The Efficient Outcome Set of a Bi-criteria Linear Programming and Application*

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7 November 2005

Abstract

We study the efficient outcome set Y_E of a bi-criteria linear programming problem (BP) and present a quite simple algorithm for generating all extreme points of Y_E. Application to optimization a scalar function $h(x)$ over the efficient set of (BP) in case of h which is a convex and dependent on the criteria is considered.

AMS 1991 Mathematics Subject Classification: 90 C29

Key words. Bi-criteria linear programming, Efficient outcome set, Optimization over the efficient set.

1 Introduction

This paper is concerned with the Bi-criteria linear programming problem

$$\mathrm{MAX}\{Cx\ ,\ x \in X\}, \qquad \text{(BP)}$$

where C is a $2 \times n$ matrix with rows c^1, c^2 and $X \subset I\!R^n$ is a nonempty compact polyhedron. It is well known that while the efficient solution set X_E for Problem (BP) is also always a connected set, generally, it is a complicated nonconvex subset of the boundary of X [8]. Therefore, the computational demands of generating all or representative portions of X_E grow rapidly with problem size (see, for example, [1] and [7]). In respond to this, in recent years some

*This paper is partially supported by the Research Program of the National Center for Natural Sciences and Technology

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methods (see, for example, [1], [3] - [5] and references therein) have been developed for generate all or portion of the *efficient outcome set* $Y_E := \{Cx | x \in X_E\}$ rather than X_E. This approach helps to reduce considerably the size of the problem when the number of the criteria is much smaller than the number of the decision variables.

In this paper we study the efficient outcome set Y_E for Problem (BP) and propose a quite simple algorithm (Algorithm 1) for generating the set of all extreme points of Y_E. Then, by simple structure of $Y_E \subset R^2$ we are able to determine the entire efficient outcome set Y_E. As a direct application, we consider the problem of maximizing a scalar function which is convex and dependent on the criteria over the efficient set X_E.

2 Efficient outcome set

Throughout the paper we will assume that in Problem (BP), the polyhedron X is given by

$$X = \{x \in R^n | Ax \leq b, x \geq 0\}, \tag{1}$$

where A is an $m \times n$ matrix and vector $b \in I\!R^m$. Denote $Y := \{Cx | x \in X\}$ the *outcome set* of Problem (BP). Note that Y is a nonempty, compact polyhedron in the outcome space $I\!R^2$ [10]. By definitions, the efficient outcome set is essentially the *set of efficient points* of Y,

$$Y_E = \{y^0 \in Y | \ \nexists y \in Y \text{ such that } y > y^0\}$$

and

$$X_E = \{x^0 \in X | Cx^0 = y^0 \in Y_E\}.$$

Let Y_{ex} denote the set of all extreme points of Y. Suppose that $Y_E \cap Y_{ex} = \{y^1, \cdots, y^k\}$, where, since Y is compact, we may assume that $k \geq 1$ [8]. A point $y^I = (y_1^I, \cdots, y_n^I) \in Y$ is said to be the *ideal efficient point* of Y if

$$y_i^I = \max\{y_i \ , \ y \in Y\}.$$

It is clear that if there is an ideal efficient point y^I of Y, then $Y_E = \{y^I\}$ and $Y_E \cap Y_{ex} = \{y^I\}$. This is a special case of problem (BP).

Let $F_i = \text{argmax}\{y_i \mid y \in Y\}, \ i = 1, 2$. Denote by y^{start} and y^{end} the optimal solution to the linear problem $\max\{y_1 \mid y \in F_2\}$

and the optimal solution to the linear problem $\max\{y_2 \mid y \in F_1\}$, respectively.

Proposition 2.1 *y^{start} and y^{end} are efficient extreme point of Y.*

Proof. We will prove only that $y^{start} \in Y_E \cap Y_{ex}$. The inclusion $y^{end} \in Y_E \cap Y_{ex}$ can be proved by an analogous way. Assume the contrary that $y^{start} \notin Y_E$. By definition, there is $y = (y_1, y_2) \in Y$ such that $y > y^{start}$. It means that either

$$(A) \qquad y_1 > y_1^{start} \text{ and } y_2 \geq y_2^{start}$$

or

$$(B) \qquad y_1 \geq y_1^{start} \text{ and } y_2 > y_2^{start}.$$

Since $y^{start} \in F_2$, the case (B) can not occur. In the case (A), one can see that $y \in F_2$. Then, as y^{start} is the optimal solution to the linear problem $\max\{y_1 \mid y \in F_2\}$, we have $y_1 \leq y_1^{start}$ that is impossible. Hence, both of these cases can not happened and we get $y^{start} \in Y_E$. Further, from linear programming theory, this implies that $y^{start} \in Y_{ex}$. The proof is complete. ■

Corollary 2.1 i) *If $\bar{y}^1$ is the unique optimal solution to problem* $\max\{y_2 \mid y \in Y\}$ *then $y^{start} = \bar{y}^1$ belongs to $Y_E \cap Y_{ex}$.*

ii) *If $\bar{y}^2$ is the unique optimal solution to problem* $\max\{y_1 \mid y \in Y\}$ *then $y^{end} = \bar{y}^2$ belongs to $Y_E \cap Y_{ex}$.*

Proof. In this case we have $F_2 = \{\bar{y}^1\}$ and $F_1 = \{\bar{y}^2\}$. The proof is straight-forward. ■

Remark 2.1 In the case $y^{start} = y^{end} = y^I$, by definition we have $Y_E = \{y^I\}$. In other words, y^I is an ideal efficient point of Y.

Recall that the efficient points set Y_E is connected and it is a subset of the boundary of Y [8]. In our considered case, $Y \subset I\!R^2$, the boundary of Y is composed of a finite number segments and Y_E is a path lying on the boundary of Y. Hence, we can represent

$$Y_E = \bigcup_{i=1}^{k-1} [y^i, y^{i+1}], \qquad (2)$$

where $y^1 := y^{start}$, $y^k := y^{end}$ and $[y^i, y^{i+1}], i = 1, \cdots, k-1$, are the efficient edges of Y .

Let $G^{i,j} = \text{conv}\{y^i, \cdots, y^j\}$ be the convex hull of the efficient extreme points $y^i, \cdots, y^j$, where $1 \leq i < j \leq k$. Then

$$G^{i,j} = \{y \in Y | \langle \ell^{i,j}, y \rangle \geq \alpha_{i,j}\} \tag{3}$$

end the set

$$\{y \in R^2 | \langle \ell^{i,j}, y \rangle = \alpha_{i,j}\}$$

is the line passing through y^i and y^j. Upon simple computation, we get

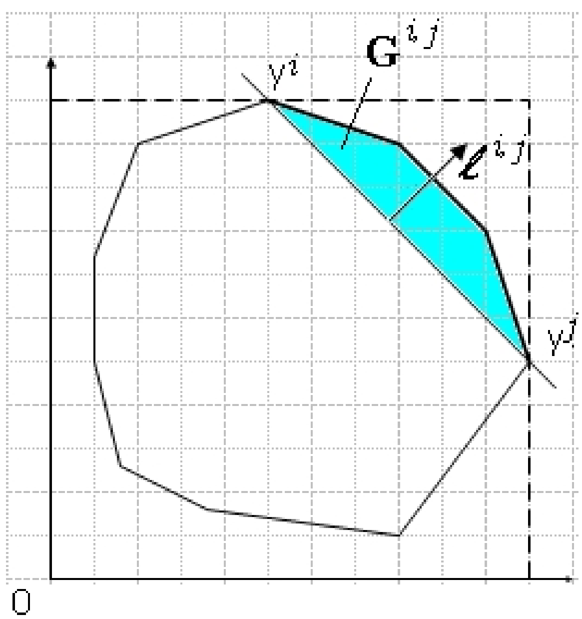
$$\ell^{i,j} = (\frac{1}{y_1^j - y_1^i}, \frac{1}{y_2^i - y_2^j}),$$

and

$$\alpha_{i,j} = \frac{y_1^i}{y_1^j - y_1^i} + \frac{y_2^i}{y_2^i - y_2^j}.$$

It is clear that vector $\ell^{i,j}$ is strictly positive, i.e, $\ell_1^{i,j} > 0$ and $\ell_2^{i,j} > 0$. Combining (1) and (3) we can obtain the explicit form of the polyhedron $G^{i,j}$

$$G^{i,j} := \{y \in R^2 : y = Cx,\ Ax \leq b,\ x \geq 0,\ \langle \ell^{i,j}, y \rangle \geq \alpha_{i,j}\}. \tag{4}$$



The following simple fact will play an important role in developing the algorithm for finding the set of all efficient outcome extreme points.

Proposition 2.2 *Let $\hat{y}$ be a optimal extreme solution of the linear programming problem* $\max\{\langle \ell^{i,j}, y\rangle : y \in G^{i,j}\}$*. If $\hat{y} \in \{y^i, y^j\}$ then $[y^i, y^j]$ is an efficient edge of Y_E, otherwise $\hat{y} \in \{y^{i+1}, \ldots, y^{j-1}\}$.*

Proof. This fact follows from the definition of $G^{i,j}$ and the linear programming theory. ■.

In the special case when $y^i = y^1 = y^{start}$ and $y^j = y^k = y^{end}$ we will denote $G = G^{1,k}$, $\ell = \ell^{1,k}$ and $\alpha = \alpha^{1,k}$. Denote by G_{ex} a the set of all extreme points of G. Then, by definition, we can see $G_{ex} = Y_E \cap Y_{ex}$. Furthermore, we have

Proposition 2.3. $Y_E = G_E$.

Proof. We can limit attention to in case of $Y_E \cap Y_{ex} = \{y^1, \cdots, y^k\}$, $k \geq 2$. First, we will show that $Y_E \subseteq G_E$. Let $y^* \in Y_E$. The, either $y^* \in \{y^1, \cdots y^k\}$ or $y^* \in [y^{i_0}, y^{i_0+1}]$, $i_0 \in \{1, \cdots, k-1\}$. Since $G = \text{conv}\{y^1, \cdots, y^k\}$, we have $y^* \in G$. Assume the contrary that $y^* \notin G_E$. Then, by the definition, there must exists $y^0 \in G \subset Y$ such that $y^0 > y^*$. This shows that $y^* \notin Y_E$ that is impossible. Hence, $y^* \in G_E$.

Now, we prove that $Y_E \supseteq G_E$. By the definition, the boundary ∂G of G can be represented in the form

$$\partial G = \bigcup_{i=1}^{k-1} [y^i, y^{i+1}] \cup [y^1, y^k]. \tag{5}$$

In the case $k = 2$, we have $G = [y^1, y^2]$,and hence, $Y_E = G_E$.

Consider the case $k \geq 3$. ¿From (2) and (5), it is sufficient to show that $(y^1, y^k) \notin G_E$. Assume the contrary that there is $y^* \in (y^1, y^k) \cap G_E$. Then

$$[y^1, y^k] := \{y \in Y | \langle \ell, y\rangle = \alpha\} \subset G_E. \tag{6}$$

(see [6]). Since vector $\ell^{1,k}$ is strictly positive and $k \geq 3$, the optimal solution $y^0 \in G$ of the problem

$$\max\{\langle \ell, y\rangle | y \in G\}$$

satisfies $\langle \ell, y^0 \rangle > \alpha$, and

$$y^0 \in \bigcup_{i=1}^{k-1} [y^i, y^{i+1}] \setminus \{y^1, y^k\}.$$

Let y^w be the unique point on the open segment (y^1, y^k) that lies on the line segment connecting y^0 and the origin point 0. Obviously, $y^0 \in y^w + R^p_+$. Therefore, by definition, $y^w \notin G_E$ which conflicts to (6). This completes the proof. ■

3 The Algorithm for Generating all Efficient Outcome Extreme Points

First, let us describe in detail procedure determining two efficient outcome extreme points y^{start} and y^{end}.

Procedure 1 *(Determine y^{start} and y^{end})*

Step 1. Find an optimal extreme point solution $\bar{y}^1$ to the linear program $\max\{y_2 \mid y \in Y\}$. If $\bar{y}^1$ is the unique optimal solution to this program, set $y^{start} = \bar{y}^1$. Otherwise, find the remain optimal extreme point solution $\bar{\bar{y}}^1$ to this program and set $y^{start} = (y^*_1, \bar{y}^1_2)$ where $y^*_1 = \max\{\bar{y}^1_1, \bar{\bar{y}}^1_1\}$.

Step 2. Find an optimal extreme point solution $\bar{y}^2$ to the linear program $\max\{y_1 \mid y \in Y\}$. If $\bar{y}^2$ is the unique optimal solution to this program, set $y^{end} = \bar{y}^2$. Otherwise, find the remain optimal extreme point solution $\bar{\bar{y}}^2$ to this program and set $y^{end} = (\bar{y}^2_1, y^*_2)$ where $y^*_2 = \max\{\bar{y}^2_2, \bar{\bar{y}}^2_2\}$.

The Algorithm 1 *(Determining $Y_E \cap Y_{ex}$)*

This algorithm will systematically generate all the vertices of Y_E from y^{end} until y^{start}, which are assumed to be given, by solving a finite sequence of simple linear programming. Since Y_E consists of a finite number of vertices, the algorithm will always be finite.

The algorithm can be described in detail as follows:

Step 0. Determine y^{start} and y^{end} by using Procedure 1.

Step 1. Set $n := 2$; $y^1 := y^{start}$; $y^2 := y^{end}$;

$k := 1;$

Step 2. **While** $n > 1$ **do**

Begin

Find an optimal extreme point solution $\hat{y}$ for problem

$$\max\{\langle \ell^{n-1,n}, y\rangle : y \in G^{n-1,n}\};$$

If $\hat{y} \notin \{y^n, y^{n-1}\}$ **Then** *($\hat{y}$ is a new efficient extreme)*

Let $y^{n+1} := y^n; y^n := \hat{y}$ and $n := n + 1$

Else Let $g^k := y^n;\ k := k + 1$ and $n := n - 1$

end

Step 3. Let $g^k = y^{start}$

For $i = 1$ to k do

$y^i := g^{k-i+1};$

In result, we obtain the set $Y_E \cap Y_{ex} = \{y^1, \cdots, y^k\}$.

Remark 3.1 According to the representation (2), after determining the set $Y_E \cap Y_{ex}$ we also obtain entire the efficient outcome set Y_E.

Examples

The following numerical examples have been computed on PC Pentium IV by experience program written in C^{++}. The computational time is not considerable.

Example 3.1. We begin with the simply example to illustrate the Algorithm 1. Consider the bi-criteria programming problem

$$\text{MAX} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$s.t. \qquad \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 0 \\ 1 \\ 7 \\ 3 \end{pmatrix}$$

$$x_1, x_2 \ge 0.$$

Determining $Y_E \cap Y_{ex}$ with Algorithm 1. First, we get $y^{start} = (1, 5)$ and $y^{end} = (6, 3)$ by computing in Procedure 1. We now show how Algorithm 1 works.

Step 1. Set $k = 1;\ n = 2;$

$y^1 = y^{start} = (1,5);\ \ y^2 = y^{end} = (6,3).$

Step 2.
Iteration 1. (with n=2) The equation of the line through y^1 and y^2 can be written in the form:

$$0.2y_1 + 0.5y_2 = 2.7.$$

Hence $G^{1,2} = \{y \in Y | \langle \ell^{1,2}, y\rangle \geq \alpha_{1,2}\}$, where $\ell^{1,2} = (0.2, 0.5)$ and $\alpha_{1,2} = 2.7$. Solve the linear programming $\max\{\langle \ell^{1,2}, y\rangle | y \in G^{1,2}\rangle\}$ we obtain $\hat{y} = (5,4) \notin \{y^1, y^2\}$. Then
$y^{n+1} = y^{2+1} = y^3 := y^n = y^2 = (6,3);$
$y^2 := \hat{y} = (5,4);$
$n := n + 1 = 2 + 1 = 3.$

Iteration 2. (with $k = 1,\ n = 3$*)* The line through $y^2 = (5,4)$ and $y^3 = (6,3)$ has the equation:

$$y_1 + y_2 = 9.$$

Solve the linear programming $\max\{\langle \ell^{2,3}, y\rangle | y \in G^{2,3}\rangle\}$ where $G^{2,3} = \{y \in Y | \langle \ell^{2,3}, y\rangle \geq \alpha_{2,3}\}$, $\ell^{2,3} = (1,1)$ and $\alpha_{2,3} = 9$. We get $\hat{y} = (6,3) \in \{y^2, y^3\}$. Then
$g^k = g^1 := y^n = y^3 = (6,3);$
$k := k + 1 = 1 + 1 = 2;$
$n := n - 1 = 3 - 1 = 2;$

Iteration 3. (with $k = 2,\ n = 2$*)* The equation of the line through $y^1 = (1,5)$ and $y^2 = (5,4)$ is

$$0.25y_1 + y_2 = 5.25.$$

Solve the linear programming $\max\{\langle \ell^{1,2}, y\rangle | y \in G^{1,2}\rangle\}$ where $G^{1,2} = \{y \in Y | \langle \ell^{1,2}, y\rangle \geq \alpha_{1,2}\}$, $\ell^{1,2} = (0.25, 1)$ and $\alpha_{1,2} = 5.25$. We get $\hat{y} = (1,5) \in \{y^1, y^2\}$. Then
$g^k = g^2 := y^n = y^2 = (5,4);$
$k := k + 1 = 2 + 1 = 3;$
$n := n - 1 = 2 - 1 = 1.$ *(Step 2 is terminated.)*

Step 3. Set
$g^k = g^3 := y^{start} = (1,5).$
We obtain

$y^1 := g^{3-1+1} = g^3 = (1.5);$
$y^2 := g^{3-2+1} = g^2 = (5,4);$
$y^3 := g^{3-3+1} = g^1 = (6,3).$
The Algorithm terminated with $Y_E \cap Y_{ex} = \{y^1, y^2, y^3\}$ and Y_E contains 2 efficient edges are $[y^1, y^2]$ and $[y^2, y^3]$.

Example 3.2. Determine the set of all efficient outcome extreme points of the problem

$$\text{MAX}\begin{pmatrix} 0 & 2 & 3 & 0 & 2 & 0 & 0 & 0 & 3 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 & 1 & 3 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{s.t.}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 8 & 6 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 5 & 7 & 0 & 1 & 3 & 0 & 0 & 0 \\ 7 & 0 & 5 & 0 & 8 & 0 & 2 & 1 & 0 & 0 \\ 0 & 7 & 0 & 4 & 3 & 0 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 & 4 & 6 & 0 & 3 & 0 & 7 \\ 0 & 0 & 8 & 0 & 7 & 0 & 0 & 0 & 0 & 6 \\ 0 & 8 & 2 & 0 & 0 & 0 & 5 & 8 & 3 & 0 \\ 5 & 3 & 8 & 0 & 0 & 0 & 6 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} \leq \begin{pmatrix} 51 \\ 56 \\ 81 \\ 77 \\ 81 \\ 93 \\ 77 \\ 76 \\ 64 \\ 100 \end{pmatrix}$$

We have $y^{start} = (10.612, 76.28)$; $y^{end} = (69.361, 3.714)$;
$Y_E \cap Y_{ex}$ contains 7 efficient outcome extreme points:
$y^1 = (10.612, 76.280)$; $y^2 = (16.419, 73.206)$; $y^3 = (52.169, 39.741)$;
$y^4 = (53.859, 37.750)$; $y^5 = (66.403, 19.410)$; $y^6 = (66.475, 19.298)$;
$y^7 = (69.361,\ 3.714)$.

Example 3.3. We find the set $Y_E \cap Y_{ex}$ for the problem (BP) where C is a 2×20 matrix, A is a 10×20 and $b \in I\!R^{10}$ determined as follows. First let $\hat{c}^j \in I\!R^2, j = 1, 2, 3, 4$ be column vectors defined by

$$\hat{c}^1 = \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}; \hat{c}^2 = \begin{pmatrix} 0.667 \\ -0.333 \end{pmatrix}; \hat{c}^3 = \begin{pmatrix} -0.75 \\ 0.25 \end{pmatrix}; \hat{c}^4 = \begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}.$$

Then C is the 2×20 matrix with columns one through four each equal to $\hat{c}^1$, columns five through eight equal to $\hat{c}^2$, columns nice and ten equal to $\hat{c}^3$ and columns 11 through 20 equal to $\hat{c}^4$. Vector $b \in I\!R^{10}$ is the vector whose entries are each equal to 1.0. And matrix A is given by

$$A = (I_{10} \vdots I_{10}),$$

where I_{10} denotes the 10×10 identity matrix. We get

$$y^{start} = (-5.5, 4.5);\ y^{end} = (2.668, -1.332);$$

$Y_E \cap Y_{ex}$ contains 4 efficient outcome extreme points:
$y^1 = (-5.500,\ 4.500)$; $y^2 = (-4.000,\ 4.000)$;
$y^3 = (-1.331,\ 2.667)$; $y^4 = (2.668,\ -1.331)$.

4 An application to Optimization over the efficient set

Consider the problem

$$\max\{h(x) \mid x \in X_E\}, \tag{P}$$

where $h(x) = \varphi(Cx)$ with φ is a convex function and X_E the efficient solution set for problem (BP). This problem has been considered by some researchers (see [7] and references therein). Since $Y = CX$, we have

$$\max\{h(x) = \varphi(Cx) \mid x \in X_E\} = \max\{\varphi(y) \mid y \in Y_E\}.$$

Since Y_E is connected and is the union of some faces of Y [8] and the function φ is convex, the problem

$$\max\{\varphi(y) \mid x \in Y_E\} \tag{$P1$}$$

attains its global solution in $Y_E \cap Y_{ex}$. So one can solve Problem (P1) by evaluation φ at each extreme point of Y_E. More precise, we have the following algorithm.

Algorithm 2 *(Solving the Problem (P))*

Step 1. Use Algorithm 1 to determine the set of all efficient extreme point $Y_E \cap Y_{ex}$.

Step 2. Find $\bar{y} \in \text{agrmax}\{\varphi(y), y \in Y_E \cap Y_{ex}\}$: *($\bar{y}$ is a global optimal solution to $(P1)$.)*

Step 3. Solve the system

$$Cx = \bar{y}, Ax \leq b, x \geq 0$$

to find a solution $\bar{x}$: *($\bar{x}$ is a global optimal solution to (P).)*

Remark 4.1 Step 1 can be executed by applying Phase I of Simplex Algorithm solving a linear programming.

Specially, when h is a linear combination of the rows of C, i.e., $h(x) = \mu_1\langle c^1, x\rangle + \mu_2\langle c^2, x\rangle$ with $\mu_1, \mu_2 \in R$ we have the following result

Proposition 4.1 *The problem (P) is equivalent to the linear programming problem*

$$\max\{\langle \mu, y\rangle \mid y \in G\}, \qquad (P2)$$

where $\mu = (\mu_1, \mu_2)$.

Proof. Recall that $G = \text{conv}\{y^1, \cdots, y^k\}$ and $G = G^{1,k}$ having explicit representation (4) with $y^1 = y^{start}, y^k = y^{end}$. This Proposition follows from Proposition 2.3 and the fact that a global optimal solution of the problem $(P1)$ is an extreme efficient point of Y_E. ■

Remark 4.2. By Proposition 3.1, a global optimal solution of the global optimal programming problem $(P1)$ can be obtained by solving only the simple scalar linear programming problem $(P2)$.

Below let us present some examples of solving the problem $(P2)$ in case of h is a linear combination of the rows of C. By Proposition 4.1, the optimal solution of $(P2)$ is also a global optimal solution of $(P1)$. Then, a global solution of (P) can be obtained by using Step 3 of the Algorithm 2.

Example 4.1. We solve the Problem (P) where X_E is the efficient set of the Problem (BP) given in Example 3.1 and $\mu = (1, 0)$. The equation of the line through $y^{start} = (1, 5)$ and $y^{end} = (6, 3)$ is

$$0.2y_1 + 0.5y_2 = 2.7.$$

The polytope G is defined by

$$G := \{y \in R^2 : y = Cx,\ Ax \leq b,\ x \geq 0,\ \langle \ell, y\rangle \geq \alpha\} \qquad (7)$$

where $\ell = (0.2, 0.5)$ and $\alpha = 2.7$. Solving the linear programming $(P2)$ we obtain the optimal solution $\bar{y} = (6, 3)$. The global optimal solution of (P) is $\bar{x} = (3, 0)$ and the optimal value is $h(\bar{x}) = 6$.

Example 4.2. We solve the Problem (P) where X_E is the efficient set of the Problem (BP) given in Example 3.2 and $\mu = (1, 1)$. The equation of the line through $y^{start} = (10.612, 76.28)$ and $y^{end} = (69.361, 3.714)$ is

$$0.017y_1 + 0.016y_2 = 1.4.$$

The polytope G is defined by (7) with $\ell = (0.017, 0.016)$ and $\alpha = 1.4$. Solving the linear programming $(P2)$ we obtain the optimal solution $\bar{y} = (52.169,\ 39.741)$. The global optimal solution of (P) is $\bar{x} = (4.527,\ 0,\ 9.333,\ 0,\ 0,\ 11.251,\ 0,\ 2.646,\ 8.056,\ 0.222)$ and the optimal value is $h(\bar{x}) = 91.91$.

Example 4.3. We solve the Problem (P) where X_E is the efficient set of the Problem (BP) given in Example 3.3 and $\mu = (1, 2)$. The equation of the line through $y^{start} = (-5.5, 4.5)$ and $y^{end} = (2.668, -1.332)$ is

$$0.122y_1 + 0.171y_2 = 1.$$

The polytope G is defined by (7) with $\ell = (0.122, 0.17)$ and $\alpha = 1$. Solving the linear programming $(P2)$ we obtain the optimal solution $\bar{y} = (-1.332, 2.668)$. The global optimal solution of (P) is $\bar{x} = (1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and the optimal value is $h(\bar{x}) = 4.004$.

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