The Effects of Adding Objectives to an Optimisation Problem on the Solution Set

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Abstract

Suppose that for a given optimisation problem (which might be multicriteria problem or a single-criteron problem), an additional objective function is introduced. How does the the set of solutions, i. e. the set of efficient points change when instead of the old problem the new multicriteria problem is considered? How does the set of properly efficient points, the set of weakly efficient points and the set of strictly efficient points change? And what kind of effects occur in the value space? This paper answers these questions under various suitable assumptions.

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1 Introduction and Preliminaries

Suppose that for a given optimisation problem (which might be multicriteria problem or a single-criteron problem), an additional objective function is introduced. How does the the set of solutions, i. e. the set of efficient points change when instead of the old problem the new multicriteria problem is considered? How does the set of properly efficient points, the set of weakly efficient points and the set of strictly efficient points change? And what kind of effects occur in the value space? This paper answers these questions under various suitable assumptions. After introducing the necessary notation in Subsection 1.1, the questions to be asked that are alluded to above and to be answered in what follows are phrased in more detail in Subsection 1.2. Section 2 considers the changes that take place in the set of weakly efficient points when an additional objective function is introduced, as well as the changes that take place in the set of strongly efficient points. Section 3 considers similar effects for the set of efficient points, while Section 4 considers the corresponding phenomena for proper efficient points.

1.1 Preliminaries

Let $X \neq \emptyset$ be an arbitrary set of alternatives (i. e. the set of feasible decisions) and suppose that there are $n \geq 1$ objective functions $f_1, \dots f_n : X \longrightarrow \mathbb{R}$ given. Define the function $f : X \longrightarrow \mathbb{R}^n$ by

$$\begin{array}{rcl} f : X & \longrightarrow & \mathbb{R}^n \\ x & \longmapsto & \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}. \end{array}$$

For the multicriteria optimisation problem defined by the set of feasible points X and the n objective functions $f_1, \dots, f_n$ (i. e. the vector-valued objective function f) to be minimiseds simultaneously, we define the set of *efficient points* (Pareto points, optimal points, minimal points) for this problem in the usual way, i. e. a point $x \in X$ is *efficient* if there does not exist a point $y \in X$ which is at least as good as x with respect to all objective functions involved and better than x with respect to at least one of the objective functions. Denote the set of efficient point by $E(n)$,

$$\begin{array}{rcl} E(n) & := & \{x \in X \mid \nexists\, y \in X : (\forall\, j \in \{1, \dots, n\} : f_j(y) \leq f_j(x) \\ & & \qquad \wedge\, \exists\, i \in \{1, \dots, n\} : f_i(y) < f_i(x))\}. \end{array}$$

Note that this set can be written as

$$E(n) = \{x \in X \mid \forall\, y \in X : f(y) = f(x) \vee \exists\, j \in \{1, \dots, n\} : f_j(x) < f_j(y)\}. \qquad (1)$$

Now suppose that we consider an additional objective function

$$f_{n+1} : X \longrightarrow \mathbb{R}.$$

With this, we can define a second multicriteria optimisation problem involving the set of feasible points X and the $n+1$ objective functions $f_1, \ldots, f_{n+1}$. Here, the solution set, i. e. the set of efficient points is given by

$$\begin{aligned} E(n+1) \quad := \quad \{x \in X \mid \nexists\, y \in X : (\forall i \in \{1, \ldots, n+1\} : f_i(y) \leq f_i(x) \\ \wedge\, \exists\, j \in \{1, \ldots, n+1\} : f_j(y) < f_j(x))\} \end{aligned}$$

which in turn can be written as

$$\begin{aligned} E(n+1) \quad = \quad \Bigg\{ x \in X \;\Bigg|\; \forall\, y \in X : \begin{bmatrix} f(y) \\ f_{n+1}(y) \end{bmatrix} = \begin{bmatrix} f(x) \\ f_{n+1}(x) \end{bmatrix} \\ \vee\, \exists\, j \in \{1, \ldots, n+1\} : f_j(x) < f_j(y) \Bigg\}. \end{aligned} \tag{2}$$

1.2 Some Open Questions

We are interested mainly in the relation between $E(n)$ and $E(n+1)$. Especially, we are interested in the following questions.

1. *How can we characterise the inclusion $E(n) \subseteq E(n+1)$ and what are sufficient conditions for this inclusion?*

   This is a situation which practicioners might want to avoid, since in this case adding another complexity level in the model under consideration, i. e. adding an additional objective function, just increases the number of solutions, which is, in a multicriteria framework, usually already rather large. On the other hand, if solutions in $E(n)$ are considered as unsatisfactory under the given circumstances, introducing an additional objective function might give decision-makers enough leeway to arrive at a decision that is satisfactory in the real world.

2. *How can we characterise the inclusion $E(n+1) \subseteq E(n)$ and what are sufficient conditions for this inclusion?*

   This is the converse situation to the one outlined under point 1. Generally, the solution set of a multicriteria optimisation problem is rather large, and reducing it by considering a more realistic model (i. e. introducing an additional objective function) seems to be an attractive alternative.

3. *How can we characterise points in the set $E(n) \cap E(n+1)$?*

   Points in the intersection of the two solution sets might be denoted as "stable" or "indifferent to the objective f_{n+1}" and might therefore be of particular interest to the decision-makers.

4. *Can similar questions like the three above be answered with respect to the set of weakly efficient points or the set of strongly efficient points?*

5. *What are the topological relationships between* $E(n)$ *and* $E(n+1)$*, i. e. under which assuptions can we conclude* $\text{cl}(E(n+1)) \subseteq E(n)$ *or even* $\text{cl}(E(n+1)) = E(n)$*, and do similar relationships exist in the value space, i. e. with respect to the sets* $f(E(n))$ *and* $f(E(n+1))$ *?*

6. *Can similar questions be answered with respect to the set of properly efficient, the set of weakly efficient, and the set of strongly efficient points?*

2 Weakly Efficient and Strongly Efficient Points

We set the stage with some simple results concerning weakly efficient and strongly efficient points. The set of *weakly efficient points* for the multicriteria problem with set of feasible points X and objective functions $f_1, \ldots, f_n$ is denoted by

$$\begin{aligned} W(n) &:= \{x \in X \mid \nexists\, y \in X : \forall\, j \in \{1, \ldots, n\} : f_j(y) < f_j(x)\} \\ &= \{x \in X \mid \forall\, y \in X : \exists\, j \in \{1, \ldots, n\} : f_j(x) \leq f_j(y)\}, \end{aligned} \tag{3}$$

while the set of *superstrongly efficient points* for the same optimisation problem is denoted by

$$S(n) := \{x \in X \mid \forall\, y \in X \setminus \{x\} : \forall\, j \in \{1, \ldots, n\} : f_j(x) \leq f_j(y)\},$$

and the set of *superstrongly efficient points* for the same optimisation problem is denoted by

$$\hat{S}(n) := \{x \in X \mid \forall\, y \in X \setminus \{x\} : \forall\, j \in \{1, \ldots, n\} : f_j(x) < f_j(y)\}.$$

It has to be noted that the set of strongly and the set of superstrongly efficient points is seldom of any interest for the decision maker. Indeed, simple examples [6, Example 11.9] show that these sets may be empty, even if the problem at hand displays all kinds of regularity that one might hope for.

The set of weakly efficient points for the multicriteria problem with set of feasible points X and objective functions $f_1, \ldots, f_{n+1}$ is denoted by

$$\begin{aligned} W(n+1) &:= \{x \in X \mid \nexists\, y \in X : \forall\, j \in \{1, \ldots, n+1\} : f_j(y) < f_j(x)\} \\ &= \{x \in X \mid \forall\, y \in X : \exists\, j \in \{1, \ldots, n+1\} : f_j(x) \leq f_j(y)\}, \end{aligned}$$

and the sets of strongly and the set of superstrongly efficient points for the same optimisation problem are denoted by

$$\begin{aligned} S(n+1) &:= \{x \in X \mid \forall\, y \in X \setminus \{x\} : \forall\, j \in \{1, \ldots, n+1\} : f_j(x) \leq f_j(y)\}, \\ \hat{S}(n+1) &:= \{x \in X \mid \forall\, y \in X \setminus \{x\} : \forall\, j \in \{1, \ldots, n+1\} : f_j(x) < f_j(y)\}, \end{aligned}$$

respectively. Of course,

$$\hat{S}(i) \subseteq S(i) \subseteq E(i) \subseteq W(i)$$

$(i = n, n+1)$, and

$$\begin{aligned} W(n) &\subseteq W(n+1), &(4)\\ S(n+1) &\subseteq S(n), &(5)\\ \hat{S}(n+1) &\subseteq \hat{S}(n), &(6) \end{aligned}$$

so the only interesting question is: when does equality in (4) or (5) or (6) occur? Necessary and sufficient conditions are given in the next propositon.

Proposition 1

1. *Suppose $W(n) = W(n+1)$. Then, all minima of f_{n+1} are in $W(n)$.*
2. *Suppose $n = 1$ and that the minima of f_{n+1} form a nonempty subset of $W(n)$. Then, $W(n) = W(n+1)$.*
3. *The equality $S(n+1) = S(n)$ holds if and only if all minima of f_{n+1} are contained in $S(n)$.*
4. *The equality $\hat{S}(n+1) = \hat{S}(n)$ holds if and only if f_{n+1} has a unique minimum, and this minimum is contained in $\hat{S}(n)$.*

Proof All minima of f_{n+1} are contained in $W(n+1)$. Due to the assumption $W(n) = W(n+1)$, the first result immediately follows. The second part of the proposition is self-evident. The third follows by noting that

$$\begin{aligned} S(n) \setminus S(n+1) &= \{x \in \hat{S}(n) \mid \exists\, y \in X \setminus \{x\} : \exists\, j \in \{1, \ldots, n+1\} : f_j(x) > f_j(y)\}\\ &= \{x \in S(n) \mid \exists\, y \in X \setminus \{x\} : f_{n+1}(x) > f_{n+1}(y)\}, \end{aligned}$$

while the fourth part follows by noting that

$$\begin{aligned} \hat{S}(n) \setminus \hat{S}(n+1) &= \{x \in \hat{S}(n) \mid \exists\, y \in X \setminus \{x\} : \exists\, j \in \{1, \ldots, n+1\} : f_j(x) \geq f_j(y)\}\\ &= \{x \in \hat{S}(n) \mid \exists\, y \in X \setminus \{x\} : f_{n+1}(x) \geq f_{n+1}(y)\}. \end{aligned}$$

■

None of the assumptions of the last proposition can be omitted, as the next two examples show.

Example 1 Define $X := \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1, x_2 \geq 0,\ x_1 x_2 \geq 1\}$ and $f_i(x_1, x_2) := x_1$ $(i = 1, 2)$. Clearly, $W(1) = \emptyset \neq W(2) = \{(x_1, x_2)^\top \in X \mid x_1 x_2 = 1\}$. $\square$

Example 2 Define $a := (3, 0, 0)^\top$, $b := (1, 1, 2)^\top$, $c := (2, 2, 1)^\top$, $X := \{a, b, c\}$, $f_i(x) := x_i$ $(i = 1, 2, 3)$. Then, $W(2) = \{a, b\} \neq W(3) = X$. $\square$

3 Efficient Points

3.1 Gaining or Loosing Efficient Points

In this subsection we want to consider the first three questions from Section 1.2. To this end, we turn our attention to the sets $E(n) \setminus E(n+1)$, $E(n+1) \setminus E(n)$ as well as $E(n) \cap E(n+1)$.

Theorem 2 *The following equations hold.*

1.

$$\begin{aligned} & E(n) \setminus E(n+1) \\ = & \ \{x \in E(n) \mid \exists\, z \in X : f(x) = f(z) \wedge f_{n+1}(z) < f_{n+1}(x)\}. \end{aligned}$$

2.

$$\begin{aligned} E(n+1) \setminus E(n) \ = \ & \{x \in E(n+1) \mid \exists\, z \in X : f(z) \neq f(x), \\ & f_{n+1}(x) < f_{n+1}(z), \forall j \in \{1, \ldots, n\} : f_j(x) \geq f_j(z)\}. \end{aligned}$$

3.

$$\begin{aligned} E(n) \cap E(n+1) \ = \ & \{x \in X \mid \forall\, y \in X : (f(x) = f(y) \wedge f_{n+1}(x) \leq f_{n+1}(y)) \\ & \vee\ \exists\, j \in \{1, \ldots, n\} : f_j(x) < f_j(y)\}. \end{aligned}$$

Proof

1. Suppose that $x \in E(n) \setminus E(n+1)$. Since $x \in E(n)$ we have that

$$\forall\, y \in X : f(y) = f(x) \text{ or } \exists\, j \in \{1, \ldots, n\} : f_j(x) < f_j(y), \tag{7}$$

and because of $x \notin E(n+1)$, there exists a $z \in X$ such that

$$\begin{bmatrix} f(z) \\ f_{n+1}(z) \end{bmatrix} \neq \begin{bmatrix} f(x) \\ f_{n+1}(x) \end{bmatrix} \text{ and } \forall\, j \in \{1, \ldots, n+1\} : f_j(x) \geq f_j(z) \tag{8}$$

holds. Assume now that for such a $z \in X$ we have $f(z) \neq f(x)$. By (7), this results in the existence of a $j \in \{1, \ldots, n\}$ with $f_j(x) < f_j(z)$, a contradiction to the second part of (8). Therefore, $f(z) = f(x)$ for all $z \in X$ fulfilling (8). Now, the first part of (8) shows us $f_{n+1}(z) \neq f_{n+1}(x)$. Invoking now the second part of (8) results in $f_{n+1}(x) > f_{n+1}(z)$. The inclusion "$\supseteq$" is trivial.

2. Suppose that we have given an $x \in E(n+1) \setminus E(n)$. Then, because of $x \notin E(n)$, we have that there exists a $z \in X$ with $f(z) \neq f(x)$ and $\forall j \in \{1, \ldots, n\} : f_j(x) \geq f_j(z)$ (cmp. (7)). But then (2) tells us that $f_{n+1}(x) < f_{n+1}(z)$. Again, the inclusion "$\supseteq$" is trivial.

3. Let $x \in E(n) \cap E(n+1)$. This holds, due to (1) and (2), if and only if for all $y \in X$ we have

$$\begin{aligned} & (f(y) = f(x) \vee \exists\, j \in \{1, \ldots, n\} : f_j(x) < f_j(y)) \\ \wedge \; & \left(\begin{bmatrix} f(y) \\ f_{n+1}(y) \end{bmatrix} = \begin{bmatrix} f(x) \\ f_{n+1}(x) \end{bmatrix} \vee \exists\, j \in \{1, \ldots, n+1\} : f_j(x) < f_j(y) \right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \left(f(y) = f(x) \wedge \begin{bmatrix} f(y) \\ f_{n+1}(y) \end{bmatrix} = \begin{bmatrix} f(x) \\ f_{n+1}(x) \end{bmatrix} \right) \\ \vee \; & \left(\exists\, j \in \{1, \ldots, n\} : f_j(x) < f_j(y) \wedge \begin{bmatrix} f(y) \\ f_{n+1}(y) \end{bmatrix} = \begin{bmatrix} f(x) \\ f_{n+1}(x) \end{bmatrix} \right) \\ \vee \; & (f(y) = f(x) \wedge \exists\, j \in \{1, \ldots, n+1\} : f_j(x) < f_j(y)) \\ \vee \; & (\exists\, j \in \{1, \ldots, n\} : f_j(x) < f_j(y) \wedge \exists\, j \in \{1, \ldots, n+1\} : f_j(x) < f_j(y)). \end{aligned}$$

Simplifying the first, third and fourth clause and noting that the second cannot hold leads to

$$\begin{aligned} & \begin{bmatrix} f(y) \\ f_{n+1}(y) \end{bmatrix} = \begin{bmatrix} f(x) \\ f_{n+1}(x) \end{bmatrix} \\ \vee \; & (f(y) = f(x) \wedge f_{n+1}(x) < f_{n+1}(y)) \\ \vee \; & (\exists\, j \in \{1, \ldots, n\} : f_j(x) < f_j(y)), \end{aligned}$$

which leads us directly to the final result.

■

Corollary 3 *We have the following.*

1. *The inclusion*
$$E(n) \subseteq E(n+1) \tag{9}$$
holds if and only if for all $x \in E(n)$ and for all $z \in X$ we have that $f(x) \neq f(z)$ or $f_{n+1}(z) \geq f_{n+1}(x)$ holds.

2. *The inclusion*
$$E(n+1) \subseteq E(n) \tag{10}$$
holds if and only if for all $x \in E(n+1)$ and all $z \in X$ we have $f(z) = f(x)$ or $f_{n+1}(x) \geq f_{n+1}(z)$ or there exists a $j \in \{1, \dots, n\}$ with $f_j(x) < f_j(z)$.

As usual, we call the function f to be injective if

$$\forall\, x, z \in X : x \neq z \Longrightarrow f(x) \neq f(z)$$

holds.

Corollary 4 *Let the function f be injective. Then, the following holds.*

1. *The inclusion $E(n) \subseteq E(n+1)$ holds if and only if the set $E(n)$ is included in the set of minimisers of f_{n+1}.*

2. *The inclusion $E(n+1) \subseteq E(n)$ holds if and only if for all $x \in E(n+1)$ and all $z \in X \setminus \{x\}$ we have $f_{n+1}(x) \geq f_{n+1}(z)$ or there exists a $j \in \{1, \dots, n\}$ with $f_j(x) < f_j(z)$.*

3. $$E(n) \cap E(n+1) = \{x \in X \mid \forall\, y \in X \setminus \{x\} : \exists\, j \in \{1, \dots, n\} : f_j(x) < f_j(y)\}.$$

In part 1 of the last corollary, the injectivity assumption on f can not be omitted, as the next example shows.

Example 3 Let $X \subset \mathbb{R}^2$ be the unit disk and let $n = 1$. Define $f(x_1, x_2) = f_1(x_1, x_2) = x_1$ as well as $f_2(x_1, x_2) = x_2$. Then, $E(1) = \{(-1, 0)^\top\}$, and $E(2)$ is the "lower left" boundary of X, i. e. the boundary of X in the third quadrant, $E(2) = \{(x_1, x_2)^\top \in \mathrm{bd}(X) \mid x_1, x_2 \leq 0\}$. The only minimum of f_2 is $(0, -1)^\top$. So, we have $E(1) \subset E(2)$, but f is not injective. □

Obviously, the injectivity assumption in the second part of Corollary 4 can be weakened to the following weaker assumption: for every $x, y \in X$ with $f(x) = f(y)$ it follows that $f_{n+1}(x) \geq f_{n+1}(y)$. Likewise, the injectivity assumption in the last part the of Corollary can be weakened to the to the following: for every $x, y \in X$ with $f(x) = f(y)$ it follows that $f_{n+1}(x) \leq f_{n+1}(y)$.

Corollary 5 *Let the function f be injective. If $E(n+1)$ is included in the set of maximisers of f_{n+1}, then $E(n+1) \subseteq E(n)$.*

Note that injectivity is not a strong assumption in decision theory, at least theoretically. Indeed, by defining the indifference relation I on X by

$$xIy \;:\Longleftrightarrow\; f(x) = f(y)$$

$(x, y \in X)$ we see that two feasible alternatives $x, y \in X$ are indifferent to each other (with respect to the multiobjective problem with the n objective functions $f_1, \ldots, f_n$) if and only if xIy, i. e. if and only if x and y are in the same equivalence class. Since I is an equivalence relation, we can define the function

$$\hat{f} : X/I \longrightarrow \mathbb{R}^n$$

on the quotient space X/I by $\hat{f}([x]) := f(x)$. This function is injective, and the set of efficient points of the multicriteria problem with the objective functions $\hat{f}_1, \ldots, \hat{f}_n$ can be lifted in a trivial way from X/I to X to obtain the set $E(n)$.

The second example of this section will show that the set $E(n) \setminus E(n+1)$, i. e. the set of those points that are efficient for the problem with n objective functions, but not efficient for the problem with one more objective function, can indeed be nonempty.

Example 4 In $\mathbb{R}^3$, let X be the cylinder with radius 2 and main axis through the point $(2, 2, 0)$, parallel to the x_3-axis, with height 1, but without the main axis, i. e.

$$X := \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid (x_1 - 2)^2 + (x_2 - 2)^2 \leq 4,\ 0 \leq x_3 \leq 1, (x_1, x_2) \neq (2, 2)\}.$$

Define $f : X \longrightarrow \mathbb{R}^2$ by

$$f(x_1, x_2, x_3) := \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \frac{1}{\left\| \begin{bmatrix} x_1 - 2 \\ x_2 - 2 \end{bmatrix} \right\|_2} \begin{bmatrix} x_1 - 2 \\ x_2 - 2 \end{bmatrix}.$$

Clearly, f projects the coordinates (x_1, x_2) on the boundary of the circle in $\mathbb{R}^2$ with center $(2, 2)^\top$ and radius 1,

$$f(X) = \{(\xi_1, \xi_2)^\top \in \mathbb{R}^2 \mid (\xi_1 - 2)^2 + (\xi_2 - 2)^2 = 1\}$$

Therefore, the image of the efficient set under the two objective functions given by f is the lower left quarter segment of this circle,

$$f(E(2)) = \{(\xi_1, \xi_2)^\top \in \mathbb{R}^2 \mid (\xi_1 - 2)^2 + (\xi_2 - 2)^2 = 1,\ \xi_1 \leq 2,\ \xi_2 \leq 2\}$$

and the set of efficient points itself is given by

$$E(2) = \{x = (x_1, x_2, x_3)^\top \in X \mid x_1 \leq 2,\ x_2 \leq 2\}.$$

Now consider a third objective function given by

$$f_3(x_1, x_2, x_3) := x_3.$$

Then,

$$f(E(3)) = \{(\xi_1, \xi_2, \xi_3,)^\top \in \mathbb{R}^3 \mid (\xi_1 - 2)^2 + (\xi_2 - 2)^2 = 1,\ \xi_1 \leq 2,\ \xi_2 \leq 2,\ \xi_3 = 0\}$$

as well as

$$E(3) = \{x = (x_1, x_2, x_3)^\top \in X \mid x_1 \leq 2,\ x_2 \leq 2,\ x_3 = 0\}$$

and therefore

$$E(3) \subset E(2).$$

Note also that the objective functions f_1, f_2, f_3 defined above are cone-convex [8, Definition 2.2.4] (generalised subconvexlike [10], nearly subconvexlike [9, 7], presubconvexlike [11]), i. e "convex where it is important": we have that $E(2)+\mathbb{R}^2_+$ as well as $E(3)+\mathbb{R}^3$ are both convex sets, and that therefore all scalarization techniques usually in use (see, e. g., [6, Chapter 11-12]) can be employed here. □

The example above shows that it is indeed possible to have a situation in which $E(n+1) \subset E(n)$ with $E(n+1) \neq E(n)$ will hold. In such a situation, we have $E(n) \setminus E(n+1) \neq \emptyset$.

Note also that it is rather easy to modify this example such that X is a polyhedral set and f is a linear function: linearity, i. e. polyhedrality, does not help here. (The corresponding linear example is only slightly more difficult to describe than the one given above.)

We can see exactly what is going on in the last example: The additional objective function f_3 gives the decision maker indeed additional information with respect to some points that are efficient for the "smaller" problem with two objective functions (to be more precise, the decision maker gains information with respect to all efficient points $(x_1, x_2, x_3)^\top \in E(2)$ with $x_3 > 0$.) These are the points $x \in E(2)$ such that there exists a $z \in X$ with $x \neq z$, $f(x) = f(z)$ but $f_3(x) > f_3(z)$, and exactly these points are deleted when replacing $E(2)$ by $E(3)$.

3.2 Stability in the Value Space

In this section, we assume that the set of feasible decisions X is endowed with a topology.

Let us consider the image of the set of efficient points under the objective functions given, i. e.

$$V(n) := f(E(n)) \subseteq \mathbb{R}^n$$

and

$$V(n+1) := \begin{bmatrix} f \\ f_{n+1} \end{bmatrix} (E(n+1)) \subseteq \mathbb{R}^{n+1}$$

These sets reside in spaces of different dimensions, so it makes a priori no sense to compare them. Therefore, we content ourselfes with the relationships between $f(E(n))$ and $f(E(n+1))$, or, to put it another way, we embed the space with the smaller dimension ($\mathbb{R}^n$) into the "larger" space and consider the set $\hat{V}(n) := V(n) \times \{0\} \subset \mathbb{R}^{n+1}$ instead of $V(n)$. So, the question under consideration can be rephrased as follows. What are the relationships between the sets $\hat{V}(n)$ and $V(n+1)$?

To answer this, define the function $F : X \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}$ by

$$F(x,u) := \begin{bmatrix} f(x) \\ uf_{n+1}(x) \end{bmatrix}.$$

Solution sets of multicriteria optimisation problems are invariant under monotone scaling of the objective functions, and therefore for $u > 0$ the multicriteria problem with set of feasible points X and objective functions $F_1(\cdot,u),\ldots,F_{n+1}(\cdot,u)$ has $E(n+1)$ as its set of efficient points. Moreover, $E(n)$ is equal to the solution set of the multicriteria problem with set of feasible points X and objective functions $F_1(\cdot,0),\ldots,F_n(\cdot,0)$. For fixed u, denote $F(X,u) := \{F(x,u) \mid x \in X\}$. For $u \geq 0$, consider the multifunction

$$N : u \mapsto \{F(x,u) \in F(X,u) \mid \forall\, \xi \in F(X,u) : \xi = F(x,u) \vee \exists\, j \in \{1,\ldots,n{+}1\} : F_j(x,u) < \xi\}$$

which maps a parameter u on the image of the solution set of the multicriteria problem with set of feasible points X and objective functions $F_1(\cdot,u),\ldots,F_{n+1}(\cdot,u)$, cmp. (1).

In what follows, we will make use of the set of weakly efficient points, see (3). Note again that $E(n) \subseteq W(n)$.

Lemma 6 *Let X be closed, let $f_1,\ldots,f_{n+1}$ be continuous, let f_{n+1} bounded, and suppose that $W(n) = E(n)$, i. e. for the problem with the n objective functions $f_1,\ldots,f_n$, all weakly efficient points are efficient. Then, the multifunction N is upper semicontinuous for $u \geq 0$.*

Moreover, if, in addition, $f_1,\ldots,f_n$ are bounded on X then the multifunction N is lower semicontinuous for $u \geq 0$ (and therefore continuous).

Proof. Since $N(u) = E(n+1)$ for $u > 0$, only the case $u = 0$ is of interest. The multifunction $Y : u \mapsto F(X, u)$ is continuous. With Theorem 4.2.1 of [8] it follows that N is upper semicontinuous.

If, in addition, all objective functions are bounded, the multifunction Y is uniform compact for $u = 0$, see Definition 2.2.2 in [8]. Moreover, since X is closed, $F(X, u)$ is a compact set and therefore $\mathbb{R}^{n+1}_+$-compact, see Definition 3.2.3 in [8]. As a consequence, Theorem 3.2.9 of [8] tells us that $N(u)$ is externally stable for all $u \geq 0$. The conclusion follows with Theorem 4.2.2 of [8]. ■

Theorem 7 *Let X be closed, let $f_1, \ldots, f_{n+1}$ be continuous, f_{n+1} bounded, and suppose that $W(n) = E(n)$, i. e. for the problem with the n objective functions $f_1, \ldots, f_n$, all weakly efficient points are efficient. Then, for any sequence $[f(x^{(k)}), f_{n+1}(x^{(k)})]^\top \in V(n+1)$ with $(f_{n+1}(x^{(k)}))_k$ bounded as well as $f(x^{(k)}) \longrightarrow f(x)$ for $k \longrightarrow \infty$ we have that $f(x) \in V(n) = f(E(n))$.*

Moreover, suppose that, in addition, $f_1, \ldots, f_n$ are bounded on X. Then, for all $f(x) \in V(n) = f(E(n))$ there exists a sequence $[f(x^{(k)}), f_{n+1}(x^{(k)})]^\top \in V(n+1)$ such that $f(x^{(k)}) \longrightarrow f(x)$ for $k \longrightarrow \infty$.

Proof. Use the last lemma, apply the definition of upper and lower semicontinuity on the situation at hand, and note again that $N(0) = F(E(n), 0) = \hat{V}(n)$ for $u = 0$ as well as $N(u) = F(E(n+1), u)$ for $u > 0$. ■

Corollary 8 *Let X be closed, let $f_1, \ldots, f_{n+1}$ be continuous, let f_{n+1} be bounded and suppose that for the problem with the n objective functions $f_1, \ldots, f_n$, we have $W(n) = E(n)$. Then,*

$$\mathrm{cl}(f(E(n+1)) \subseteq f(E(n)).$$

Moreover, suppose that, in addition, $f_1, \ldots, f_n$ are bounded on X. Then,

$$\mathrm{cl}(f(E(n+1)) = f(E(n)).$$

The simple example $X := \{(x_1, x_2)^\top \mid x_1, x_2 \geq 0, x_1 + x_2 \geq 1\}$, $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = x_2$, $n = 1$, shows clearly that the assumption $W(n) = E(n)$ cannot be omitted.

Theorem 9 *Let $F(X, 1)$ be closed and suppose $W(n) = E(n)$. Then, the following holds. For any sequence $[f(x^{(k)}), f_{n+1}(x^{(k)})]^\top \in f(E(n+1))$ with $f(x^{(k)}) \longrightarrow f(x)$ for $k \longrightarrow \infty$ we have that $f(x) \in f(E(n))$.*

Proof. As usual, denote by $\mathbb{R}^n_+$ the nonnegative orthant of the $\mathbb{R}^n$. Consider the multifunction $D : u \mapsto \mathrm{co}(\mathbb{R}^n_+ \times [-u, u])$, lower (but not upper) semicontinuous at $u = 0$. The result follows with Theorem 4.3.1 of [8]. ■

Corollary 10 *If $F(X, 1)$ is closed and W(n) = E(n), then*

$$\mathrm{cl}(f(E(n+1))) \subseteq f(E(n)).$$

3.3 Stability in the Decision Space

Again, we assume that the set of feasible decisions X is endowed with a topology.

We return our attention to the sets $E(n)$ and $E(n+1)$ residing in the decision space. Now we try to apply the same tools as in the last section. Let us therefore consider the multifunction

$$M : u \mapsto \{x \in X \mid \forall \xi \in F(X, u) : \xi = F(x, u) \vee \exists\, j \in \{1, \ldots, n+1\} : F_j(x, u) < \xi\}.$$

Lemma 11 *Let X be compact and let all objective functions $f_1, \ldots, f_{n+1}$ be continuous. Suppose also that $W(n) = E(n)$, i. e. for the problem with the n objective functions $f_1, \ldots, f_n$, all weakly efficient points are efficient. Then, the multifunction M is upper semicontinuous in $u = 0$.*

Moreover, if, in addition, f is bijective on X, then the multifunction M is lower semicontinuous in $u = 0$.

Proof. Apply Theorem 4.4.1 of [8] along the same lines as in the proof of the last lemma yields the upper semicontinuity. Theorem 3.2.9 and Theorem 4.4.2 of [8] yield the lower semicontinuity in an analogous way. ■

Note that a bijective function f occurs rather seldom in multicriteria optimisation. Indeed, in the linear case with a full-dimensional set of feasible points such an occurence means that the number of criteria is at least as large as the number of decision variables.

Theorem 12 *Let X be compact and let all objective functions $f_1, \ldots, f_{n+1}$ be continuous. Suppose also that for the problem with the n objective functions $f_1, \ldots, f_n$ we have $W(n) = E(n)$, i. e. for the problem with the n objective functions $f_1, \ldots, f_n$, all weakly efficient points are efficient. Then, for any sequence $x^{(k)} \in E(n+1)$ with $x^{(k)} \longrightarrow x$ for $k \longrightarrow \infty$ we have that $x \in E(n)$.*

Moreover, suppose that, in addition, f is bijective on X. Then, for all $x \in E(n)$ there exists a sequence $x^{(k)} \in E(n+1)$ with $x^{(k)} \longrightarrow x$ for $k \longrightarrow \infty$.

Proof. Analogous to the last theorem. ∎

Corollary 13 *Let X be compact and let all objective functions $f_1, \ldots, f_{n+1}$ be continuous. Suppose also that for the problem with the n objective functions $f_1, \ldots, f_n$, we have $W(n) = E(n)$. Then,*

$$\text{cl}(E(n+1)) \subseteq E(n).$$

Moreover, suppose that, in addition, f is bijective on X. Then,

$$\text{cl}(E(n+1)) = E(n).$$

The compactness assumption in the first part of the last corollary can be relaxed to mere closedness of the set X. Then, a corresponding argument analogous to the proof of Theorem 9 can be used, where Theorem 4.4.3 of [8] is invoked at the appropriate place.

4 Properly Efficient Points

Define the set of proper efficient points $P(n)$ for the multicriteria optimisation problem with the objective functions $f_1, \ldots, f_n$ in the usual way: a point $x \in X$ is called proper efficient if x is a solution to an optimisation problem of the form

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \omega_i f_i(x) \\ \text{subject to} \quad & x \in X \end{aligned}$$

with $\omega_1, \ldots, \omega_n > 0$. The set of all proper efficient points for the optimisation problem given is denoted by $P(n)$,

$$P(n) = \bigcup_{\omega_1, \ldots, \omega_n > 0} \arg\min_{x \in X} \sum_{i=1}^{n} \omega_i f_i(x).$$

Likewise,

$$P(n+1) = \bigcup_{\omega_1, \ldots, \omega_{n+1} > 0} \arg\min_{x \in X} \sum_{i=1}^{n+1} \omega_i f_i(x).$$

We then have

$$P(n) \subseteq E(n) \text{ and } P(n+1) \subseteq E(n+1),$$

see [5].

Up to now, no unreasonable assumptions on the set of feasible alternatives X or the objective functions $f_1, \dots, f_{n+1}$ where needed. (The only debatable assumption up to now has been the injectivity of f, and this has only been invoked to provide for an occasional stronger result.) Now we turn our attention to convex problems, i. e. problems in which the set X as well as all objective functions are convex. First of all, we have the following well-known result

Theorem 14 *Let X be a convex closed subset of a Banach space and let $f_1, \dots, f_{n+1}$ be convex. Then,*

$$E(n) \subseteq \mathrm{cl}(P(n)) \text{ and } E(n+1) \subseteq \mathrm{cl}(P(n+1)).$$

For a proof, see [1] or [8, p. 74]. Moreover, equality holds for finite X and $f_i(x) = x_i$ $(i = 1, \dots, n+1)$, see [1], or if X and all f_i $(i = 1, \dots, n+1)$ are polyhedral, see [4, 3].

From these results it is clear that the set of properly efficient points is of prime importance in multicriteria optimisation. Indeed, in applications the set of properly efficient points is often used as a surrogate for the set actually searched for, i. e. the set of efficient points.

We assume in what follows that X is a subset of some Banach space. Let us turm our attention to the interplay between $P(n)$ and $P(n+1)$.

Theorem 15 *Suppose that X is closed, that $f_1, \dots, f_{n+1}$ are continuous. Furthermore, suppose that for all $\hat{\omega} \in \mathbb{R}^n$ with $\hat{\omega}_i > 0$ $(i = 1, \dots, n)$ there exists an $\alpha \in \mathbb{R}$ and a compact set $C \subseteq X$ such that for every $\omega \in \mathbb{R}^{n+1}$ in a neighbourhood of $(\hat{\omega}, 0) \in \mathbb{R}^{n+1}$ we have*

$$\emptyset \neq \left\{ x \in X \;\middle|\; \sum_{i=1}^{n+1} \omega_i f_i(x) \leq \alpha \right\} \subseteq C.$$

Then,

$$\mathrm{cl}(P(n)) \cap \mathrm{cl}(P(n+1)) \neq \emptyset.$$

Proof. This follows from Proposition 4.4 in [2] by noting that under the assumptions of the theorem, the multifunction

$$\omega \mapsto \arg\min_{x \in X} \sum_{i=1}^{n+1} \omega_i f_i(x)$$

is upper semicontinuous at all $(\hat{\omega}, 0)$ considered. ∎

In other words, if X is closed and all objective functions are continuous and the level sets of all scalar problems considered are bounded, then $P(n)$ and $P(n+1)$ are connected with each other.

Corollary 16 *let X be closed and $f_1, \ldots, f_{n+1}$ are continuous. If X is compact or $f_1, \ldots, f_{n+1}$ are coercive, then*

$$\mathrm{cl}(P(n)) \cap \mathrm{cl}(P(n+1)) \neq \emptyset.$$

Like in Subsection 3.2 and 3.3, let us consider the function

$$F(x, u) := \begin{bmatrix} f(x) \\ u f_{n+1}(x) \end{bmatrix},$$

where u plays the role of a perturbation parameter. Define the sets

$$Q(n) := f(P(n))$$

and

$$Q(n+1) := \begin{bmatrix} f \\ f_{n+1} \end{bmatrix} (P(n+1))$$

and the multifunction S by

$$S : u \mapsto \bigcup_{\omega_1, \ldots, \omega_{n+1} > 0} F\left(\arg\min_{x \in X} \sum_{i=1}^{n+1} \omega_i F_i(x, u) \right).$$

Lemma 17 *If $f_1, \ldots, f_n$ are continuous, f_{n+1} is bounded and $W(n) = P(n)$, i. e. all weakly efficient points for the problem with the objective functions $f_1, \ldots, f_n$ are properly efficient, then S is upper semicontinuous.*

Let X be compact and let $f_1, \ldots, f_{n+1}$ be continuous. Then, S is lower semicontinuous.

Proof. Again, only the case $u = 0$ is of any interest. In the first case, the multifunction $Y : u \mapsto F(X, u)$ is continuous. The result follows with Theorem 4.5.1 of [8]. In the second case, Y is continuous, closed, and uniformly compact in a neighbourhood of $u = 0$. The result follows with Theorem 4.5.2 of [8]. ■

Theorem 18 *If $f_1, \ldots, f_n$ are continuous, f_{n+1} is bounded and all weakly efficient points for the problem with the objective functions $f_1, \ldots, f_n$ are properly efficient, then the following holds. For any sequence $[f(x^{(k)}), f_{n+1}(x^{(k)})]^\top \in Q(n+1)$ with $f(x^{(k)}) \longrightarrow f(x)$ for $k \longrightarrow \infty$ we have that $f(x) \in f(P(n))$.*

Let X be compact and let $f_1, \ldots, f_{n+1}$ be continuous. Then, for any point $f(x) \in f(P(n))$ there exists a sequence $[f(x^{(k)}), f_{n+1}(x^{(k)})]^\top \in Q(n+1)$ such that $f(x^{(k)}) \longrightarrow f(x)$ for $k \longrightarrow \infty$.

Proof. Analogously to Theorem 7. ∎

Corollary 19 *If $f_1, \ldots, f_n$ are continuous, f_{n+1} is bounded and $W(n) = P(n)$, then*

$$\mathrm{cl}(f(P(n+1))) \subseteq Q(n).$$

Let X be compact and $f_1, \ldots, f_{n+1}$ be continuous, then,

$$\mathrm{cl}(f(P(n+1))) = Q(n).$$

Theorem 20 *Let $F(X, 1)$ be closed and suppose $W(n) = P(n)$. Then, the following holds. For any sequence $[f(x^{(k)}), f_{n+1}(x^{(k)})]^\top \in Q(n+1)$ with $f(x^{(k)}) \longrightarrow f(x)$ for $k \longrightarrow \infty$ we have that $f(x) \in f(P(n))$.*

Proof. Analogous to the proof of Theorem 9, invoking Theorem 4.5.3 of [8] at the appropriate place. ∎

Corollary 21 *If $F(X, 1)$ is closed and $W(n) = P(n)$, then*

$$\mathrm{cl}(f(P(n+1))) \subseteq Q(n).$$

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