An LMI description for the cone of Lorentz-positive maps

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Abstract

Let L_n be the n-dimensional second order cone. A linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called positive if the image of L_m under this map is contained in L_n. For any pair (n, m) of dimensions, the set of positive maps forms a convex cone. We construct a linear matrix inequality (LMI) that describes this cone. Namely, we show that its dual cone, the cone of Lorentz-Lorentz separable elements, is a section of some cone of positive semidefinite complex hermitian matrices. Therefore the cone of positive maps is a projection of a positive semidefinite matrix cone. The construction of the LMI is based on the spinor representations of the groups $\mathrm{Spin}_{1,n-1}(\mathbb{R})$, $\mathrm{Spin}_{1,m-1}(\mathbb{R})$. We also show that the positive cone is not hyperbolic for $\min(n, m) \geq 3$.

1 Introduction

Let K, K' be regular convex cones (closed convex cones, containing no lines, with non-empty interior), residing in finite-dimensional real vector spaces E, E'. A linear map from E to E' that takes K to K' is called *K-to-K' positive* or just *positive*, if it is clear which cones are meant [8],[5],[17]. The set of positive maps forms a regular convex cone. Cones of positive maps frequently appear in applications [19],[11],[15], but are in general hard to describe. For instance, to decide whether a given linear map takes the cone of positive semidefinite matrices to itself is an NP-hard problem [2]. However, in a few low-dimensional cases a description of the positive cone in form of a linear matrix inequality (LMI) is at hand. This is the case for maps that take the cone of positive semidefinite real symmetric 2×2 matrices to another positive semidefinite matrix cone or a Lorentz cone. An LMI description is also available for those maps that take the cone of positive semidefinite complex hermitian 2×2 matrices to a Lorentz cone or a positive semidefinite matrix cone of size 3×3 [23]. Recently the author constructed also an LMI description for maps that take the cone of positive semidefinite real symmetric 3×3 matrices to a Lorentz cone. Note that the cone of positive semidefinite real symmetric (complex hermitian) 2×2 matrices is isomorphic to the Lorentz cone in dimension 3 (4). Related results are the LMI description of the set of quadratic forms that are positive on the intersection of an ellipsoid with a half-space and generalisations of the $\mathcal{S}$-lemma to the matrix case [18],[13].

One of the simplest cones used in conic programming is the *second order cone* or *Lorentz cone*. In this contribution we consider positivity of maps with respect to two Lorentz cones. The corresponding cones of positive maps are in a somewhat peculiar situation. On the one hand, a description of these cones by a *nonlinear* matrix inequality, which is based on the $\mathcal{S}$-lemma, is known for decades. Moreover, these cones are efficiently computable [14]. This means that for a given point in the space of linear maps one can readily decide whether the point represents a positive map and if not, furnish a hyperplane that separates it from the cone of positive maps. This allows the use of polynomial-time black-box algorithms such as the ellipsoid algorithm [3] for solving optimisation problems over this cone of positive maps.

However, a description of these cones by a *linear* matrix inequality was long elusive. Moreover, it was not even known whether the cones of positive maps defined with respect to two Lorentz cones of dimension ≥ 5 possess an LMI description or whether they or their duals are hyperbolic (i.e. their boundary is defined by the zero set of a hyperbolic polynomial [1]). In this contribution we construct such a description. We show that the dual to the cone of positive maps, namely the cone of Lorentz-Lorentz separable elements, is a section of a cone of positive semidefinite matrices. The cone of positive maps itself is then a projection of a positive semidefinite matrix cone. It follows that the cone of Lorentz-Lorentz separable elements is hyperbolic, because its boundary is defined by the zero

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set of the determinant of the constructed LMI. We present a simple description of the corresponding determinantal polynomial. We also show that the cone of positive maps itself is not hyperbolic, even for Lorentz cones in dimension 3.

The paper is structured as follows. At the end of this introductory section we give a precise definition of the studied objects. In the next section we provide the necessary mathematical tools by giving an overview over Clifford algebras, special orthogonal groups and spin groups, their representations and Lie algebras, and relations between these objects. In Section 3 we consider the known description of the cone of positive maps in terms of a nonlinear matrix inequality and scaling of positive maps by the automorphism group of the Lorentz cone. The aim of this section is to construct a polynomial whose zero set defines the boundary of the Lorentz-Lorentz separable cone. At the same time we show that such a polynomial does not exist for the cone of positive maps itself, which implies that the latter is not hyperbolic. In Section 4 we show that a power of the polynomial describing the boundary of the Lorentz-Lorentz separable cone coincides with the determinant of an LMI that is based on the spinor representations of a spin group. This polynomial is hence hyperbolic. We construct the LMI explicitly. In the last section we draw some conclusions.

Definitions

Throughout the paper m, n are positive natural numbers.

Definition 1.1. The cone $L_n \subset \mathbb{R}^n$ defined by

$$L_n = \left\{ (x_0, x_1, \dots, x_{n-1})^T \in \mathbb{R}^n \;\middle|\; x_0 \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2} \right\}$$

is called the n-dimensional *second order cone* or *Lorentz cone*.

The Lorentz cone can equivalently be described by the inequalities

$$x_0 \geq 0, \qquad x_0^2 - x_1^2 - \cdots - x_{n-1}^2 \geq 0. \tag{1}$$

It is a regular (closed pointed with nonempty interior) convex cone. This cone is *self-dual*, i.e. it coincides with its dual cone with respect to the standard Euclidean scalar product on $\mathbb{R}^n$.

Remark 1.2. Let E be a real vector space equipped with a scalar product $\langle \cdot, \cdot \rangle$ and let $K \subset E$ be a convex cone. Then the dual cone K^* is defined as the set of elements $y \in E$ such that $\langle x, y \rangle \geq 0$ for all $x \in K$.

Definition 1.3. A linear map $M : \mathbb{R}^m \to \mathbb{R}^n$ is called *positive* if $M[L_m] \subset L_n$.

The self-duality of L_n and L_m implies that $M : \mathbb{R}^m \to \mathbb{R}^n$ is positive if and only if $M^T : \mathbb{R}^n \to \mathbb{R}^m$ is positive.

We will represent linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ by real $n \times m$ matrices. The elements of these matrices will carry two indices which run from 0 to $n-1$ and $m-1$, respectively. So, the upper left corner element of the matrix M will be denoted by M_{00}. The set of positive maps forms a regular convex cone in the nm-dimensional real vector space of real $n \times m$ matrices. We denote this cone by $P_{n,m}$.

Definition 1.4. A real $n \times m$ matrix is called *separable* if it can be represented as a sum of rank 1 matrices of the form xy^T, where $x \in L_n$, $y \in L_m$.

The empty sum is defined to be the zero matrix here. The set of separable matrices also forms a regular convex cone in the space of $n \times m$ real matrices. We will denote this cone by $Sep_{n,m}$.

On the vector space of $n \times m$ real matrices we have the usual Euclidean scalar product defined by $\langle M_1, M_2 \rangle = tr\,(M_1 M_2^T)$. Using the self-duality of L_n and L_m, it is straightforward to show that the cones $P_{n,m}$ and $Sep_{n,m}$ are dual to each other with respect to this scalar product.

The ultimate goal of the present contribution is to construct a *linear matrix inequality* describing $P_{n,m}$. More precisely, we want to find a finite number of equally sized complex hermitian matrices $A_1, \dots, A_N$, $N \geq nm$ such that a real $n \times m$ matrix M is contained in $P_{n,m}$ if and only if

$$\exists\, c_1, \dots, c_{N-nm} \in \mathbb{R} \;:\; \qquad \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} M_{kl} A_{nl+k+1} + \sum_{k=1}^{N-nm} c_k A_{k+nm} \succeq 0.$$

By self-duality of the cone of positive semidefinite matrices an LMI description of $P_{n,m}$ furnishes also an LMI description of $Sep_{n,m}$ and vice versa. We will construct an LMI describing $Sep_{n,m}$ which does not contain free variables, i.e. for which $N = nm$ in above terms.

Note that for $n \leq 2$ the Lorentz cone L_n is polyhedral with n generating rays. Hence for $\min(n,m) \leq 2$ the cone $P_{n,m}$ is a direct product of $\min(n,m)$ Lorentz cones of dimension $\max(n,m)$. In this case an LMI description is readily available. Throughout the paper we will hence assume $\min(n,m) \geq 3$.

An identity matrix of size $k \times k$ will be denoted by I_k. A zero matrix of size $k \times l$ will be denoted by $0_{k\times l}$ or just by 0. The canonical basis elements of $\mathbb{R}^n$ will be denoted by e_k^n, $k = 0, \dots, n-1$. The canonical basis elements of the space of real $n \times m$ matrices will be denoted by e_{kl}, $k = 0, \dots, n-1$, $l = 0, \dots, m-1$.

By $int\, X$ we denote the interior of the set X, and by ∂X its boundary.

2 Clifford algebras and spin groups

In this section we provide the mathematical tools to study the symmetry groups of the cones $P_{n,m}$ and $Sep_{n,m}$. The material contained in this section is from standard sources on Clifford and Lie algebras and groups and their representations [16],[12],[4],[21],[22].

As can be seen from (1), the definition of L_n involves a nonsingular indefinite quadratic form of signature $(1, n-1)$. Let us denote the matrix of this form by J_n,

$$J_n = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}.$$

From (1) we have the following lemma.

Lemma 2.1. *If $x \in \partial L_n$, then $x^T J_n x = 0$, and if $x \in int\, L_n$, then $x^T J_n x > 0$. On the other hand, $x^T J_n x = 0$ implies $x \in \partial L_n$ or $-x \in \partial L_n$ and $x^T J_n x > 0$ implies $x \in int\, L_n$ or $-x \in int\, L_n$.* □

The real *generalized orthogonal group* $O_{1,n-1}(\mathbb{R})$ associated with the form J_n is the group of real $n \times n$ matrices A that leave J_n invariant, i.e. that obey the relation $AJ_nA^T = J_n$. This relation implies $AJ_nA^TJ_nA = J_nJ_nA = A$, which in turn yields $A^TJ_nA = J_n$. Hence $O_{1,n-1}(\mathbb{R})$ is invariant with respect to transposition. This group is a Lie group consisting of 4 connected components. The connected component of the identity I_n is called the *restricted special orthogonal group* $SO^+_{1,n-1}(\mathbb{R})$. It consists of those elements from $O_{1,n-1}(\mathbb{R})$ which have determinant 1 and leave the Lorentz cone L_n invariant. There are two connected components of $O_{1,n-1}(\mathbb{R})$ whose elements map L_n onto itself, one with elements with determinant 1 and the other with elements with determinant -1. These two components also form a Lie group, which is a subgroup of $O_{1,n-1}(\mathbb{R})$. We will call this group $\mathrm{Aut}(L_n)$. Actually, this is a little abuse of notation because the automorphism group of the Lorentz cone is the direct product of $\mathrm{Aut}(L_n)$ with the multiplicative group $\mathbb{R}_+$. However, $\mathrm{Aut}(L_n)$ consists of those automorphisms of L_n which leave J_n invariant. The group $SO^+_{1,n-1}(\mathbb{R})$ serves also as the connected component of the identity of the group $\mathrm{Aut}(L_n)$ and shares its Lie algebra with it. Thus it is not surprising that $SO^+_{1,n-1}(\mathbb{R})$ and $SO^+_{1,m-1}(\mathbb{R})$ play a major role in describing the symmetries of the cones $P_{n,m}$ and $Sep_{n,m}$. Let us investigate these groups in detail.

The group $SO^+_{1,n-1}(\mathbb{R})$ is not simply connected, rather its fundamental group equals that of the special orthogonal group $SO_{n-1}(\mathbb{R})$, which is $\mathbb{Z}$ for $n = 3$ and $\mathbb{Z}_2$ for $n \geq 4$. Closely related to the special orthogonal group is the so-called *spin group*, which is defined via the concept of *Clifford algebras*.

Definition 2.2. [22] Let Q be a non-degenerate quadratic form on a vector space V. The *Clifford algebra* $Cl(V,Q)$ is the universal associative algebra with 1 which contains and is generated by V subject to the condition $v^2 = Q(v)$ for all $v \in V$. If V is a real vector space and Q has signature (p,q), then the Clifford algebra is denoted by $Cl_{p,q}(\mathbb{R})$. For $q = 0$, $p = n$, we just write $Cl_n(\mathbb{R})$.

We will need only Clifford algebras over real vector spaces. Let $n = p + q$ be the dimension of V and let $e_1, \dots, e_n$ be a basis of V in which Q becomes a diagonal matrix with p 1's and q -1's on the diagonal. Then the elements of this basis anticommute and a basis of the whole Clifford algebra is given by the ensemble of ordered products $e_{k_1}e_{k_2}\cdots e_{k_l}$ with $1 \leq k_1 < k_2 < \cdots < k_l \leq n$, $l = 0, \dots, n$. The Clifford algebra has hence 2^n dimensions. We extend Q to $Cl_{p,q}(\mathbb{R})$ by letting this basis be orthogonal and setting $Q(e_{k_1}e_{k_2}\cdots e_{k_l}) := Q(e_{k_1})Q(e_{k_2})\cdots Q(e_{k_l})$.

The Clifford algebra $Cl(V,Q)$ contains a subalgebra, namely the algebra generated by products $e_{k_1}e_{k_2}\cdots e_{k_l}$ with l even. This subalgebra has 2^{n-1} dimensions and is denoted by $Cl^0(V,Q)$. The subspace spanned by products $e_{k_1}e_{k_2}\cdots e_{k_l}$ with l odd is denoted by $Cl^1(V,Q)$. Any element $x \in Cl(V,Q)$ can be written as a sum $x = x^0 + x^1$ with $x^0 \in Cl^0(V,Q)$, $x^1 \in Cl^1(V,Q)$. For any $x \in Cl(V,Q)$, decomposed in such a way, let us define $\hat{x} = x^0 - x^1$. The element $\hat{x}$ is called the *involute* of x, and the map $x \mapsto \hat{x}$ defines an automorphism of $Cl(V,Q)$.

Denote the even subalgebra of $Cl_{p,q}(\mathbb{R})$ by $Cl^0_{p,q}(\mathbb{R})$.

Theorem 2.3. *[16, Corollary 13.34] Let $n = p + q$, $p > 0$ and consider the Clifford algebras $Cl_{p,q}(\mathbb{R})$ with generators $e_0, \ldots, e_{n-1}$ ($e_0^2 = e_1^2 = \cdots = e_{p-1}^2 = 1$, $e_p^2 = \cdots = e_{n-1}^2 = -1$) and $Cl_{q,p-1}(\mathbb{R})$ with generators $f_1, \ldots, f_{n-1}$ ($f_1^2 = \cdots = f_q^2 = 1$, $f_{q+1}^2 = \cdots = f_{n-1}^2 = -1$). Define an algebra homomorphism $\theta : Cl_{q,p-1}(\mathbb{R}) \to Cl_{p,q}(\mathbb{R})$ by putting $\theta(f_k) = e_{n-k}e_0$ on the generators. Then the image of θ is the subalgebra $Cl^0_{p,q}(\mathbb{R})$ and θ defines an isomorphism*

$$\theta : Cl_{q,p-1}(\mathbb{R}) \cong Cl^0_{p,q}(\mathbb{R}). \tag{2}$$

One can define a *transposition* on $Cl_{p,q}(\mathbb{R})$. It acts on the basis vectors by reversing the order of the factors, $e_{k_1}e_{k_2}\cdots e_{k_l} \mapsto e_{k_l}e_{k_{l-1}}\cdots e_{k_1}$. We denote the transpose of an element x by x^t. The transposition is an anti-automorphism, i.e. $(xy)^t = y^t x^t$.

Lemma 2.4. *Assume the notations of the previous theorem. For any $x \in Cl_{q,p-1}(\mathbb{R})$, we have $\theta(\hat{x}^t) = \widehat{\theta(x)}^t$.*

Proof. Since the map $x \mapsto \hat{x}^t$ is an anti-automorphism, it suffices to check the relation one the generators f_k. We have $\theta(\hat{f}_k^t) = \theta(-f_k) = -e_{n-k}e_0 = (\widehat{e_{n-k}e_0})^t = \widehat{\theta(f_k)}^t$. □

Definition 2.5. The *Clifford group* $\Gamma \subset Cl(V, Q)$ is defined to be the set of invertible elements x of the Clifford algebra such that $xv\hat{x}^{-1} \in V$ for all $v \in V$.

Denote the linear map $v \mapsto xv\hat{x}^{-1}$ by ρ_x.

Proposition 2.6. *[16, Proposition 13.37] For any $x \in \Gamma$, the map $\rho_x : V \to V$ is orthogonal.*

Hence the map $\rho : x \mapsto \rho_x$ is a group homomorphism from Γ to the generalized orthogonal group of V.

We can repeat this construction for the subspace $Y = \mathbb{R} \oplus V \subset Cl(V, Q)$, where $\mathbb{R} \subset Cl(V, Q)$ is the one-dimensional subspace spanned by 1. Define a quadratic form Q' on Y by $Q'(y) = \hat{y}y$, $y \in Y$. We define the group $\Omega \subset Cl(V, Q)$ as the set of invertible elements x such that $xy\hat{x}^{-1} \in Y$ for all $y \in Y$. Denote the map $y \mapsto xy\hat{x}^{-1}$ by ρ'_x. Then the map $\rho' : x \mapsto \rho'_x$ is a group homomorphism from Ω to the generalized orthogonal group of the space Y, equipped with the quadratic form Q' [16, Proposition 13.40].

The Clifford group contains a subgroup Γ^0 of index 2, namely the intersection $\Gamma \cap Cl^0_{p,q}(\mathbb{R})$.

Theorem 2.7. *([16, Proposition 13.49]) Let $n = p+q$, $p > 0$ and consider again the Clifford algebras $Cl_{p,q}(\mathbb{R})$ with generators $e_0, \ldots, e_{n-1}$ and $Cl_{q,p-1}(\mathbb{R})$ with generators $f_1, \ldots, f_{n-1}$. Let $V_{p,q}$ be the vector space underlying $Cl_{p,q}(\mathbb{R})$ and $V_{q,p-1}$ be the vector space underlying $Cl_{q,p-1}(\mathbb{R})$, and let θ be defined as in the previous theorem. Let $Y_{q,p-1} = \mathbb{R} \oplus V_{q,p-1} \subset Cl_{q,p-1}(\mathbb{R})$, equipped with the corresponding quadratic form $Q'(y) = \hat{y}y$, and denote by $\Omega_{q,p-1}$ the group of invertible elements in $Cl_{q,p-1}(\mathbb{R})$ such that $xy\hat{x}^{-1} \in Y_{q,p-1}$ for all $y \in Y_{q,p-1}$. Let further $\Gamma^0_{p,q}$ be the intersection of the Clifford group of $Cl_{p,q}(\mathbb{R})$ with the even subalgebra $Cl^0_{p,q}(\mathbb{R})$. Define a linear map u from $Y_{q,p-1}$ to $V_{p,q}$ by putting $u(y) = \theta(y)e_0$, $y \in Y_{q,p-1}$. Then*

(i) the map u is an orthogonal isomorphism,
(ii) for any $x \in \Omega_{q,p-1}$, $\theta(x) \in \Gamma^0_{p,q}$ and $u \circ \rho'_x = \rho_{\theta(x)} \circ u$,
(iii) the map $\Theta : \Omega_{q,p-1} \to \Gamma^0_{p,q}$ defined by $\Theta : x \mapsto \theta(x)$ is a group isomorphism.

Definition 2.8. The spin group $\mathrm{Spin}_{p,q}(\mathbb{R})$ is the subgroup of the group $\Gamma^0_{p,q}$ of elements for which $x^{-1} = x^t$.

The Lie algebra of the spin group is represented by the algebra of bivectors $e_k e_l$ of the Clifford algebra. For elements x of the spin group we have $\hat{x}^{-1} = x^t$. Denote the restriction of the group homomorphism ρ to $\mathrm{Spin}_{p,q}(\mathbb{R})$ by ς, $\varsigma : x \mapsto \varsigma_x$. Then the spin group $\mathrm{Spin}_{p,q}(\mathbb{R})$ acts on $V_{p,q}$ by

$$\varsigma_x : v \mapsto xvx^t, \quad x \in \mathrm{Spin}_{p,q}(\mathbb{R}), \quad v \in V_{p,q}, \tag{3}$$

and this action defines a group homomorphism from $\mathrm{Spin}_{p,q}(\mathbb{R})$ to the generalized orthogonal group $O_{p,q}(\mathbb{R})$. Moreover, the image $\varsigma[\mathrm{Spin}_{p,q}(\mathbb{R})]$ is exactly the restricted special orthogonal group $SO^+_{p,q}(\mathbb{R})$ and we have the following theorem.

Theorem 2.9. *(cf. [16, Proposition 13.48]) The group $\mathrm{Spin}_{p,q}(\mathbb{R})$ is a two-fold cover of the restricted special orthogonal group $SO^+_{p,q}(\mathbb{R})$, the covering map being defined by ς. Its kernel is ± 1.*

In particular, the spin group has the same Lie algebra as the generalized orthogonal group. Since the fundamental group of $SO^+_{1,n-1}(\mathbb{R})$ is $\mathbb{Z}_2$ for $n \geq 4$, the corresponding spin group $\mathrm{Spin}_{1,n-1}(\mathbb{R})$ will be simply connected and is the universal cover group of $SO^+_{1,n-1}(\mathbb{R})$ (likewise for $SO_n(\mathbb{R})$ for $n \geq 3$).

Consider the group isomorphism Θ defined in Theorem 2.7. Let $\Omega' \subset Cl_{q,p-1}(\mathbb{R})$ be the image of $\mathrm{Spin}_{p,q}(\mathbb{R})$ under Θ^{-1}, and let $Y = \mathbb{R} \oplus V \subset Cl_{q,p-1}(\mathbb{R})$ be equipped with the quadratic form $Q'(y) = \hat{y}y$ of signature (p,q). Then we have the following theorem.

Theorem 2.10. *For any $x \in \Omega'$, define the map $\varsigma'_x : Y \to Y$ by $\varsigma'_x : y \mapsto xyx^t$. Then*

(i) for any $x \in \Omega'$, the map ς'_x is an orthogonal automorphism of Y,
(ii) the group Ω' is isomorphic to $\mathrm{Spin}_{p,q}(\mathbb{R})$,
(iii) the map $\varsigma' : x \mapsto \varsigma'_x$ defines a surjective group homomorphism from $\Omega' \cong \mathrm{Spin}_{p,q}(\mathbb{R})$ to $SO^+_{p,q}(\mathbb{R})$, the restricted special orthogonal group of Y,
(iv) the Lie algebra of the subgroup Ω' is spanned by the generators f_k of $Cl_{q,p-1}(\mathbb{R})$ and their bi-products $f_k f_l$.

Proof. By Lemma 2.4 and Definition 2.8, the map ς' coincides on its domain of definition with the map ρ' appearing in Theorem 2.7. Items (i)–(iii) of the theorem now follow from Theorems 2.7 and 2.9. The Lie algebra of Ω' is the image of the Lie algebra of $\mathrm{Spin}_{p,q}(\mathbb{R})$, that is the subspace spanned by the bivectors $e_k e_l$, under Θ^{-1}. □

Theorem 2.11. *[22] For even n the spin group $\mathrm{Spin}_{p,q}(\mathbb{R})$ has two irreducible representations of dimension $2^{n/2-1}$. These representations are called Weyl representations. For odd n $\mathrm{Spin}_{p,q}(\mathbb{R})$ has an irreducible representation of dimension $2^{(n-1)/2}$. It is called spinor representation.*

These representations are induced by matrix representations of the Clifford algebra itself.

3 Scaling of positive maps

Symmetries facilitate the study of mathematical objects. If properties of an ensemble of objects are to be studied that are preserved under the action of a transformation group, it is often useful to consider only orbit representatives which have some particularly simple canonical form. The large symmetry group of the Lorentz cone will allow us to restrict the study of positive maps to those which have a particularly simple structure, namely which are diagonal. Moreover, the intersections of the cones $P_{n,m}$ and $Sep_{n,m}$ with the subspace of diagonal maps turn out to be polyhedral and hence can be readily described. In addition, the large symmetry group allows us to extract a full set of invariants and to relate these invariants with the diagonal entries of the canonical orbit representative in a simple way. The invariants will depend polynomially on the elements of the positive map. This in turn will enable us to construct a polynomial whose zero set describes the boundary of $Sep_{n,m}$, which will be the ultimate goal of this section.

We start with a well-known description of $P_{n,m}$ by a nonlinear matrix inequality.

Lemma 3.1. *A map*

$$M = \begin{pmatrix} 1 & h \\ v & A \end{pmatrix} \in \mathbb{R}^{n\times m} \tag{4}$$

is positive if and only if

$$\exists\, \lambda \geq 0 : \quad M^T J_n M \succeq \lambda J_m, \quad |h| \leq 1,$$

or equivalently,

$$\exists\, \lambda' \geq 0 : \quad M J_m M^T \succeq \lambda J_n, \quad |v| \leq 1.$$

Here h is a row vector of length $m-1$, v is a column vector of length $n-1$ and A is an $(n-1)\times(m-1)$ matrix. Note that the upper left corner element of a nonzero positive map must always be positive. Since $P_{n,m}$ is a cone, we do not restrict generality if we put this element equal to 1.

Proof. By definition, M is positive if for any $x_0 \geq 0$, $x \in \mathbb{R}^{m-1}$ such that $x_0 \geq |x|$ we have $y_0 \geq |y|$, where

$$\begin{pmatrix} y_0 \\ y \end{pmatrix} = M \begin{pmatrix} x_0 \\ x \end{pmatrix}.$$

We can rewrite this equivalently as

$$\forall\, x_0, x \;\middle|\; x_0 \geq 0,\ (x_0\ x^T) J_m \begin{pmatrix} x_0 \\ x \end{pmatrix} \geq 0 : \quad x_0 + hx \geq 0, \quad (x_0\ x^T) M^T J_n M \begin{pmatrix} x_0 \\ x \end{pmatrix} \geq 0.$$

By the $\mathcal{S}$-lemma [6],[24] this is equivalent to the conditions

$$(1\ h)^T \in L_n, \quad \exists\, \lambda \geq 0 : \quad M^T J_n M \succeq \lambda J_m,$$

which gives the first set of conditions claimed by the lemma. The second set is obtained by considering the adjoint map M^T. $\square$

Lemma 3.2. *Let $U \in \mathrm{Aut}(L_n)$, $V \in \mathrm{Aut}(L_m)$ be any two automorphisms, represented by square matrices of appropriate sizes. Let M be an $n \times m$ matrix. Then UMV^T is in $P_{n,m}$ if and only if M is in $P_{n,m}$, and UMV^T is in $Sep_{n,m}$ if and only if M is in $Sep_{n,m}$.*

The proof of the lemma is trivial.

The lemma states that $\mathrm{Aut}(L_n) \times \mathrm{Aut}(L_m)$ is a subgroup of the automorphism groups of $P_{n,m}$ and $Sep_{n,m}$ (an automorphism outside $\mathrm{Aut}(L_n) \times \mathrm{Aut}(L_m)$ for $n = m$ is the transposition of M). Here the pair $(U, V) \in \mathrm{Aut}(L_n) \times \mathrm{Aut}(L_m)$ acts as $M \mapsto UMV^T$.

Definition 3.3. The action defined by $M \mapsto UMV^T$ of the group $\mathrm{Aut}(L_n) \times \mathrm{Aut}(L_m)$ is called *scaling*.

Scaling of positive maps between symmetric cones often allows to make the concerned map *doubly stochastic* [7]. This means that the map as well as its adjoint preserve the central ray of the concerned symmetric cones. The central ray of the Lorentz cone is the ray generated by e_0, so in our case doubly stochastic maps are represented by matrices whose first row and first column are given by $(e_0^m)^T$ and e_0^n, respectively. In our case, however, we can show an even stronger result.

Theorem 3.4. *Let M be in the interior of the cone $P_{n,m}$. Then there exists an element $(U, V) \in \mathrm{Aut}(L_n) \times \mathrm{Aut}(L_m)$ such that UMV^T is diagonal with nonnegative diagonal elements.*

Before proceeding to the proof we state the following obvious result.

Lemma 3.5. *A map M is in the interior of $P_{n,m}$ if and only if it maps every nonzero point of L_m into the interior of L_n.* $\square$

Proof of the theorem. Let $M \in int\, P_{n,m}$. We consider several cases.

1. M has rank 1. Then it must have the form $M = xy^T$, where $x \in int\, L_n, y \in int\, L_m$. But for any vector x in the interior of L_n there exists an element from $\mathrm{Aut}(L_n)$ which takes this vector to a point on the central ray, namely to the point $\sqrt{x^T J_n x} e_0$. This is because the Lorentz cones are homogeneous [20]. Hence we find elements $U \in \mathrm{Aut}(L_n)$, $V \in \mathrm{Aut}(L_m)$ such that Ux and Vy are on the central rays of L_n and L_m, respectively. It is not hard to see that UMV^T will be proportional to e_{00}, which proves the theorem for this case.

2. M is of rank strictly greater than 1. Define two functions $p, q : \mathbb{R}^m \to \mathbb{R}$ by $p(x) = x^T J_m x$, $q(x) = x^T M^T J_n M x$. Then the set $N = \{(p(x), q(x)) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}^m\}$ is called the *joint numerical range* of the matrices J, $M^T J_n M$ underlying the quadratic forms p, q. It is known [6] that the set N is a convex cone. Lemma 3.1 states the existence of a number $\lambda \geq 0$ such that $q(x) \geq \lambda p(x)$ for all $x \in \mathbb{R}^m$. Let λ^* be the maximal such λ. Since M takes the interior of L_m to the interior of L_n, the set N has a non-empty intersection with the open first orthant. Therefore λ^* exists. Moreover, $\lambda^* > 0$, otherwise we would have $\lambda^* = 0$ and the matrix $M^T J_n M$ would be positive semidefinite, which is not possible if the rank of M is strictly greater than 1. We have $M^T J_n M - \lambda^* J_m \succeq 0$ and $M^T J_n M - \lambda^* J_m - \delta J_m \not\succeq 0$ for any $\delta > 0$. Hence there exists $x^* \neq 0$ such that $q(x^*) - \lambda^* p(x^*) = (x^*)^T (M^T J_n M - \lambda^* J_m) x^* = 0$ and $p(x^*) = (x^*)^T J_m x^* \geq 0$. If $p(x^*) = 0$, then either $x^* \in \partial L_m$ or $x^* \in -\partial L_m$ by Lemma 2.1. In the second case we can replace x^* by $-x^*$, so let us assume $x^* \in \partial L_m$. But then $Mx^* \in int\, L_n$ and $q(x^*) > 0$, which contradicts $q(x^*) - \lambda^* p(x^*) = 0$. Hence $p(x^*) > 0$.

Without restriction of generality we can choose x^* such that $x^* \in int\, L_m$ (by replacing x^* by $-x^*$ if $x^* \in -int\, L_m$) and $p(x^*) = 1$. Denote Mx^* by y^*. Since M takes $int\, L_m$ to $int\, L_n$, we have $y^* \in int\, L_n$. In fact, $(y^*)^T J_n y^* = q(x^*) = \lambda^* > 0$. Let $U \in \mathrm{Aut}(L_m)$, $V \in \mathrm{Aut}(L_n)$ such that $Ux^* = e_0^m$ and $Vy^* = \sqrt{\lambda^*} e_0^n$.

Define a map $\tilde{M} = (\lambda^*)^{-1/2} VMU^{-1}$. By the positivity of M this map is also positive. We have $\tilde{M} e_0^m = (\lambda^*)^{-1/2} VMU^{-1}Ux^* = (\lambda^*)^{-1/2} Vy^* = e_0^n$. Since x^* is contained in the nullspace of the positive semidefinite matrix $M^T J_n M - \lambda^* J_m$, we have $M^T J_n M x^* = M^T J_n y^* = \lambda^* J_m x^*$. It follows that

$$\begin{aligned} \tilde{M}^T e_0^n &= \tilde{M}^T (J_n e_0^n) = (\lambda^*)^{-1/2} U^{-T} M^T V^T (V^{-T} J_n V^{-1}) e_0^n = (\lambda^*)^{-1/2} U^{-T} M^T J_n [(\lambda^*)^{-1/2} y^*] \\ &= (\lambda^*)^{-1} U^{-T} [\lambda^* J_m x^*] = U^{-T} J_m x^* = J_m U x^* = e_0^m. \end{aligned}$$

Hence $\tilde{M}$ is doubly stochastic and can be written in the form (4) with $h = v = 0$. Let now $A = V'DU'$ be the singular value decomposition of the submatrix A of $\tilde{M}$. Then the map

$$\begin{pmatrix} 1 & 0 \\ 0 & V' \end{pmatrix}^{-1} VMU^{-1} \begin{pmatrix} 1 & 0 \\ 0 & U' \end{pmatrix}^{-1} = \sqrt{\lambda^*} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$$

has the form required by the theorem. $\square$

Remark 3.6. A similar statement for cones of maps that take the positive semidefinite cone to the positive semidefinite cone was proven by Leonid Gurvits [7].

Corollary 3.7. *Let M be in the interior of the cone $P_{n,m}$. Then there exists an element $(U,V) \in SO^+_{1,n-1}(\mathbb{R}) \times SO^+_{1,m-1}(\mathbb{R})$ such that UMV^T is diagonal.*

Proof. Recall that $SO^+_{1,n-1}(\mathbb{R})$ consists of exactly those elements of $\mathrm{Aut}(L_n)$ which have determinant 1. It is hence sufficient to compose the maps $U \in \mathrm{Aut}(L_n)$, $V \in \mathrm{Aut}(L_m)$ obtained by application of Theorem 3.4, if they have determinant -1, with a reflection of the last coordinate. This operation will preserve the diagonal structure of UMV^T. ☐

Lemma 3.8. *The eigenvalues of the matrix $J_n M J_m M^T$ are invariant with respect to scalings.*

Proof. Let $U \in \mathrm{Aut}(L_n)$, $V \in \mathrm{Aut}(L_m)$. Since elements of $\mathrm{Aut}(L_n)$ preserve the form J_n, we have

$$\begin{aligned} &\det(J_n(UMV^T)J_m(UMV^T)^T - \lambda I_n) = \det(J_n UM(V^T J_m V)M^T U^T - \lambda I_n) \\ = \;& \det(U^{-T} J_n M J_m M^T U^T - \lambda I_n) = \det(J_n M J_m M^T - \lambda I_n). \quad \square \end{aligned}$$

Note that the eigenvalues of $J_n M J_m M^T$ are the squares of the diagonal elements of the canonical form obtained in Theorem 3.4, appended with $n-m$ zeros if $n > m$. Note also that the first $\min(n,m)$ eigenvalues of this product are invariant under cyclic permutations of the factors.

Corollary 3.9. *The coefficients of the characteristic polynomial of $J_n M J_m M^T$, which are polynomial in the entries of M, are invariant under scalings.* ☐

Lemma 3.10. *Let M be a diagonal matrix. Then M is in $P_{n,m}$ if and only if $M_{00} \geq |M_{kk}|$ for all $k = 1, \dots, \min(n,m) - 1$.*

Proof. By Lemma 3.1 positivity of M is equivalent to nonnegativity of M_{00} and the existence of $\lambda \geq 0$ such that $M_{00}^2 \geq \lambda \geq M_{kk}^2$ for all $k = 1, \dots, \min(n,m) - 1$. The lemma now follows readily. ☐

The intersection of the cone $P_{n,m}$ with the subspace of diagonal matrices is therefore a cone with (hyper-)cubic section.

Lemma 3.11. *The cone $Sep_{n,m}$ is contained in the cone $P_{n,m}$. If $M \in Sep_{n,m}$ and $M \in \partial P_{n,m}$, then M must be of the form $M = x'x^T + y(y')^T$, where $x \in L_m$, $y \in L_n$, $x' \in \partial L_n$ and $y' \in \partial L_m$.*

Proof. The first statement follows easily from the self-duality of L_n and L_m. Let us prove the second one. Suppose $M \in Sep_{n,m}$. Then M can be represented as a finite sum $\sum_{k=1}^N x_k y_k^T$ with $x_k \in L_n$, $y_k \in L_m$. If now $M \in \partial P_{n,m}$, then by Lemma 3.5 there exist nonzero vectors $\tilde{x} \in \partial L_n$, $\tilde{y} \in \partial L_m$ such that $\tilde{x}^T M \tilde{y} = \sum_{k=1}^N (\tilde{x}^T x_k)(y_k^T \tilde{y}) = 0$. But by self-duality of L_n we have $\tilde{x}^T x_k \geq 0$, and by self-duality of L_m $\tilde{y}^T y_k \geq 0$ for all k. Therefore for any k either $\tilde{x}^T x_k = 0$ or $\tilde{y}^T y_k = 0$, which implies that either x_k is proportional to $x' = J_n \tilde{x}$ or y_k is proportional to $y' = J_m \tilde{y}$. If we plug this into the representation $M = \sum_{k=1}^N x_k y_k^T$, we obtain the required form for M. ☐

Lemma 3.12. *Let M be a diagonal matrix. Then M is in $Sep_{n,m}$ if and only if $M_{00} \geq \sum_{k=1}^{\min(n,m)-1} |M_{kk}|$. If $M_{00} > \sum_{k=1}^{\min(n,m)-1} |M_{kk}|$, then $M \in int\, Sep_{n,m}$.*

Proof. By Lemma 3.10 the first inequality is necessary for separability of M. Let us proof sufficiency by constructing an explicit separable representation. If M is diagonal with $M_{00} \geq \sum_{k=1}^{\min(n,m)-1} |M_{kk}|$, then it can be represented as a convex conic combination of diagonal matrices $M_{k\pm} = e_{00} \pm e_{kk}$, $k = 1, \dots, \min(n,m) - 1$. However, any of the matrices $M_{k\pm}$ is representable as the sum of two separable rank 1 matrices, namely

$$2M_{k+} = (e_0^n + e_k^n)(e_0^m + e_k^m)^T + (e_0^n - e_k^n)(e_0^m - e_k^m)^T, \quad 2M_{k-} = (e_0^n + e_k^n)(e_0^m - e_k^m)^T + (e_0^n - e_k^n)(e_0^m + e_k^m)^T.$$

This proves the first part of the lemma.

Let now M be diagonal with $M_{00} > \sum_{k=1}^{\min(n,m)-1} |M_{kk}|$ and assume that $M \in \partial Sep_{n,m}$. Then there exists a nonzero linear functional L which separates M from $Sep_{n,m}$. This functional will be an element of $P_{n,m}$, because $P_{n,m}$ is dual to $Sep_{n,m}$. On the other hand, M is contained in the relative interior of the intersection of $Sep_{n,m}$ with the subspace of diagonal matrices. Therefore the diagonal elements of L have to be zero, in particular the corner element L_{00}. It follows that $L = 0$, which leads to a contradiction. Thus $M \in int\, Sep_{n,m}$. ☐

The intersection of the cone $Sep_{n,m}$ with the subspace of diagonal matrices is hence a cone with (hyper-)octahedral section.

Let M be an $n \times m$ real matrix. We now define a homogeneous polynomial $\mathcal{P}(M)$ of degree $2^{\min(n,m)-1}$ in the nm entries of M. To this end we first consider the following homogeneous polynomial of degree $2^{\min(n,m)-1}$ in $\min(n,m)$ variables $\sigma_0, \dots, \sigma_{\min(n,m)-1}$:

$$p(\sigma_0, \dots, \sigma_{\min(n,m)-1}) = \prod_{s \in \{-1,1\}^{\min(n,m)-1}} \left(\sigma_0 + \sum_{k=1}^{\min(n,m)-1} s_k \sigma_k \right) = \prod (\sigma_0 \pm \sigma_1 \pm \sigma_2 \pm \cdots \pm \sigma_{\min(n,m)-1}).$$

Here s_k are the elements of the vector s, which runs through the vertices of the $(\min(n,m)-1)$-dimensional hypercube. For $\min(n,m) \geq 3$ (which we assume) the polynomial p is symmetric and even. We can hence define a symmetric homogeneous polynomial q of degree $2^{\min(n,m)-2}$ in $\min(n,m)$ variables by setting $q(\sigma_0^2, \dots, \sigma_{\min(n,m)-1}^2) \equiv p(\sigma_0, \dots, \sigma_{\min(n,m)-1})$. We then define the polynomial $\mathcal{P}$ by $\mathcal{P}(M) = q(\lambda_1, \dots, \lambda_{\min(n,m)})$, where $\lambda_1, \dots, \lambda_{\min(n,m)}$ are the first $\min(n,m)$ eigenvalues of $J_n M J_m M^T$. Since q is symmetric and homogeneous, $\mathcal{P}$ is a homogeneous polynomial of degree $2^{\min(n,m)-2}$ in the entries of $J_n M J_m M^T$ and hence a homogeneous polynomial of degree $2^{\min(n,m)-1}$ in the entries of M. The following statement follows from Corollary 3.9.

Lemma 3.13. *The polynomial $\mathcal{P}$ is invariant under scalings.* $\square$

Theorem 3.14. *If $M \in int\, Sep_{n,m}$, then $\mathcal{P}(M) > 0$. If $M \in \partial Sep_{n,m}$, then $\mathcal{P}(M) = 0$.*

Proof. If $M \in int\, Sep_{n,m}$, then $M \in int\, P_{n,m}$ by Lemma 3.11. Hence by Theorem 3.4 we find a scaling that transforms M in a diagonal matrix D, and $\mathcal{P}(M) = \mathcal{P}(D)$. By construction we have

$$\mathcal{P}(D) = p(D_{00}, \dots, D_{\min(n,m)-1,\min(n,m)-1}) = \prod_{s \in \{-1,1\}^{\min(n,m)-1}} \left(D_{00} + \sum_{k=1}^{\min(n,m)-1} s_k D_{kk} \right). \tag{5}$$

$M \in int\, Sep_{n,m}$ also implies $D \in int\, Sep_{n,m}$ and $D_{00} > \sum_{k=1}^{\min(n,m)-1} |D_{kk}|$ by Lemma 3.12. But then $\mathcal{P}(M) > 0$ since all factors on the right-hand side of (5) are positive.

Let now $M \in \partial Sep_{n,m}$. We consider two cases.

1. If $M \in int\, P_{n,m}$, then we can follow the same lines of reasoning as above. There exists a diagonal matrix $D \in \partial Sep_{n,m}$ with $\mathcal{P}(M) = \mathcal{P}(D) = p(D_{00}, \dots, D_{\min(n,m)-1,\min(n,m)-1})$. However, by Lemma 3.12 we have $D_{00} = \sum_{k=1}^{\min(n,m)-1} |D_{kk}|$. This means that at least one of the factors on the right-hand side of (5) is zero, which yields $\mathcal{P}(M) = 0$.

2. If $M \in \partial P_{n,m}$, then by Lemma 3.11 $M = x' x^T + y (y')^T$, where $x' \in \partial L_n$, $y' \in \partial L_m$, $x \in L_m$ and $y \in L_n$. By appropriate normalization of x', y' (at the expense of multiplying x, y by nonnegative constants) and rotations in the subspace spanned by the basis vectors $e_1^n, \dots, e_{n-1}^n$ or $e_1^m, \dots, e_{m-1}^m$, respectively, (these rotations being scalings and hence not altering the value of $\mathcal{P}$) we can achieve that $x' = e_0^n + e_1^n$, $y' = e_0^m + e_1^m$; and x and y lie in the subspaces spanned by e_0^m, e_1^m, e_2^m and e_0^n, e_1^n, e_2^n, respectively. We then obtain

$$M = \begin{pmatrix} x_0 + y_0 & x_1 + y_0 & x_2 & 0_{1\times(n-3)} \\ x_0 + y_1 & x_1 + y_1 & x_2 & 0_{1\times(n-3)} \\ y_2 & y_2 & 0 & 0_{1\times(n-3)} \\ 0_{(n-3)\times 1} & 0_{(n-3)\times 1} & 0_{(n-3)\times 1} & 0_{(n-3)\times(n-3)} \end{pmatrix}.$$

Therefore at most three eigenvalues of $J_n M J_m M^T$ are nonzero, say $\lambda_1, \lambda_2, \lambda_3$. We then can compute $\mathcal{P}(M)$ explicitly. We have

$$\begin{aligned} p(\sigma_0, \sigma_1, \sigma_2, 0, \dots, 0) &= [(\sigma_0 + \sigma_1 + \sigma_2)(\sigma_0 + \sigma_1 - \sigma_2)(\sigma_0 - \sigma_1 + \sigma_2)(\sigma_0 - \sigma_1 - \sigma_2)]^{2^{\min(n,m)-3}} \\ &= [\sigma_0^4 + \sigma_1^4 + \sigma_2^4 - 2\sigma_0^2\sigma_1^2 - 2\sigma_0^2\sigma_2^2 - 2\sigma_1^2\sigma_2^2]^{2^{\min(n,m)-3}} \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{P}(M) &= [\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3]^{2^{\min(n,m)-3}} \\ &= [(\lambda_1 + \lambda_2 + \lambda_3)^2 - 4(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)]^{2^{\min(n,m)-3}}. \end{aligned}$$

Replacing the symmetric functions in the eigenvalues by the appropriate polynomials in the coefficients of the matrix $J_n M J_m M^T$, which is given by

$$\begin{pmatrix} (x_0 - x_1)(2y_0 + x_0 + x_1) - x_2^2 & (x_0 - x_1)(x_0 + x_1 + y_0 + y_1) - x_2^2 & (x_0 - x_1)y_2 & 0 \\ -(x_0 - x_1)(x_0 + x_1 + y_0 + y_1) + x_2^2 & -(x_0 - x_1)(2y_1 + x_0 + x_1) + x_2^2 & -(x_0 - x_1)y_2 & 0 \\ -(x_0 - x_1)y_2 & -(x_0 - x_1)y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we obtain a polynomial that is identically zero in the remaining variables $x_0, x_1, x_2, y_0, y_1, y_2$.

This completes the proof. $\square$

Therefore the boundary of $Sep_{n,m}$ is described by the zero set of the polynomial $\mathcal{P}$. More precisely, consider the complement of the zero set of $\mathcal{P}$. Then the interior of $Sep_{n,m}$ is the connected component of this complement which contains the matrix e_{00}.

Lemma 3.15. *Let* $\min(n,m) \geq 3$. *Then the interior of* $P_{n,m}$ *is described by the nonlinear matrix inequality*

$$\exists\, \lambda: \ MJ_mM^T \succ \lambda J_n, \quad M_{00} > 0. \tag{6}$$

Proof. From Lemma 3.1 it follows that the interior of $P_{n,m}$ is described by the nonlinear matrix inequality

$$\exists \lambda \geq 0: \quad MJ_mM^T \succ \lambda J_n, \quad Me_0^m \in int\, L_n.$$

This matrix inequality implies (6). Let us show the converse implication. Suppose M fulfills (6). If $\lambda \leq 0$, then both matrices MJ_mM^T and $-\lambda J_n$ have at most one positive eigenvalue. Since $m \geq 3$, the matrix $MJ_mM^T - \lambda J_n$ cannot be positive definite and (6) is violated. It follows that $\lambda > 0$. For any non-zero vector $v \in L_n$ we then have $v^TMJ_mM^Tv > \lambda v^TJ_nv \geq 0$. Hence either $M^Tv \in int\, L_m$ or $M^Tv \in -int\, L_m$. Since $L_n \setminus \{0\}$ is connected, the map M^T maps the whole set $L_n \setminus \{0\}$ either to $int\, L_m$ or to $-int\, L_m$. By Lemma 3.5 M^T is hence either in $int\, P_{m,n}$ or in $-int\, P_{m,n}$. The inequality $M_{00} > 0$ now yields $M^T \in int\, P_{m,n}$ and hence $M \in int\, P_{n,m}$. $\square$

Lemma 3.16. *Let* A, B *be real symmetric matrices of size* $n \times n$, $n \geq 2$. *Suppose that the line* $L = \{A + \lambda B \,|\, \lambda \in \mathbb{R}\}$ *intersects the cone of positive semidefinite matrices, but not its interior. Then the polynomial* $p(\lambda) = \det(A + \lambda B)$ *has a real multiple root.*

Proof. Assume without restriction of generality that A is positive semidefinite of rank $k < n$. If $k = 0$, then $p(\lambda) = \lambda^n \det B$ and the lemma is proven. Assume hence that $k > 0$. By conjugation of both A, B with a nonsingular matrix we can bring A to the form $diag(1, \dots, 1, 0, \dots, 0)$. Partition the matrix B accordingly,

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{pmatrix}.$$

Then we have $p(\lambda) = \det(I_k + \lambda B_{11}) \det(\lambda B_{22} - \lambda^2 B_{12}(I_k + \lambda B_{11})^{-1} B_{12}^T) = \lambda^{n-k} \det(I_k + \lambda B_{11}) \det(B_{22} - \lambda B_{12}(I_k + \lambda B_{11})^{-1} B_{12}^T)$. If $n - k > 1$, then $p(\lambda)$ has a multiple zero at 0. Assume hence that $k = n - 1$.

Both the cone of positive semidefinite matrices and the line L are convex, hence there exists a linear functional that separates them. In other words, there exists a nonzero positive semidefinite matrix C such that $\langle A, C \rangle = \langle B, C \rangle = 0$. But then C must be of rank 1. Since the images of A and C are orthogonal, the only nonzero element of C is the lower right corner element. This implies $B_{22} = 0$ and $p(\lambda) = -\lambda^2 \det(I_{n-1} + \lambda B_{11})(B_{12}(I_{n-1} + \lambda B_{11})^{-1} B_{12}^T)$ also has a multiple zero at 0. This completes the proof. $\square$

By Lemma 3.15 $M \in int\, P_{n,m}$ if and only if $M_{00} > 0$ and the line $\{MJ_mM^T + \lambda J_n \,|\, \lambda \in \mathbb{R}\}$ intersects the interior of the cone of positive semidefinite matrices. Let now $M \in \partial P_{n,m}$. If $M_{00} = 0$, then $M = 0$ and the polynomial $d_M(\lambda) = \det(MJ_mM^T + \lambda J_n)$ has a multiple zero at $\lambda = 0$. If $M_{00} > 0$, then the line $\{MJ_mM^T + \lambda J_n \,|\, \lambda \in \mathbb{R}\}$ intersects the cone of positive semidefinite matrices, but not its interior (otherwise we would have $M \in int\, P_{n,m}$). Then by the previous lemma the polynomial $d_M(\lambda)$ has a multiple real zero.

Therefore $\partial P_{n,m}$ is contained in the zero set of the discriminant of $d_M(\lambda)$. This discriminant $D(M)$ is now a polynomial in the entries of M. But $D(M)$ is also zero at points in the interior of $P_{n,m}$, e.g. the point e_{00}, because for this point $d_{e_{00}}(\lambda)$ has a multiple zero at $\lambda = 0$. We use this fact to show that for $\min(n,m) \geq 3$ there exists no polynomial in the entries of M that fulfills an analogue of Theorem 3.14 for the cone $P_{n,m}$.

Lemma 3.17. *Let* $\min(n,m) \geq 3$. *Then there does not exist a polynomial* $Q(M)$ *in the entries of* M *such that if* $M \in int\, P_{n,m}$, *then* $Q(M) > 0$ *and if* $M \in \partial P_{n,m}$, *then* $Q(M) = 0$.

Proof. Let first $n = m = 3$ and consider the following 2-dimensional affine subspace of the space of real 3×3 matrices.

$$L = \left\{ \left. \begin{pmatrix} 2 & 0 & a \\ a & 1+b & 0 \\ 0 & 0 & 1-b \end{pmatrix} \right| a, b \in \mathbb{R} \right\}.$$

We will identify the matrices in this subspace with the corresponding points $(a,b) \in \mathbb{R}^2$. Let $P_L = L \cap P_{3,3}$ be the intersection of the positive cone with this subspace. It is a compact set, which is invariant with respect to reflections around the coordinate axes, i.e. the maps $a \mapsto -a$ and $b \mapsto -b$. We have

$$\begin{aligned} & \det\left[\begin{pmatrix} 2 & 0 & a \\ a & 1+b & 0 \\ 0 & 0 & 1-b \end{pmatrix} J_3 \begin{pmatrix} 2 & a & 0 \\ 0 & 1+b & 0 \\ a & 0 & 1-b \end{pmatrix} + \lambda J_3\right] \\ = \; & \lambda^3 + 2\left(3 - a^2 + b^2\right)\lambda^2 + \left(9 - 2a^2 + 6b^2 + (a^2-b^2)^2\right)\lambda + 4(1-b^2)^2. \end{aligned}$$

The discriminant of this polynomial is a polynomial $D(a,b)$ of degree 10 which contains only even powers of a, b. We introduce new variables α, β and define the quintic polynomial $d(\alpha,\beta)$ by $d(a^2,b^2) \equiv D(a,b)$. It is given by

$$\begin{aligned} d(\alpha,\beta) \;=\; & -12\beta^3\alpha + 5\beta^2\alpha^2 + 12\beta^2\alpha - 7\beta^3\alpha^2 + 7\beta^2\alpha^3 + 4\beta^4\alpha - 117\beta\alpha^2 + 252\beta\alpha + 18\beta\alpha^3 - 4\beta\alpha^4 \\ & -118\beta^3 + 180\beta^2 + 20\beta^4 - 81\beta - \beta^5 - 9\alpha^2 + 23\alpha^3 - 4\alpha^4 + \alpha^5. \end{aligned}$$

Define further a map $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ by $(a,b) \mapsto (a^2,b^2)$. Then the image $\mu[P_L]$ will be a compact set in the closed positive orthant, bounded by the coordinate axes and an arc γ which lies in the positive orthant and links the point $(\alpha_*,0)$ to the point $(0,1)$, where $\alpha_* \approx 0.4186$ is the real root of the polynomial $\alpha_*^3 - 4\alpha_*^2 + 23\alpha_* - 9$. The image $\mu[\partial P_L]$ is the closure of γ. Therefore the polynomial $d(\alpha,\beta)$ is zero on γ. Note also that $d(0,0) = 0$. In fact, the zero set of $d(\alpha,\beta)$ is a connected algebraic curve γ' containing two ordinary cusps (singularities of the type $f(x,y) = -x^2 + y^3$). These two cusps are the only real solutions of the algebraic equation system $\{d(\alpha,\beta) = 0, \nabla d(\alpha,\beta) = 0\}$. Hence, if we complexify $\mathbb{R}^2$ to $\mathbb{C}^2$, then the curve γ' turns out to be the intersection of the real space $\mathbb{R}^2$ with a single Riemann surface S_d.

Suppose now that there exists a polynomial $Q(M)$ that fulfills the conditions of the lemma. Then its restriction to L will be a polynomial $Q_L(a,b)$ in two variables. Define the polynomial $q_L(a,b) = Q_L(a,b) + Q_L(-a,b) + Q_L(a,-b) + Q_L(-a,-b)$. This polynomial contains only even powers of each variable and we have $q_L > 0$ on $int\, P_L$ and $q_L = 0$ on ∂P_L, due to the symmetry of P_L. Finally, define a polynomial $q(\alpha,\beta)$ by $q(a^2,b^2) \equiv q_L(a,b)$. Then we have $q(\alpha,\beta) = 0$ on the arc γ and $q(0,0) > 0$. But this is impossible, because if an analytic function on $\mathbb{C}^2$ is zero on γ, then by an analytic continuation argument it has to be zero on the whole Riemann surface S_d, to which the point $(0,0)$ belongs.

This proves the lemma for the case $n = m = 3$.

If a polynomial $Q(M)$ with the properties enumerated in the lemma would exist for some pair (n,m) with $\min(n,m) \geq 3$, we could construct such a polynomial also for $n = m = 3$ by restricting $Q(M)$ to the subspace of matrices spanned by e_{kl}, $k,l = 0,1,2$. As just shown, this is impossible. □

Corollary 3.18. *The cones $P_{n,m}$, $\min(n,m) \geq 3$, are not hyperbolic.* □

Let us resume the results of this section. We studied the possibility to simplify positive maps by scaling. We showed that the interiors of $P_{n,m}$ and $Sep_{n,m}$ consist of orbits containing diagonal matrices. Moreover, the intersections of the subspace composed of the diagonal matrices with $P_{n,m}$ and $Sep_{n,m}$, respectively, are polyhedral with a particularly simple description. This enabled us to construct a polynomial $\mathcal{P}(M)$ whose zero set contains the boundary of $Sep_{n,m}$ in a way such that $Sep_{n,m}$ is a connected component of the complement of this zero set.

4 Construction of the LMI describing $Sep_{n,m}$

In this section we show that the polynomial $\mathcal{P}$ constructed in the previous section is hyperbolic, with $Sep_{n,m}$ as the corresponding hyperbolic cone. This will be accomplished by furnishing a linear mapping $\mathcal{A} : M \mapsto \mathcal{A}(M)$ which takes values in some space $\mathcal{S}$ of complex hermitian matrices, such that the determinant $\det \mathcal{A}(M)$ coincides with a power of the polynomial $\mathcal{P}(M)$. We first give a sketch of the proof to clarify the main line of reasoning.

We need two ingredients for the proof. First, we have to show that the determinant of $\mathcal{A}(M)$ coincides with a power of $\mathcal{P}(M)$ on the subspace composed of the diagonal matrices M. Second, we have to find a representation $\mathcal{G}$ of the symmetry group $SO^+_{1,n-1}(\mathbb{R}) \times SO^+_{1,m-1}(\mathbb{R})$ as a group of automorphisms of the space $\mathcal{S}$. This subgroup of $\mathrm{Aut}(\mathcal{S})$ has to preserve the determinant on $\mathcal{S}$. Moreover, the representation $\mathcal{G}$ has to mimic the action of $SO^+_{1,n-1}(\mathbb{R}) \times SO^+_{1,m-1}(\mathbb{R})$ on the entries of M. This means that for any $n \times m$ matrix M and any $(U,V) \in SO^+_{1,n-1}(\mathbb{R}) \times SO^+_{1,m-1}(\mathbb{R})$ the relation $\mathcal{G}(U,V)[\mathcal{A}(M)] = \mathcal{A}(UMV^T)$ holds. Note that the action of the product group $SO^+_{1,n-1}(\mathbb{R}) \times SO^+_{1,m-1}(\mathbb{R})$ on the space of real $n \times m$ matrices is the tensor product of the actions of the individual restricted special orthogonal groups on $\mathbb{R}^n, \mathbb{R}^m$, respectively. Since the spin group $\mathrm{Spin}_{1,n-1}(\mathbb{R})$ is a twofold cover of $SO^+_{1,n-1}(\mathbb{R})$ with kernel ± 1, it therefore suggests itself to look for representations of $\mathrm{Spin}_{1,n-1}(\mathbb{R})$, $\mathrm{Spin}_{1,m-1}(\mathbb{R})$ as subgroups G_n, G_m of appropriate special linear groups and to construct $\mathcal{A}$ as some tensor product $\mathcal{B}_n \otimes \mathcal{B}_m$, where $\mathcal{B}_n, \mathcal{B}_m$ are linear mappings from $\mathbb{R}^n$, $\mathbb{R}^m$ to smaller factor spaces of complex hermitian matrices. This means that we define $\mathcal{A}$ on rank 1 matrices $M = xy^T$ as $\mathcal{A}(xy^T) = \mathcal{B}_n(x) \otimes \mathcal{B}_m(y)$. An element $g \in G_n$ would then act on the first factor space like $B \mapsto gBg^*$, and this action would have to mimic the action of $SO^+_{1,n-1}(\mathbb{R})$ on $\mathbb{R}^n$; similar for the second factor space. Such an action arises in a natural way, namely as action (3) on the vector space V underlying the Clifford algebra $Cl_{1,n-1}(\mathbb{R})$. It is then obvious to use a representation of the whole Clifford algebra $Cl_{1,n-1}(\mathbb{R})$ itself. But then we have to represent the elements of V as *hermitian* matrices, whereas there are elements of V that square to -1. Hence such a direct approach is not possible.

Nevertheless, this situation can be remedied by making use of Theorem 2.10. We are interested in the representations of the spin group $\mathrm{Spin}_{1,s-1}(\mathbb{R})$ $(s = n, m)$. Let hence $p = 1$, $q = s-1$. By Theorem 2.10, $\mathrm{Spin}_{1,s-1}(\mathbb{R})$ can be considered as a subgroup of the Clifford algebra $Cl_{s-1}(\mathbb{R})$, namely Ω'. This algebra also contains the space $\mathbb{R}^s$, equipped with a quadratic form of signature $(1, s-1)$, where the spin group is acting on. This is the subspace Y spanned by 1 and the generators of the algebra. Since we have the two different values n, m of s in mind, we will attach a subscript to Y and Ω' in the sequel.

We thus look for a complex matrix representation of the Clifford algebra $Cl_{s-1}(\mathbb{R})$ such that its generators are represented by hermitian matrices and the image Ω'_s of the spin group is a subgroup of the special linear group. The former property also guarantees that the transposition in $Cl_{s-1}(\mathbb{R})$ becomes the ordinary hermitian transpose. Let us present such a representation.

We start by defining the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

These matrices generate the well-known *Pauli algebra*, which is simply the algebra $\mathbb{C}(2)$ of complex 2×2 matrices and is isomorphic to the Clifford algebra $Cl_3(\mathbb{R})$. The Pauli matrices play the role of the generators of $Cl_3(\mathbb{R})$, in particular they anticommute and square to the identity. For convenience we also define $\sigma_0 = I_2$. Further we define multi-indexed matrices by $\sigma_{k_1k_2\dots k_\nu} = \sigma_{k_1} \otimes \sigma_{k_2} \otimes \cdots \otimes \sigma_{k_\nu}$, where $\otimes$ denotes the Kronecker product. Hence $\sigma_{k_1k_2\dots k_\nu}$ will be of size $2^\nu \times 2^\nu$. Multiplication of multi-indexed matrices happens index by index, $\sigma_{k_1k_2\dots k_\nu}\sigma_{l_1l_2\dots l_\nu} = (\sigma_{k_1}\sigma_{l_1}) \otimes (\sigma_{k_2}\sigma_{l_2}) \otimes \cdots \otimes (\sigma_{k_\nu}\sigma_{l_\nu})$. Observe that any multi-indexed matrix is hermitian and squares to the identity $\sigma_{0\dots 0}$ (because the σ_k are hermitian and square to the identity). Moreover, any two multi-indexed matrices either commute or anticommute, depending on the number of anticommuting index pairs. In particular, we have the following lemma.

Lemma 4.1. *For a fixed length ν of the index set, all Kronecker squares $\sigma_{k_1k_2\dots k_\nu} \otimes \sigma_{k_1k_2\dots k_\nu} = \sigma_{k_1k_2\dots k_\nu k_1k_2\dots k_\nu}$ commute and hence share a common orthogonal set of eigenvectors.* □

The trace of a multi-indexed matrix $\sigma_{k_1k_2\dots k_\nu}$ is the product $\prod_{j=1}^{\nu}(tr\, \sigma_{k_j})$. Since the Pauli matrices are traceless, the trace of a multi-indexed matrix will be zero unless all indices k_j are zero.

Let us now define a representation $\mathcal{R}_s$ of the Clifford algebra $Cl_{s-1}(\mathbb{R})$ by such multi-indexed matrices. The length ν of the index set will be $\nu = [(s-1)/2]$. The representation maps the generators $f_1, \dots, f_{s-1}$ of the algebra as

$$1 \mapsto \sigma_{0\dots 0}; \qquad f_k \mapsto \sigma_0^{\otimes(\nu-(k+1)/2)} \otimes \sigma_1 \otimes \sigma_3^{\otimes(k-1)/2}, \quad \text{if } 2 \nmid k, k < s-1;$$

$$f_k \mapsto \sigma_0^{\otimes(\nu-k/2)} \otimes \sigma_2 \otimes \sigma_3^{\otimes(k/2-1)}, \quad \text{if } 2 \mid k; \qquad f_{s-1} \mapsto \sigma_{3\dots 3}, 2 \mid s.$$

Here $\sigma_j^{\otimes k}$ denotes the k-fold Kronecker product $\sigma_j \otimes \sigma_j \otimes \cdots \otimes \sigma_j$. For example, for $n = 5$ $(n = 6)$ we have

$$1 \mapsto \sigma_{00}, f_1 \mapsto \sigma_{01}, f_2 \mapsto \sigma_{02}, f_3 \mapsto \sigma_{13}, f_4 \mapsto \sigma_{23}, (f_5 \mapsto \sigma_{33}).$$

For even s we define also another representation $\mathcal{R}'_s$, which differs from $\mathcal{R}_s$ by a sign change in the representation of the last generator, $\mathcal{R}'_s : f_{s-1} \mapsto -\sigma_{3\dots 3}$.

Remark 4.2. The representations $\mathcal{R}_s \circ \Theta^{-1}, \mathcal{R}'_s \circ \Theta^{-1}$ are the irreducible representations of $\mathrm{Spin}_{1,s-1}(\mathbb{R})$ mentioned in Theorem 2.11.

Lemma 4.3. *The representations $\mathcal{R}_s, \mathcal{R}'_s$ map the space $Y_s \subset Cl_{s-1}(\mathbb{R})$ to subspaces of hermitian matrices of size $2^{[(s-1)/2]} \times 2^{[(s-1)/2]}$. The subgroup Ω'_s is mapped to subgroups G_s, G'_s of the special linear group $SL(2^{[(s-1)/2]}, \mathbb{C})$.*

For any (pseudo-)rotation $U \in SO^+_{1,s-1}(\mathbb{R})$ of Y_s there exists an element $x \in \Omega'_s$ such that for any $y \in Y_s$ we have $\mathcal{R}_s(U(y)) = g\mathcal{R}_s(y)g^*$, $\mathcal{R}'_s(U(y)) = g'\mathcal{R}'_s(y)g'^*$, where $g = \mathcal{R}_s(x) \in G_s$, $g' = \mathcal{R}'_s(x) \in G'_s$. Moreover, $\det \mathcal{R}_s(U(y)) = \det \mathcal{R}_s(y)$ and $\det \mathcal{R}'_s(U(y)) = \det \mathcal{R}'_s(y)$.

Here g^* denotes the complex conjugate transpose of g.

Proof. The subspace $Y_s \subset Cl_{s-1}(\mathbb{R})$ is spanned by 1 and the generators f_k, which are mapped to hermitian multi-indexed matrices $\sigma_{k_1\dots k_\nu}$. Hence $\mathcal{R}_s[Y_s]$, $\mathcal{R}'_s[Y_s]$ are subspaces of hermitian matrices. By Theorem 2.10 the Lie algebra of Ω'_s is represented in $Cl_{s-1}(\mathbb{R})$ by the subspace spanned by the f_k and their products $f_k f_l$. However, among the matrices $\mathcal{R}_s(f_k), \mathcal{R}_s(f_k)\mathcal{R}_s(f_l)$ there are no multiples of the identity, and therefore the image of the Lie algebra under $\mathcal{R}_s$ and $\mathcal{R}'_s$ is traceless. This implies that Ω'_s is represented as a subgroup of the special linear group.

Let $U \in SO^+_{1,s-1}(\mathbb{R})$ and choose an element $x \in \Omega'_s$ such that $\varsigma'_x = U$. This is possible by Theorem 2.10. Then $U(y) = xyx^t$ for all $y \in Y_s$ and $g^* = \mathcal{R}_s(x^t)$, $g'^* = \mathcal{R}'_s(x^t)$. Note that by the above $\det g = \det g' = 1$. The second paragraph of the lemma then follows from the fact that $\mathcal{R}_s$, $\mathcal{R}'_s$ are representations of $Cl_{s-1}(\mathbb{R})$. $\square$

The action of $\mathrm{Aut}(L_n) \times \mathrm{Aut}(L_m)$ on the space of real $n \times m$ matrices, which is isomorphic to $\mathbb{R}^n \otimes \mathbb{R}^m$, can be represented as tensor product of the action of $\mathrm{Aut}(L_n)$ on $\mathbb{R}^n$ with the action of $\mathrm{Aut}(L_m)$ on $\mathbb{R}^m$. In other words, if $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $M = x \otimes y = xy^T$ and $U \in \mathrm{Aut}(L_n)$, $V \in \mathrm{Aut}(L_n)$, then the result of the action of (U, V) on M is just $UMV^T = Ux(Vy)^T = (Ux) \otimes (Vy)$.

Let us identify the space of $n \times m$ matrices with $Y_n \otimes Y_m$, in a way such that the quadratic forms on Y_n and Y_m coincide with the quadratic forms J_n and J_m on the factor spaces $\mathbb{R}^n$, $\mathbb{R}^m$. Namely, we identify e_{kl}, $k = 0, \dots, n-1$, $l = 0, \dots, m-1$, with $f^n_k \otimes f^m_l$, where for convenience we put $f^n_0 = 1 \in Cl_{n-1}(\mathbb{R})$, $f^m_0 = 1 \in Cl_{m-1}(\mathbb{R})$; f^n_k, $k = 1, \dots, n-1$ are the generators of $Cl_{n-1}(\mathbb{R})$ and f^m_l, $l = 1, \dots, m-1$ are the generators of $Cl_{m-1}(\mathbb{R})$. Define $\nu_n = [(n-1)/2]$ and $\nu_m = [(m-1)/2]$. The tensored map $\mathcal{R}_n \otimes \mathcal{R}_m$ then maps $Y_n \otimes Y_m$ to the space of $2^{\nu_n+\nu_m} \times 2^{\nu_n+\nu_m}$ complex hermitian matrices. The preceding lemma then yields the following lemma.

Lemma 4.4. *Let $U \in SO^+_{1,n-1}(\mathbb{R})$, $V \in SO^+_{1,m-1}(\mathbb{R})$. Then there exist elements $x_U \in \Omega'_n$, $x_V \in \Omega'_m$ such that $(\mathcal{R}_n \otimes \mathcal{R}_m)(UMV^T) = g[(\mathcal{R}_n \otimes \mathcal{R}_m)(M)]g^*$, where $g = \mathcal{R}_n(x_U) \otimes \mathcal{R}_m(x_V) \in G_n \otimes G_m$. Moreover, $\det g = 1$ and $\det[(\mathcal{R}_n \otimes \mathcal{R}_m)(UMV^T)] = \det[(\mathcal{R}_n \otimes \mathcal{R}_m)(M)]$. A similar statement holds for the tensor map $\mathcal{R}_n \otimes \mathcal{R}'_m$.* $\square$

We construct the mapping $\mathcal{A}$ as follows.

For odd $\min(n, m)$, $\mathcal{A} = \mathcal{R}_n \otimes \mathcal{R}_m$. Writing the definition out, we get

$$\mathcal{A}(M) = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} M_{kl}\mathcal{R}_n(f^n_k) \otimes \mathcal{R}_m(f^m_l).$$

For even $\min(n, m)$ and $n \geq m$, $\mathcal{A}$ will be defined as the direct sum $(\mathcal{R}_n \otimes \mathcal{R}_m) \oplus (\mathcal{R}_n \otimes \mathcal{R}'_m)$. This means that for any M the matrix $\mathcal{A}(M)$ is block-diagonal with two blocks, one of them being the image of M under $\mathcal{R}_n \otimes \mathcal{R}_m$ and the other the image under $\mathcal{R}_n \otimes \mathcal{R}'_m$. Let $\mathcal{A}^1, \mathcal{A}^2$ be the mappings taking M to the individual blocks. Writing the definitions out, we get

$$\mathcal{A}^1(M) = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} M_{kl}\mathcal{R}(f^n_k) \otimes \mathcal{R}(f^m_l), \quad \mathcal{A}^2(M) = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} M_{kl}\mathcal{R}(f^n_k) \otimes \mathcal{R}'(f^m_l).$$

For even $\min(n, m)$ and $n < m$, $\mathcal{A}$ will be defined as the direct sum $(\mathcal{R}_n \otimes \mathcal{R}_m) \oplus (\mathcal{R}'_n \otimes \mathcal{R}_m)$.

By Lemmas 3.13, 4.4 and Corollary 3.7 we have the following lemma.

Lemma 4.5. *Let $M \in int\, P_{n,m}$. Then there exists a diagonal matrix $M' \in int\, P_{n,m}$ such that $\mathcal{P}(M) = \mathcal{P}(M')$ and $\det \mathcal{A}(M) = \det \mathcal{A}(M')$.* $\square$

It now rests to show that $\det \mathcal{A}(M)$ is a power of $\mathcal{P}(M)$ on the subspace of diagonal matrices M.

In the sequel we will consider operators $\mathcal{A}$ for different pairs of dimensions (n, m). Let $\mathcal{A}_s$ be the operator $\mathcal{A}$ corresponding to square matrices of size $s \times s$. Define $\mathcal{A}^1_s$, $\mathcal{A}^2_s$ similarly. However, we will identify notationally the matrices e_{kk} of different sizes to avoid an overloading with indices. The sizes of e_{kk} will be determined by the operators that are acting on them, so a confusion is not possible.

Lemma 4.6. *Let $s = \min(n, m)$. For all $k = 0, \dots, s-1$, the matrices $\mathcal{A}(e_{kk})$ share the common Kronecker factor $I_{2^{|\nu_n-\nu_m|}}$ and can after a possible common rearrangement of rows and columns be represented as Kronecker product $I_{2^{|\nu_n-\nu_m|}} \otimes \mathcal{A}_s(e_{kk})$.*

Proof. Let us introduce the following notation. If A, B are hermitian matrices and there exists a permutation matrix P such that $B = PAP^T$, then we write $A \sim B$. Note that $A \sim B$ yields $\det A = \det B$ and that the rearrangement of factors in a Kronecker product amounts to conjugation with a permutation matrix.

An exchange of n and m amounts to an exchange of Kronecker factors, so we assume without restriction of generality that $n \geq m$. Let us consider two cases.

1. $s = \min(n, m) = m$ is odd. From the definition of $\mathcal{R}_n, \mathcal{R}_m$ we have that $\mathcal{R}_n(f_k^n) = I_{2^{\nu_n-\nu_m}} \otimes \mathcal{R}_m(f_k^m)$ for all $k = 0, \ldots, m-1$. But then

$$\mathcal{A}(e_{kk}) \quad = \quad \mathcal{R}_n(f_k^n) \otimes \mathcal{R}_m(f_k^m) = I_{2^{\nu_n-\nu_m}} \otimes \mathcal{R}_m(f_k^m) \otimes \mathcal{R}_m(f_k^m) = I_{2^{\nu_n-\nu_m}} \otimes \mathcal{A}_m(e_{kk}),$$

which proves the lemma for this case.

2. $s = \min(n, m) = m$ is even. If $n = m$, then the lemma holds trivially. Assume $n > m$. This implies $\nu_n > \nu_m$. We have again $\mathcal{R}_n(f_k^n) = I_{2^{\nu_n-\nu_m}} \otimes \mathcal{R}_m(f_k^m) = I_{2^{\nu_n-\nu_m-1}} \otimes \sigma_0 \otimes \mathcal{R}_m(f_k^m)$, but only for all $k = 0, \ldots, m-2$. For $k = m-1$ we have $\mathcal{R}_n(f_{m-1}^n) = I_{2^{\nu_n-\nu_m-1}} \otimes \sigma_1 \otimes \mathcal{R}_m(f_{m-1}^m)$. Let us rearrange the Kronecker factors in order to make the singled out σ_0 or σ_1 the first factor. Then we obtain

$$\begin{aligned}
\mathcal{R}_n(f_k^n) &= [I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}_m(f_k^m)] \oplus [I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}_m(f_k^m)] \\
&= [I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}_m(f_k^m)] \oplus [I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}'_m(f_k^m)], \quad \text{if } k \leq m-2, \\
\mathcal{R}_n(f_{m-1}^n) &\sim [I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}_m(f_{m-1}^m)] \oplus [-I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}_m(f_{m-1}^m)] \\
&= [I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}_m(f_{m-1}^m)] \oplus [I_{2^{\nu_n-\nu_m-1}} \otimes \mathcal{R}'_m(f_{m-1}^m)].
\end{aligned}$$

After a further rearrangement of blocks in the obtained block-diagonal structure this gives the relation $\mathcal{R}_n \sim I_{2^{\nu_n-\nu_m-1}} \otimes (\mathcal{R}_m \oplus \mathcal{R}'_m)$. Since the rearrangement of blocks was the same for all $k = 0, \ldots, m-1$, this relation is valid on the whole subspace of diagonal matrices M. We get further

$$\begin{aligned}
\mathcal{A}^1 &= \mathcal{R}_n \otimes \mathcal{R}_m \sim I_{2^{\nu_n-\nu_m-1}} \otimes (\mathcal{R}_m \oplus \mathcal{R}'_m) \otimes \mathcal{R}_m = I_{2^{\nu_n-\nu_m-1}} \otimes ((\mathcal{R}_m \otimes \mathcal{R}_m) \oplus (\mathcal{R}'_m \otimes \mathcal{R}_m)) \\
&\sim I_{2^{\nu_n-\nu_m-1}} \otimes (\mathcal{A}_m^1 \oplus \mathcal{A}_m^2), \\
\mathcal{A}^2 &= \mathcal{R}_n \otimes \mathcal{R}'_m \sim I_{2^{\nu_n-\nu_m-1}} \otimes (\mathcal{R}_m \oplus \mathcal{R}'_m) \otimes \mathcal{R}'_m = I_{2^{\nu_n-\nu_m-1}} \otimes ((\mathcal{R}_m \otimes \mathcal{R}'_m) \oplus (\mathcal{R}'_m \otimes \mathcal{R}'_m)) \\
&= I_{2^{\nu_n-\nu_m-1}} \otimes (\mathcal{A}_m^2 \oplus \mathcal{A}_m^1).
\end{aligned}$$

The last relation is due to the equality $\mathcal{R}'_m \otimes \mathcal{R}'_m = \mathcal{R}_m \otimes \mathcal{R}_m$, which is valid on all e_{kk}, $k = 0, \ldots, m-1$ and hence on the whole subspace of diagonal matrices. Summarizing, we get

$$\mathcal{A} \quad = \quad \mathcal{A}^1 \oplus \mathcal{A}^2 \sim I_{2^{\nu_n-\nu_m-1}} \otimes (\mathcal{A}_m^1 \oplus \mathcal{A}_m^2 \oplus \mathcal{A}_m^2 \oplus \mathcal{A}_m^1) \sim I_{2^{\nu_n-\nu_m}} \otimes (\mathcal{A}_m^1 \oplus \mathcal{A}_m^2) = I_{2^{\nu_n-\nu_m}} \otimes \mathcal{A}_m.$$

This proves the lemma also for this case. □

Corollary 4.7. *Let M be a diagonal matrix of size $n \times m$. Put $s = \min(n, m)$ and define M_{tr} as the truncation of M to a square matrix of size $s \times s$. Then, after a possible rearrangement of rows and columns, $\mathcal{A}(M)$ can be written as $I_{2^{|\nu_n-\nu_m|}} \otimes \mathcal{A}_s(M_{tr})$. In particular, $\det \mathcal{A}(M) = (\det \mathcal{A}_s(M_{tr}))^{(2^{|\nu_n-\nu_m|})}$.* □

In view of this corollary, it suffices to consider the case when $n = m = \min(n, m)$ if we study the determinant of $\mathcal{A}(M)$. We will hence assume $m = n$ in the sequel.

By Lemma 4.1 it follows that the matrices $\mathcal{A}_n(e_{kk})$, $k = 0, \ldots, n-1$, share a common orthogonal set $\mathcal{V}$ of eigenvectors. In the basis defined by $\mathcal{V}$ the matrix $\mathcal{A}_n(M)$ will be diagonal for all diagonal matrices M. The eigenvectors in $\mathcal{V}$ are Kronecker products of elements of the common set of eigenvectors of $\sigma_{00}, \sigma_{11}, \sigma_{22}, \sigma_{33}$, which is given by the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

In the corresponding basis σ_{ll} $(l = 0, 1, 2, 3)$ becomes $diag(d_l)$, where the d_l are the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}. \tag{7}$$

The vector of diagonal elements of $\mathcal{A}_n(e_{kk})$ in the basis defined by $\mathcal{V}$ will be given by Kronecker products of the d_l, $l = 0, 1, 2, 3$. Hence the eigenvalues of $\mathcal{A}_n(e_{kk})$ can only be ± 1.

Let λ_k^v be the eigenvalue of $\mathcal{A}_n(e_{kk})$ associated to the common eigenvector $v \in \mathcal{V}$.

Lemma 4.8. *All eigenvalues of $\mathcal{A}_n(e_{00})$ equal 1. The vector $(\lambda_1^v, \ldots, \lambda_{n-1}^v)$ runs through all 2^{n-1} possible combinations of ± 1 if v runs through the set $\mathcal{V}$.*

Proof. The first assertion of the lemma is clear. The second one will be proven by induction over n.

We begin with $n = 3$. The representation $\mathcal{R}_3$ is defined by $1 \mapsto \sigma_0$, $f_1 \mapsto \sigma_1$, $f_2 \mapsto \sigma_2$. Hence $\mathcal{A}_3(e_{00}) = \sigma_{00}$, $\mathcal{A}_3(e_{11}) = \sigma_{11}$, $\mathcal{A}_3(e_{22}) = \sigma_{22}$. The eigenvalues of these three matrices are given by the first 3 columns of matrix (7). While the first column, which corresponds to $\mathcal{A}_3(e_{00})$, consists entirely of 1's, the next two columns indeed present all 4 possible combinations of ± 1.

Induction step from odd $n-1$ to even n. Let n be even, and suppose the lemma holds for $n-1$. Then we have two representations $\mathcal{R}_n, \mathcal{R}'_n$ of the Clifford algebra $Cl_{n-1}(\mathbb{R})$ and two subblocks $\mathcal{A}^1_n, \mathcal{A}^2_n$ of the matrix $\mathcal{A}_n$. Since $\nu_{n-1} = \nu_n$, we have $\mathcal{R}_n(f^n_k) = \mathcal{R}'_n(f^n_k) = \mathcal{R}_{n-1}(f^{n-1}_k)$ for $k = 0, \dots, n-2$. Therefore $\mathcal{A}_{n-1}(e_{kk}) = \mathcal{A}^1_n(e_{kk}) = \mathcal{A}^2_n(e_{kk})$ for $k = 0, \dots, n-2$. Further, we have $\mathcal{A}^1_n(e_{n-1,n-1}) = -\mathcal{A}^2_n(e_{n-1,n-1}) = \sigma_3^{\otimes(n-2)}$. The matrices $\mathcal{A}^1_n(e_{kk}), \mathcal{A}^2_n(e_{kk})$, $k = 0, \dots, n-1$ therefore have the same set of eigenvectors, but the eigenvalues of $\mathcal{A}^1_n(e_{n-1,n-1}), \mathcal{A}^2_n(e_{n-1,n-1})$ have each time opposite sign, whereas the eigenvalues of $\mathcal{A}^1_n(e_{kk}), \mathcal{A}^2_n(e_{kk})$, $k = 1, \dots, n-2$ are the same. But $\mathcal{A}_{n-1}(e_{kk}) = \mathcal{A}^1_n(e_{kk}) = \mathcal{A}^2_n(e_{kk})$, hence by the assumption of the induction the eigenvalues of $\mathcal{A}^1_n(e_{kk}), \mathcal{A}^2_n(e_{kk})$, $k = 1, \dots, n-2$ both run through all possible combinations of ± 1. Therefore the vector $(\lambda^v_1, \dots, \lambda^v_{n-2})$ runs two times through all possible combinations of ± 1 if v runs through $\mathcal{V}$, whereas $\lambda^v_{n-1}, \lambda^{v'}_{n-1}$ have opposite signs whenever $(\lambda^v_1, \dots, \lambda^v_{n-2}) = (\lambda^{v'}_1, \dots, \lambda^{v'}_{n-2})$. This proves the assertion of the lemma.

Induction step from odd $n-2$ to odd n. Let n be odd, and suppose the lemma holds for $n-2$. Let $\mathcal{W}$ be the common set of eigenvectors of $\mathcal{A}_{n-2}(f^{n-2}_k)$, $k = 1, \dots, n-3$ and $\sigma_3^{\otimes(n-3)}$. Let us enumerate the vectors $w \in \mathcal{W}$ in some way and define a $2^{n-3} \times (n-3)$ matrix $(\mathbf{W}_{jk})$ in the following way. The element $\mathbf{W}_{jk}$ will be the eigenvalue of the matrix $\mathcal{A}_{n-2}(f^{n-2}_k)$ associated with the eigenvector $w_j \in \mathcal{W}$. By the assumption of the induction the rows of $\mathbf{W}$ run through all 2^{n-3} possible combinations of ± 1. Define further a column vector $\mathbf{w}$ of length 2^{n-3} putting $\mathbf{w}_j$ equal to the eigenvalue of $\sigma_3^{\otimes(n-3)}$ associated with the eigenvector $w_j \in \mathcal{W}$.

We have by definition $\mathcal{R}_n(f^n_k) = \sigma_0 \otimes \mathcal{R}_{n-2}(f^{n-2}_k)$ for $k = 0, \dots, n-3$ and $\mathcal{R}_n(f^n_{n-2}) = \sigma_1 \otimes \sigma_3^{\otimes((n-3)/2)}$, $\mathcal{R}_n(f^n_{n-1}) = \sigma_2 \otimes \sigma_3^{\otimes((n-3)/2)}$. A common rearrangement of Kronecker factors then yields $\mathcal{A}_n(f^n_k) \sim \sigma_{00} \otimes \mathcal{A}_{n-2}(f^{n-2}_k)$ for $k = 0, \dots, n-3$ and $\mathcal{A}_n(f^n_{n-2}) \sim \sigma_{11} \otimes \sigma_3^{\otimes(n-3)}$, $\mathcal{A}_n(f^n_{n-1}) \sim \sigma_{22} \otimes \sigma_3^{\otimes(n-3)}$. Thus the vectors in $\mathcal{V}$ can be arranged in such a way that the $2^{n-1} \times n$ matrix $(\mathbf{V}_{jk})$ defined componentwise by $\mathbf{V}_{jk} = \lambda^{v_j}_k$ has the form

$$\mathbf{V} = (d_0 \otimes \mathbf{W} \quad d_1 \otimes \mathbf{w} \quad d_2 \otimes \mathbf{w}).$$

Observing, as for the case $n = 3$, than d_1, d_2 form all possible combinations of the two remaining ± 1, one completes the proof. □

The following corollary now follows immediately from the definition (5) of $\mathcal{P}(M)$ for diagonal matrices M.

Corollary 4.9. *Let $n = m \geq 3$. Then on the subspace of diagonal matrices M we have* $\det \mathcal{A}(M) \equiv \mathcal{P}(M)$. □

From Corollary 4.7 we obtain the following.

Corollary 4.10. *For general $n, m \geq 3$, on the subspace of diagonal matrices M we have* $\det \mathcal{A}(M) \equiv \mathcal{P}(M)^{(2^{|\nu_n - \nu_m|})}$. □

Theorem 4.11. *Let* $\min(n,m) \geq 3$. *A matrix M is contained in $Sep_{n,m}$ if and only if $\mathcal{A}(M) \succeq 0$. The cone $Sep_{n,m}$ is hyperbolic, defined by the hyperbolic polynomial $\mathcal{P}(M)$. The power $\mathcal{P}(M)^{(2^{|\nu_n - \nu_m|})}$ of this polynomial coincides with the determinant of the matrix $\mathcal{A}(M)$. For odd* $\min(n,m)$ *the matrix $\mathcal{A}(M)$ has size $2^{\nu_m + \nu_n}$. For even* $\min(n,m)$ *it decomposes in two subblocks $\mathcal{A}_1(M), \mathcal{A}_2(M)$ of size $2^{\nu_m + \nu_n}$ each. Here $\nu_m = [\frac{m-1}{2}]$, $\nu_n = [\frac{n-1}{2}]$.*

Proof. By Lemma 4.5, the relation $\det \mathcal{A}(M) \equiv \mathcal{P}(M)^{(2^{|\nu_n - \nu_m|})}$ holds on the whole interior of the cone $P_{n,m}$, and hence on the whole nm-dimensional space of real $n \times m$ matrices. Clearly $\det \mathcal{A}(M)$ is a hyperbolic polynomial. It follows that $\mathcal{P}$ is also hyperbolic, and their positive cones coincide. The interior of the cone $Sep_{n,m}$ is by Theorem 3.14 a connected component of the complement to the zero set of $\mathcal{P}$. But $e_{00} \in int\, Sep_{n,m}$ and $\mathcal{A}(e_{00})$ is the identity matrix and hence positive definite. Therefore $Sep_{n,m}$ must coincide with the positive cone defined by $\det \mathcal{A}(M)$, and hence by $\mathcal{P}$. □

In particular, $\mathcal{A}[Sep_{n,m}]$ is a section of the positive semidefinite matrix cone. From this it is straightforward to construct an LMI defining $P_{n,m}$. For odd $\min(n,m)$ this cone can be described as a projection of the cone of $2^{\nu_m + \nu_n} \times 2^{\nu_m + \nu_n}$ positive semidefinite complex hermitian matrices. For even $\min(n,m)$ it is a sum of two projections of the cone of $2^{\nu_m + \nu_n} \times 2^{\nu_m + \nu_n}$ positive semidefinite hermitian matrices.

5 Conclusions

In this contribution we have constructed LMI's describing the cone of Lorentz-Lorentz separable elements and the cone of linear maps taking a Lorentz cone L_m to another Lorentz cone L_n. The size of the LMI's, however, is exponential in the dimensions of the Lorentz cones. This is due to the fact that the LMI was constructed using exact representations of the spin group $\mathrm{Spin}_{1,n-1}(\mathbb{R})$, which is a two-fold cover (and a universal cover for $n \geq 4$) of the connected subgroup of $\mathrm{Aut}(L_n)$, the automorphism group of the Lorentz cone. This spin group has only exact representations of exponential size, whereas the group $\mathrm{Aut}(L_n)$ itself trivially has representations of the same dimension as the Lorentz cone. However, embeddings of the automorphism group in the automorphism group of a positive semidefinite matrix cone are of quadratic nature (the group element acting as $v \mapsto gvg^*$) and therefore naturally bring the two-fold covering group into play.

The result suggests that algebraic properties of symmetry groups, in particular their representations, play a major role in the study of separable cones. Let us stress, however, that the presented result relies on the exceptional richness of the scaling group for the case of Lorentz cones, which allowed to reduce the dimensions of the considered cones from nm to $\min(n, m)$. Such a reduction is not possible for separable matrix cones. For $\min(n, m) = 3, 4$ the constructed LMI coincides with the well-known semidefinite descriptions for 2×2 bipartite separable matrices in the real symmetric and the complex hermitian case. It also lets these descriptions appear from a new angle of view, linking them to the irreducible representations of the spin group and explaining the additional appearance of the partial transpose in the complex hermitian case with the existence of a second irreducible representation of the spin group $\mathrm{Spin}_{1,3}(\mathbb{R})$. For this case there exist many proofs, even solely by dimensional arguments [10]. Leonid Gurvits provided a proof that is based on scaling [9].

The question whether there exists an LMI description of $P_{n,m}$ and $Sep_{n,m}$ of polynomial size remains open, however. In the case $n = m$ the presented description of $Sep_{n,m}$ is the simplest one if one restricts the search to *sections* of positive semidefinite matrix cones, because the hyperoctahedral section of $Sep_{n,n}$ corresponding to diagonal matrices has already 2^{n-1} faces of codimension 1. The cone $P_{n,m}$ cannot at all be described as such a section for $\min(n, m) \geq 3$ and is not hyperbolic.

References

[1] Heinz H. Bauschke, Osman Güler, Adrian S. Lewis, and Hristo S. Sendov. Hyperbolic polynomials and convex analysis. *Canadian Journal of Mathematics*, 53(3):470–488, 2001.

[2] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Math. Oper. Res.*, 23:769–805, 1998.

[3] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15 of *SIAM Stud. Appl. Math.* SIAM, 1994.

[4] E. Cartan. *The Theory of Spinors.* Dover, New York, 1981.

[5] M.-D. Choi. A Schwarz inequality for positive linear maps on C-algebras. *Illinois J. Math.*, 18(4):565–574, 1974.

[6] Lloyd L. Dines. On linear combinations of quadratic forms. *Bull. Am. Math. Soc.*, 49:388–393, 1943.

[7] Leonid Gurvits. Classical deterministic complexity of edmonds' problem and quantum entanglement. arXiv quant-ph e-print 0303055, 2003.

[8] Leonid Gurvits and Howard Barnum. Separable balls around the maximally mixed multipartite quantum states. arXiv quant-ph e-print 0302102, 2003.

[9] Leonid Gurvits and Howard Barnum. Revisiting separability of 2x2 quantum systems. Personal communication, 2004.

[10] Roland Hildebrand. On the exactness of sums of squares relaxations involving 3x3 matrices. Submitted to Mathematical Programming Series A, October 2004.

[11] Ryszard Horodecki, Pawel Horodecki, and Michal Horodecki. Separability of mixed states: necessary and sufficient conditions. *Physics Letters*, 223(1):1–8, 1996.

[12] P. Lounesto. *Clifford Algebras and Spinors.* Cambridge University Press, Cambridge, 2001.

[13] Zhi-Quan Luo, Jos F. Sturm, and Shuzhong Zhang. Multivariate nonnegative quadratic mappings. *SIAM J. Optim.*, 14(4):1140–1162, 2004.

[14] Arkadi Nemirovski. A separation oracle for the cone of Lorentz-positive matrices. Private communication, Georgia Tech School of Industrial and Systems Engineering, 2005.

[15] A. Peres. Separability criterion for density matrices. *Phys. Rev. Lett.*, 77:1413–1415, 1996.

[16] Ian Porteous. *Topological geometry.* van Nostrand, New York, 1969.

[17] Erling Størmer. Positive linear maps of operator algebras. *Acta Mathematica*, 110:233–278, 1963.

[18] Jos F. Sturm and Shuzhong Zhang. On cones of nonnegative quadratic functions. *Math. Oper. Res.*, 28(2):246–267, 2003.

[19] F.J. Terpstra. Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung. *Mathematische Annalen*, 116:166–180, 1939.

[20] E. B. Vinberg. The theory of convex homogeneous cones. *Transactions of Moscow Mathematical Society*, 12:340–403, 1963.

[21] Frank Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Springer, New York, Berlin, Heidelberg, Tokyo, 1983.

[22] Wikipedia. Clifford algebras. http://en.wikipedia.org/wiki/Clifford_algebra, 2005.

[23] S.L. Woronowicz. Positive maps of low dimensional matrix algebras. *Reports on Mathematical Physics*, 10:165–183, 1976.

[24] V.A. Yakubovitch. Factorization of symmetric matrix polynomials. *Doklady Akademii Nauk SSSR*, 194(3):1261–1264, 1970.