

# The Rate of Convergence of the Augmented Lagrangian Method for Nonlinear Semidefinite Programming\*

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## Abstract

We analyze the rate of local convergence of the augmented Lagrangian method for nonlinear semidefinite optimization. The presence of the positive semidefinite cone constraint requires extensive tools such as the singular value decomposition of matrices, an implicit function theorem for semismooth functions, and certain variational analysis on the projection operator in the symmetric-matrix space. Without requiring strict complementarity, we prove that, under the constraint nondegeneracy condition and the strong second order sufficient condition, the rate is proportional to  $1/c$ , where  $c$  is the penalty parameter that exceeds a threshold  $\bar{c} > 0$ .

**Key words:** The augmented Lagrangian method, nonlinear semidefinite programming, rate of convergence, variational analysis.

## 1 Introduction

We are concerned with the local convergence rate of the augmented Lagrangian method for solving the following optimization problem

$$(OP) \quad \min f(x) \quad \text{s.t. } h(x) = 0, g(x) \in K, \quad (1.1)$$

where  $f : X \mapsto \mathfrak{R}$ ,  $h : X \mapsto \mathfrak{R}^m$ , and  $g : X \mapsto Y$  are twice continuously differentiable functions,  $X$  and  $Y$  are two finite-dimensional real Hilbert spaces equipped with a scalar product  $\langle \cdot, \cdot \rangle$  and its induced norm denoted by  $\|\cdot\|$ , and  $K$  is a closed convex cone in  $Y$ . For

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any given  $\bar{x} \in X$  and  $\varepsilon > 0$ , let the open ball  $\mathbb{B}_\varepsilon(\bar{x}) := \{x \in X \mid \|x - \bar{x}\| < \varepsilon\}$ . Suppose that  $X'$  and  $Y'$  are two finite-dimensional real Hilbert spaces and that  $F : X \times X' \mapsto Y'$ . If  $F$  is Fréchet-differentiable at  $(x, x') \in X \times X'$ , then we use  $\mathcal{J}F(x, x')$  (respectively,  $\mathcal{J}_x F(x, x')$ ) to denote the Fréchet-derivative of  $F$  at  $(x, x')$  (respectively, the partial Fréchet-derivative of  $F$  at  $(x, x')$  with respect to  $x$ ) and  $\nabla F(x, x') := \mathcal{J}F(x, x')^*$ , the adjoint of  $\mathcal{J}F(x, x')$  (respectively,  $\nabla_x F(x, x') := \mathcal{J}_x F(x, x')^*$ , the adjoint of  $\mathcal{J}_x F(x, x')$ ). Moreover, if  $F$  is twice Fréchet-differentiable at  $(x, x') \in X \times X'$ , we define

$$\mathcal{J}^2 F(x, x') := \mathcal{J}(\mathcal{J}F)(x, x'), \quad \mathcal{J}_{xx}^2 F(x, x') := \mathcal{J}_x(\mathcal{J}_x F)(x, x')$$

and

$$\nabla^2 F(x, x') := \mathcal{J}(\nabla F)(x, x'), \quad \nabla_{xx}^2 F(x, x') := \mathcal{J}_x(\nabla_x F)(x, x').$$

A feasible point  $x \in X$  to (OP) is called a stationary point if there exist  $\zeta \in \mathfrak{R}^m$  and  $\xi \in Y$  such that the following Karush-Kuhn-Tucker (KKT) condition is satisfied at  $(x, \zeta, \xi)$ :

$$\nabla_x L_0(x, \zeta, \xi) = 0, \quad h(x) = 0, \quad g(x) \in K, \quad \xi \in K^*, \quad \text{and} \quad \langle g(x), \xi \rangle = 0, \quad (1.2)$$

where the Lagrangian function  $L_0 : X \times \mathfrak{R}^m \times Y \mapsto \mathfrak{R}$  is defined as

$$L_0(x, \zeta, \xi) := f(x) + \langle \zeta, h(x) \rangle - \langle \xi, g(x) \rangle \quad (1.3)$$

and  $K^*$  is the dual cone of  $K$ , i.e.,

$$K^* := \{v \in Y \mid \langle v, z \rangle \geq 0 \quad \forall z \in K\}.$$

Any point  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y$  satisfying (1.2) is called a KKT point and the corresponding point  $(\zeta, \xi)$  is called a Lagrange multiplier at  $x$ . Let  $\mathcal{M}(x)$  (maybe empty) be the set of all Lagrangian multipliers at  $x$ .

Let  $c > 0$  be a parameter. The corresponding augmented Lagrangian function with the penalty parameter  $c$  for problem (OP) is defined as (cf. [36, Section 11.K])

$$L_c(x, \zeta, \xi) := f(x) + \langle \zeta, h(x) \rangle + \frac{c}{2} \|h(x)\|^2 + \frac{1}{2c} [\|\Pi_{K^*}(\xi - cg(x))\|^2 - \|\xi\|^2], \quad (1.4)$$

where  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y$  and  $\Pi_{K^*}(\cdot)$  denotes the metric projection operator onto the set  $K^*$ . The function  $L_c(\cdot)$  can be derived in the following way. Problem (OP) can be rewritten as

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, \quad v - g(x) = 0, \quad (x, v) \in X \times K. \quad (1.5)$$

Define  $\tilde{L}_c : X \times Y \times \mathfrak{R}^m \times Y \mapsto \mathfrak{R}$  by

$$\tilde{L}_c(x, v, \zeta, \xi) := f(x) + \langle \zeta, h(x) \rangle + \frac{c}{2} \|h(x)\|^2 + \langle \xi, v - g(x) \rangle + \frac{c}{2} \|v - g(x)\|^2. \quad (1.6)$$

Since for any given  $(x, \xi) \in X \times Y$ , the unique optimal solution to the following problem

$$\inf_{v \in K} \langle \xi, v - g(x) \rangle + \frac{c}{2} \|v - g(x)\|^2$$

is given by

$$\tilde{v}(x, \xi, c) := \Pi_K(g(x) - c^{-1}\xi) = (g(x) - c^{-1}\xi) + c^{-1}\Pi_{K^*}(\xi - cg(x)), \quad (1.7)$$

we have for any  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y$  that

$$\begin{aligned}
& \inf_{v \in K} \tilde{L}_c(x, v, \zeta, \xi) \\
&= f(x) + \langle \zeta, h(x) \rangle + \frac{c}{2} \|h(x)\|^2 + \inf_{v \in K} \langle \xi, v - g(x) \rangle + \frac{c}{2} \|v - g(x)\|^2 \\
&= f(x) + \langle \zeta, h(x) \rangle + \frac{c}{2} \|h(x)\|^2 + \langle \xi, \tilde{v}(x, \xi, c) - g(x) \rangle + \frac{c}{2} \|\tilde{v}(x, \zeta, \xi) - g(x)\|^2 \\
&= \tilde{L}_c(x, \tilde{v}(x, \xi, c), \zeta, \xi) \\
&= L_c(x, \zeta, \xi).
\end{aligned} \tag{1.8}$$

The augmented Lagrangian function for the equality constrained optimization problem

$$\min f(x) \quad \text{s.t. } h(x) = 0 \tag{1.9}$$

turns to be

$$L_c(x, \zeta) = f(x) + \langle \zeta, h(x) \rangle + \frac{c}{2} \|h(x)\|^2, \quad (x, \zeta) \in X \times \mathfrak{R}^m, \tag{1.10}$$

which was introduced by Arrow and Solow [2] in the study of a differential equation method for solving (1.9). The augmented Lagrangian method was initiated by Hestenes [17] and Powell [27] for solving the equality constrained problem (1.9) and was generalized by Rockafellar [30] to the following nonlinear programming problem

$$\text{(NLP)} \quad \min f(x) \quad \text{s.t. } h(x) = 0, \quad g(x) \geq 0, \tag{1.11}$$

where  $f : \mathfrak{R}^n \mapsto \mathfrak{R}$ ,  $h : \mathfrak{R}^n \mapsto \mathfrak{R}^m$ , and  $g : \mathfrak{R}^n \mapsto \mathfrak{R}^p$  are twice continuously differentiable. Problem (NLP) is a special case of (OP) with  $X := \mathfrak{R}^n$ ,  $Y := \mathfrak{R}^p$ , and  $K := \mathfrak{R}_+^p$ .

For the equality constrained optimization problem (1.9), Powell sketched a proof in [27] to show that if the linear independence constraint qualification and the second-order sufficient condition are satisfied, then the augmented Lagrangian method can converge locally at a linear rate without having  $c \rightarrow \infty$ . For convex programming, Rockafellar [30] established a saddle point theorem in terms of  $L_c(\cdot)$  and Rockafellar [31] and Tretyakov [43] proved the global convergence of the augmented Lagrangian method for any  $c > 0$ .

The augmented Lagrangian method for solving (OP) can be stated as follows. Let  $c_0 > 0$  be given. Let  $(\zeta^0, \xi^0) \in \mathfrak{R}^m \times K^*$  be the initial estimated Lagrange multiplier. At the  $k$ th iteration,

$$\text{determine } x^k \text{ by minimizing } L_{c_k}(x, \zeta^k, \xi^k), \tag{1.12}$$

compute  $(\zeta^{k+1}, \xi^{k+1})$  by

$$\begin{cases} \zeta^{k+1} := \zeta^k + c_k h(x^k), \\ \xi^{k+1} := \Pi_{K^*}(\xi^k - c_k g(x^k)), \end{cases} \tag{1.13}$$

and update  $c_{k+1}$  by

$$c_{k+1} := c_k \quad \text{OR} \quad c_{k+1} := \kappa c_k \tag{1.14}$$

according to certain rules, where  $\kappa > 1$  is a preselected positive number. If the sequence of parameters  $\{c_k\}$  is chosen to satisfy  $c_k \rightarrow +\infty$ , then the global convergence of the augmented Lagrangian method can be similarly discussed to the penalty function method [4]. If  $c_k$  has a finite limit, then there exists a positive integer  $N_0$  such that  $c_k \equiv c$  for  $k \geq N_0$  and some

positive number  $c$ . In this paper, instead of considering global convergence properties, we consider the local convergence properties of the augmented Lagrangian method for (OP) when the second case occurs; namely the case in which  $c_k \equiv c$  for all sufficient large  $k$ . For the sake of simplicity in our analysis, for  $k$  sufficiently large, we choose  $x^k$  as an exact local solution of  $L_c(\cdot, \zeta^k, \xi^k)$ .

In [3] (also see [4, Section 2.2]), Bertsekas established the following important result on the rate of convergence of the augmented Lagrangian method for the equality constrained problem (1.9).

**Theorem 1.1** *Let  $(\bar{x}, \bar{\zeta}) \in X \times \mathfrak{R}^m$  be a KKT point of the equality constrained optimization problem (1.9). Suppose that  $\mathcal{J}h(\bar{x}) : X \mapsto \mathfrak{R}^m$  is onto and the second order sufficient condition is satisfied at  $(\bar{x}, \bar{\zeta})$ . Let  $\bar{c} > 0$  be such that*

$$\langle d, \nabla_{xx}^2 L_{\bar{c}}(\bar{x}, \bar{\zeta})d \rangle > 0 \quad \forall 0 \neq d \in X. \quad (1.15)$$

*Then there exist positive scalars  $\delta, \varepsilon$ , and  $\varrho_0$  such that*

(i) *For all  $(\zeta, c)$  in the set  $D$  defined as*

$$D := \{(\zeta, c) \in \mathfrak{R}^m \times \mathfrak{R} \mid \|\zeta - \bar{\zeta}\| < \delta c, \bar{c} \leq c\}, \quad (1.16)$$

*the problem*

$$\min L_c(x, \zeta) \quad \text{s.t. } x \in \mathbb{B}_\varepsilon(\bar{x}) \quad (1.17)$$

*has a unique solution denoted  $x(\zeta, c)$ . The function  $x(\cdot, \cdot)$  is continuously differentiable in the interior of  $D$ , and, for all  $(\zeta, c) \in D$ , we have*

$$\|x(\zeta, c) - \bar{x}\| \leq \varrho_0 \|\zeta - \bar{\zeta}\|/c. \quad (1.18)$$

(ii) *For all  $(\zeta, c) \in D$ , we have*

$$\|\tilde{\zeta}(\zeta, c) - \bar{\zeta}\| \leq \varrho_0 \|\zeta - \bar{\zeta}\|/c, \quad (1.19)$$

*where*

$$\tilde{\zeta}(\zeta, c) := \zeta + ch(x(\zeta, c)).$$

In [4, Chapter 3], Bertsekas also discussed similar results to Theorem 1.1 for nonlinear programming (NLP) when the strict complementarity condition holds. On the other hand, for (NLP) many authors, for examples, Conn et al. [11], Contesse-Becker[12], and Ito and Kunisch [19] proved the linear convergence rate of the augmented Lagrangian method without assuming the strict complementarity condition. For more on the augmented Lagrangian method for nonlinear programming, see the two monographs [4, 16] and the survey paper [35].

The results in Theorem 1.1 on the rate of convergence of the augmented Lagrangian method for the equality constrained problem (1.9) can be used to deal with optimization problem (OP) when it can be equivalently converted to an equality constrained problem that satisfies the conditions given in Theorem 1.1. To see this, let us consider the following nonlinear semidefinite programming

$$(\text{NLSDP}) \quad \min f(x) \quad \text{s.t. } h(x) = 0, g(x) \in \mathcal{S}_+^p, \quad (1.20)$$

where  $\mathcal{S}_+^p$  is the cone of all positive semidefinite matrices in  $\mathcal{S}^p$ , the linear space of all  $p$  by  $p$  symmetric matrices in  $\mathfrak{R}^{p \times p}$ . We write  $C \succeq 0$  and  $C \succ 0$  to mean that  $C$  is a symmetric positive semidefinite matrix and a symmetric positive definite matrix in  $\mathcal{S}^p$ , respectively. For any two matrices  $C$  and  $D$  in  $\mathfrak{R}^{m \times n}$ , we write

$$\langle C, D \rangle := \text{Tr}(C^T D)$$

for the Frobenius inner product between  $C$  and  $D$ , where “Tr” denotes the trace of a square matrix. Since in this case  $\mathcal{S}_+^p = \{z^2 \mid z \in \mathcal{S}^p\}$ , (NLSDP) can be transformed into the following equality constrained optimization problem:

$$\min f(x) \quad \text{s.t. } h(x) = 0, z^2 - g(x) = 0, (x, z) \in X \times \mathcal{S}^p. \quad (1.21)$$

Let  $\widehat{L}_c(x, z, \zeta, \xi)$  be the augmented Lagrangian function for (1.21) and  $\widetilde{L}_c(\cdot)$  be defined as in (1.6). Then for any  $(x, z, \zeta, \xi) \in X \times \mathcal{S}^p \times \mathfrak{R}^m \times \mathcal{S}^p$ , we have

$$\begin{aligned} \widehat{L}_c(x, z, \zeta, \xi) &= \widetilde{L}_c(x, z^2, \zeta, \xi) \\ &= f(x) + \langle \zeta, h(x) \rangle + \frac{c}{2} \|h(x)\|^2 + \langle \xi, z^2 - g(x) \rangle + \frac{c}{2} \|z^2 - g(x)\|^2. \end{aligned} \quad (1.22)$$

Thus, by (1.7), (1.8), and the fact that  $\mathcal{S}_+^p$  is a self-dual cone, for any  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times \mathcal{S}^p$  we have

$$L_c(x, \zeta, \xi) = \widehat{L}_c(x, z_c(x, \zeta, \xi), \zeta, \xi) = \widetilde{L}_c(x, z_c^2(x, \zeta, \xi), \zeta, \xi), \quad (1.23)$$

where  $z_c(x, \zeta, \xi)$  is the square root of  $\Pi_{\mathcal{S}_+^p}(g(x) - c^{-1}\xi)$ , i.e.,

$$z_c(x, \zeta, \xi) := \sqrt{\Pi_{\mathcal{S}_+^p}(g(x) - c^{-1}\xi)}. \quad (1.24)$$

For any  $(\zeta, \xi) \in \mathfrak{R}^m \times \mathcal{S}^p$ , let

$$\|(\zeta, \xi)\| := \sqrt{\langle \zeta, \zeta \rangle + \langle \xi, \xi \rangle}.$$

Define  $\widehat{h} : X \times \mathcal{S}^p \mapsto \mathfrak{R}^m \times \mathcal{S}^p$  by

$$\widehat{h}(x, z) := \begin{pmatrix} h(x) \\ z^2 - g(x) \end{pmatrix}, \quad (x, z) \in X \times \mathcal{S}^p.$$

Then we have the following conclusion on the rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming.

**Theorem 1.2** *Consider (NLSDP) and its equivalent problem (1.21). Let  $(\bar{x}, \bar{z}, \bar{\zeta}, \bar{\xi}) \in X \times \mathcal{S}^p \times \mathfrak{R}^m \times \mathcal{S}_+^p$  be a KKT point of (1.21) with  $\bar{z} := \sqrt{g(\bar{x})}$ . Suppose that  $\mathcal{J}\widehat{h}(\bar{x}, \bar{z}) : X \times \mathcal{S}^p \mapsto \mathfrak{R}^m \times \mathcal{S}^p$  is onto and that the second order sufficient condition is satisfied at  $(\bar{x}, \bar{z}, \bar{\zeta}, \bar{\xi})$ . Let  $\hat{c}$  be a positive scalar such that*

$$\left\langle d, \nabla_{(x,z)(x,z)}^2 \widehat{L}_{\hat{c}}(\bar{x}, \bar{z}, \bar{\zeta}, \bar{\xi}) d \right\rangle > 0 \quad \forall 0 \neq d \in X \times \mathcal{S}^p. \quad (1.25)$$

*Then there exist positive scalars  $\bar{c} \geq \hat{c}$ ,  $\delta$ ,  $\varepsilon$ , and  $\varrho_0$  such that*

(i) For all  $(\zeta, \xi, c)$  in the set  $D \subset \mathfrak{R}^m \times \mathcal{S}^p \times \mathfrak{R}$  defined as

$$D := \{(\zeta, \xi, c) \in \mathfrak{R}^m \times \mathcal{S}^p \times \mathfrak{R} \mid \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\| < \delta c, \bar{c} \leq c\}, \quad (1.26)$$

the problem

$$\min L_c(x, \zeta, \xi) \quad \text{s.t. } x \in \mathbb{B}_\varepsilon(\bar{x}) \quad (1.27)$$

has a unique solution denoted  $x(\zeta, \xi, c)$ . The function  $x(\cdot, \cdot, \cdot)$  is continuously differentiable in the interior of  $D$ , and, for all  $(\zeta, \xi, c) \in D$ , we have

$$\|x(\zeta, \xi, c) - \bar{x}\| \leq \varrho_0 \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\|/c. \quad (1.28)$$

(ii) For all  $(\zeta, \xi, c) \in D$ , we have

$$\|(\tilde{\zeta}(\zeta, \xi, c), \tilde{\xi}(\zeta, \xi, c)) - (\bar{\zeta}, \bar{\xi})\| \leq \varrho_0 \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\|/c, \quad (1.29)$$

where

$$\tilde{\zeta}(\zeta, \xi, c) := \zeta + ch(x(\zeta, \xi, c)), \quad \tilde{\xi}(\zeta, \xi, c) := \Pi_{\mathcal{S}_+^p}(\xi - cg(x(\zeta, \xi, c))). \quad (1.30)$$

**Proof.** It follows from Theorem 1.1 that there exist positive numbers  $\hat{\delta}$ ,  $\hat{\varepsilon}$ , and  $\varrho_0$  such that for all  $(\zeta, \xi, c)$  in the set  $\hat{D} \subset \mathfrak{R}^m \times \mathcal{S}^p \times \mathfrak{R}$  defined as

$$\hat{D} := \{(\zeta, \xi, c) \mid \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\| < \hat{\delta}c, \hat{c} \leq c\}, \quad (1.31)$$

the problem

$$\min \hat{L}_c(x, z, \zeta, \xi) \quad \text{s.t. } (x, z) \in \mathbb{B}_{\hat{\varepsilon}}(\bar{x}) \times \mathbb{B}_{\hat{\varepsilon}}(\bar{z}) \quad (1.32)$$

has a unique solution denoted  $(\hat{x}(\zeta, \xi, c), \hat{z}(\zeta, \xi, c))$  satisfying

$$\begin{aligned} \|(\hat{x}(\zeta, \xi, c), \hat{z}(\zeta, \xi, c)) - (\bar{x}, \bar{z})\| &\leq \varrho_0 \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\|/c, \\ \|(\hat{\zeta}(\zeta, \xi, c), \hat{\xi}(\zeta, \xi, c)) - (\bar{\zeta}, \bar{\xi})\| &\leq \varrho_0 \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\|/c, \end{aligned} \quad (1.33)$$

where

$$\hat{\zeta}(\zeta, \xi, c) := \zeta + ch(\hat{x}(\zeta, \xi, c)), \quad \hat{\xi}(\zeta, \xi, c) := \xi + c(\hat{z}^2(\zeta, \xi, c) - g(\hat{x}(\zeta, \xi, c))). \quad (1.34)$$

Assume that  $\text{rank}(g(\bar{x})) = r_0$  and  $g(\bar{x})$  has the following spectral decomposition

$$g(\bar{x}) = P \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_0 \end{bmatrix} P^T,$$

where  $\Lambda_0 \in \mathcal{S}^{r_0}$  is a diagonal matrix whose diagonal elements are the  $r_0$  positive eigenvalues of  $g(\bar{x})$  and  $P$  is an orthogonal matrix. Then the mapping  $G : \mathcal{S}^{r_0} \mapsto \mathcal{S}^{r_0}$  defined as

$$G(\Lambda) := \sqrt{\sqrt{\Lambda^2}}, \quad \Lambda \in \mathcal{S}^{r_0}$$

is analytic at  $\Lambda_0$  [45, Theorem 3.1]. Therefore there exists a positive number  $\hat{\varepsilon}_1 \in (0, \hat{\varepsilon})$  such that for any  $\Lambda \in \mathcal{S}^{r_0}$  with  $\|\Lambda - \Lambda_0\| \leq 2\hat{\varepsilon}_1$ ,  $\Lambda$  is positive definite and

$$\|\sqrt{\Lambda} - \sqrt{\Lambda_0}\| = \|G(\Lambda) - G(\Lambda_0)\| \leq 2\|\mathcal{J}G(\Lambda_0)\|\|\Lambda - \Lambda_0\|. \quad (1.35)$$

Let  $\hat{\varepsilon}_2 \in (0, \hat{\varepsilon}_1]$  be such that

$$3\sqrt{p}\hat{\varepsilon}_2 + 16\|\mathcal{J}G(\Lambda_0)\|^2\hat{\varepsilon}_2^2 \leq \hat{\varepsilon}_1^2. \quad (1.36)$$

Since  $g$  is continuously differentiable, there exists  $l_g > 0$  such that

$$\|g(x) - g(\bar{x})\| \leq l_g\|x - \bar{x}\| \quad \forall x \in \mathbb{B}_{\hat{\varepsilon}}(\bar{x}).$$

Let  $\varepsilon \in (0, \hat{\varepsilon}]$ ,  $\delta \in (0, \hat{\delta}]$ , and  $\bar{c} \geq \hat{c}$  be such that

$$l_g\varepsilon + \delta + \|\bar{\xi}\|/\bar{c} \leq \hat{\varepsilon}_2. \quad (1.37)$$

Define

$$D := \{(\zeta, \xi, c) \in \mathfrak{R}^m \times \mathcal{S}^p \times \mathfrak{R} \mid \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\| < \delta c, \bar{c} \leq c\}.$$

Then, for any  $(\zeta, \xi, c) \in D$  and  $x \in X$  such that  $\|x - \bar{x}\| \leq \varepsilon$ , it follows from (1.24) and (1.37) that

$$\begin{aligned} \|z_c^2(x, \zeta, \xi) - g(\bar{x})\| &= \|\Pi_{\mathcal{S}_+^p}(g(x) - \xi/c) - \Pi_{\mathcal{S}_+^p}(g(\bar{x}))\| \\ &\leq \|g(x) - g(\bar{x}) - (\xi - \bar{\xi})/c - \bar{\xi}/c\| \\ &\leq \|g(x) - g(\bar{x})\| + \|(\xi - \bar{\xi})/c\| + \|\bar{\xi}/c\| \\ &\leq l_g\|x - \bar{x}\| + \delta + \|\bar{\xi}\|/c \leq \hat{\varepsilon}_2. \end{aligned} \quad (1.38)$$

Let  $v \in \mathcal{S}_+^p$  be such that  $v \in \mathbb{B}_{\hat{\varepsilon}_2}(g(\bar{x}))$ . Define  $Z_{11} \in \mathcal{S}^{p-r_0}$ ,  $Z_{12} \in \mathfrak{R}^{(p-r_0) \times r_0}$ , and  $Z_{22} \in \mathcal{S}^{r_0}$  by

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} := P^T \sqrt{v} P.$$

Then

$$\left\| \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}^2 - \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_0 \end{bmatrix} \right\| = \|P^T v P - P^T g(\bar{x}) P\| = \|v - g(\bar{x})\| \leq \hat{\varepsilon}_2.$$

Therefore,

$$\langle Z_{11}^2, Z_{11}^2 \rangle \leq \hat{\varepsilon}_2^2, \quad \langle Z_{12} Z_{12}^T, Z_{12} Z_{12}^T \rangle \leq \hat{\varepsilon}_2^2, \quad \langle Z_{22}^2 + Z_{12}^T Z_{12} - \Lambda_0, Z_{22}^2 + Z_{12}^T Z_{12} - \Lambda_0 \rangle \leq \hat{\varepsilon}_2^2,$$

which, implies

$$\text{Tr}(Z_{11}^2) \leq \sqrt{p}\hat{\varepsilon}_2, \quad \text{Tr}(Z_{12} Z_{12}^T) = \text{Tr}(Z_{12}^T Z_{12}) \leq \sqrt{p}\hat{\varepsilon}_2. \quad (1.39)$$

Let  $\Gamma_0 := Z_{22}^2 - \Lambda_0$ . Then

$$\|\Gamma_0\| = \|(Z_{22}^2 + Z_{12}^T Z_{12} - \Lambda_0) - Z_{12}^T Z_{12}\| \leq \hat{\varepsilon}_2 + \|Z_{12} Z_{12}^T\| \leq 2\hat{\varepsilon}_2.$$

Hence, by (1.35),

$$\|Z_{22} - \sqrt{\Lambda_0}\| = \|\sqrt{\Lambda_0 + \Gamma_0} - \sqrt{\Lambda_0}\| \leq 2\|\mathcal{J}G(\Lambda_0)\|\|\Gamma_0\| \leq 4\hat{\varepsilon}_2\|\mathcal{J}G(\Lambda_0)\|. \quad (1.40)$$

Therefore, by (1.39), (1.40), and (1.36), we have

$$\begin{aligned} \|\sqrt{v} - \sqrt{g(\bar{x})}\|^2 &= \left\| \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_0 \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} - \Lambda_0 \end{bmatrix} \right\|^2 \\ &= \text{Tr}(Z_{11}^2 + Z_{12}Z_{12}^T) + \text{Tr}(Z_{12}^TZ_{12}) + \text{Tr}((Z_{22} - \Lambda_0)^2) \\ &\leq 3\sqrt{p}\hat{\varepsilon}_2 + 16\hat{\varepsilon}_2^2\|\mathcal{J}G(\Lambda_0)\|^2 \leq \hat{\varepsilon}_1^2, \end{aligned}$$

which implies that for any  $v \in \mathcal{S}_+^p$  satisfying  $v \in \mathbb{B}_{\hat{\varepsilon}_2}(g(\bar{x}))$  we have

$$\|\sqrt{v} - \bar{z}\| \leq \hat{\varepsilon}_1.$$

In particular, by (1.38), for any  $(\zeta, \xi, c) \in D$  and  $x \in X$  such that  $\|x - \bar{x}\| \leq \varepsilon$ , we have

$$z_c(x, \zeta, \xi) \in \mathbb{B}_{\hat{\varepsilon}_1}(\bar{z}).$$

Thus, from (1.23) we know that for any  $(\zeta, \xi, c) \in D$ ,

$$\begin{aligned} \min_{x \in \mathbb{B}_\varepsilon(\bar{x})} L_c(x, \zeta, \xi) &= \min_{x \in \mathbb{B}_\varepsilon(\bar{x})} \widehat{L}_c(x, z_c(x, \zeta, \xi), \zeta, \xi) \\ &\geq \min_{(x, z) \in \mathbb{B}_\varepsilon(\bar{x}) \times \mathbb{B}_{\hat{\varepsilon}_1}(\bar{z})} \widehat{L}_c(x, z, \zeta, \xi) \\ &\geq \min_{x \in \mathbb{B}_\varepsilon(\bar{x})} \left( \min_{z \in \mathcal{S}^p} \widehat{L}_c(x, z, \zeta, \xi) \right) \\ &= \min_{x \in \mathbb{B}_\varepsilon(\bar{x})} \left( \min_{v \in \mathcal{S}_+^p} \widetilde{L}_c(x, v, \zeta, \xi) \right) \\ &= \min_{x \in \mathbb{B}_\varepsilon(\bar{x})} L_c(x, \zeta, \xi). \end{aligned} \quad (1.41)$$

Let

$$x(\zeta, \xi, c) := \hat{x}(\zeta, \xi, c), \quad (\zeta, \xi, c) \in D.$$

Then, by (1.41), for any  $(\zeta, \xi, c) \in D$ ,  $x(\zeta, \xi, c)$  is a solution to problem (1.27). The uniqueness of  $x(\zeta, \xi, c)$  follows from the uniqueness of  $(\hat{x}(\zeta, \xi, c), \hat{z}(\zeta, \xi, c))$ .

For any  $(\zeta, \xi, c) \in D$ , by using (1.34), (1.41), and the fact that  $(\hat{x}(\zeta, \xi, c), \hat{z}(\zeta, \xi, c))$  is the unique solution to (1.32), we know that

$$\hat{\zeta}(\zeta, \xi, c) = \zeta + ch(\hat{x}(\zeta, \xi, c)) = \zeta + ch(x(\zeta, \xi, c)) = \widetilde{\zeta}(\zeta, \xi, c)$$

and

$$\begin{aligned} \hat{\xi}(\zeta, \xi, c) &= \xi + c(\hat{z}^2(\zeta, \xi, c) - g(\hat{x}(\zeta, \xi, c))) \\ &= \xi + cz_c^2(x(\zeta, \xi, c), \zeta, \xi) - cg(x(\zeta, \xi, c)) \\ &= \xi - cg(x(\zeta, \xi, c)) + c\Pi_{\mathcal{S}_+^p}(g(x(\zeta, \xi, c)) - c^{-1}\xi) \\ &= \Pi_{\mathcal{S}_+^p}(\xi - cg(x(\zeta, \xi, c))) = \widetilde{\xi}(\zeta, \xi, c). \end{aligned}$$



Finally, the estimates (1.28) and (1.29) follow from (1.33).  $\square$

Note that one can adapt the proof used in Theorem 1.2 to deal with (NLP). By doing so, one can actually give a corrected proof of the approach sketched in Bertsekas [4] for (NLP) (compared with the proof given here, the missing parts in Bertsekas' approach [4, Section 3.1] can be readily seen). Just as in the case for (NLP) [4], the second order sufficient condition in Theorem 1.2 automatically implies the strict complementarity condition. See more on this at the end of Section 4. For (NLP), when the strict complementarity condition holds, an approach to derive results similar to (1.28) and (1.29) was also suggested by Golshtein and Tretyakov [16, Chapter 7].

The main objective of this paper is to study the rate of convergence of the augmented Lagrangian method for solving nonlinear semidefinite programming (NLSDP) without assuming the strict complementarity condition. The difficulty for achieving this objective lies in the facts that the positive semidefinite cone  $\mathcal{S}_+^n$  is non-polyhedral for  $n > 1$  and very few theory exists for dealing with the rate of convergence of the augmented Lagrangian method for (OP) of this type. The only exception is perhaps Pennanen's local convergence analysis of proximal point methods for the inclusion problem

$$0 \in \mathcal{T}(x), \tag{1.42}$$

where  $\mathcal{T}$  is a set-valued mapping from a Hilbert space  $X'$  to itself [26]. Based in part on Rockafellar's convergence analysis for the inclusion problem (1.42) with monotone operators [33, 34], Pennanen [26] established local linear convergence results of the proximal point methods under the condition that  $\mathcal{T}^{-1}$  has a Lipschitz localization property at a solution  $\bar{x}$  to (1.42). One interesting part of Pennanen's results is that he used his theory to establish the local linear convergence of the proximal point method of multipliers (the regularized augmented Lagrangian method) for solving (NLP) without assuming the strict complementarity condition. This suggests that one may be able to do the same for (NLSDP). However, by focusing on the optimization problem (OP) instead of the more general inclusion problem (1.42), we hope to gain more by making use of the rich symmetry structure uniquely possessed by this optimization problem. Indeed, we not only prove that the local rate of convergence is proportional to  $1/c$  with the penalty parameter  $c$  exceeding a threshold  $\bar{c} > 0$ , but also provide nice properties on the generalized Hessian of the dual function used in our analysis (cf. Proposition 4.2) for (NLSDP) that relate the augmented Lagrangian method to an approximate generalized Newton method.

The organization of this paper is as follows. In Section 2, we discuss several technical results used in our convergence analysis. In Section 3, we develop a general theory on the rate of convergence of the augmented Lagrangian method for a class of constrained optimization problems under two basic assumptions. Section 4 is devoted to applying the theory developed in Section 3 to nonlinear nonlinear semidefinite programming. Finally, we give our conclusions in Section 5.

## 2 Preliminaries

Let  $X$  and  $Y$  be two finite dimensional real vector spaces. Let  $\mathcal{O}$  be an open set in  $X$  and  $\Phi : \mathcal{O} \subseteq X \rightarrow Y$  be a locally Lipschitz continuous function on the open set  $\mathcal{O}$ . By

Rademacher's theorem,  $\Phi$  is almost everywhere Fréchet-differentiable in  $\mathcal{O}$ . We denote by  $\mathcal{D}_\Phi$  the set of Fréchet-differentiable points of  $\Phi$  in  $\mathcal{O}$ . Then, the Bouligand-subdifferential of  $\Phi$  at  $x \in \mathcal{O}$ , denoted  $\partial_B \Phi(x)$ , is

$$\partial_B \Phi(x) := \left\{ \lim_{k \rightarrow \infty} \mathcal{J}\Phi(x^k) \mid x^k \in \mathcal{D}_\Phi, x^k \rightarrow x \right\}.$$

Clarke's generalized Jacobian of  $\Phi$  at  $x$  is the convex hull of  $\partial_B \Phi(x)$  (see [10]), i.e.,

$$\partial \Phi(x) = \text{conv} \{ \partial_B \Phi(x) \}.$$

The following concept of semismoothness was first introduced by Mifflin [24] for functionals and was extended by Qi and Sun [28] to vector valued functions.

**Definition 2.1** *Let  $\Phi : \mathcal{O} \subseteq X \mapsto Y$  be a locally Lipschitz continuous function on the open set  $\mathcal{O}$ . We say that  $\Phi$  is semismooth at a point  $x \in \mathcal{O}$  if*

- (i)  $\Phi$  is directionally differentiable at  $x$ ; and
- (ii) for any  $\Delta x \in X$  and  $V \in \partial \Phi(x + \Delta x)$  with  $\Delta x \rightarrow 0$ ,

$$\Phi(x + \Delta x) - \Phi(x) - V(\Delta x) = o(\|\Delta x\|). \quad (2.1)$$

Furthermore,  $\Phi$  is said to be strongly semismooth at  $x \in \mathcal{O}$  if  $\Phi$  is semismooth at  $x$  and for any  $\Delta x \in X$  and  $V \in \partial \Phi(x + \Delta x)$  with  $\Delta x \rightarrow 0$ ,

$$\Phi(x + \Delta x) - \Phi(x) - V(\Delta x) = O(\|\Delta x\|^2). \quad (2.2)$$

By combining Clarke's implicit function theorem for locally Lipschitz continuous functions [10, Section 7.1] with [39, Theorem 1.1] and [21, Lemma 2], we can get the following result directly.

**Lemma 2.1** *Suppose that  $H : X \times Y \mapsto X$  is a locally Lipschitz continuous function in an open neighborhood of  $(\bar{x}, \bar{y}) \in X \times Y$  with  $H(\bar{x}, \bar{y}) = 0$ . If every element in  $\pi_x \partial H(\bar{x}, \bar{y})$ , the projection of  $\partial H(\bar{x}, \bar{y})$  onto the space  $X$ , is nonsingular, then there exist an open neighborhood  $\mathcal{O}_Y$  of  $\bar{y}$  and a locally Lipschitz continuous function  $x(\cdot) : \mathcal{O}_Y \mapsto X$  satisfying  $x(\bar{y}) = \bar{x}$  such that for every  $y \in \mathcal{O}_Y$ ,*

$$H(x(y), y) = 0.$$

Furthermore, if  $H$  is (strongly) semismooth at every point in the open neighborhood of  $(\bar{x}, \bar{y})$ , then  $x(\cdot)$  is (strongly) semismooth at every point in  $\mathcal{O}_Y$ .

The following two lemmas on the Bouligand-subdifferential of composite functions are useful in determining  $\pi_x \partial_B(\nabla_x L_c)(\cdot)$ . The first one is proved in [40, Lemma 2.1] and the second one needs a proof, which will be given here.

**Lemma 2.2** *Let  $F : X \mapsto Y$  be a continuously differentiable function on an open neighborhood  $\mathcal{O}$  of  $\bar{x} \in X$  and  $\Phi : \mathcal{O}_Y \subseteq Y \mapsto X'$  be a locally Lipschitz continuous function on an open set  $\mathcal{O}_Y$  containing  $\bar{y} := F(\bar{x})$ , where  $X'$  is a finite-dimensional real vector space. Suppose that  $\Phi$  is directionally differentiable at every point in  $\mathcal{O}_Y$  and that  $\mathcal{J}F(\bar{x}) : X \rightarrow Y$  is onto. Then it holds that*

$$\partial_B(\Phi \circ F)(\bar{x}) = \partial_B \Phi(\bar{y}) \mathcal{J}F(\bar{x}). \quad (2.3)$$

**Lemma 2.3** Let  $F : X \mapsto Y$  be a continuously differentiable function on an open neighborhood  $\mathcal{O}$  and  $\Phi : X \rightarrow X'$  be a locally Lipschitz continuous function on  $\mathcal{O}$ , where  $X'$  is a finite-dimensional real vector space. Suppose that  $\Phi$  is semismooth at every point in  $\mathcal{O}$ . Let  $\Psi : X \mapsto Y'$  be defined as

$$\Psi(x) := F(x)\Phi(x) \equiv F(x) \cdot \Phi(x), \quad x \in X,$$

where  $Y'$  is a finite-dimensional real vector space and “ $\cdot$ ” is a bi-linear operator from  $Y \times X'$  to  $Y'$ . Then for every  $x \in \mathcal{O}$  and  $\Delta x \in X$ ,

$$\partial_B \Psi(x)(\Delta x) = \mathcal{J}F(x)(\Delta x)\Phi(x) + F(x)\partial_B \Phi(x)(\Delta x). \quad (2.4)$$

**Proof.** Let  $x \in \mathcal{O}$  and  $\Delta x \in X$  be two arbitrary but fixed points. By using the fact that if  $\Phi$  is Fréchet differentiable at  $y \in \mathcal{O}$ , then  $\Psi$  is also Fréchet differentiable at  $y$  we obtain

$$\partial_B \Psi(x)(\Delta x) \supseteq \mathcal{J}F(x)(\Delta x)\Phi(x) + F(x)\partial_B \Phi(x)(\Delta x).$$

Conversely, let  $W \in \partial_B \Psi(x)$ . Then there exists a sequence of Fréchet differentiable points  $\{x^k\} \subseteq \mathcal{O}$  converging to  $x$  such that  $W = \lim_{k \rightarrow \infty} \mathcal{J}\Psi(x^k)$ . Since  $\Phi$  is assumed to be semismooth at each  $x^k$ , we have

$$\Phi'(x^k; \Delta x) \in \partial_B \Phi(x^k)(\Delta x).$$

Thus, for any  $k \geq 1$ ,

$$\begin{aligned} \mathcal{J}\Psi(x^k)(\Delta x) &= \mathcal{J}F(x^k)(\Delta x)\Phi(x^k) + F(x^k)\Phi'(x^k; \Delta x) \\ &\in \mathcal{J}F(x^k)(\Delta x)\Phi(x^k) + F(x^k)\partial_B \Phi(x^k)(\Delta x), \end{aligned}$$

which, together with the upper semicontinuity of  $\partial_B \Phi(\cdot)$ , implies

$$W(\Delta x) = \lim_{k \rightarrow \infty} \mathcal{J}\Psi(x^k)(\Delta x) \in \mathcal{J}F(x)(\Delta x)\Phi(x) + F(x)\partial_B \Phi(x)(\Delta x).$$

Consequently, (2.4) holds. □

Let  $K$  be a closed convex set in  $Y$ . It is well known [46] that the metric projector  $\Pi_K(\cdot)$  is Lipschitz continuous with the Lipschitz constant as 1. Then for any  $y \in Y$ ,  $\partial \Pi_K(y)$  is well defined. Below is a lemma on the general properties of  $\partial \Pi_K(\cdot)$ .

**Lemma 2.4** [23, Proposition 1] Let  $K \subseteq Y$  be a closed convex set. Then, for any  $y \in Y$  and  $V \in \partial \Pi_K(y)$ , it holds that

- (i)  $V$  is self-adjoint.
- (ii)  $\langle d, Vd \rangle \geq 0 \quad \forall d \in Y$ .
- (iii)  $\langle Vd, d - Vd \rangle \geq 0 \quad \forall d \in Y$ .

For discussions on nonlinear semidefinite programming we need more properties about  $\partial_B \Pi_{\mathcal{S}_+^p}(\cdot)$ . Let  $\bar{Z} \in \mathcal{S}^p$  and  $\bar{Z}_+ := \Pi_{\mathcal{S}_+^p}(\bar{Z})$ . Suppose that  $\bar{Z}$  has the following spectral decomposition

$$\bar{Z} = \bar{P}\Lambda\bar{P}^T, \quad (2.5)$$

where  $\Lambda$  is the diagonal matrix of eigenvalues of  $\bar{Z}$  and  $\bar{P}$  is a corresponding orthogonal matrix of the orthonormal eigenvectors. Then

$$\bar{Z}_+ = \bar{P}\Lambda_+\bar{P}^T,$$

where  $\Lambda_+$  is the diagonal matrix whose diagonal entries are the nonnegative parts of the respective diagonal entries of  $\Lambda$  [18, 44]. Define three index sets of positive, zero, and negative eigenvalues of  $\bar{Z}$ , respectively, as

$$\alpha := \{i \mid \lambda_i > 0\}, \quad \beta := \{i \mid \lambda_i = 0\}, \quad \gamma := \{i \mid \lambda_i < 0\}.$$

Write

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_\gamma \end{bmatrix} \quad \text{and} \quad \bar{P} = [\bar{P}_\alpha \quad \bar{P}_\beta \quad \bar{P}_\gamma]$$

with  $\bar{P}_\alpha \in \mathfrak{R}^{p \times |\alpha|}$ ,  $\bar{P}_\beta \in \mathfrak{R}^{p \times |\beta|}$ , and  $\bar{P}_\gamma \in \mathfrak{R}^{p \times |\gamma|}$ . Let  $\Theta$  be any matrix in  $\mathcal{S}^p$  with entries

$$\begin{cases} \Theta_{ij} = \frac{\max\{\lambda_i, 0\} + \max\{\lambda_j, 0\}}{|\lambda_i| + |\lambda_j|} & \text{if } (i, j) \notin \beta \times \beta, \\ \Theta_{ij} \in [0, 1] & \text{if } (i, j) \in \beta \times \beta. \end{cases} \quad (2.6)$$

The projection operator  $\Pi_{\mathcal{S}_+^p}(\cdot)$  is directionally differentiable everywhere in  $\mathcal{S}^p$  [5, 6] and is a strongly semismooth matrix-valued function [41]. For any  $H \in \mathcal{S}^p$ , we have

$$\Pi'_{\mathcal{S}_+^p}(\bar{Z}; H) = \bar{P} \begin{bmatrix} \bar{P}_\alpha^T H \bar{P}_\alpha & \bar{P}_\alpha^T H \bar{P}_\beta & \Theta_{\alpha\gamma} \circ \bar{P}_\alpha^T H \bar{P}_\gamma \\ \bar{P}_\beta^T H \bar{P}_\alpha & \Pi_{\mathcal{S}_+^{|\beta|}}(\bar{P}_\beta^T H \bar{P}_\beta) & 0 \\ \bar{P}_\gamma^T H \bar{P}_\alpha \circ \Theta_{\gamma\alpha} & 0 & 0 \end{bmatrix} \bar{P}^T, \quad (2.7)$$

where “ $\circ$ ” denotes the Hadamard product [25, 41]. When  $\beta = \emptyset$ ,  $\Pi_{\mathcal{S}_+^p}(\cdot)$  is Fréchet-differentiable at  $\bar{Z}$  and (2.7) reduces to the classical result of Löwner [22]:

$$\mathcal{J}\Pi_{\mathcal{S}_+^p}(\bar{Z})H = \bar{P} \begin{bmatrix} \bar{P}_\alpha^T H \bar{P}_\alpha & \Theta_{\alpha\gamma} \circ \bar{P}_\alpha^T H \bar{P}_\gamma \\ \bar{P}_\gamma^T H \bar{P}_\alpha \circ \Theta_{\gamma\alpha} & 0 \end{bmatrix} \bar{P}^T \quad \forall H \in \mathcal{S}^p. \quad (2.8)$$

The tangent cone of  $\mathcal{S}_+^p$  at  $\bar{Z}_+$ , denoted  $\mathcal{T}_{\mathcal{S}_+^p}(\bar{Z}_+)$ , can be completely characterized as follows

$$\mathcal{T}_{\mathcal{S}_+^p}(\bar{Z}_+) = \{B \in \mathcal{S}^p \mid B = \Pi'_{\mathcal{S}_+^p}(\bar{Z}_+; B)\} = \{B \in \mathcal{S}^p \mid [\bar{P}_\beta \quad \bar{P}_\gamma]^T B [\bar{P}_\beta \quad \bar{P}_\gamma] \succeq 0\}. \quad (2.9)$$

The characterization of  $\mathcal{T}_{\mathcal{S}_+^p}(\bar{Z}_+)$  was first obtained by Arnold [1] by using a different approach from the above. The lineality space of  $\mathcal{T}_{\mathcal{S}_+^p}(\bar{Z}_+)$ , i.e., the largest linear space in  $\mathcal{T}_{\mathcal{S}_+^p}(\bar{Z}_+)$ , denoted by  $\text{lin}(\mathcal{T}_{\mathcal{S}_+^p}(\bar{Z}_+))$ , takes the following form:

$$\text{lin}(\mathcal{T}_{\mathcal{S}_+^p}(\bar{Z}_+)) = \{B \in \mathcal{S}^p \mid [\bar{P}_\beta \ \bar{P}_\gamma]^T B [\bar{P}_\beta \ \bar{P}_\gamma] = 0\}. \quad (2.10)$$

The critical cone of  $\mathcal{S}_+^p$  at  $\bar{Z} \in \mathcal{S}^p$  associated with  $\bar{Z}_+$  is given by

$$\mathcal{C}(\bar{Z}; \mathcal{S}_+^p) = \left\{ B \in \mathcal{S}^p \mid \bar{P}_\beta^T B \bar{P}_\beta \succeq 0, \bar{P}_\beta^T B \bar{P}_\gamma = 0, \bar{P}_\gamma^T B \bar{P}_\gamma = 0 \right\}. \quad (2.11)$$

The affine hull of  $\mathcal{C}(\bar{Z}; \mathcal{S}_+^p)$ , denoted by  $\text{aff}(\mathcal{C}(\bar{Z}; \mathcal{S}_+^p))$ , can thus be written as

$$\text{aff}(\mathcal{C}(\bar{Z}; \mathcal{S}_+^p)) = \left\{ B \in \mathcal{S}^p \mid \bar{P}_\beta^T B \bar{P}_\gamma = 0, \bar{P}_\gamma^T B \bar{P}_\gamma = 0 \right\}. \quad (2.12)$$

The following lemma on  $\partial_B \Pi_{\mathcal{S}_+^p}(\bar{Z})$  is part of [40, Proposition 4], which is based on [25, Lemma 11].

**Lemma 2.5** *Let  $\Theta \in \mathcal{S}^p$  satisfy (2.6). Then  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{Z})$  if and only if there exists  $W_0 \in \partial_B \Pi_{\mathcal{S}_+^{|\beta|}}(0)$  such that*

$$W(H) = \bar{P} \begin{bmatrix} \bar{P}_\alpha^T H \bar{P}_\alpha & \bar{P}_\alpha^T H \bar{P}_\beta & \Theta_{\alpha\gamma} \circ \bar{P}_\alpha^T H \bar{P}_\gamma \\ \bar{P}_\beta^T H \bar{P}_\alpha & W_0(\bar{P}_\beta^T H \bar{P}_\beta) & 0 \\ \bar{P}_\gamma^T H \bar{P}_\alpha \circ \Theta_{\gamma\alpha} & 0 & 0 \end{bmatrix} \bar{P}^T \quad \forall H \in \mathcal{S}^p. \quad (2.13)$$

Let  $\mathcal{Q}$  be the set of all orthogonal matrices of order  $|\beta| \times |\beta|$ . Let

$$\mathcal{P} := \{P \in \mathfrak{R}^{p \times p} \mid P = [P_\alpha \ P_\beta \ P_\gamma] = [\bar{P}_\alpha \ (\bar{P}_\beta Q) \ \bar{P}_\gamma], Q \in \mathcal{Q}\}. \quad (2.14)$$

Note that all  $P \in \mathcal{P}$  have the same  $P_\alpha$  and  $P_\gamma$ . From the definition of  $\partial_B \Pi_{\mathcal{S}_+^{|\beta|}}(0)$  and (2.8) we know that if  $W_0 \in \partial_B \Pi_{\mathcal{S}_+^{|\beta|}}(0)$ , then there exist matrices  $Q \in \mathcal{Q}$  and  $\Omega \in \mathcal{S}^{|\beta|}$  with entries  $\Omega_{ij} \in [0, 1]$  such that

$$W_0(D) = Q(\Omega \circ (Q^T D Q))Q^T \quad \forall D \in \mathcal{S}^{|\beta|}.$$

For an extension on the above result, see [9, Lemma 4.7]. Thus, by using Lemma 2.5 we obtain the following useful lemma, which does not need further explanation.

**Lemma 2.6** *For any  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{Z})$ , there exist two matrices  $P \in \mathcal{P}$  and  $\Theta \in \mathcal{S}^p$  satisfying (2.6) such that*

$$W(H) = P(\Theta \circ (P^T H P))P^T \quad \forall H \in \mathcal{S}^p. \quad (2.15)$$

The following result, due to Debreu [13], is useful for the study of the Bouligand-subdifferential of  $\nabla_x L_c(\cdot)$ .

**Lemma 2.7** Let  $\phi : X \mapsto \Re$  be continuous and positive homogeneous of degree two:

$$\phi(td) = t^2\phi(d) \quad \forall t \geq 0 \text{ and } d \in X.$$

Suppose that there exists a positive number  $\mu_0 > 0$  such that for any  $d$  satisfying  $\mathcal{L}d = 0$ , one has  $\phi(d) \geq \eta_0\|d\|^2$ , where  $\mathcal{L} : X \mapsto Y$  is a given linear operator. Then there exist positive numbers  $\underline{\eta} \in (0, \eta_0]$  and  $c_0 > 0$  such that for any  $c \geq c_0$ ,

$$\phi(d) + c\langle \mathcal{L}d, \mathcal{L}d \rangle \geq \underline{\eta}\langle d, d \rangle \quad \forall d \in X. \quad (2.16)$$

Next, we provide a technical result used in Section 4.

**Lemma 2.8** Let  $a, b, c$ , and  $c_0$  be four positive scalars with  $c \geq c_0$ . Let

$$\psi(t; c, a, b, c_0) := a - \frac{1}{c}t + \frac{t^2}{b + (c - c_0)t}, \quad t \in [0, 1]. \quad (2.17)$$

Then, for any  $c \geq \max\{c_0, (b - c_0)^2/c_0\}$ ,  $\psi(\cdot; c, a, b, c_0)$  is a convex function on  $[0, 1]$ ,

$$\min_{t \in [0, 1]} \psi(t; c, a, b, c_0) = a - \frac{1}{c} \frac{b}{(\sqrt{c} + \sqrt{c_0})^2}, \quad (2.18)$$

and

$$\max_{t \in [0, 1]} \psi(t; c, a, b, c_0) = \max\left\{\psi(0; c, a, b, c_0), \psi(1; c, a, b, c_0)\right\}. \quad (2.19)$$

**Proof.** By simple calculations for any  $t \in [0, 1]$  we have

$$\nabla_t \psi(t; c, a, b, c_0) = \frac{c_0}{c(c - c_0)} - \frac{b^2}{(c - c_0)(b + (c - c_0)t)^2}$$

and

$$\nabla_{tt}^2 \psi(t; c, a, b, c_0) = \frac{2b^2}{(b + (c - c_0)t)^3}.$$

Then, since for  $c \geq c_0$ ,  $\nabla_{tt}^2 \psi(t, c, a, b, c_0) \geq 0$  for all  $t \in [0, 1]$ ,  $\psi(\cdot; c, a, b, c_0)$  is a convex function on  $[0, 1]$ . Consequently, (2.19) holds.

Let  $\underline{t} := b/(c_0 + \sqrt{cc_0})$ . Then  $\nabla_t \psi(\underline{t}; c, a, b, c_0) = 0$ . Since for any  $c \geq \max\{c_0, (b - c_0)^2/c_0\}$ ,  $\underline{t} \in (0, 1]$  and  $\psi(\cdot; c, a, b, c_0)$  is convex on  $[0, 1]$ , we have

$$\min_{t \in [0, 1]} \psi(t; c, a, b, c_0) = \psi(\underline{t}; c, a, b, c_0),$$

which, implies that (2.18) holds.  $\square$

### 3 General discussions on the rate of convergence

In this section, we always assume that the cone  $K$  presented in the optimization problem (OP) is a self-dual cone, i.e,  $K = K^*$  and that  $\Pi_K(\cdot)$  is semismooth everywhere. In particular, this is the case for any closed symmetric cone because a closed symmetric cone is always self-dual [15] and  $\Pi_K(\cdot)$  is strongly semismooth everywhere [42]. The cones  $\Re_+^p$

and  $\mathcal{S}_+^p$  are special cases of symmetric cones. For more on symmetric cones, see Faraut and Korányi [15].

Let  $c > 0$  and  $\bar{x}$  be a stationary point of (OP). Then  $\mathcal{M}(\bar{x})$ , the set of Lagrange multipliers at  $\bar{x}$ , is nonempty. Since  $f, h$ , and  $g$  are assumed to be twice continuously differentiable, we know from (1.4) and [46] that the augmented Lagrangian function  $L_c(\cdot)$  is continuously differentiable and for any  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y$ ,

$$\nabla_x L_c(x, \zeta, \xi) = \nabla f(x) + \nabla h(x)(\zeta + ch(x)) - \nabla g(x)\Pi_K(\xi - cg(x)). \quad (3.1)$$

Therefore, from (1.2) and [14], we have  $\nabla_x L_c(\bar{x}, \zeta, \xi) = 0$  for any  $(\zeta, \xi) \in \mathcal{M}(\bar{x})$ .

Define  $F_c : X \times \mathfrak{R}^m \times Y \mapsto Y$  by

$$F_c(x, \zeta, \xi) = \xi - cg(x), \quad (x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y.$$

Since  $\Pi_K(\cdot)$  is assumed to be semismooth everywhere,  $\Pi_K(\cdot)$  is directionally differentiable at any point  $y \in Y$ . Hence, by using the fact that for any  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y$ ,  $\mathcal{J}F_c(x, \zeta, \xi) : X \times \mathfrak{R}^m \times Y \mapsto Y$  is onto, we know from Lemma 2.2 that

$$\partial_B(\Pi_K \circ F_c)(x, \zeta, \xi) = \partial_B \Pi_K(\xi - cg(x))\mathcal{J}F_c(x, \zeta, \xi). \quad (3.2)$$

For any  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y$ , let

$$\Psi_c(x, \zeta, \xi) := \nabla g(x)(\Pi_K \circ F_c)(x, \zeta, \xi) = \nabla g(x)\Pi_K(\xi - cg(x)).$$

Let  $(x, \zeta, \xi) \in X \times \mathfrak{R}^m \times Y$ . Then from the semismoothness of  $\Pi_K(\cdot)$  and Lemma 2.3 we obtain that for any  $(\Delta x, \Delta \zeta, \Delta \xi) \in X \times \mathfrak{R}^m \times Y$ ,

$$\begin{aligned} & \partial_B \Psi_c(x, \zeta, \xi)(\Delta x, \Delta \zeta, \Delta \xi) \\ &= \nabla^2 g(x)(\Delta x)\Pi_K(\xi - cg(x)) + \nabla g(x)\partial_B(\Pi_K \circ F_c)(x, \zeta, \xi)(\Delta x, \Delta \zeta, \Delta \xi). \end{aligned} \quad (3.3)$$

From (3.1) and the definition of  $\Psi_c(\cdot)$  we know that

$$\begin{aligned} \partial_B(\nabla_x L_c)(x, \zeta, \xi) &= (\nabla^2 f(x), 0, 0) + \left( \sum_{i=1}^m (\zeta_i + ch_i(x))\nabla^2 h_i(x) + c\nabla h(x)\mathcal{J}h(x), \nabla h(x), 0 \right) \\ &\quad - \partial_B \Psi_c(x, \zeta, \xi), \end{aligned}$$

which, together with (3.2) and (3.3), implies that for any  $\Delta x \in X$ ,

$$\begin{aligned} (\pi_x \partial_B(\nabla_x L_c)(x, \zeta, \xi))(\Delta x) &= \nabla_{xx}^2 L_0(x, \zeta + ch(x), \Pi_K(\xi - cg(x)))(\Delta x) \\ &\quad + c\nabla h(x)\mathcal{J}h(x)(\Delta x) + c\nabla g(x)\partial_B \Pi_K(\xi - cg(x))\mathcal{J}g(x)(\Delta x), \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} \nabla_{xx}^2 L_0(x, \zeta + ch(x), \Pi_K(\xi - cg(x)))(\Delta x) &= \nabla^2 f(x)(\Delta x) + \nabla^2 h(x)(\Delta x)(\zeta + ch(x)) \\ &\quad - \nabla^2 g(x)(\Delta x)\Pi_K(\xi - cg(x)). \end{aligned}$$

Let  $(\bar{\zeta}, \bar{\xi}) \in \mathcal{M}(\bar{x})$  be a Lagrange multiplier at  $\bar{x}$ . For any  $W : Y \mapsto Y$ , let

$$\mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W) := \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\xi}) + c\nabla h(\bar{x})\mathcal{J}h(\bar{x}) + c\nabla g(\bar{x})W\mathcal{J}g(\bar{x}). \quad (3.5)$$

Then for any  $\Delta x \in X$ ,

$$(\pi_x \partial_B(\nabla_x L_c)(\bar{x}, \bar{\zeta}, \bar{\xi})) (\Delta x) = \{ \mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W)(\Delta x) \mid W \in \partial_B \Pi_K(\bar{\xi} - c g(\bar{x})) \}. \quad (3.6)$$

Next, we make two basic assumptions for the constrained optimization problem (OP). The first one is about the positive definiteness of  $\mathcal{A}_c(\bar{\zeta}, \bar{\xi}, \cdot)$ .

**Assumption B1.** Suppose that  $(\bar{\zeta}, \bar{\xi})$  is the unique Lagrange multiplier at  $\bar{x}$ , i.e.,  $\mathcal{M}(\bar{x}) = \{(\bar{\zeta}, \bar{\xi})\}$  and that there exist two positive numbers  $c_0$  and  $\underline{\eta}$  such that for any  $c \geq c_0$  and  $W \in \partial_B \Pi_K(\bar{\xi} - c g(\bar{x}))$ ,

$$\langle d, \mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W)d \rangle \geq \underline{\eta} \langle d, d \rangle \quad \forall d \in X.$$

For the remaining part of this section, we suppose that Assumption B1 is satisfied. Let  $\bar{y} := (\bar{\zeta}, \bar{\xi})$ . Then  $\nabla_x L_c(\bar{x}, \bar{y}) = 0$ . Let  $c_0$  and  $\eta$  be two positive numbers defined in Assumption B1 and  $c \geq c_0$  be a positive number. Since by (3.6) and Assumption B1, every element in  $\pi_x \partial_B(\nabla_x L_c)(\bar{x}, \bar{y})$  is positive definite, we know from Lemma 2.1 that there exist an open neighborhood  $\mathcal{O}_{\bar{y}}$  of  $\bar{y}$  and a locally Lipschitz continuous function  $x_c(\cdot)$  defined on  $\mathcal{O}_{\bar{y}}$  such that for any  $y \in \mathcal{O}_{\bar{y}}$ ,  $\nabla_x L_c(x_c(y), y) = 0$ . Furthermore, since  $\Pi_K(\cdot)$  is assumed to be semismooth everywhere,  $x_c(\cdot)$  is semismooth (strongly semismooth if  $\nabla^2 f, \nabla^2 g$ , and  $\nabla^2 h$  are locally Lipschitz continuous and  $\Pi_K(\cdot)$  is strongly semismooth everywhere) at any point in  $\mathcal{O}_{\bar{y}}$ . Moreover, there exist two positive numbers  $\varepsilon > 0$  and  $\delta_0 > 0$  (both depending on  $c$ ) such that for any  $x \in \mathbb{B}_\varepsilon(\bar{x})$  and  $y \in \mathbb{B}_{\delta_0}(\bar{y}) := \{y \in \mathfrak{R}^m \times Y \mid \|y - \bar{y}\| < \delta_0\} \subset \mathcal{O}_{\bar{y}}$ , every element in  $\pi_x \partial_B(L_c)(x, y)$  is positive definite. Thus, for any  $y \in \mathbb{B}_{\delta_0}(\bar{y})$ ,  $x_c(y)$  is the unique minimizer of  $L_c(\cdot, y)$  over  $\mathbb{B}_\varepsilon(\bar{x})$ . For ease of reference, we write these conclusions in the following proposition.

**Proposition 3.1** *Suppose that Assumption B1 is satisfied. Let  $c \geq c_0$ . Then there exist two positive numbers  $\varepsilon > 0$  and  $\delta_0 > 0$  (both depending on  $c$ ) and a locally Lipschitz continuous function  $x_c(\cdot)$  defined on the open ball  $\mathbb{B}_{\delta_0}(\bar{y})$  such that the following conclusions hold:*

- (i) *The function  $x_c(\cdot)$  is semismooth at any point in  $\mathbb{B}_{\delta_0}(\bar{y})$ .*
- (ii) *If  $\nabla^2 f, \nabla^2 g$ , and  $\nabla^2 h$  are locally Lipschitz continuous and  $\Pi_K(\cdot)$  is strongly semismooth everywhere, then  $x_c(\cdot)$  is strongly semismooth at any point in  $\mathbb{B}_{\delta_0}(\bar{y})$ .*
- (iii) *For any  $x \in \mathbb{B}_\varepsilon(\bar{x})$  and  $y \in \mathbb{B}_{\delta_0}(\bar{y})$ , every element in  $\pi_x \partial_B(L_c)(x, y)$  is positive definite.*
- (iv) *For any  $y \in \mathbb{B}_{\delta_0}(\bar{y})$ ,  $x_c(y)$  is the unique optimal solution to*

$$\min L_c(x, y) \quad \text{s.t. } x \in \mathbb{B}_\varepsilon(\bar{x}).$$

Let  $\vartheta_c : \mathfrak{R}^m \times Y \mapsto \mathfrak{R}$  be defined as

$$\vartheta_c(\zeta, \xi) := \min_{x \in \mathbb{B}_\varepsilon(\bar{x})} L_c(x, \zeta, \xi), \quad (\zeta, \xi) \in \mathfrak{R}^m \times Y. \quad (3.7)$$



Since for each fixed  $x \in X$ ,  $L_c(x, \cdot)$  is a concave function,  $\vartheta_c(\cdot)$  is also a concave function as it is the minimum function of a family of concave functions. By using the fact that for any  $y \in \mathbb{B}_{\delta_0}(\bar{y})$ ,  $x_c(y)$  is the unique minimizer of  $L_c(\cdot, y)$  over  $\mathbb{B}_\varepsilon(\bar{x})$ , we have

$$\vartheta_c(y) = L_c(x_c(y), y), \quad y \in \mathbb{B}_{\delta_0}(\bar{y}). \quad (3.8)$$

For any  $y \in \mathbb{B}_{\delta_0}(\bar{y})$  with  $y = (\zeta, \xi) \in \mathfrak{R}^m \times Y$ , let

$$\begin{pmatrix} \zeta_c(y) \\ \xi_c(y) \end{pmatrix} := \begin{pmatrix} \zeta + ch(x_c(y)) \\ \Pi_K(\xi - cg(x_c(y))) \end{pmatrix}. \quad (3.9)$$

Then we have

$$\nabla_x L_0(x_c(y), \zeta_c(y), \xi_c(y)) = \nabla_x L_c(x_c(y), y) = 0, \quad y \in \mathbb{B}_{\delta_0}(\bar{y}). \quad (3.10)$$

**Proposition 3.2** *Suppose that Assumption B1 is satisfied. Let  $c \geq c_0$ . Then the concave function  $\vartheta_c(\cdot)$  defined by (3.7) is continuously differentiable on  $\mathbb{B}_{\delta_0}(\bar{y})$  with*

$$\nabla \vartheta_c(y) = \begin{pmatrix} h(x_c(y)) \\ -c^{-1}\xi + c^{-1}\Pi_K(\xi - cg(x_c(y))) \end{pmatrix}, \quad y = (\zeta, \xi) \in \mathbb{B}_{\delta_0}(\bar{y}). \quad (3.11)$$

Moreover,  $\nabla \vartheta_c(\cdot)$  is semismooth at any point in  $\mathbb{B}_{\delta_0}(\bar{y})$ . It is strongly semismooth at any point in  $\mathbb{B}_{\delta_0}(\bar{y})$  if  $\nabla^2 f$ ,  $\nabla^2 g$ , and  $\nabla^2 h$  are locally Lipschitz continuous and  $\Pi_K(\cdot)$  is strongly semismooth everywhere.

**Proof.** Let  $y = (\zeta, \xi) \in \mathbb{B}_{\delta_0}(\bar{y})$ . Then from (3.10) and [10, Theorem 2.6.6] we have for any  $(\Delta\zeta, \Delta\xi) \in \mathfrak{R}^m \times Y$  that

$$\begin{aligned} \partial \vartheta_c(y)(\Delta\zeta, \Delta\xi) &= \mathcal{J}_x L_c(x_c(y), y)(\partial x_c(y)(\Delta\zeta, \Delta\xi)) \\ &\quad + \mathcal{J}_\zeta L_c(x_c(y), y)(\Delta\zeta) + \mathcal{J}_\xi L_c(x_c(y), y)(\Delta\xi) \\ &= \langle h(x_c(y)), \Delta\zeta \rangle - c^{-1} \langle \xi, \Delta\xi \rangle + \langle c^{-1} \Pi_K(\xi - cg(x_c(y))), \Delta\xi \rangle. \end{aligned}$$

Thus,  $\partial \vartheta_c(y)(\Delta\zeta, \Delta\xi)$  is a singleton for each  $(\Delta\zeta, \Delta\xi) \in \mathfrak{R}^m \times Y$ . This implies that  $\partial \vartheta_c(y)$  is a singleton. Therefore,  $\vartheta_c(\cdot)$  is Fréchet-differentiable at  $y$  and  $\nabla \vartheta_c(y)$  is given by (3.11). The continuity of  $\nabla \vartheta_c(\cdot)$  follows from the continuity of  $x_c(\cdot)$ .

The properties on the (strong) semismoothness of  $\nabla \vartheta_c(\cdot)$  at  $y$  follows directly from (3.11) and Proposition 3.1.  $\square$

For any  $c \geq c_0$  and  $\Delta y := (\Delta\zeta, \Delta\xi) \in \mathfrak{R}^m \times Y$ , define

$$\begin{aligned} \bar{\mathcal{V}}_c(\Delta y) : &= \left\{ \begin{pmatrix} \mathcal{J}h(\bar{x}) \\ -W \mathcal{J}g(\bar{x}) \end{pmatrix} \mathcal{A}_c(\bar{y}, W)^{-1} (-\nabla h(\bar{x})(\Delta\zeta) + \nabla g(\bar{x})W(\Delta\xi)) \right. \\ &\quad \left. + \begin{pmatrix} 0 \\ -c^{-1}\Delta\xi + c^{-1}W(\Delta\xi) \end{pmatrix} \mid W \in \partial_B \Pi_K(\bar{\xi} - cg(\bar{x})) \right\}. \end{aligned} \quad (3.12)$$

Since by Assumption B1,  $\mathcal{A}_c(\bar{y}, W)$  is positive definite for any  $W \in \partial_B \Pi_K(\bar{\xi} - cg(\bar{x}))$ ,  $\bar{\mathcal{V}}_c(\cdot)$  is well defined. The next proposition establishes an important relationship between  $\bar{\mathcal{V}}_c(\cdot)$  and  $\partial_B(\nabla \vartheta_c)(\bar{y})(\cdot)$ . Note that the function  $\nabla \vartheta_c(\cdot)$  given by (3.11) involves two nonsmooth functions  $\Pi_K(\cdot)$  and  $x_c(\cdot)$ , which are related to each other.

**Proposition 3.3** *Suppose that Assumption B1 is satisfied. Let  $c \geq c_0$ . Then for any  $\Delta y := (\Delta\zeta, \Delta\xi) \in \mathfrak{R}^m \times Y$ ,*

$$\partial_B(\nabla\vartheta_c)(\bar{y})(\Delta y) \subseteq \bar{\mathcal{V}}_c(\Delta y). \quad (3.13)$$

**Proof.** Let  $\Delta y = (\Delta\zeta, \Delta\xi) \in \mathfrak{R}^m \times Y$  be an arbitrary but fixed point. From Proposition 3.2, we know that  $\nabla\vartheta_c(\cdot)$  is semismooth at any point  $y \in \mathbb{B}_{\delta_0}(\bar{y})$ . Let  $\mathcal{D}_{\nabla\vartheta_c}$  denote the set of all Fréchet-differentiable points of  $\nabla\vartheta_c(\cdot)$  in  $\mathbb{B}_{\delta_0}(\bar{y})$ . Then for any  $y = (\zeta, \xi) \in \mathcal{D}_{\nabla\vartheta_c}$ , we have

$$\nabla^2\vartheta_c(y)(\Delta y) = \left( \begin{array}{c} \mathcal{J}h(x_c(y))(x_c)'(y; \Delta y) \\ -c^{-1}\Delta\xi + c^{-1}\Pi'_K(\xi - cg(x_c(y)); \Delta\xi - c\mathcal{J}g(x_c(y))(x_c)'(y; \Delta y)) \end{array} \right). \quad (3.14)$$

Let  $y \in \mathbb{B}_{\delta_0}(\bar{y})$ . Now, we derive the formula for  $(x_c)'(y; \Delta y)$ . From (3.10) and (3.9) we have

$$\begin{aligned} 0 &= \nabla_{xx}L_0(x_c(y), \zeta_c(y), \xi_c(y))(x_c)'(y; \Delta y) + c\nabla h(x_c(y))\mathcal{J}h(x_c(y))(x_c)'(y; \Delta y) \\ &\quad + \nabla h(x_c(y))(\Delta\zeta) - \nabla g(x_c(y))\Pi'_K(\xi - cg(x_c(y)); \Delta\xi - c\mathcal{J}g(x_c(y))(x_c)'(y; \Delta y)). \end{aligned} \quad (3.15)$$

Since  $\Pi_K(\cdot)$  is semismooth everywhere, there exists an element  $\widehat{W} \in \partial_B\Pi_K(\xi - cg(x_c(y)))$  such that

$$\Pi'_K(\xi - cg(x_c(y)); \Delta\xi - c\mathcal{J}g(x_c(y))(x_c)'(y; \Delta y)) = \widehat{W}(\Delta\xi - c\mathcal{J}g(x_c(y))(x_c)'(y; \Delta y)). \quad (3.16)$$

For any  $W \in \partial_B\Pi_K(\xi - cg(x_c(y)))$ , let

$$\begin{aligned} \mathcal{A}_c(y, W) : &= \nabla_{xx}^2L_0(x_c(y), \zeta_c(y), \xi_c(y)) + c\nabla h(x_c(y))\mathcal{J}h(x_c(y)) \\ &\quad + c\nabla g(x_c(y))W\mathcal{J}g(x_c(y)). \end{aligned} \quad (3.17)$$

From (3.4) and the definition of  $\delta_0$ ,  $\mathcal{A}_c(y, W)$  is positive definite for any  $W \in \partial_B\Pi_K(\xi - cg(x_c(y)))$ . Then from (3.15) and (3.16) we obtain that

$$(x_c)'(y; \Delta y) = \mathcal{A}_c(y, \widehat{W})^{-1} \left( -\nabla h(x_c(y))(\Delta\zeta) + \nabla g(x_c(y))\widehat{W}(\Delta\xi) \right). \quad (3.18)$$

Therefore, we have from (3.18) and (3.14) that for any  $y = (\zeta, \xi) \in \mathcal{D}_{\nabla\vartheta_c}$ ,

$$\begin{aligned} \nabla^2\vartheta_c(y)(\Delta y) \in & \left\{ \left( \begin{array}{c} \mathcal{J}h(x_c(y)) \\ -W\mathcal{J}g(x_c(y)) \end{array} \right) \mathcal{A}_c(y, W)^{-1} \left( -\nabla h(x_c(y))(\Delta\zeta) + \nabla g(x_c(y))W(\Delta\xi) \right) \right. \\ & \left. + \left( \begin{array}{c} 0 \\ -c^{-1}(\Delta\xi) + c^{-1}W(\Delta\xi) \end{array} \right) \middle| W \in \partial_B\Pi_K(\xi - cg(x_c(y))) \right\}, \end{aligned}$$

which, together with the continuity of  $x_c(\cdot)$  and the upper semicontinuity of  $\partial_B\Pi_K(\cdot)$ , implies that for any  $V \in \partial_B(\nabla\vartheta_c)(\bar{y})$ , one has  $V(\Delta y) \in \bar{\mathcal{V}}_c(\Delta y)$ . Consequently, (3.13) holds.  $\square$

The second basic assumption needed in this section is stated below.

**Assumption B2.** There exist positive numbers  $\bar{c} \geq c_0$ ,  $\mu_0 > 0$ ,  $\varrho_0 > 0$ , and  $\tau > 1$  such that for any  $c \geq \bar{c}$  and  $\Delta y \in \mathfrak{R}^m \times Y$ ,

$$\|(x_c)'(\bar{y}; \Delta y)\| \leq \varrho_0 \|\Delta y\|/c \quad (3.19)$$

and

$$\langle V(\Delta y) + c^{-1}\Delta y, \Delta y \rangle \in \mu_0 [-1, 1] \|\Delta y\|^2/c^\tau \quad \forall V(\Delta y) \in \bar{\mathcal{V}}_c(\Delta y). \quad (3.20)$$

Under Assumptions B1 and B2, we are ready to give the main result on the rate of convergence of the augmented Lagrangian method for the constrained optimization problem (OP).

**Theorem 3.1** *Suppose that  $K$  is a self-dual cone and that  $\Pi_K(\cdot)$  is semismooth everywhere. Let Assumptions B1 and B2 be satisfied. Let  $c_0$ ,  $\underline{\eta}$ ,  $\bar{c}$ ,  $\mu_0$ ,  $\varrho_0$ , and  $\tau$  be the positive numbers defined in these assumptions. Define*

$$\varrho_1 := 2\varrho_0 \quad \text{and} \quad \varrho_2 := 4\mu_0. \quad (3.21)$$

*Then for any  $c \geq \bar{c}$ , there exist two positive numbers  $\varepsilon$  and  $\delta$  (both depending on  $c$ ) such that for any  $(\zeta, \xi) \in \mathbb{B}_\delta(\bar{\zeta}, \bar{\xi})$ , the problem*

$$\min L_c(x, \zeta, \xi) \quad \text{s.t. } x \in \mathbb{B}_\varepsilon(\bar{x}) \quad (3.22)$$

*has a unique solution denoted  $x_c(\zeta, \xi)$ . The function  $x_c(\cdot, \cdot)$  is locally Lipschitz continuous on  $\mathbb{B}_\delta(\bar{\zeta}, \bar{\xi})$  and is semismooth at any point in  $\mathbb{B}_\delta(\bar{\zeta}, \bar{\xi})$ , and for any  $(\zeta, \xi) \in \mathbb{B}_\delta(\bar{\zeta}, \bar{\xi})$ , we have*

$$\|x_c(\zeta, \xi) - \bar{x}\| \leq \varrho_1 \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\|/c \quad (3.23)$$

and

$$\|(\zeta_c(\zeta, \xi), \xi_c(\zeta, \xi)) - (\bar{\zeta}, \bar{\xi})\| \leq \varrho_2 \|(\zeta, \xi) - (\bar{\zeta}, \bar{\xi})\|/c^{\tau-1}, \quad (3.24)$$

where  $\zeta_c(\zeta, \xi)$  and  $\xi_c(\zeta, \xi)$  are defined by (3.9), i.e.,

$$\zeta_c(\zeta, \xi) := \zeta + ch(x_c(\zeta, \xi)) \quad \text{and} \quad \xi_c(\zeta, \xi) := \Pi_K(\xi - cg(x_c(\zeta, \xi))).$$

**Proof.** Let  $c \geq \bar{c}$ . From Proposition 3.1 we have already known that there exist two positive numbers  $\varepsilon > 0$  and  $\delta_0 > 0$  (both depending on  $c$ ) and a locally Lipschitz continuous function  $x_c(\cdot, \cdot)$  defined on  $\mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\xi})$  such that the function  $x_c(\cdot, \cdot)$  is semismooth at any point in  $\mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\xi})$  and for any  $(\zeta, \xi) \in \mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\xi})$ ,  $x_c(\zeta, \xi)$  is the unique solution to (3.22).

Denote  $y := (\zeta, \xi) \in \mathfrak{R}^m \times Y$ . Since  $x_c(\cdot)$  is locally Lipschitz continuous on  $\mathbb{B}_{\delta_0}(\bar{y})$  and is directionally differentiable at  $\bar{y}$ , by [37] we know that  $x_c(\cdot)$  is Bouligand-differentiable at  $\bar{y}$ , i.e.,  $x_c(\cdot)$  is directionally differentiable at  $\bar{y}$  and

$$\lim_{y \rightarrow \bar{y}} \frac{\|x_c(y) - x_c(\bar{y}) - (x_c)'(\bar{y}; y - \bar{y})\|}{\|y - \bar{y}\|} = 0.$$

By Proposition 3.2,  $\nabla \vartheta_c(\cdot)$  is semismooth at  $\bar{y}$ , and thus is also Bouligand-differentiable at  $\bar{y}$ . Then there exists  $\delta \in (0, \delta_0]$  such that for any  $y \in \mathbb{B}_\delta(\bar{y})$ ,

$$\|x_c(y) - x_c(\bar{y}) - (x_c)'(\bar{y}; y - \bar{y})\| \leq \varrho_0 \|y - \bar{y}\|/c \quad (3.25)$$

and

$$\|\nabla\vartheta_c(y) - \nabla\vartheta_c(\bar{y}) - (\nabla\vartheta_c)'(\bar{y}; y - \bar{y})\| \leq \mu_0\|y - \bar{y}\|/c^\tau. \quad (3.26)$$

Let  $y := (\zeta, \xi) \in \mathbb{B}_\delta(\bar{y})$  be an arbitrary point. From (3.19), (3.25), and the fact that  $x_c(\bar{y}) = \bar{x}$ , we have

$$\|x_c(y) - \bar{x}\| \leq \|(x_c)'(\bar{y}; y - \bar{y})\| + \varrho_0\|y - \bar{y}\|c = \varrho_1\|y - \bar{y}\|/c,$$

which, shows that (3.23) holds.

Since  $\nabla\vartheta_c(\cdot)$  is semismooth at  $\bar{y}$ , there exists an element  $V \in \partial_B(\nabla\vartheta_c)(\bar{y})$  such that  $(\nabla\vartheta_c)'(\bar{y}; y - \bar{y}) = V(y - \bar{y})$ . By using the fact that  $V$  is self-adjoint (see Lemma 2.4), we know from (3.20) in Assumption B2 and Proposition 3.3 that

$$\|V(y - \bar{y}) + c^{-1}(y - \bar{y})\| \leq 3\mu_0\|y - \bar{y}\|/c^\tau. \quad (3.27)$$

Therefore, we have from (3.26) and (3.27)

$$\begin{aligned} & \|y + c\nabla\vartheta_c(y) - \bar{y}\| \\ &= c\|\nabla\vartheta_c(y) - \nabla\vartheta_c(\bar{y}) - (\nabla\vartheta_c)'(\bar{y}; y - \bar{y}) + (\nabla\vartheta_c)'(\bar{y}; y - \bar{y}) + c^{-1}(y - \bar{y})\| \\ &\leq c\|\nabla\vartheta_c(y) - \nabla\vartheta_c(\bar{y}) - (\nabla\vartheta_c)'(\bar{y}; y - \bar{y})\| + c\|V(y - \bar{y}) + c^{-1}(y - \bar{y})\| \\ &\leq \mu_0\|y - \bar{y}\|/c^{\tau-1} + 3\mu_0\|y - \bar{y}\|/c^{\tau-1} = \varrho_2\|y - \bar{y}\|/c^{\tau-1}, \end{aligned}$$

which, together with (3.11) and the definitions of  $\zeta_c(\zeta, \xi)$  and  $\xi_c(\zeta, \xi)$ , proves (3.24). The proof is completed.  $\square$

Under Assumptions B1 and B2, Theorem 3.1 shows that if for all  $k$  sufficiently large with  $c_k \equiv c$  larger than a threshold and if  $(x^k, \zeta^k, \xi^k)$  is sufficiently close to  $(\bar{x}, \bar{\zeta}, \bar{\xi})$ , then the augmented Lagrangian method can locally be regarded as the gradient ascent method applied to the dual problem

$$\max \vartheta_c(\zeta, \xi) \quad \text{s.t. } (\zeta, \xi) \in \mathfrak{R}^m \times Y \quad (3.28)$$

with a constant step-length  $c$ , i.e., for all  $k$  sufficiently large

$$\begin{pmatrix} \zeta^{k+1} \\ \xi^{k+1} \end{pmatrix} = \begin{pmatrix} \zeta^k \\ \xi^k \end{pmatrix} + c\nabla\vartheta_c(\zeta^k, \xi^k).$$

By (3.19) in Assumption B2, we see that locally the augmented Lagrangian method can also be treated as an approximate generalized Newton method applied to the following nonsmooth equation

$$\nabla\vartheta_c(\zeta, \xi) = 0$$

with  $-c^{-1}\mathcal{I}$  as a good estimate to elements in  $\partial\nabla\vartheta_c(\zeta^k, \xi^k)$  for all  $(\zeta^k, \xi^k)$  sufficiently close to  $(\bar{\zeta}, \bar{\xi})$  as every element in  $\partial\nabla\vartheta_c(\bar{\zeta}, \bar{\xi})$  is in the form of  $-c^{-1}I + O(c^{-\tau})$ , where  $\mathcal{I}$  is the identity operator in  $\mathfrak{R}^m \times Y$ . Since  $\nabla\vartheta_c(\cdot, \cdot)$  is semismooth at  $(\bar{\zeta}, \bar{\xi})$  (cf. Proposition 3.2), the fast local convergence of the augmented Lagrangian method comes no surprise for those who are familiar with the theory developed by Kummer [20] and Qi and Sun [28] on the superlinear convergence of the generalized Newton method for semismooth equations.

The local rate of convergence for  $\{(\zeta^k, \xi^k)\}$  established in Theorem 3.1 is proportional to  $1/c^{\tau-1}$ , which tends to zero as  $c \rightarrow \infty$ . However, to increase the value of  $c$  may force the convergence sphere to shrink. In the next section, we shall check whether Assumptions B1 and B2 imposed in this section can be satisfied by nonlinear semidefinite programming.

## 4 The case for nonlinear semidefinite programming

This section is devoted to studying the following nonlinear semidefinite programming

$$(NLSDP) \quad \min f(x) \quad \text{s.t.} \quad h(x) = 0, \quad g(x) \in \mathcal{S}_+^p, \quad (4.1)$$

where  $f : \mathfrak{R}^n \mapsto \mathfrak{R}$ ,  $h : \mathfrak{R}^n \mapsto \mathfrak{R}^m$ , and  $g : \mathfrak{R}^n \mapsto \mathcal{S}^p$  are twice continuously differentiable. Nonlinear semidefinite programming (NLSDP) is a special case of (OP) with  $X := \mathfrak{R}^n$ ,  $Y := \mathcal{S}^p$  and  $K := \mathcal{S}_+^p$ . The Lagrangian function for (NLSDP) is

$$L_0(x, \zeta, \Xi) = f(x) + \langle \zeta, h(x) \rangle - \langle \Xi, g(x) \rangle, \quad (x, \zeta, \Xi) \in \mathfrak{R}^n \times \mathfrak{R}^m \times \mathcal{S}^p,$$

where we use  $\Xi$  instead of  $\xi$  to represent the Lagrange multiplier corresponding to the constraint  $g(x) \in \mathcal{S}_+^p$ . Then for any  $(x, \zeta, \Xi) \in \mathfrak{R}^n \times \mathfrak{R}^m \times \mathcal{S}^p$ ,

$$\nabla_x L_0(x, \zeta, \Xi) = \nabla f(x) + \nabla h(x)\zeta - \nabla g(x)\Xi.$$

Let  $(\bar{x}, \bar{\zeta}, \bar{\Xi}) \in \mathfrak{R}^n \times \mathfrak{R}^m \times \mathcal{S}^p$  be a given KKT point. Then,  $(\bar{x}, \bar{\zeta}, \bar{\Xi})$  satisfies

$$\nabla_x L_0(\bar{x}, \bar{\zeta}, \bar{\Xi}) = 0, \quad h(\bar{x}) = 0, \quad \bar{\Xi} \succeq 0, \quad g(\bar{x}) \succeq 0, \quad \text{and} \quad \langle \bar{\Xi}, g(\bar{x}) \rangle = 0. \quad (4.2)$$

Let  $\bar{Z} := \bar{\Xi} - g(\bar{x})$ . Suppose that  $\bar{Z}$  has the spectral decomposition as in (2.5), i.e.,

$$\bar{Z} = \bar{P}\Lambda\bar{P}^T, \quad (4.3)$$

where  $\Lambda$  is the diagonal matrix of eigenvalues of  $\bar{Z}$  and  $\bar{P}$  is a corresponding orthogonal matrix of orthonormal eigenvectors. Define three index sets of positive, zero, and negative eigenvalues of  $\bar{Z}$ , respectively, as

$$\alpha := \{i \mid \lambda_i > 0\}, \quad \beta := \{i \mid \lambda_i = 0\}, \quad \gamma := \{i \mid \lambda_i < 0\}.$$

Write

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_\gamma \end{bmatrix} \quad \text{and} \quad \bar{P} = [\bar{P}_\alpha \quad \bar{P}_\beta \quad \bar{P}_\gamma]$$

with  $\bar{P}_\alpha \in \mathfrak{R}^{p \times |\alpha|}$ ,  $\bar{P}_\beta \in \mathfrak{R}^{p \times |\beta|}$ , and  $\bar{P}_\gamma \in \mathfrak{R}^{p \times |\gamma|}$ . From (4.2), we know that  $\bar{\Xi}g(\bar{x}) = g(\bar{x})\bar{\Xi} = 0$ . Thus, we have

$$\bar{\Xi} = \bar{P} \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{P}^T, \quad g(\bar{x}) = \bar{P} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Lambda_\gamma \end{bmatrix} \bar{P}^T \quad (4.4)$$

and for any  $t \in \mathfrak{R}$ ,

$$\bar{\Xi} - tg(\bar{x}) = \bar{P} \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t\Lambda_\gamma \end{bmatrix} \bar{P}^T. \quad (4.5)$$

Let

$$\underline{\nu}_0 := \min_{i \in \alpha, j \in \gamma} \lambda_i / |\lambda_j|, \quad \bar{\nu}_0 := \max_{i \in \alpha, j \in \gamma} \lambda_i / |\lambda_j|. \quad (4.6)$$

Let  $\mathcal{Q}$  be the set of all orthogonal matrices of order  $|\beta| \times |\beta|$ . Define  $\mathcal{P}$  by (2.14), i.e.,

$$\mathcal{P} = \{P \in \mathfrak{R}^{p \times p} \mid P = [\bar{P}_\alpha (\bar{P}_\beta Q) \bar{P}_\gamma], Q \in \mathcal{Q}\}.$$

We now introduce the conditions needed in this section.

**Assumption (nlsdp-A1).** The constraint nondegeneracy condition holds at  $\bar{x}$ :

$$\begin{pmatrix} \mathcal{J}h(\bar{x}) \\ \mathcal{J}g(\bar{x}) \end{pmatrix} \mathfrak{R}^n + \begin{pmatrix} \{0\} \\ \text{lin}(\mathcal{T}_{\mathcal{S}_+^p}(g(\bar{x}))) \end{pmatrix} = \begin{pmatrix} \mathfrak{R}^m \\ \mathcal{S}^p \end{pmatrix}. \quad (4.7)$$

Assumption (nlsdp-A1) is the analogue to the linear independence constraint qualification for nonlinear programming [29, 38]. It also implies that  $\mathcal{M}(\bar{x})$  is a singleton [8, Proposition 4.50].

**Assumption (nlsdp-A2)** The strong second order sufficient condition holds at  $\bar{x}$  [40]:

$$\langle d, \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\Xi})d \rangle + \Upsilon_{g(\bar{x})}(\bar{\Xi}, \mathcal{J}g(\bar{x})d) > 0 \quad \forall d \in \text{app}(\bar{\zeta}, \bar{\Xi}) \setminus \{0\}, \quad (4.8)$$

where

$$\text{app}(\bar{\zeta}, \bar{\Xi}) := \{d \mid \mathcal{J}h(\bar{x})d = 0, \mathcal{J}g(\bar{x})d \in \text{aff}(\mathcal{C}(g(\bar{x}) - \bar{\Xi}; \mathcal{S}_+^p))\} \quad (4.9)$$

and for any given  $B \in \mathcal{S}^p$ , the linear-quadratic function  $\Upsilon_B(\cdot, \cdot)$  is defined as

$$\Upsilon_B(\Gamma, C) := 2 \langle \Gamma, CB^\dagger C \rangle, \quad (\Gamma, C) \in \mathcal{S}^p \times \mathcal{S}^p \quad (4.10)$$

with  $B^\dagger$  being the Moore-Penrose pseudo-inverse of  $B$ .

Let  $P \in \mathcal{P}$ . Then there exists  $Q \in \mathcal{Q}$  such that  $P = [\bar{P}_\alpha (\bar{P}_\beta Q) \bar{P}_\gamma]$ . For index sets  $\chi, \chi' \in \{\alpha, \beta, \gamma\}$ , let

$$C_{(\chi, \chi')}(P) := \left( \text{vec}(P_\chi^T \mathcal{J}_{x_1} g(\bar{x}) P_{\chi'}) \cdots \text{vec}(P_\chi^T \mathcal{J}_{x_n} g(\bar{x}) P_{\chi'}) \right) \quad (4.11)$$

and

$$\widehat{C}_{(\chi, \chi)}(P) := \left( \text{svec}(P_\chi^T \mathcal{J}_{x_1} g(\bar{x}) P_\chi) \cdots \text{svec}(P_\chi^T \mathcal{J}_{x_n} g(\bar{x}) P_\chi) \right), \quad (4.12)$$

where  $\text{vec}(B)$  denotes the vector obtained by stacking up all the columns of a given matrix  $B$  and  $\text{svec}(B)$  denotes the vector obtained by stacking up all the columns of the upper triangular part of a given symmetric matrix  $B$ . Since  $P_\alpha = \bar{P}_\alpha$  and  $P_\gamma = \bar{P}_\gamma$ , we write  $C_{(\chi, \chi')}$  and  $\widehat{C}_{(\chi, \chi)}$  instead of  $C_{(\chi, \chi')}(P)$  and  $\widehat{C}_{(\chi, \chi)}(P)$ , respectively if  $\chi, \chi' \in \{\alpha, \gamma\}$ . Define

$$n_1 := m + |\alpha|(|\alpha| + 1)/2, \quad n_2 := n_1 + |\beta|(|\beta| + 1)/2 + |\alpha||\beta|, \quad n_3 := n - n_2,$$

and

$$A(P) := \begin{pmatrix} \mathcal{J}h(\bar{x}) \\ -\widehat{C}_{(\alpha, \alpha)} \\ -\widehat{C}_{(\beta, \beta)}(P) \\ -C_{(\alpha, \beta)}(P) \end{pmatrix}. \quad (4.13)$$

Suppose that Assumption (nlsdp-A1) holds. Then by (4.7) in Assumption (nlsdp-A1) we know that  $A(P)$  is of full row rank<sup>1</sup>. Let  $A(P)$  have the following singular value decomposition:

$$A(P) = U[\Sigma(P) \ 0]R^T, \quad (4.14)$$

where  $U \in \mathfrak{R}^{n_2 \times n_2}$  and  $R \in \mathfrak{R}^{n \times n}$  are orthogonal matrices,  $\Sigma(P) = \text{Diag}(\sigma_1(A(P)), \dots, \sigma_{n_2}(A(P)))$ , and  $\sigma_1(A(P)) \geq \sigma_2(A(P)) \geq \dots \geq \sigma_{n_2}(A(P)) > 0$  are the singular values of  $A(P)$ . Let

$$\underline{\sigma} := \min \left\{ 1, \min_{P \in \mathcal{P}} \min_{1 \leq i \leq n_2} \sigma_i^{-2}(A(P)) \right\} \text{ and } \bar{\sigma} := \max \left\{ 1, \max_{P \in \mathcal{P}} \max_{1 \leq i \leq n_2} \sigma_i^{-2}(A(P)) \right\}. \quad (4.15)$$

Then, since  $\mathcal{P}$  is a compact set and  $\Sigma(P)$  changes continuously with respect to  $P$ , both  $\underline{\sigma}$  and  $\bar{\sigma}$  are finite positive numbers. Thus there exist two positive numbers  $\underline{\nu}$  and  $\bar{\nu}$  such that for any  $P \in \mathcal{P}$  and  $s \in \mathfrak{R}^{|\alpha|+|\gamma|}$ ,

$$\underline{\nu} \|s\|^2 \leq \max \left\{ \left\langle s, \tilde{C}_{(\alpha, \gamma)}(P) (\tilde{C}_{(\alpha, \gamma)}^T(P) s) \right\rangle, \left\langle s, C_{(\alpha, \gamma)} C_{(\alpha, \gamma)}^T s \right\rangle \right\} \leq \bar{\nu} \|s\|^2, \quad (4.16)$$

where

$$\tilde{C}_{(\alpha, \gamma)}(P) := C_{(\alpha, \gamma)} R \begin{bmatrix} \Sigma(P)^{-1} U^T & 0 \\ 0 & I_{n_3} \end{bmatrix}.$$

When no ambiguity arises, we often drop  $P$  from  $A(P)$ ,  $C_{(\chi, \chi')}(P)$ , and  $\tilde{C}_{(\alpha, \gamma)}(P)$ .

Let  $c > 0$  and  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{\Xi} - cg(\bar{x}))$ . Define  $\lambda_c \in \mathfrak{R}^p$  as

$$(\lambda_c)_i := \begin{cases} \lambda_i & \text{if } i \in \alpha \cup \beta, \\ c\lambda_i & \text{if } i \in \gamma. \end{cases}$$

Then it follows from Lemma 2.6 that there exist two matrices  $Q \in \mathcal{Q}$  with  $P = [\bar{P}_\alpha (\bar{P}_\beta Q) \bar{P}_\gamma]$  and  $\Theta_c \in \mathcal{S}^p$  such that

$$W(H) = P (\Theta_c \circ (P^T H P)) P^T \quad \forall H \in \mathcal{S}^p \quad (4.17)$$

with the entries of  $\Theta_c$  being given by

$$\begin{cases} (\Theta_c)_{ij} = \frac{\max\{(\lambda_c)_i, 0\} + \max\{(\lambda_c)_j, 0\}}{|\lambda_c)_i| + |\lambda_c)_j|} & \text{if } (i, j) \notin \beta \times \beta, \\ (\Theta_c)_{ij} \in [0, 1] & \text{if } (i, j) \in \beta \times \beta. \end{cases} \quad (4.18)$$

For index sets  $\chi, \chi' \in \{\alpha, \beta, \gamma\}$ , we introduce the following notation:

$$(\Theta_c)_{(\chi, \chi')} = \text{Diag}(\text{vec}((\Theta_c)_{\chi\chi'})) \text{ , } (\hat{\Theta}_c)_{(\chi, \chi)} = \text{Diag}(\text{svec}((\Theta_c)_{\chi\chi} \circ E_{\chi\chi})) \text{ ,}$$

where “ $\circ$ ” is the Hadamard product and  $E$  is a matrix in  $\mathcal{S}^p$  with entries being given by

$$E_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \neq j. \end{cases}$$

<sup>1</sup>One may consult [8, Proposition 5.71] for a proof, where Bonnans and Shapiro only considered the case that  $g(x) \in \mathcal{S}_+^p$ . However, it is easy to modify their arguments to include the equality constraint  $h(x) = 0$ .

Let

$$D_c := \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & (\widehat{\Theta}_c)_{(\alpha,\alpha)} & 0 & 0 \\ 0 & 0 & (\widehat{\Theta}_c)_{(\beta,\beta)} & 0 \\ 0 & 0 & 0 & 2I_{|\alpha||\beta|} \end{bmatrix}. \quad (4.19)$$

Let  $\mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W)$  be defined as (3.5), i.e,

$$\mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W) = \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\xi}) + c \nabla h(\bar{x}) \mathcal{J}h(\bar{x}) + c \nabla g(\bar{x}) W \mathcal{J}g(\bar{x}).$$

A compact formula for  $\mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W)$  is given in the next lemma.

**Lemma 4.1** *The matrix  $\mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W)$  can be expressed equivalently as*

$$\begin{aligned} \mathcal{A}_c(\bar{\zeta}, \bar{\xi}, W) &= \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\xi}) + c \left( \nabla h(\bar{x}) \mathcal{J}h(\bar{x}) + \widehat{C}_{(\alpha,\alpha)}^T (\widehat{\Theta}_c)_{(\alpha,\alpha)} \widehat{C}_{(\alpha,\alpha)} \right. \\ &\quad \left. + 2C_{(\alpha,\beta)}^T C_{(\alpha,\beta)} + 2C_{(\alpha,\gamma)}^T (\Theta_c)_{(\alpha,\gamma)} C_{(\alpha,\gamma)} + \widehat{C}_{(\beta,\beta)}^T (\widehat{\Theta}_c)_{(\beta,\beta)} \widehat{C}_{(\beta,\beta)} \right). \end{aligned} \quad (4.20)$$

**Proof.** Let  $d$  be an arbitrary point in  $\mathfrak{R}^n$ . By the definition of  $\nabla g(\bar{x})$ , we have for  $H := \mathcal{J}g(\bar{x})d$  that

$$\nabla g(\bar{x})W(H) = \begin{bmatrix} \langle \mathcal{J}_{x_1} g(\bar{x}), W(H) \rangle \\ \langle \mathcal{J}_{x_2} g(\bar{x}), W(H) \rangle \\ \vdots \\ \langle \mathcal{J}_{x_n} g(\bar{x}), W(H) \rangle \end{bmatrix}. \quad (4.21)$$

Noting that from (4.17) and (4.18), for any  $1 \leq l \leq p$ ,

$$\begin{aligned} \langle \mathcal{J}_{x_l} g(\bar{x}), W(H) \rangle &= \left\langle P^T \mathcal{J}_{x_l} g(\bar{x}) P, \sum_{i=1}^p P^T W(\mathcal{J}_{x_i} g(\bar{x}) d_i) P \right\rangle \\ &= \left\langle P^T \mathcal{J}_{x_l} g(\bar{x}) P, (\Theta_c) \circ \left( \sum_{i=1}^p P^T \mathcal{J}_{x_i} g(\bar{x}) P d_i \right) \right\rangle \\ &= \langle (C_{(\alpha,\alpha)})_l, C_{(\alpha,\alpha)} d \rangle + \langle (C_{(\beta,\alpha)})_l, C_{(\beta,\alpha)} d \rangle + \langle (C_{(\alpha,\beta)})_l, C_{(\alpha,\beta)} d \rangle \\ &\quad + \langle (C_{(\gamma,\alpha)})_l, (\Theta_c)_{(\gamma,\alpha)} C_{(\gamma,\alpha)} d \rangle + \langle (C_{(\alpha,\gamma)})_l, (\Theta_c)_{(\alpha,\gamma)} C_{(\alpha,\gamma)} d \rangle \\ &\quad + \langle (C_{(\beta,\beta)})_l, (\Theta_c)_{(\beta,\beta)} C_{(\beta,\beta)} d \rangle \\ &= \langle (\widehat{C}_{(\alpha,\alpha)})_l, (\widehat{\Theta}_c)_{(\alpha,\alpha)} \widehat{C}_{(\alpha,\alpha)} d \rangle + \langle (C_{(\beta,\alpha)})_l, C_{(\beta,\alpha)} d \rangle + \langle (C_{(\alpha,\beta)})_l, C_{(\alpha,\beta)} d \rangle \\ &\quad + \langle (C_{(\gamma,\alpha)})_l, (\Theta_c)_{(\gamma,\alpha)} C_{(\gamma,\alpha)} d \rangle + \langle (C_{(\alpha,\gamma)})_l, (\Theta_c)_{(\alpha,\gamma)} C_{(\alpha,\gamma)} d \rangle \\ &\quad + \langle (\widehat{C}_{(\beta,\beta)})_l, (\widehat{\Theta}_c)_{(\beta,\beta)} \widehat{C}_{(\beta,\beta)} d \rangle, \end{aligned}$$

we have from (4.21) that

$$\begin{aligned} \nabla g(\bar{x})W(\mathcal{J}g(\bar{x})d) &= \left( \widehat{C}_{(\alpha,\alpha)}^T (\widehat{\Theta}_c)_{(\alpha,\alpha)} \widehat{C}_{(\alpha,\alpha)} + 2C_{(\alpha,\beta)}^T C_{(\alpha,\beta)} \right. \\ &\quad \left. + 2C_{(\alpha,\gamma)}^T (\Theta_c)_{(\alpha,\gamma)} C_{(\alpha,\gamma)} + \widehat{C}_{(\beta,\beta)}^T (\widehat{\Theta}_c)_{(\beta,\beta)} \widehat{C}_{(\beta,\beta)} \right) d. \end{aligned}$$



Since  $d$  is arbitrarily chosen, (4.20) holds.  $\square$

Lemma 4.1 shows that  $\mathcal{A}_c(\bar{\zeta}, \bar{\Xi}, W)$  can be written as

$$\mathcal{A}_c(\bar{\zeta}, \bar{\Xi}, W) = \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\Xi}) + cA^T D_c A + 2cC_{(\alpha, \gamma)}^T(\Theta_c)_{(\alpha, \gamma)} C_{(\alpha, \gamma)}. \quad (4.22)$$

For any  $c', c > 0$ , let

$$\mathcal{B}_{c', c}(\bar{\zeta}, \bar{\Xi}, W) := \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\Xi}) + c'A^T D_c A + 2cC_{(\alpha, \gamma)}^T(\Theta_c)_{(\alpha, \gamma)} C_{(\alpha, \gamma)}. \quad (4.23)$$

The following proposition shows that, under Assumptions (nlsdp-A1) and (nlsdp-A2), the basic Assumption B1 made in Section 3 is satisfied by nonlinear semidefinite programming.

**Proposition 4.1** *Suppose that Assumptions (nlsdp-A1) and (nlsdp-A2) are satisfied. Then there exist two positive numbers  $c_0$  and  $\underline{\eta}$  such that for any  $c \geq c_0$  and  $W \in \partial_B \Pi_{S_+^p}(\bar{\Xi} - cg(\bar{x}))$ ,*

$$\langle d, \mathcal{A}_c(\bar{\zeta}, \bar{\Xi}, W)d \rangle \geq \langle d, \mathcal{B}_{c_0, c}(\bar{\zeta}, \bar{\Xi}, W)d \rangle \geq \underline{\eta} \langle d, d \rangle.$$

**Proof.** It follows from Assumption (nlsdp-A2) that there exists  $\eta_0 > 0$  such that

$$\langle d, \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\Xi})d \rangle + \Upsilon_{g(\bar{x})}(\bar{\Xi}, \mathcal{J}g(\bar{x})d) \geq \eta_0 \|d\|^2 \quad \forall d \in \text{app}(\bar{\zeta}, \bar{\Xi}). \quad (4.24)$$

By (2.12), (4.5), (4.9), and the fact that  $g(\bar{x}) - \bar{\Xi} = -\bar{Z}$ , we have

$$\text{app}(\bar{\zeta}, \bar{\Xi}) = \{d \mid \mathcal{J}h(\bar{x})d = 0, \bar{P}_\alpha^T(\mathcal{J}g(\bar{x})d)\bar{P}_\alpha = 0, \bar{P}_\alpha^T(\mathcal{J}g(\bar{x})d)\bar{P}_\beta = 0\}$$

or equivalently

$$\text{app}(\bar{\zeta}, \bar{\Xi}) = \{d \mid \mathcal{J}h(\bar{x})d = 0, \widehat{C}_{(\alpha, \alpha)}d = 0, C_{(\alpha, \beta)}(\bar{P})d = 0\}. \quad (4.25)$$

Since (4.24) and (4.25) hold, it follows from Lemma 2.7 that there exist two positive numbers  $c_1$  and  $\underline{\eta} \in (0, \eta_0/2]$  such that for any  $c \geq c_1$ ,

$$\begin{aligned} & \langle d, \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\Xi})d \rangle + \Upsilon_{g(\bar{x})}(\bar{\Xi}, \mathcal{J}g(\bar{x})d) \\ & + c\|\mathcal{J}h(\bar{x})d\|^2 + c\|\widehat{C}_{(\alpha, \alpha)}d\|^2 + c\|C_{(\alpha, \beta)}(\bar{P})d\|^2 \geq 2\underline{\eta}\|d\|^2 \quad \forall d \in \mathfrak{R}^n. \end{aligned} \quad (4.26)$$

Let  $c_0 \geq c_1$  be such that for any  $c \geq c_0$ ,

$$\max_{1 \leq l \leq n} \|\mathcal{J}_{x_l} g(\bar{x})\|^2 \sum_{i \in \gamma, j \in \alpha} \frac{\lambda_j^2}{|\lambda_i|(\lambda_j + c|\lambda_i|)} \leq \underline{\eta}/2. \quad (4.27)$$

Let  $c \geq c_0$  and  $W \in \partial_B \Pi_{S_+^p}(\bar{\Xi} - cg(\bar{x}))$ . Then there exist two matrices  $Q \in \mathcal{Q}$  with  $P = [\bar{P}_\alpha \ (\bar{P}_\beta Q) \ \bar{P}_\gamma]$  and  $\Theta_c \in S^p$  satisfying (4.18) such that (4.17) holds, i.e.,

$$W(H) = P(\Theta_c \circ (P^T H P)) P^T \quad \forall H \in S^p.$$

It is easy to see from (4.27) that for any  $c \geq c_0$  and  $d \in \mathfrak{R}^n$  we have

$$\begin{aligned}
& \Upsilon_{g(\bar{x})}(\bar{\Xi}, \mathcal{J}g(\bar{x})d) - 2c \left\langle d, C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)} C_{(\alpha,\gamma)} d \right\rangle \\
&= 2 \left\langle \bar{\Xi}, \mathcal{J}g(\bar{x})dg(\bar{x})^\dagger \mathcal{J}g(\bar{x})d \right\rangle - 2c \left\langle d, C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)} C_{(\alpha,\gamma)} d \right\rangle \\
&= 2 \sum_{i \in \gamma, j \in \alpha} \frac{\lambda_j}{|\lambda_i|} \left( \sum_{l=1}^n P_i^T \mathcal{J}_{x_l} g(\bar{x}) P_j d_l \right)^2 - 2c \sum_{i \in \gamma, j \in \alpha} \frac{\lambda_j}{\lambda_j + c|\lambda_i|} \left( \sum_{l=1}^n P_i^T \mathcal{J}_{x_l} g(\bar{x}) P_j d_l \right)^2 \\
&\leq 2 \sum_{i \in \gamma, j \in \alpha} \left[ \frac{\lambda_j^2}{|\lambda_i|(\lambda_j + c|\lambda_i|)} \sum_{l=1}^n \|\mathcal{J}_{x_l} g(\bar{x})\|^2 \|P_i\|^2 \|P_j\|^2 d_l^2 \right] \\
&\leq 2 \max_{1 \leq l \leq n} \|\mathcal{J}_{x_l} g(\bar{x})\|^2 \sum_{i \in \gamma, j \in \alpha} \frac{\lambda_j^2}{|\lambda_i|(\lambda_j + c|\lambda_i|)} \|d\|^2 \\
&\leq \underline{\eta} \|d\|^2,
\end{aligned}$$

which, together with (4.26), implies that for any  $c \geq c_0$  we have

$$\begin{aligned}
& \left\langle d, \nabla_{xx}^2 L_0(\bar{x}, \bar{\zeta}, \bar{\Xi}) d \right\rangle + 2c \left\langle d, C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)} C_{(\alpha,\gamma)} d \right\rangle \\
&+ c_0 \|\mathcal{J}h(\bar{x})d\|^2 + c_0 \|\widehat{C}_{(\alpha,\alpha)} d\|^2 + c_0 \|C_{(\alpha,\beta)}(\bar{P})d\|^2 \geq \underline{\eta} \|d\|^2 \quad \forall d \in \mathfrak{R}^n.
\end{aligned} \tag{4.28}$$

Let “ $\otimes$ ” denote the Kronecker product. Since for any  $d \in \mathfrak{R}^n$ ,

$$\begin{aligned}
& \|C_{(\alpha,\beta)}(P)d\|^2 = \left\langle C_{(\alpha,\beta)}(P)d, C_{(\alpha,\beta)}(P)d \right\rangle \\
&= \left\langle \sum_{l=1}^n \text{vec}(P_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) P_\beta) d_l, \sum_{l=1}^n \text{vec}(P_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) P_\beta) d_l \right\rangle \\
&= \left\langle \sum_{l=1}^n \text{vec}(\bar{P}_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) \bar{P}_\beta Q) d_l, \sum_{l=1}^n \text{vec}(\bar{P}_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) \bar{P}_\beta Q) d_l \right\rangle \\
&= \left\langle (Q^T \otimes I_{|\alpha|}) \sum_{l=1}^n \text{vec}(\bar{P}_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) \bar{P}_\beta) d_l, (Q^T \otimes I_{|\alpha|}) \sum_{l=1}^n \text{vec}(\bar{P}_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) \bar{P}_\beta) d_l \right\rangle \\
&= \left\langle \sum_{l=1}^n \text{vec}(\bar{P}_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) \bar{P}_\beta) d_l, \sum_{l=1}^n \text{vec}(\bar{P}_\alpha^T \mathcal{J}_{x_l} g(\bar{x}) \bar{P}_\beta) d_l \right\rangle \\
&= \left\langle C_{(\alpha,\beta)}(\bar{P})d, C_{(\alpha,\beta)}(\bar{P})d \right\rangle = \|C_{(\alpha,\beta)}(\bar{P})d\|^2,
\end{aligned}$$

from (4.23), (4.28), and the fact that  $\widehat{C}_{(\beta,\beta)}^T(\widehat{\Theta}_c)_{(\beta,\beta)} \widehat{C}_{(\beta,\beta)} \succeq 0$ , we can see that for any  $c \geq c_0$ ,

$$\left\langle d, \mathcal{B}_{c_0,c}(\bar{\zeta}, \bar{\Xi}, W) d \right\rangle \geq \underline{\eta} \|d\|^2 \quad \forall d \in \mathfrak{R}^n.$$

By noting the fact that

$$\mathcal{A}_c(\bar{\zeta}, \bar{\Xi}, W) = \mathcal{B}_{c_0,c}(\bar{\zeta}, \bar{\Xi}, W) + (c - c_0) A^T D_c A,$$

we complete the proof.  $\square$

Let Assumptions (nlsdp-A1) and (nlsdp-A2) be satisfied. Let the two positive numbers  $c_0$  and  $\eta$  be defined as in Proposition 4.1. Let  $c \geq c_0$ . Then by Propositions 3.1 and 4.1 and the fact that  $\Pi_{\mathcal{S}_+^p}(\cdot)$  is strongly semismooth everywhere there exist two positive numbers  $\varepsilon > 0$  and  $\delta_0 > 0$  (both depending on  $c$ ) and a locally Lipschitz continuous function  $x_c(\cdot, \cdot)$  defined on  $\mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\Xi})$  such that for any  $(\zeta, \Xi) \in \mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\Xi})$ ,  $x_c(\zeta, \Xi)$  is the unique minimizer of  $L_c(\cdot, \zeta, \Xi)$  over  $\mathbb{B}_\varepsilon(\bar{x})$  and  $x_c(\cdot, \cdot)$  is semismooth at  $(\zeta, \Xi)$ . Let  $\vartheta_c : \mathfrak{R}^m \times \mathcal{S}^p \mapsto \mathfrak{R}$  be defined as (3.7), i.e.,

$$\vartheta_c(\zeta, \Xi) := \min_{x \in \mathbb{B}_\varepsilon(\bar{x})} L_c(x, \zeta, \Xi), \quad (\zeta, \Xi) \in \mathfrak{R}^m \times \mathfrak{R}^p.$$

Then it holds that

$$\vartheta_c(\zeta, \Xi) = L_c(x_c(\zeta, \Xi), \zeta, \Xi), \quad (\zeta, \Xi) \in \mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\Xi}).$$

Furthermore, it follows from Propositions 3.2 and 4.1 that the concave function  $\vartheta_c(\cdot, \cdot)$  is continuously differentiable on  $\mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\Xi})$  with

$$\nabla \vartheta_c(\zeta, \Xi) = \begin{pmatrix} h(x_c(\zeta, \Xi)) \\ c^{-1} \left( -\Xi + \Pi_{\mathcal{S}_+^p}(\xi - cg(x_c(\zeta, \Xi))) \right) \end{pmatrix}, \quad (\zeta, \Xi) \in \mathbb{B}_{\delta_0}(\bar{\zeta}, \bar{\Xi}). \quad (4.29)$$

For any  $(\Delta\zeta, \Delta\Xi) \in \mathfrak{R}^m \times \mathcal{S}^p$ , let  $\bar{\mathcal{V}}_c(\Delta\zeta, \Delta\Xi)$  be defined as in (3.12). By Propositions 3.3 and 4.1, we have for any  $(\Delta\zeta, \Delta\Xi) \in \mathfrak{R}^m \times \mathcal{S}^p$  that

$$\partial_B(\nabla \vartheta_c)(\bar{\zeta}, \bar{\Xi})(\Delta\zeta, \Delta\Xi) \subseteq \bar{\mathcal{V}}_c(\Delta\zeta, \Delta\Xi). \quad (4.30)$$

Since

$$\lim_{c \rightarrow \infty} c(\Theta_c)_{ij} = \lim_{c \rightarrow \infty} c \frac{\lambda_i}{\lambda_i + c|\lambda_j|} = \frac{\lambda_i}{|\lambda_j|} \quad \forall (i, j) \in \alpha \times \gamma,$$

we know that there exists a positive number  $\bar{\eta}$  such that

$$\langle d, \mathcal{B}_{c_0, c}(\bar{\zeta}, \bar{\Xi}, W)d \rangle \leq \bar{\eta} \langle d, d \rangle \quad \forall d \in \mathfrak{R}^n, \quad c \geq c_0, \quad \text{and } W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{\Xi} - cg(\bar{x})). \quad (4.31)$$

Let  $c \geq c_0$  and  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{\Xi} - cg(\bar{x}))$ . Then there exist two matrices  $Q \in \mathcal{Q}$  with  $P = [\bar{P}_\alpha \quad (\bar{P}_\beta Q) \quad \bar{P}_\gamma]$  and  $\Theta_c \in \mathcal{S}^p$  satisfying (4.18) such that (4.17) holds. Let  $A(P)$  have the singular value decomposition as in (4.14), i.e.,

$$A(P) = U[\Sigma(P) \quad 0]R^T. \quad (4.32)$$

Let  $\bar{y} := (\bar{\zeta}, \bar{\Xi})$ . Then we have the following result for  $\mathcal{A}_c(\bar{y}, W)$ .

**Lemma 4.2** *Let  $c \geq c_0$ . Suppose that Assumptions (nlsdp-A1) and (nlsdp-A2) are satisfied. Then we have*

$$\mathcal{A}_c(\bar{y}, W)^{-1} \preceq R \begin{bmatrix} \Sigma^{-1} U^T \left( \underline{\sigma} \underline{\eta} I_{n_2} + (c - c_0) D_c \right)^{-1} U \Sigma^{-1} & 0 \\ 0 & \underline{\sigma}^{-1} \underline{\eta}^{-1} I_{n_3} \end{bmatrix} R^T \quad (4.33)$$

and

$$\mathcal{A}_c(\bar{y}, W)^{-1} \succeq R \begin{bmatrix} \Sigma^{-1} U^T \left( \bar{\sigma} \bar{\eta} I_{n_2} + (c - c_0) D_c \right)^{-1} U \Sigma^{-1} & 0 \\ 0 & \bar{\sigma}^{-1} \bar{\eta}^{-1} I_{n_3} \end{bmatrix} R^T, \quad (4.34)$$

where  $\Sigma := \Sigma(P)$ .

**Proof.** Let  $\hat{c} := c - c_0$ . By (4.22), (4.23), and the singular value decomposition (4.32) of  $A := A(P)$ , we have

$$\begin{aligned}
\mathcal{A}_c(\bar{y}, W)^{-1} &= \left( \mathcal{B}_{c_0, c}(\bar{y}, W) + \hat{c} A^T D_c A \right)^{-1} \\
&= \left( \mathcal{B}_{c_0, c}(\bar{y}, W) + \hat{c} R [\Sigma \quad 0]^T U^T D_c U [\Sigma \quad 0] R^T \right)^{-1} \\
&= R \left( R^T \mathcal{B}_{c_0, c}(\bar{y}, W) R + \hat{c} \begin{bmatrix} \Sigma & 0 \\ 0 & I_{n_3} \end{bmatrix} \begin{bmatrix} U^T D_c U & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & I_{n_3} \end{bmatrix} \right)^{-1} R^T \\
&= R \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix} \left( \mathcal{G}_{c_0}(\bar{y}, W) + \hat{c} \begin{bmatrix} U^T D_c U & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix} R^T, \tag{4.35}
\end{aligned}$$

where

$$\mathcal{G}_{c_0}(\bar{y}, W) := \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix} R^T \mathcal{B}_{c_0, c}(\bar{y}, W) R \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix}.$$

It follows from Proposition 4.1, the definitions of  $\underline{\sigma}$  and  $\bar{\sigma}$ , and (4.31) that

$$\mathcal{G}_{c_0}(\bar{y}, W) \succeq \underline{\eta} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix}^2 \succeq \underline{\sigma} \underline{\eta} I_n$$

and

$$\mathcal{G}_{c_0}(\bar{y}, W) \preceq \bar{\eta} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix}^2 \preceq \bar{\sigma} \bar{\eta} I_n.$$

Therefore, (4.33) and (4.34) follow from (4.35). The proof is completed.  $\square$

Let

$$\bar{c} := \max \left\{ c_0, (\bar{\sigma} \bar{\eta} - c_0)^2 / c_0, (\underline{\sigma} \underline{\eta} / 2 - c_0)^2 / c_0 \right\} \tag{4.36}$$

and

$$\varrho_0 := \left( \max \left\{ 2\bar{\sigma} \max \left\{ 1, \frac{4c_0^2}{\bar{\sigma}^2 \bar{\eta}^2} \right\}, 4\bar{\nu} \underline{\sigma}^{-2} \underline{\eta}^{-2} \bar{\nu}_0^2 \right\} \right)^{1/2}. \tag{4.37}$$

**Proposition 4.2** *Suppose that Assumptions (nlsdp-A1) and (nlsdp-A2) are satisfied. Then there exists a positive number  $\mu_0$  such that for any  $c \geq \bar{c}$  and  $\Delta y \in \mathfrak{R}^m \times \mathcal{S}^p$ ,*

$$\|(x_c)'(\bar{y}; \Delta y)\| \leq \varrho_0 \|\Delta y\| / c \tag{4.38}$$

and

$$\langle V(\Delta y) + c^{-1} \Delta y, \Delta y \rangle \in \mu_0 [-1, 1] \|\Delta y\|^2 / c^2 \quad \forall V(\Delta y) \in \bar{\mathcal{V}}_c(\Delta y). \tag{4.39}$$

**Proof.** Let  $c \geq \bar{c}$ . Let  $\Delta y := (\Delta \zeta, \Delta \Xi) \in \mathfrak{R}^m \times \mathcal{S}^p$ . From the proof of Proposition 3.3 we know that there exists an element  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{\Xi} - cg(\bar{x}))$  such that

$$(x_c)'(\bar{y}; \Delta y) = \mathcal{A}_c(\bar{y}, W)^{-1} (-\nabla h(\bar{x})(\Delta \zeta) + \nabla g(\bar{x})W(\Delta \Xi)). \tag{4.40}$$

For this  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{\Xi} - cg(\bar{x}))$ , there exist two matrices  $P \in \mathcal{P}$  and  $\Theta_c \in \mathcal{S}^p$  satisfying (4.18) such that

$$W(H) = P (\Theta_c \circ (P^T H P)) P^T \quad \forall H \in \mathcal{S}^p.$$

Let  $A := A(P)$  have the singular value decomposition as in (4.14), i.e.,

$$A = U[\Sigma \ 0]R^T, \quad (4.41)$$

where  $\Sigma := \Sigma(P)$ . For any two index sets  $\chi, \chi' \in \{\alpha, \beta, \gamma\}$ , let

$$\omega_{(\chi, \chi')} := \text{vec}(P_\chi^T \Delta \Xi P_{\chi'}), \quad \widehat{\omega}_{(\chi, \chi)} := \text{svec}(P_\chi^T \Delta \Xi P_\chi).$$

Define

$$\Delta d_0 := \begin{pmatrix} \Delta \zeta \\ \widehat{\omega}_{(\alpha, \alpha)} \\ \widehat{\omega}_{(\beta, \beta)} \\ \omega_{(\alpha, \beta)} \end{pmatrix}, \quad \Delta d := \begin{pmatrix} \Delta d_0 \\ \omega_{(\alpha, \gamma)} \end{pmatrix}.$$

Then, from (4.40), we have

$$\begin{aligned} & \langle (x_c)'(\bar{y}; \Delta y), (x_c)'(\bar{y}; \Delta y) \rangle \\ &= \left\langle [A^T D_c \ 2C_{(\alpha, \gamma)}^T(\Theta_c)_{(\alpha, \gamma)}] \Delta d, \mathcal{A}_c(\bar{y}, W)^{-2} [A^T D_c \ 2C_{(\alpha, \gamma)}^T(\Theta_c)_{(\alpha, \gamma)}] \Delta d \right\rangle \\ &\leq 2 \left\langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-2} A^T D_c \Delta d_0 \right\rangle \\ &\quad + 8 \left\langle C_{(\alpha, \gamma)}^T(\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}, \mathcal{A}_c(\bar{y}, W)^{-2} C_{(\alpha, \gamma)}^T(\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)} \right\rangle. \end{aligned} \quad (4.42)$$

Let

$$\underline{\mathcal{E}}_c := (\bar{\sigma} \bar{\eta} I_{n_2} + (c - c_0) D_c)^{-1}, \quad \bar{\mathcal{E}}_c := (\underline{\sigma} \underline{\eta} I_{n_2} + (c - c_0) D_c)^{-1} \quad (4.43)$$

and

$$\underline{\mathcal{H}}_c := \begin{bmatrix} \underline{\mathcal{E}}_c & 0 \\ 0 & \bar{\sigma}^{-1} \bar{\eta}^{-1} I_{n_3} \end{bmatrix}, \quad \bar{\mathcal{H}}_c := \begin{bmatrix} \bar{\mathcal{E}}_c & 0 \\ 0 & \underline{\sigma}^{-1} \underline{\eta}^{-1} I_{n_3} \end{bmatrix}. \quad (4.44)$$

By using (4.41) and Lemma 4.2, we obtain

$$0 \preceq A \mathcal{A}_c(\bar{y}, W)^{-2} A^T \preceq \bar{\mathcal{E}}_c (U \Sigma^{-2} U^T) \bar{\mathcal{E}}_c.$$

Thus, we have

$$\begin{aligned} & \left\langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-2} A^T D_c \Delta d_0 \right\rangle \\ &\leq \left\langle D_c \Delta d_0, \bar{\mathcal{E}}_c (U \Sigma^{-2} U^T) \bar{\mathcal{E}}_c D_c \Delta d_0 \right\rangle \\ &\leq \bar{\sigma} \left\langle \bar{\mathcal{E}}_c D_c \Delta d_0, \bar{\mathcal{E}}_c D_c \Delta d_0 \right\rangle \\ &\leq \bar{\sigma} (\bar{\sigma} \bar{\eta} / 2 + (c - c_0))^{-2} (\|(\Delta \zeta, \omega_{(\alpha, \alpha)}, \omega_{(\beta, \beta)})\|^2 + 2\|\omega_{(\alpha, \beta)}\|^2) \\ &\leq \frac{1}{2} \varrho_0^2 c^{-2} (\|(\Delta \zeta, \omega_{(\alpha, \alpha)}, \omega_{(\beta, \beta)})\|^2 + 2\|\omega_{(\alpha, \beta)}\|^2). \end{aligned} \quad (4.45)$$

By recalling that

$$\tilde{C}_{(\alpha, \gamma)} = C_{(\alpha, \gamma)} R \begin{bmatrix} \Sigma^{-1} U^T & 0 \\ 0 & I_{n_3} \end{bmatrix},$$

we know from Lemma 4.2, (4.44), (4.16), and (4.6) that

$$\begin{aligned}
& \left\langle C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}\omega_{(\alpha,\gamma)}, \mathcal{A}_c(\bar{y}, W)^{-2}C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}\omega_{(\alpha,\gamma)} \right\rangle \\
& \leq \left\langle \tilde{C}_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}\omega_{(\alpha,\gamma)}, (\bar{\mathcal{H}}_c)^2\tilde{C}_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}\omega_{(\alpha,\gamma)} \right\rangle \\
& \leq \underline{\sigma}^{-2}\underline{\eta}^{-2} \left\langle \tilde{C}_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}\omega_{(\alpha,\gamma)}, \tilde{C}_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}\omega_{(\alpha,\gamma)} \right\rangle \\
& \leq \bar{\nu}\underline{\sigma}^{-2}\underline{\eta}^{-2} \|(\Theta_c)_{(\alpha,\gamma)}\omega_{(\alpha,\gamma)}\|^2 \\
& \leq \bar{\nu}\underline{\sigma}^{-1}\underline{\eta}^{-1} \left( \max_{i \in \alpha, j \in \gamma} \lambda_i / (\lambda_i + c|\lambda_j|) \right)^2 \|\omega_{(\alpha,\gamma)}\|^2 \\
& \leq \bar{\nu}\underline{\sigma}^{-2}\underline{\eta}^{-2}\bar{\nu}_0^2(\bar{\nu}_0 + c)^{-2} \|\omega_{(\alpha,\gamma)}\|^2 \\
& \leq \bar{\nu}\underline{\sigma}^{-2}\underline{\eta}^{-2}\bar{\nu}_0^2c^{-2} \|\omega_{(\alpha,\gamma)}\|^2, \\
& \leq \frac{1}{8}\varrho_0^2c^{-2}(2\|\omega_{(\alpha,\gamma)}\|^2), \tag{4.46}
\end{aligned}$$

which, together with (4.42) and (4.45), implies

$$\langle (x_c)'(\bar{y}; \Delta y), (x_c)'(\bar{y}; \Delta y) \rangle \leq \varrho_0^2 \|\Delta y\|^2 / c^2. \tag{4.47}$$

Thus (4.38) holds.

Let  $V(\Delta y) \in \bar{\mathcal{V}}_c(\Delta y)$ . Then from the definition of  $\bar{\mathcal{V}}_c(\Delta y)$ , there exists  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{\Xi} - cg(\bar{x}))$  such that

$$\begin{aligned}
V(\Delta y) &= \begin{pmatrix} \mathcal{J}h(\bar{x}) \\ -W\mathcal{J}g(\bar{x}) \end{pmatrix} \mathcal{A}_c(\bar{y}, W)^{-1} \begin{pmatrix} -\nabla h(\bar{x})\Delta\zeta + \nabla g(\bar{x})W(\Delta\Xi) \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -c^{-1}\Delta\Xi + c^{-1}W(\Delta\Xi) \end{pmatrix}.
\end{aligned}$$

For notational convenience, we assume that this  $W \in \partial_B \Pi_{\mathcal{S}_+^p}(\bar{\Xi} - cg(\bar{x}))$  is the same as the previous one. After direct calculations (cf. Lemma 4.1), we obtain

$$\begin{aligned}
& -\langle V(\Delta y), \Delta y \rangle \\
&= \left\langle [A^T D_c \quad 2C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}] \Delta d, \mathcal{A}_c(\bar{y}, W)^{-1} [A^T D_c \quad 2C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)}] \Delta d \right\rangle \\
&\quad + c^{-1} \|\Delta\Xi\|^2 - c^{-1} \langle \Delta\Xi, W(\Delta\Xi) \rangle \\
&= \langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-1} A^T D_c \Delta d_0 \rangle + 4 \left\langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-1} C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)} \omega_{(\alpha,\gamma)} \right\rangle \\
&\quad + 4 \left\langle C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)} \omega_{(\alpha,\gamma)}, \mathcal{A}_c(\bar{y}, W)^{-1} C_{(\alpha,\gamma)}^T(\Theta_c)_{(\alpha,\gamma)} \omega_{(\alpha,\gamma)} \right\rangle \\
&\quad + c^{-1} \|\Delta\Xi\|^2 - c^{-1} \langle \Delta\Xi, W(\Delta\Xi) \rangle. \tag{4.48}
\end{aligned}$$

Next, we estimate the lower and upper bounds of the right had side of (4.48). Then by using (4.41) and Lemma 4.2 we obtain

$$\underline{\mathcal{E}}_c \preceq A \mathcal{A}_c(\bar{y}, W)^{-1} A^T \preceq \bar{\mathcal{E}}_c. \tag{4.49}$$

Thus, we have

$$\begin{aligned}
& \langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-1} A^T D_c \Delta d_0 \rangle \geq \langle D_c \Delta d_0, \underline{\mathcal{E}}_c D_c \Delta d_0 \rangle \\
& \geq (\bar{\sigma}\eta + (c - c_0))^{-1} \|(\Delta\zeta, \omega_{(\alpha, \alpha)})\|^2 + 4(\bar{\sigma}\eta + 2(c - c_0))^{-1} \|\omega_{(\alpha, \beta)}\|^2 \\
& \quad + \left\langle (\widehat{\Theta}_c)_{(\beta, \beta)} \widehat{\omega}_{(\beta, \beta)}, \left( \bar{\sigma}\eta I_{|\beta|(|\beta|+1)/2} + (c - c_0)(\widehat{\Theta}_c)_{(\beta, \beta)} \right)^{-1} (\widehat{\Theta}_c)_{(\beta, \beta)} \widehat{\omega}_{(\beta, \beta)} \right\rangle \\
& \geq (\bar{\sigma}\eta + (c - c_0))^{-1} \|(\Delta\zeta, \omega_{(\alpha, \alpha)})\|^2 + 4(\bar{\sigma}\eta + 2(c - c_0))^{-1} \|\omega_{(\alpha, \beta)}\|^2 \\
& \quad + \left\langle (\Theta_c)_{(\beta, \beta)} \omega_{(\beta, \beta)}, (\bar{\sigma}\eta I_{|\beta|} + (c - c_0)(\Theta_c)_{(\beta, \beta)})^{-1} (\Theta_c)_{(\beta, \beta)} \omega_{(\beta, \beta)} \right\rangle
\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
& \langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-1} A^T D_c \Delta d_0 \rangle \leq \langle D_c \Delta d_0, \bar{\mathcal{E}}_c D_c \Delta d_0 \rangle \\
& \leq (\underline{\sigma}\eta/2 + (c - c_0))^{-1} \|(\Delta\zeta, \omega_{(\alpha, \alpha)})\|^2 + 4(\underline{\sigma}\eta + 2(c - c_0))^{-1} \|\omega_{(\alpha, \beta)}\|^2 \\
& \quad + \left\langle (\widehat{\Theta}_c)_{(\beta, \beta)} \widehat{\omega}_{(\beta, \beta)}, \left( \underline{\sigma}\eta I_{|\beta|(|\beta|+1)/2} + (c - c_0)(\widehat{\Theta}_c)_{(\beta, \beta)} \right)^{-1} (\widehat{\Theta}_c)_{(\beta, \beta)} \widehat{\omega}_{(\beta, \beta)} \right\rangle \\
& \leq (\underline{\sigma}\eta/2 + (c - c_0))^{-1} \|(\Delta\zeta, \omega_{(\alpha, \alpha)})\|^2 + 4(\underline{\sigma}\eta + 2(c - c_0))^{-1} \|\omega_{(\alpha, \beta)}\|^2 \\
& \quad + \left\langle (\Theta_c)_{(\beta, \beta)} \omega_{(\beta, \beta)}, ((\underline{\sigma}\eta/2) I_{|\beta|} + (c - c_0)(\Theta_c)_{(\beta, \beta)})^{-1} (\Theta_c)_{(\beta, \beta)} \omega_{(\beta, \beta)} \right\rangle,
\end{aligned} \tag{4.51}$$

which, together with the fact that  $(\Theta_c)_{ij} \in [0, 1]$  for all  $(i, j) \in \beta \times \beta$ , implies

$$\begin{aligned}
& \langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-1} A^T D_c \Delta d_0 \rangle \\
& \leq (\underline{\sigma}\eta/2 + (c - c_0))^{-1} (\|(\Delta\zeta, \omega_{(\alpha, \alpha)}, \omega_{(\beta, \beta)})\|^2 + 2\|\omega_{(\alpha, \beta)}\|^2).
\end{aligned} \tag{4.52}$$

From Lemma 4.2, (4.44), (4.16), and (4.6) we know that

$$\begin{aligned}
& \left\langle C_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}, \mathcal{A}_c(\bar{y}, W)^{-1} C_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)} \right\rangle \\
& \geq \left\langle \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}, \underline{\mathcal{H}}_c \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)} \right\rangle \\
& \geq (\bar{\sigma}\eta + 2(c - c_0))^{-1} \left\langle \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}, \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)} \right\rangle \\
& \geq \underline{\nu} (\bar{\sigma}\eta + 2(c - c_0))^{-1} \|(\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}\|^2 \\
& \geq \underline{\nu} (\bar{\sigma}\eta + 2(c - c_0))^{-1} \left( \min_{i \in \alpha, j \in \gamma} \lambda_i / (\lambda_i + c|\lambda_j|) \right)^2 \|\omega_{(\alpha, \gamma)}\|^2 \\
& \geq \underline{\nu} (\bar{\sigma}\eta + 2(c - c_0))^{-1} \underline{\nu}_0^2 (\underline{\nu}_0 + c)^{-2} \|\omega_{(\alpha, \gamma)}\|^2,
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
& \left\langle C_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}, \mathcal{A}_c(\bar{y}, W)^{-1} C_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)} \right\rangle \\
& \leq \left\langle \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}, \bar{\mathcal{H}}_c \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)} \right\rangle \\
& \leq \underline{\sigma}^{-1} \underline{\eta}^{-1} \left\langle \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}, \tilde{C}_{(\alpha, \gamma)}^T (\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)} \right\rangle \\
& \leq \bar{\nu} \underline{\sigma}^{-1} \underline{\eta}^{-1} \|(\Theta_c)_{(\alpha, \gamma)} \omega_{(\alpha, \gamma)}\|^2 \\
& \leq \bar{\nu} \underline{\sigma}^{-1} \underline{\eta}^{-1} \left( \max_{i \in \alpha, j \in \gamma} \lambda_i / (\lambda_i + c|\lambda_j|) \right)^2 \|\omega_{(\alpha, \gamma)}\|^2 \\
& \leq \bar{\nu} \underline{\sigma}^{-1} \underline{\eta}^{-1} \bar{\nu}_0^2 (\bar{\nu}_0 + c)^{-2} \|\omega_{(\alpha, \gamma)}\|^2,
\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}
& \left\langle C_{(\alpha,\gamma)}^T(\Theta_c)(\alpha,\gamma)\omega_{(\alpha,\gamma)}, C_{(\alpha,\gamma)}^T(\Theta_c)(\alpha,\gamma)\omega_{(\alpha,\gamma)} \right\rangle \\
& \leq \bar{\nu} \|\Theta_c(\alpha,\gamma)\omega_{(\alpha,\gamma)}\|^2 \\
& \leq \bar{\nu} \left( \max_{i \in \alpha, j \in \gamma} \lambda_i / (\lambda_i + c|\lambda_j|) \right)^2 \|\omega_{(\alpha,\gamma)}\|^2 \\
& \leq \bar{\nu} \bar{\nu}_0^2 c^{-2} \|\omega_{(\alpha,\gamma)}\|^2.
\end{aligned} \tag{4.55}$$

By using (4.45) and (4.55) we have

$$\begin{aligned}
& \left| \left\langle A^T D_c \Delta d_0, \mathcal{A}_c(\bar{y}, W)^{-1} C_{(\alpha,\gamma)}^T(\Theta_c)(\alpha,\gamma)\omega_{(\alpha,\gamma)} \right\rangle \right| \\
& \leq \|\mathcal{A}_c(\bar{y}, W)^{-1} A^T D_c \Delta d_0\| \|C_{(\alpha,\gamma)}^T(\Theta_c)(\alpha,\gamma)\omega_{(\alpha,\gamma)}\| \\
& \leq \frac{\varrho_0}{\sqrt{2}} c^{-1} (\|(\Delta\zeta, \omega_{(\alpha,\alpha)}, \omega_{(\beta,\beta)})\|^2 + 2\|\omega_{(\alpha,\beta)}\|^2)^{1/2} (\bar{\nu}_0 \sqrt{\bar{\nu}} c^{-1} \|\omega_{(\alpha,\gamma)}\|) \\
& \leq \frac{\varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}}}{4} c^{-2} (\|(\Delta\zeta, \omega_{(\alpha,\alpha)}, \omega_{(\beta,\beta)})\|^2 + 2\|\omega_{(\alpha,\beta)}\|^2 + 2\|\omega_{(\alpha,\gamma)}\|^2).
\end{aligned} \tag{4.56}$$

By direct calculations we have

$$\begin{aligned}
& \|\Delta\Xi\|^2 - \langle \Delta\Xi, W(\Delta\Xi) \rangle \\
& = (\|\omega_{(\gamma,\gamma)}\|^2 + 2\|\omega_{(\beta,\gamma)}\|^2) + 2(\|\omega_{(\alpha,\gamma)}\|^2 - \langle \omega_{(\alpha,\gamma)}, (\Theta_c)_{(\alpha,\gamma)} \omega_{(\alpha,\gamma)} \rangle) \\
& \quad + (\|\omega_{(\beta,\beta)}\|^2 - \langle \omega_{(\beta,\beta)}, (\Theta_c)_{(\beta,\beta)} \omega_{(\beta,\beta)} \rangle).
\end{aligned} \tag{4.57}$$

Now we are ready to estimate the lower and upper bounds of  $-\langle V(\Delta y), \Delta y \rangle$ . In light of (4.48), (4.50), (4.53), (4.56), and (4.57), we have

$$\begin{aligned}
-\langle V(\Delta y), \Delta y \rangle & \geq c^{-1} (\|\omega_{(\gamma,\gamma)}\|^2 + 2\|\omega_{(\beta,\gamma)}\|^2) \\
& \quad + \kappa_1(c) \|(\Delta\zeta, \omega_{(\alpha,\alpha)})\|^2 + 2\kappa_2(c) \|\omega_{(\alpha,\beta)}\|^2 \\
& \quad + 2\kappa_3(c) \|\omega_{(\alpha,\gamma)}\|^2 + \kappa_4(c) \|\omega_{(\beta,\beta)}\|^2,
\end{aligned} \tag{4.58}$$

where

$$\begin{aligned}
\kappa_1(c) & := (\bar{\sigma}\bar{\eta} + (c - c_0))^{-1} - \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2}, \\
\kappa_2(c) & := (\bar{\sigma}\bar{\eta}/2 + (c - c_0))^{-1} - \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2} \\
\kappa_3(c) & := c^{-1} [1 - \bar{\nu}_0(\bar{\nu}_0 + c)^{-1}] + 2\underline{\nu} (\bar{\sigma}\bar{\eta} + 2(c - c_0))^{-1} \underline{\nu}_0^2 (\underline{\nu}_0 + c)^{-2} - \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2},
\end{aligned}$$

and

$$\kappa_4(c) := \min_{t \in [0,1]} \psi(t; c, a_c, b_c, c_0)$$

with  $\psi(\cdot; \cdot)$  being defined as (2.17) in Lemma 2.8 and

$$a_c := c^{-1} - \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2}, \quad b_c := \bar{\sigma}\bar{\eta}.$$

It follows from (2.18) in Lemma 2.8 that for  $c \geq \bar{c}$ ,

$$\kappa_4(c) = c^{-1} - \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2} - \frac{\bar{\sigma}\bar{\eta}}{c(\sqrt{c} + \sqrt{c_0})^2}. \tag{4.59}$$



Thus, there exists a positive number  $\mu_1$  such that for  $c \geq \bar{c}$  we have

$$\min\{\kappa_1(c), \kappa_2(c), \kappa_3(c), \kappa_4(c)\} \geq c^{-1} - \mu_1 c^{-2}.$$

Therefore, from (4.58) we have

$$-\langle V(\Delta y), \Delta y \rangle \geq \min\{c^{-1}, \kappa_1(c), \kappa_2(c), \kappa_3(c), \kappa_4(c)\} \|\Delta y\|^2 \geq (c^{-1} - \mu_1 c^{-2}) \|\Delta y\|^2. \quad (4.60)$$

On the other hand, in light of (4.48), (4.51), (4.54), (4.56), and (4.57), we have

$$\begin{aligned} -\langle V(\Delta y), \Delta y \rangle &\leq c^{-1} (\|\omega_{(\gamma, \gamma)}\|^2 + 2\|\omega_{(\beta, \gamma)}\|^2) \\ &\quad + \mu_1(c) \|\Delta \zeta, \omega_{(\alpha, \alpha)}\|^2 + 2\mu_2(c) \|\omega_{(\alpha, \beta)}\|^2 \\ &\quad + 2\mu_3(c) \|\omega_{(\alpha, \gamma)}\|^2 + \mu_4(c) \|\omega_{(\beta, \beta)}\|^2, \end{aligned} \quad (4.61)$$

where

$$\begin{aligned} \mu_1(c) &:= (\underline{\sigma}\eta/2 + (c - c_0))^{-1} + \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2}, \\ \mu_2(c) &:= \mu_1(c), \\ \mu_3(c) &:= c^{-1} [1 - \underline{\nu}_0(\underline{\nu}_0 + c)^{-1}] + 2\bar{\nu}\underline{\sigma}^{-1}\eta^{-1}\bar{\nu}_0^2(\bar{\nu}_0 + (c - c_0))^{-2} + \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2}, \end{aligned}$$

and

$$\mu_4(c) := \max_{t \in [0, 1]} \psi(t; c, a'_c, b'_c, c_0)$$

with

$$a'_c := c^{-1} + \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2}, \quad b'_c := \underline{\sigma}\eta/2.$$

It follows from (2.19) in Lemma 2.8 that for  $c \geq \bar{c}$ ,

$$\begin{aligned} \mu_4(c) &= \max\{\psi(0; c, a'_c, b'_c, c_0), \psi(1; c, a'_c, b'_c, c_0)\} \\ &= \varrho_0 \bar{\nu}_0 \sqrt{\bar{\nu}} c^{-2} + \max\{c^{-1}, (\underline{\sigma}\eta/2 + (c - c_0))^{-1}\}. \end{aligned} \quad (4.62)$$

Then there exists a positive number  $\mu_0 \geq \mu_1$  such that for  $c \geq \bar{c}$  we have

$$\max\{\mu_1(c), \mu_2(c), \mu_3(c), \mu_4(c)\} \leq c^{-1} + \mu_0 c^{-2}.$$

Therefore, from (4.61) we have

$$-\langle V(\Delta y), \Delta y \rangle \leq \max\{c^{-1}, \mu_1(c), \mu_2(c), \mu_3(c), \mu_4(c)\} \|\Delta y\|^2 \leq (c^{-1} + \mu_0 c^{-2}) \|\Delta y\|^2. \quad (4.63)$$

By (4.60) and (4.63) we obtain that

$$\mu_0 c^{-2} \|\Delta y\|^2 \geq -\langle V(\Delta y) + c^{-1} \Delta y, \Delta y \rangle \geq -\mu_0 c^{-2} \|\Delta y\|^2. \quad (4.64)$$

This shows that (4.39) holds. The proof is completed.  $\square$

Now we are ready to state our main result on the rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming.

**Theorem 4.1** *Suppose that Assumptions (nlsdp-A1) and (nlsdp-A2) are satisfied. Let  $c_0$  and  $\eta$  be two positive numbers obtained by Proposition 4.1. Let  $\bar{\eta}$ ,  $\bar{c}$ , and  $\varrho_0$  be defined as in (4.31), (4.36), and (4.37), respectively. Let  $\mu_0$  be obtained by Proposition 4.2. Define*

$$\varrho_1 := 2\varrho_0 \quad \text{and} \quad \varrho_2 := 4\mu_0.$$

*Then for any  $c \geq \bar{c}$ , there exist two positive numbers  $\varepsilon$  and  $\delta$  (both depending on  $c$ ) such that for any  $(\zeta, \Xi) \in \mathbb{B}_\delta(\bar{\zeta}, \bar{\Xi})$ , the problem*

$$\min L_c(x, \zeta, \Xi) \quad \text{s.t. } x \in \mathbb{B}_\varepsilon(\bar{x}) \tag{4.65}$$

*has a unique solution denoted  $x_c(\zeta, \Xi)$ . The function  $x_c(\cdot, \cdot)$  is locally Lipschitz continuous on  $\mathbb{B}_\delta(\bar{\zeta}, \bar{\Xi})$  and is semismooth at any point in  $\mathbb{B}_\delta(\bar{\zeta}, \bar{\Xi})$ , and for any  $(\zeta, \Xi) \in \mathbb{B}_\delta(\bar{\zeta}, \bar{\Xi})$ , we have*

$$\|x_c(\zeta, \Xi) - \bar{x}\| \leq \varrho_1 \|(\zeta, \Xi) - (\bar{\zeta}, \bar{\Xi})\|/c \tag{4.66}$$

*and*

$$\|(\zeta_c(\zeta, \Xi), \Xi_c(\zeta, \Xi)) - (\bar{\zeta}, \bar{\Xi})\| \leq \varrho_2 \|(\zeta, \Xi) - (\bar{\zeta}, \bar{\Xi})\|/c, \tag{4.67}$$

*where  $\zeta_c(\zeta, \Xi)$  and  $\Xi_c(\zeta, \Xi)$  are defined as*

$$\zeta_c(\zeta, \Xi) := \zeta + ch(x_c(\zeta, \Xi)) \quad \text{and} \quad \Xi_c(\zeta, \Xi) := \Pi_{\mathcal{S}_+^p}(\xi - cg(x_c(\zeta, \Xi))).$$

**Proof.** If Assumptions (nlsdp-A1) and (nlsdp-A2) are satisfied, then from Propositions 4.1 and 4.2 we know that both Assumption B1 and Assumption B2 (with  $\tau = 2$ ) made in Section 3 are satisfied. Then the conclusions in this theorem follow from Theorem 3.1.  $\square$

Note that the conditions imposed in Theorem 1.2 are equivalent to Assumptions (nlsdp-A1) and (nlsdp-A2) plus the strict complementarity condition, i.e.,  $\beta = \emptyset$ . Compared Theorem 4.1 with Theorem 1.2, we can see that there is no loss on the rate of convergence of the augmented Lagrangian method for (NLSDP) even the strict complementarity condition fails to hold. However, different from Theorem 1.2, the convergence region in Theorem 4.1 may depend on  $c$ . It would be interesting to know if this dependence can be removed under Assumptions (nlsdp-A1) and (nlsdp-A2) only.

## 5 Conclusions

This paper provides an analysis on the rate of convergence of the augmented Lagrangian method for solving nonlinear semidefinite programming. By assuming that  $K$  is a self-dual cone and that  $\Pi_K(\cdot)$  is semismooth everywhere, we first establish a general result on the rate of convergence of the augmented Lagrangian method for a class of general optimization problems. Then we apply this general result to nonlinear semidefinite programming under the the constraint nondegeneracy condition and the strong second order sufficient condition. This suggests that our result may be used to deal with other optimization problems. For example, it seems possible to apply our general result to nonlinear second order cone programming by using the strong second order sufficient condition recently proposed in [7] for second order cone programming.

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