \tag{10}$$

$$\{\lambda_1, \mu_1, \ldots, \lambda_T, \mu_T\} \text{ is SOSD.} \tag{11}$$

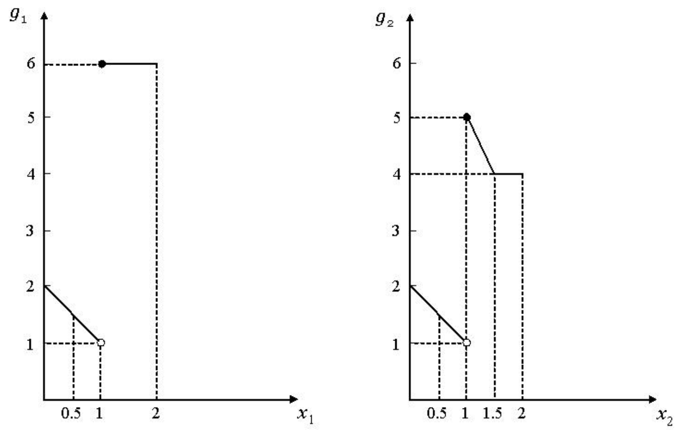

Figure 2: Upper semi-continuous PLFs

The LP relaxation of the SOSD model consists of (2), (3), (6), and (8). We now show that when the PLFs are lower semi-continuous, the LP relaxation bound given by the SOSD model is equal to the one given by the LP relaxation of the MIP.

Proposition 1 *Suppose that the PLFs of the PLO are lower semi-continuous. Then, the optimal value of the LP relaxation of the SOSD model is equal to the optimal value of the LP relaxation of the (CGM) MIP.*

Proof Constraints (5) and (8) imply that (λ^*, μ^*) is a feasible solution to the LP relaxation of the SOSD model iff (λ^*, μ^*, y^*) is a feasible solution to the LP relaxation of the (CGM) MIP, where $y_i^* = \lambda_i^* + \mu_i^* \ \forall i$. On the other hand, because both models include (2) (i.e. the MIP model does not contain any y_i variable in the objective function), their objective function values are the same. □

In the SOSD approach we enforce constraint (11) by branching on the SOSD. Specifically, suppose that (λ^*, μ^*) does not satisfy (11) and $x^* \in [a_i, a_{i+1})$ for some $i \in \{1, \ldots, T-1\}$. If $\lambda_k^* > 0$ or $\mu_k^* > 0$ for some $k < i$, then we impose $\lambda_i = \mu_i = \cdots = \lambda_T = \mu_T = 0$ in one branch, and $\lambda_1 = \mu_1 = \cdots = \lambda_{i-1} = \mu_{i-1} = 0$ in the other branch; if $\lambda_k^* = \mu_k^* = 0 \ \forall k < i$, then we impose $\lambda_{i+1} = \mu_{i+1} = \cdots = \lambda_T = \mu_T = 0$ in one branch, and $\lambda_1 = \mu_1 = \cdots = \lambda_i = \mu_i = 0$ in the other branch. In any case, (λ^*, μ^*) is cut off in both children nodes. The motivation is to branch around a line segment, as guided by the optimal solution x^* of the LP relaxation. In general, this is significantly more efficient than branching on a single binary variable y_i, see for example [2, 6, 10].

It is easy to see that when the PLFs of the PLO are lower semi-continuous, constraints (9) and (10) are always satisfied by the LP relaxation optimal solution in both (CGM) and SOSD models. However, as shown earlier in Example 1, this may not happen when some of the PLFs are not lower semi-continuous. In this case, the SOSD approach once more branches on the SOSD. In addition, the SOSD approach adds one more criterion for fathoming a node in the branch-and-bound tree.

Suppose that after solving the LP relaxation of the SOSD model, we obtain $\lambda_i^* = 1$ for some $i \in L$. We then impose $\lambda_i = \mu_i = \cdots = \lambda_T = \mu_T = 0$ in one branch and $\lambda_1 = \mu_1 = \cdots = \lambda_{i-1} = \mu_{i-1} = 0$ in the other branch. (The branching rule for $\mu_i^* = 1$, $i \in M$, is similar.) Now, let (S_1, S_2) be a partition of the feasible set of the SOSD model. It is clear that the subproblem over S_1, for example, may not have an optimal solution, even if it is feasible and bounded. In this case, S_1 does not contain any optimal solution to PLO. So, we can fathom the node of the branch-and-bound tree with feasible set S_1. This means that we can add to the branch-and-bound algorithm the following node fathoming criterion FC_{SOSD}. Suppose that, in the domain of a node N in the branch-and-bound tree, for each $j \in \{1, \ldots, n\}$ there exists $i(j) \in \{1, \ldots, T_j\}$ such that $\lambda_{i(j)}^j$ and $\mu_{i(j)}^j$ are the only free variables corresponding to x_j, i.e. $\lambda_s^j = \mu_s^j = 0 \ \forall s \neq i(j)$. Suppose also that for some $k \in \{1, \ldots, n\}$, $c_{i(k)}^k a_{i(k)-1}^k + f_{i(k)}^k < c_{i(k)}^k a_{i(k)}^k + f_{i(k)}^k$, but $g_k(a_{i(k)-1}^k) \neq c_{i(k)}^k a_{i(k)-1}^k + f_{i(k)}^k$. Then, we fathom node N. Likewise, we fathom N when $c_{i(k)}^k a_{i(k)-1}^k + f_{i(k)}^k > c_{i(k)}^k a_{i(k)}^k + f_{i(k)}^k \neq g_k(a_{i(k)}^k)$. Note that if PLO does not have an optimal solution, the fathoming criterion FC_{SOSD} guarantees that the branch-and-bound algorithm ends with a proof of that. Also, FC_{SOSD} guarantees finite termination of the branch-and-bound algorithm even though the branching rule for enforcing (9) and (10) does not guarantee that (λ^*, μ^*) is cut off in both children nodes.

In summary, we have that:

Theorem 1 *The SOSD approach to PLO ends within a finite number of nodes, either with an optimal solution or with a proof that PLO has no optimal solution.* □

Example 1 (Continued) The optimal solution of the LP relaxation of the SOSD model is $\mu_1^{1*} = \mu_1^{2*} = 1$ and $\lambda_i^{j*} = \mu_r^{s*} = 0$ otherwise, with objective function value 2. We then branch to enforce $\mu_1^2 \neq 1$. In one branch we impose $\lambda_2^2 = \mu_2^2 = \lambda_3^2 = \mu_3^2 = 0$, giving node N_1; in the other branch we impose $\lambda_1^2 = \mu_1^2 = 0$, giving node N_2. The optimal solution of the LP relaxation of node N_2 is $\lambda_1^{1*} = \mu_1^{1*} = \frac{1}{2}$, $\mu_2^{2*} = 1$, and $\lambda_i^{j*} = \mu_r^{s*} = 0$ otherwise, with objective function value 5.5. This solution satisfies (9), (10), and (11), and therefore we fathom N_2 for optimality.

The optimal solution of the LP relaxation of node N_1 is the same as the optimal solution of the root node LP relaxation. We then branch to enforce $\mu_1^1 \neq 1$. In one branch we impose $\lambda_2^1 = \mu_2^1 = 0$, giving node N_3; in the other branch we impose $\lambda_1^1 = \mu_1^1 = 0$, giving node N_4. The optimal solution of the LP relaxation of node N_3 is the same as the optimal solution of the root node LP relaxation, and we fathom N_3 for FC_{SOSD}. The optimal solution of the LP relaxation of node N_4 is $\lambda_2^{1*} = 1$, $\mu_1^{2*} = 1$, and $\lambda_i^{j*} = \mu_r^{s*} = 0$ otherwise, with objective function value 7. Thus, we can fathom N_4 for either FC_{SOSD} or bound, and the optimal solution to the PLO is $\lambda_1^{1*} = \mu_1^{1*} = \frac{1}{2}$, $\mu_2^{2*} = 1$, and $\lambda_i^{j*} = \mu_r^{s*} = 0$ otherwise, with objective function value 5.5. □

The SOSD approach for discontinuous PLO contains all advantages of SOS2 over MIP for continuous PLO, and in addition it solves PLO even when some of the PLFs are not lower semi-continuous. However, it has been established that for continuous PLO branch-and-cut improves considerably over the computational time of branch-and-bound [9, 15]. In continuation to this research, we are studying the polyhedron defined by the feasible set of the SOSD formulation. In particular, we are studying inequalities that take constraints (9) and (10) into account. Such inequalities may help reduce the enumeration effort to solve discontinuous PLOs for which the number of constraints (9) and (10) is large. Finally, we are studying efficient policies to select the constraint (9) or (10) to be enforced in the next branching.

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