Single-Product Pricing via Robust Optimization

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Abstract

We present a robust optimization approach to the problem of pricing a capacitated product over a finite time horizon in the presence of demand uncertainty. This technique does not require the knowledge of the underlying probability distributions, which in practice are difficult to estimate accurately, and instead models random variables as uncertain parameters belonging to a polyhedral uncertainty set. A novelty of the proposed approach is that, instead of imposing an upper bound on the number of uncertain parameters that can reach their worst-case value, which is known as a budget-of-uncertainty constraint and has received much attention in the robust optimization literature, we introduce a budget of resource consumption by the uncertainty. This budget limits the amount of the resource that can be used by the random part of the cumulative demand, and allows us to derive key insights on the structure of the optimal solution for a broad class of nominal demand functions. We establish the existence of a reference price for the product and show that this new parameter plays a crucial role in understanding the impact of uncertainty on the optimal prices. In particular, it is not always optimal to decrease prices when demand is uncertain. Whether it is optimal or not will instead depend on whether the price at a given time period is above or below the reference price, and whether the maximal amount of uncertainty at that time exceeds a threshold. We compare the optimal solution in the cases of additive and multiplicative uncertainty and analyze the problem with linear nominal demand in detail. Numerical results are encouraging.

1 Introduction

Demand uncertainty in revenue management has traditionally been addressed by assuming specific probability distributions and maximizing the expected profit. This was, and indeed remains, justified in a slow-changing environment where abundant historical data is available to quantify the randomness (see Talluri and van Ryzin [17] for a comprehensive review of these models.) In recent years however, the pace of technological change and the volatility in customers' tastes have led to a dramatic decrease in product life cycles, affecting a wide range of industries from blue-chip

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manufacturing to fashion retail. Applying probabilistic techniques in these settings makes the solution vulnerable to estimation errors, and raises concerns about the practical relevance of a strategy relying on unverifiable assumptions. The first attempt to address this issue is due to Scarf [15], who considers the classical newsvendor problem when only the first two moments of the demand are known. His work was later extended by Gallego [11] and Moon and Gallego [12]. An approach which does not require any forecasting has been proposed by van Ryzin and McGill [20] in the context of airline protection levels; however, the performance of their algorithm in numerical studies was mixed. Another drawback of the traditional framework is that it considers the decision-maker to be risk-neutral. While few object to this claim when the decision-maker observes the uncertainty repeatedly, for instance in airline revenue management (where the same flight takes off every day, if not several times a day), this does not capture the risk preferences of many managers in other sectors. Sophisticated methods to incorporate risk in the decision-making process, in particular through the use of expected utility (Eeckhoudt et. al. [10], Agrawal and Seshadri [1]) and value-at-risk constraints (Levin et. al. [13]), face practical challenges as well, as most individuals are unable to articulate their own utility function and probabilities remain difficult to estimate.

In contrast, the robust optimization paradigm we propose does not require any probabilistic knowledge besides the mean and the support of the distributions, and incorporates risk aversion through a single parameter. The field of robust optimization was pioneered by Soyster [16] to address parameter uncertainty in convex programming, and is concerned with optimizing a problem over the worst-case value of the uncertain parameters within a set. Ben-Tal and Nemirovski [2, 3, 4] have focused on ellipsoidal uncertainty sets, which yield an elegant theoretical framework but increase the complexity of the problem considered. In contrast, polyhedral uncertainty sets, which have been investigated by Bertsimas et. al. [6] and Bertsimas and Sim [7], do not change the structure of the problem at hand; for instance, the robust counterpart of a linear programming problem remains linear. This technique was first applied to random variables with unknown probabilities in Bertsimas and Thiele [9] in the context of inventory management. The robust optimization models in the present paper are motivated by Bertsimas and Thiele [9], and in particular are implemented on a rolling horizon basis (i.e., the problem is re-solved through the planning period and the optimal decision variables are updated repeatedly - see Thiele [18] for an approach that yields robust policies instead, and is based on allocating budgets of uncertainty over time to either protect the present against uncertainty or save the whole budget for future use.) However, the models here also differ from Bertsimas and Sim [7] and Bertsimas and Thiele [9] in an important way, as we do not limit the number of uncertain parameters that can equal their worst-case value, which is commonly known as a budget-of-uncertainty constraint. Instead, we introduce the concept of a budget of resource consumption by the uncertainty, that is, an upper bound on the amount of the resource that can be allocated to the uncertain (as opposed to nominal) part of the aggregate demand. This is motivated by the specific structure of the pricing problem. To the best of our knowledge, this is the first paper that specifically investigates polyhedral uncertainty sets that do not fit the budget-of-uncertainty framework.

The focus of this paper is on pricing a single product over time in the presence of demand uncertainty and without replenishment. The multi-product multi-resource problem is addressed in a companion paper [19]. Our contributions can be summarized as follows:

1. We provide a tractable framework based on robust optimization that incorporates uncertainty through just one new decision variable and no new constraint. This framework is convex when the uncertainty is additive, and can be solved efficiently under mild conditions in the case of multiplicative uncertainty.

2. We establish the existence of a *reference price,* independent of time, which is equal to one of optimal prices. This parameter plays a crucial role in understanding the impact of uncertainty on the solution, as the optimal price at time t converges further towards the reference price when demand uncertainty at that time period increases.

3. We compare the impact of additive and multiplicative uncertainty on the optimal solution, and apply these insights to the case of nominal demand linear in the price, for which we obtain closed-form expressions that only depend on the system parameters and on the position (higher than, equal to, lesser than) of the optimal prices with respect to the reference price.

In Section 2, we develop the robust pricing model in the presence of additive uncertainty. We analyze its counterpart in the case of multiplicative uncertainty in Section 3, and provide numerical results in Section 4. Finally, Section 5 contains some concluding remarks.

2 Single-Product Pricing With Additive Uncertainty

2.1 The Deterministic Model

We consider here the problem of pricing a single product over a finite horizon of length T, given an initial inventory of C items, without replenishment. Throughout the paper, we assume that the average demand $\overline{d}_t$ at time t, $t = 0, \ldots, T-1$, as a function of prices is fully known and only depends on the prices at that time period. We further assume that the average demand, resp. the revenue, at time t is a convex, resp. concave, function of the prices. This is formulated as:

$$\overline{d}_t(p_t) = f_t(p_t),\ t = 0, \ldots, T-1, \tag{1}$$

for some function f_t which is convex, strictly decreasing, and such that $p_t \mapsto p_t\, f_t(p_t)$, is concave. A widely used choice for f_t in practical implementations is: $f_t(p_t) = a_t - b_t\, p_t$, with $a_t,\, b_t > 0$, which corresponds to an average demand linear in the price at each time period.

We first review the properties of the nominal problem, which will be useful in the analysis of its robust counterpart. When the demand is deterministic, the problem of finding the optimal prices to maximize revenue is formulated as a convex programming problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} p_t \, \overline{d}_t(p_t) \\ \text{s.t.} \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C, \\ & p_t^{\min} \leq p_t \leq p_t^{\max}, \quad \forall t, \end{aligned} \tag{2}$$

where $p_t^{\min}$ and p_t^{max} are lower and upper bounds on the prices. In particular, these bounds enforce that p_t, $\overline{d}_t(p_t) \geq 0$ at each time period. The optimal solution can be characterized as follows.

Theorem 2.1 (Optimal Solution) *Let $\overline{\lambda}^* \geq 0$ be the optimal Lagrange multiplier for the capacity constraint and $\overline{\mu}_t^{*\min}$, $\overline{\mu}_t^{*\max} \geq 0$ the optimal Lagrange multipliers for the bound constraints at time t, $t = 0, \ldots, T-1$. Then the optimal price p_t^* at t satisfies:*

$$\overline{d}_t(p_t^*) + (p_t^* - \overline{\lambda}^*)\, \overline{d}_t'(p_t^*) + \overline{\mu}_t^{*\min} - \overline{\mu}_t^{*\max} = 0. \tag{3}$$

In particular, if $p_t^{\min} < p_t^* < p_t^{\max}$, we have:

$$\overline{d}_t(p_t^*) + (p_t^* - \overline{\lambda}^*)\, \overline{d}_t'(p_t^*) = 0. \tag{4}$$

Proof: Follows by applying the Karush-Kuhn-Tucker conditions to Problem (2) (see Bertsekas [5] for an introduction to nonlinear programming.) By complementarity slackness, $\overline{\mu}_t^{\min} = 0$ if $p_t > p_t^{\min}$ and $\overline{\mu}_t^{\max} = 0$ if $p_t < p_t^{\max}$. □

2.2 Description of the Uncertainty

In practice, demand is subject to uncertainty, and the decision-maker has only limited knowledge on the underlying probability distributions. Following the approach developed by Bertsimas and Sim [7] for uncertain data coefficients and Bertsimas and Thiele [9] for random variables with unknown probabilities, we model the random demands d_t as uncertain parameters of known mean and support. In this section, we assume that the uncertainty is *additive*, i.e., the random demand d_t is modeled as:

$$d_t(p_t) = \overline{d}_t(p_t) + \delta_t, \ \forall p_t, \tag{5}$$

where $\overline{d}_t$ verifies Equation (1), and δ_t is a zero-mean random variable independent of the prices. For notational convenience, we present the approach when the support of δ_t is symmetric, i.e., $\delta_t \in [-\widehat{\delta}_t, \widehat{\delta}_t]$ for some $\widehat{\delta}_t > 0$, although the results can be extended to the asymmetric case without difficulty. This yields a box-type description of the uncertainty:

$$d_t(p_t) = \overline{d}_t(p_t) + \widehat{\delta}_t \, z_t, \ |z_t| \leq 1, \tag{6}$$

where $\overline{d}_t$ verifies Equation (1). The scalar z_t is called the *scaled deviation* of the demand from its nominal value at time t. To avoid overprotecting the system, we impose an additional constraint on the scaled deviations, specifically:

$$\left|\sum_{t=0}^{T-1} \widehat{\delta}_t \, z_t\right| \leq \Delta, \tag{7}$$

which limits the *total consumption of the resource by the uncertainty*. Note that $\sum_{t=0}^{T-1} \widehat{\delta}_t \, z_t$ is the uncertain component of the aggregate demand, as $\sum_{t=0}^{T-1} d_t(p_t) = \sum_{t=0}^{T-1} \overline{d}_t(p_t) + \sum_{t=0}^{T-1} \widehat{\delta}_t \, z_t$. The parameter Δ is chosen in $\left[0, \sum_{t=0}^{T-1} \widehat{\delta}_t\right]$ using the historical data available to the decision-maker, and is the *maximum allowable impact of the uncertainty* on the resource, or *budget of uncertainty impact*. This represents a departure from the polyhedral uncertainty sets presented in the literature (Bertsimas and Sim [7, 8], Bertsimas and Thiele [9]), which bound the number of uncertain parameters that can deviate from their nominal values. The choice of Constraint (7) is motivated in multiperiod pricing by the specific impact of the uncertainty on the formulation.

2.3 The Robust Optimization Approach

We start by defining the robust problem, and in a second step propose an equivalent convex formulation, which can thus be solved efficiently. The definition of the robust problem is not straightforward; indeed, if we replace the deterministic demands in Problem (2) by the uncertain parameters in Equation (6), we obtain:

1. an objective value of $\sum_{t=0}^{T-1} p_t \, (\overline{d}_t(p_t) + \widehat{\delta}_t \, z_t)$, which is negatively affected by the uncertainty when there is *less* demand than expected,
2. a constraint on the available capacity of $\sum_{t=0}^{T-1} (\overline{d}_t(p_t) + \widehat{\delta}_t \, z_t) \leq C$, which is negatively affected by the uncertainty when there is *more* demand than expected.

An approach considering these two worst cases simultaneously, i.e., maximizing the smallest revenue while guaranteeing feasibility for all possible realizations of the demands, will therefore overprotect the system, since no realization of the random demand yields the worst objective *and* resource utilization. To address this issue and connect the robust optimization approach more tightly to the worst-case value of the uncertainty, we consider instead the problem of maximizing the worst-case profit over the set of scaled deviations *that are feasible for the capacity constraint*, i.e.:

$$\begin{array}{lll} \max & \left[\displaystyle\sum_{t=0}^{T-1} p_t \, \overline{d}_t(p_t) + \right. & \min \quad \left.\displaystyle\sum_{t=0}^{T-1} p_t \, \widehat{\delta}_t \, z_t\right] \\ & & \text{s.t.} \quad \displaystyle\sum_{t=0}^{T-1} \widehat{\delta}_t z_t \leq C - \sum_{t=0}^{T-1} \overline{d}_t(p_t), \\ & & \qquad -\Delta \leq \displaystyle\sum_{t=0}^{T-1} \widehat{\delta}_t z_t \leq \Delta, \\ & & \qquad |z_t| \leq 1 \ \forall t, \\ \text{s.t.} & p_t^{\min} \leq p_t \leq p_t^{\max}, \ \forall t. & \end{array} \tag{8}$$

The following theorem provides a tractable equivalent formulation to the max-min problem given in Equation (8).

Theorem 2.2 (Robust Formulation) *The robust counterpart to Problem (8) can be formulated as a convex programming problem with only one new decision variable, called the* reference price, *and no new constraint:*

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t |p_t - x|\right] \\ s.t. \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta, \\ & p_t^{\min} \leq p_t \leq p_t^{\max}, \ \forall t. \end{aligned} \tag{9}$$

Hence, it can be solved as efficiently as its deterministic counterpart.

Proof: The inner minimization problem in Problem (8) is feasible if and only if the constraint:

$$\sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta \tag{10}$$

holds. Furthermore, the feasible set is obviously bounded and the worst-case scaled deviations correspond to having $\sum_{t=0}^{T-1} \widehat{\delta}_t\, z_t$ as low as possible, so that none of the upper bound constraints on $\sum_{t=0}^{T-1} \widehat{\delta}_t\, z_t$ will be tight at optimality. Hence, we can discard these constraints without affecting the optimal solution. By strong duality, Problem (8) can then be reformulated as:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} (y_t^+ + y_t^-)\right] \\ \text{s.t.} \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta, \\ & \widehat{\delta}_t\, x - y_t^+ + y_t^- = p_t\, \widehat{\delta}_t, \qquad \forall t, \\ & p_t^{\min} \leq p_t \leq p_t^{\max}, \ y_t^+, y_t^- \geq 0, \qquad \forall t. \end{aligned} \tag{11}$$

Problem (9) follows from interpreting y_t^+, resp. y_t^- as the positive, resp. negative, component of $\widehat{\delta}_t(x - p_t)$, and therefore $y_t^+ + y_t^-$ as its absolute value. □

Analysis: The objective in the robust problem (9) has two components: (i) the nominal revenue, and (ii) a penalty term, which penalizes the deviations (both upside and downside) of the decision variables from a **reference price** x, common to all time periods. The unit penalty is equal to the maximum amount $\widehat{\delta}_t$ of demand uncertainty faced in the time period considered. The constraints are the same as in the deterministic problem where the capacity of the resource has become $C + \Delta$. Section 2.4 provides further theoretical insights.

2.4 Theoretical Insights

Throughout this section, we will assume that no bound constraint is binding, i.e., $p_t^{\min} < p_t^* < p_t^{\max}$ for all t, and that the capacity constraint is tight at optimality $\sum_{t=0}^{T-1} \overline{d}_t(p_t^*) = C + \Delta$.

2.4.1 Optimal reference price

Let $p^*_{(t)}$, $t = 0, \ldots, T-1$, be the optimal prices in Problem (9) ranked in increasing order ($p^*_{(0)} \leq \ldots \leq p^*_{(T-1)}$). Theorem 2.3 establishes that at optimality, the reference price is equal to the price of the product for some time period.

Theorem 2.3 (Optimal reference price) *At optimality, $x^* = p^*_{(s)}$, where s is the smallest integer such that:*

$$\sum_{t|p^*_t \leq p^*_{(s)}} \widehat{\delta}_t > \frac{1}{2}\left(\sum_{t=0}^{T-1} \widehat{\delta}_t - \Delta\right). \tag{12}$$

Proof: Let $p^*_{(s)} \leq x \leq p^*_{(s+1)}$ for some s. ($p^*_{(-1)} = -\infty$ and $p^*_{(T)} = \infty$ by convention.) Then the slope in x of the objective function in Problem (9) is: $-\Delta + \sum_{t|p^*_t \geq p^*_{(s+1)}} \widehat{\delta}_t - \sum_{t|p^*_t \leq p^*_{(s)}} \widehat{\delta}_t$, i.e., $-\Delta + \sum_{t=0}^{T-1} \widehat{\delta}_t - 2\sum_{t|p^*_t \leq p^*_{(s)}} \widehat{\delta}_t$, which decreases as x increases. Hence, the maximum over all real numbers is reached at $p_{(s)}$, with s such that the slope is nonnegative on $[p_{(s-1)}, p_{(s)}]$ and negative on $[p_{(s)}, p_{(s+1)}]$. Equation (12) follows immediately. □

Remarks:

1. If the decision-maker is very risk-averse and plans for the maximal amount of uncertainty $\Delta = \sum_{t=0}^{T-1} \widehat{\delta}_t$, the optimal reference x^* is equal to the smallest price $p^*_{(0)}$.
2. The optimal reference price never exceeds the τ-th smallest price ($x^* \leq p^*_{(\tau)}$ for any Δ), where τ is the smallest integer s verifying $\sum_{t|p^*_t \leq p^*_{(s)}} \widehat{\delta}_t > \frac{1}{2}\sum_{t=0}^{T-1} \widehat{\delta}_t$. (This is because $\Delta \geq 0$.)
3. If $\widehat{\delta}_t = \widehat{\delta}$ for all t, the optimal reference price for the item is equal to $p^*_{(s)}$ with $s = \left\lfloor \frac{1}{2}\left(T - \frac{\Delta}{\widehat{\delta}}\right)\right\rfloor$.

2.4.2 Preliminary results

Let $\mathcal{T}$ be the set of time periods t for which $p^*_t = x^*$ at optimality (we know from Theorem 2.3 that $\mathcal{T}$ is nonempty), and let $\lambda^* \geq 0$ be the optimal Lagrange multiplier associated with the capacity constraint in Problem (9). We only consider changes small enough so that the set $\mathcal{T}$ is not affected. We first need the following lemma.

Lemma 2.4
(a) For all $t \notin \mathcal{T}$, p^*_t satisfies:

$$(p^*_t - \lambda^*)\,\overline{d}'_t(p^*_t) + \overline{d}_t(p^*_t) = \widehat{\delta}_t\, sgn(p^*_t - x^*). \tag{13}$$

Furthermore, x^* satisfies:

$$(x^* - \lambda^*)\sum_{t\in\mathcal{T}} \overline{d}'_t(x^*) + \sum_{t\in\mathcal{T}} d_t(x^*) = \Delta, \tag{14}$$

as well as:

$$-\widehat{\delta}_t \leq (x^* - \lambda^*)\,\overline{d}'_t(x^*) + \overline{d}_t(x^*) \leq \widehat{\delta}_t,\ \ \forall t \in \mathcal{T}. \tag{15}$$

(b) All prices exceed the marginal value of the resource λ^* at optimality.

(c) Let $\phi_t(p_t,\lambda) = (p_t - \lambda)\,\overline{d}_t'(p_t) + \overline{d}_t(p_t)$ and $\psi(x,\lambda) = (x-\lambda)\sum_{t\in\mathcal{T}} \overline{d}_t'(x) + \sum_{t\in\mathcal{T}} d_t(x)$. Then:

(c-i) $\phi_t(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$ decrease at $\lambda \geq 0$ given.

(c-ii) $\phi_t(p_t,\cdot)$, resp. $\psi(x,\cdot)$, increases at p_t, resp. x, given.

Proof: (a) follows from applying the Karush-Kuhn-Tucker optimality conditions to Problem (9). Equation (13) is obtained by maximizing in p_t the unconstrained objective: $(p_t - \lambda^*)\,\overline{d}_t(p_t) - \widehat{\delta}_t\, sgn(p_t^* - x^*)\,(p_t - x^*)$. Equation (14) is obtained by maximizing in x the unconstrained objective: $(x - \lambda^*)\sum_{t\in\mathcal{T}} \overline{d}_t(x) - \Delta\, x$. Equation (15) is obtained by writing the conditions for x^* to be the global maximum of the unconstrained function nondifferentiable at x^*, with $t \in \mathcal{T}$: $(p_t - \lambda^*)\,\overline{d}_t(p_t) - \widehat{\delta}_t |p_t - x^*|$.

(b) Demand is always nonnegative, including in the worst case. Therefore, $(\lambda^* - p_t^*)\,\overline{d}_t'(p_t^*)$ (for $t \notin \mathcal{T}$) and $(\lambda^* - x^*)\,\overline{d}_t'(x^*)$ (for $t \in \mathcal{T}$) are nonnegative. The fact that demand decreases in prices allows us to conclude.

(c) At λ given, $\phi_t(\cdot,\lambda)$ is the derivative of $(p_t - \lambda)\,\overline{d}_t(p_t)$, which is a concave function (revenue is concave, demand is convex and λ is nonnegative.) This yields (i). Moreover, $\phi_t(p_t,\cdot)$ is linear in λ, with slope $-\overline{d}_t'(p_t)$, which is nonnegative since demand decreases in price. This yields (ii). The proof for ψ is similar. □

Remark: Once the set of $\mathcal{T}$ has been determined, the specific amount of uncertainty $\widehat{\delta}_t$ at the time periods in $\mathcal{T}$ does not affect the optimal reference price x^*.

We now characterize the optimal prices. These results will be particularly useful when we investigate the dependence of the optimal solution on the uncertainty in Section 2.4.3.

Lemma 2.5

(a) The optimal reference and product prices are a function of the uncertainty as follows:

(i) For each $t \notin \mathcal{T}$, there exists a function F_{1t} such that:

$$p_t^* = F_{1t}(\widehat{\delta}_t\, sgn(p_t^* - x^*), \lambda^*). \tag{16}$$

Moreover, there exists a function F_1 such that:

$$x^* = F_1(\Delta, \lambda^*). \tag{17}$$

(ii) There exists a function F_2 such that:

$$\lambda^* = F_2((\widehat{\delta}_s\, sgn(p_s^* - x^*))_{s=0,\ldots,T-1}, \Delta). \tag{18}$$

(iii) For each t, there exists a function F_{3t} such that:

$$p_t^* = F_{3t}((\widehat{\delta}_s\, sgn(p_s^* - x^*))_{s=0,\ldots,T-1}, \Delta). \tag{19}$$

(b) The functions F_{1t} (for any $t \notin \mathcal{T}$) and F_1 are monotonic in both arguments. Specifically,

$$\frac{\partial F_{1t}}{\partial u_1}(u_1, u_2) = \left(2\,\overline{d}_t'[F_{1t}(u_1, u_2)] + (F_{1t}(u_1, u_2) - u_2)\,\overline{d}_t''[F_{1t}(u_1, u_2)]\right)^{-1},\ t \notin \mathcal{T}, \tag{20}$$

$$\frac{\partial F_1}{\partial u_1}(u_1, u_2) = \left(2 \sum_{t \in \mathcal{T}} \overline{d}_t'[F_1(u_1, u_2)] + (F_1(u_1, u_2) - u_2) \sum_{t \in \mathcal{T}} \overline{d}_t''[F_1(u_1, u_2)]\right)^{-1}, \tag{21}$$

which are both nonpositive, and:

$$\frac{\partial F_{1t}}{\partial u_2}(u_1, u_2) = \left(2\,\overline{d}_t'[F_{1t}(u_1, u_2)] + (F_{1t}(u_1, u_2) - u_2)\,\overline{d}_t''[F_{1t}(u_1, u_2)]\right)^{-1} \overline{d}_t'[F_{1t}(u_1, u_2)],\ t \notin \mathcal{T}, \tag{22}$$

$$\frac{\partial F_1}{\partial u_2}(u_1, u_2) = \left(2 \sum_{t \in \mathcal{T}} \overline{d}_t'[F_1(u_1, u_2)] + (F_1(u_1, u_2) - u_2) \sum_{t \in \mathcal{T}} \overline{d}_t''[F_1(u_1, u_2)]\right)^{-1} \sum_{t \in \mathcal{T}} \overline{d}_t'[F_1(u_1, u_2)], \tag{23}$$

which are both nonnegative.

Proof: (a-i) and (a-ii) We know by concavity of the objective function that Equations (13) and (14) have a unique solution, so that F_{1t} (for all $t \notin \mathcal{T}$) and F_1 are well defined. A similar argument applies to F_2 using that $\sum_{t=0}^{T-1} \overline{d}_t(p_t^*) = C + \Delta$ and the fact that the demand is a convex function.
(a-iii) combines Equations (16), (17) and (18).
(b) follows from differentiating Equations (13), resp. (14), with respect to $u_1 = \widehat{\delta}_t\, sgn(p_t^* - x^*)$, resp. $u_1 = \Delta$, and $u_2 = \lambda^*$. The functions $(p_t - \lambda^*)\, d_t(p_t)$, for $t \notin \mathcal{T}$, and $(x - \lambda^*) \sum_{t \in \mathcal{T}} d_t(x)$ are concave at $\lambda^* \geq 0$ given (as sum of concave functions), and the demand is nonincreasing in the price, which yields the sign of the partial derivatives. □

Lemma 2.5 provides a high-level algorithm to express the optimal prices as a function of the system parameters for a given set $\mathcal{T}$: (i) express p_t^*, $t \notin \mathcal{T}$ and x^* as a function of λ^*, (ii) use the fact that the capacity constraint is tight to obtain λ^*, (iii) reinject into the expression of the prices. These closed-form expressions provide valuable insights into the impact of the problem parameters on the optimal solution. Section 2.5 illustrates this point when the nominal demand is linear in the prices.

2.4.3 Impact of uncertainty

In this section, we investigate the impact of the uncertainty (measured either in terms of budget Δ or variability $\widehat{\delta}_t$ at a specific time period t) on the optimal prices and the marginal value of the resource. We make the following assumptions to simplify the analysis:

Assumption 2.6 (Impact of the uncertainty) *Increasing the uncertainty decreases the value of the resource, in the following sense:*

(i) The marginal value λ^* of the resource in the robust framework is always less than or equal to the marginal value $\overline{\lambda}^*$ of the resource in the nominal model.

(ii) The marginal value λ^* of the resource in the robust framework is nonincreasing as the budget Δ of resource consumption by the uncertainty increases.

Remark: We will show below that the condition $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$ is equivalent to:

$$\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \leq 0. \tag{24}$$

By concavity of the revenue function, we already know that:

$$\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \leq - \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*), . \tag{25}$$

where the right-hand side is positive since demand decreases with price. This motivates the claim that Assumption 2.6 (ii) imposes only mild restrictions on the demand function. In particular, Equation (24) is trivially satisfied when the nominal demand is linear in the prices, and it is easy to check whether Assumption 2.6 (ii) holds in any application by injecting the specific demand function into the equations defining λ^*. (We only use Assumption 2.6 (ii) to study the impact of Δ on the prices.) Furthermore, if Assumption 2.6 (i) does not hold, then $\lambda^* > \overline{\lambda}^*$, which simply changes the sign of some inequalities, so that our analysis can easily be extended to that case as well.

We first analyze the direction of change in the prices when uncertainty is incorporated into the deterministic model.

Theorem 2.7 (Comparison with nominal prices)
(a) Optimal robust prices are always smaller than their nominal counterparts if they strictly exceed the reference price.
(b) Optimal robust prices that fall strictly below the reference price are smaller than their nominal counterparts if and only if the uncertainty remains below a threshold, specifically:

$$\widehat{\delta}_t \leq -(\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*). \tag{26}$$

Proof: Using the notations of Lemma 2.4, the optimal price at time t, $t \notin \mathcal{T}$, satisfies: $\phi_t(p_t^*, \lambda^*) = \widehat{\delta}_t\, sgn(p_t^* - x^*)$ with $\phi_t(p_t^*, \lambda^*) = \phi_t(p_t^*, \overline{\lambda}^*) + (\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*)$. Therefore, we have: $\phi_t(p_t^*, \overline{\lambda}^*) = \widehat{\delta}_t\, sgn(p_t^* - x^*) - (\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*)$. Since $\phi_t(\overline{p}_t^*, \overline{\lambda}^*) = 0$ and $\phi_t(\cdot, \lambda)$ decreases (from Lemma 2.4 (c)), $p_t^* \leq \overline{p}_t^*$ if and only if:

$$\widehat{\delta}_t\, sgn(p_t^* - x^*) - (\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*) \geq 0. \tag{27}$$

(a) and (b) follow by distinguishing between $p_t^* < x^*$ and $p_t^* > x^*$, using that nominal demand decreases in price and invoking Assumption 2.6 (i). □

Remark: Low-priced items with high uncertainty see a price *increase* from their nominal values. Intuitively, the decision-maker reduces the sales at that time so that capacity can be reallocated to more profitable time periods.

Theorem 2.8 analyzes the dependence of the optimal prices and the marginal value of the resource on the budget Δ, as the $\widehat{\delta}_t$ are kept constant. In this context, increasing Δ can be interpreted as increasing the risk aversion of the decision-maker. The key insight of Theorem 2.8 is that, provided that, for any t, the position of p_t^* with respect to x^* (less than, equal to, higher than) does not change, the optimal prices, the reference price and the marginal value of the resource all decrease as the decision-maker's degree of conservatism increases.

Theorem 2.8 (Impact of the budget of resource consumption by the uncertainty)
(a) The marginal value of the resource satisfies:

$$\frac{\partial \lambda^*}{\partial \Delta} = \left[\frac{\partial F_1}{\partial u_2}(\Delta, \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}'_t(x^*) + \sum_{t \notin \mathcal{T}} \frac{\partial F_{1t}}{\partial u_2}(\widehat{\delta}_t\, sgn(p_t^* - x^*), \lambda^*)\, \overline{d}'_t(p_t^*) \right]^{-1} \left(1 - \frac{\partial F_1}{\partial u_1}(\Delta, \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}'_t(x^*) \right), \tag{28}$$

where $\frac{\partial F_1}{\partial u_1}$, $\frac{\partial F_{1t}}{\partial u_2}$ and $\frac{\partial F_1}{\partial u_2}$ are defined in Equations (21)-(23). From Assumption 2.6, the marginal value of the resource decreases in the budget of resource consumption by the uncertainty.
(b) The optimal price at time t (both for $t \in \mathcal{T}$ and $t \notin \mathcal{T}$) satisfies:

$$\frac{\partial p_t^*}{\partial \Delta} \leq 0 \ \forall t. \tag{29}$$

Specifically,

$$\frac{\partial p_t^*}{\partial \Delta} = \frac{\partial F_{1t}}{\partial u_2}(\widehat{\delta}_t\, sgn(p_t^* - x^*), \lambda^*)\, \frac{\partial \lambda^*}{\partial \Delta}, \ t \notin \mathcal{T}, \tag{30}$$

and:

$$\frac{\partial x^*}{\partial \Delta} = \frac{\partial F_1}{\partial u_1}(\Delta, \lambda^*) + \frac{\partial F_1}{\partial u_2}(\Delta, \lambda^*)\, \frac{\partial \lambda^*}{\partial \Delta}, \tag{31}$$

where $\frac{\partial F_1}{\partial u_1}$, $\frac{\partial F_{1t}}{\partial u_2}$ and $\frac{\partial F_1}{\partial u_2}$ are defined in Equations (21)-(23) and $\frac{\partial \lambda^*}{\partial \Delta}$ is defined in Equation (28). A higher degree of risk aversion will decrease all optimal prices.

Proof: We will prove (a) and (b) simultaneously. Differentiating Equations (13) and (14) with respect to Δ yields Equations (30) and (31). The sign of $\frac{\partial p_t^*}{\partial \Delta}$ follows from injecting $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$ from Assumption 2.6 and the sign of the partial derivatives from Lemma 2.5 into Equations (30) and (31). Furthermore, differentiating $\sum_{t=0}^{T-1} \overline{d}_t(p_t^*) = C + \Delta$ with respect to Δ yields:

$$\frac{\partial x^*}{\partial \Delta} \sum_{t \in \mathcal{T}} \overline{d}'_t(x^*) + \sum_{t \notin \mathcal{T}} \frac{\partial p_t^*}{\partial \Delta}\, \overline{d}'_t(p_t^*) = 1. \tag{32}$$

We obtain Equation (28) by reinjecting Equations (30) and (31) into Equation (32). □

Remark: The robust problem with $\Delta = 0$ is not equivalent to the nominal problem since the choice of $\Delta = 0$ does not set the $\widehat{\delta}_t$ to 0. This explains why, although $\frac{\partial p_t^*}{\partial \Delta} \leq 0$ for all t, the optimal prices in the robust framework are not always smaller than the optimal prices in the nominal model, as formalized in Theorem 2.7.

Finally, Theorem 2.9 establishes that, for a given value of the budget Δ, increasing the uncertainty at time t does not necessarily decrease the optimal price p_t^*. Instead, the optimal price converges further towards the reference price.

Theorem 2.9 (Impact of the uncertainty at each time period)
(a) The marginal value of the resource increases, resp. decreases, when the demand uncertainty increases at a time where the price is strictly above, resp. strictly below, the reference price.
(b) For $t \notin \mathcal{T}$, the optimal price p_t^* converges towards the reference price x^* as $\widehat{\delta}_t$ increases.

Proof: (a) Differentiating $\sum_{s \notin \mathcal{T}} \overline{d}_s[F_{1s}(\widehat{\delta}_s\, sgn(p_s^* - x^*), \lambda^*)] + \sum_{s \in \mathcal{T}} \overline{d}_s[F_1(\Delta, \lambda^*)] = C + \Delta$ with respect to $\widehat{\delta}_t$ for some $t \notin \mathcal{T}$ yields (after dropping the arguments of the partial derivatives for notational convenience):

$$sgn(p_t^* - x^*)\, \frac{\partial F_{1t}}{\partial u_1} \cdot \overline{d}_t'(p_t^*) + \frac{\partial \lambda^*}{\partial \widehat{\delta}_t} \cdot \left[\sum_{s \notin \mathcal{T}} \frac{\partial F_{1s}}{\partial u_2} \cdot \overline{d}_s'(p_s^*) + \frac{\partial F_1}{\partial u_2} \sum_{s \in \mathcal{T}} \overline{d}_s'(x^*) \right] = 0, \tag{33}$$

or equivalently:

$$\frac{\partial \lambda^*}{\partial \widehat{\delta}_t} = -sgn(p_t^* - x^*)\, \frac{\frac{\partial F_{1t}}{\partial u_1} \cdot \overline{d}_t'(p_t^*)}{\sum_{s \notin \mathcal{T}} \frac{\partial F_{1s}}{\partial u_2} \cdot \overline{d}_s'(p_s^*) + \frac{\partial F_1}{\partial u_2} \sum_{s \in \mathcal{T}} \overline{d}_s'(x^*)}. \tag{34}$$

Let α_t be such that $\frac{\partial \lambda^*}{\partial \widehat{\delta}_t} = -sgn(p_t^* - x^*)\, \alpha_t$ in Equation (34). From Lemma 2.5 (b) and the fact that the demand decreases in the prices, we have: $\alpha_t \leq 0$. (a) follows immediately.
(b) Differentiating Equation (16) with respect to $\widehat{\delta}_t$ and injecting Equation (34) yields:

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_t} = sgn(p_t^* - x^*)\, \frac{\partial F_{1t}}{\partial u_1} \left(1 - \frac{\frac{\partial F_{1t}}{\partial u_2} \cdot \overline{d}_t'(p_t^*)}{\sum_{s \notin \mathcal{T}} \frac{\partial F_{1s}}{\partial u_2} \cdot \overline{d}_s'(p_s^*) + \frac{\partial F_1}{\partial u_2} \sum_{s \in \mathcal{T}} \overline{d}_s'(x^*)} \right). \tag{35}$$

From Lemma 2.5 (b), the coefficient β_t defined such that $\frac{\partial p_t^*}{\partial \widehat{\delta}_t} = sgn(p_t^* - x^*)\, \beta_t$ is nonpositive. □

2.5 Example

In this section, we illustrate the results above on the example of additive nominal demand, i.e., nominal demand that is linear in the prices:

$$\overline{d}_t(p_t) = a_t - b_t\, p_t, \text{ with } a_t,\, b_t > 0,\ \forall t. \tag{36}$$

Because the mathematical expressions follow directly from injecting Equation (36) into the framework developed in Sections 2.3 and 2.4, we state them without proof. As before, all the insights (with the exception of the robust formulation) are derived assuming that the capacity constraint is tight at optimality and that the prices do not reach their bounds.

2.5.1 Robust formulation

The robust problem when the average demand is linear in the prices and uncertainty is additive can be formulated as a convex programming problem with linear constraints:

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} p_t\,(a_t - b_t\,p_t) - \left[\Delta\,x + \sum_{t=0}^{T-1} \widehat{\delta}_t |p_t - x|\right] \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} b_t\,p_t \geq \sum_{t=0}^{T-1} a_t - (C + \Delta), \\
& p_t^{\min} \leq p_t \leq p_t^{\max}, \;\forall t.
\end{aligned} \tag{37}
$$

2.5.2 Optimal prices

Let $\mathcal{T} = \{t | p_t^* = x^*\}$. The optimal price at time t satisfies:

$$
p_t^* = \begin{cases} \dfrac{a_t - \widehat{\delta}_t\, sgn(p_t^* - x^*)}{2\,b_t} + \dfrac{\lambda^*}{2}, & \text{if } t \notin \mathcal{T}, \\ \dfrac{\sum_{t\in\mathcal{T}} a_t - \Delta}{2\sum_{t\in\mathcal{T}} b_t} + \dfrac{\lambda^*}{2}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{38}
$$

This defines the functions F_{1t} and F_1 in Equations (16) and (17). The impact of uncertainty on the optimal prices at each time period has two (additive) components:

- a term that is specific to that time period and depends on the maximal deviation allowed $\widehat{\delta}_t$,
- a term that depends on the marginal value of the capacity and is common to all time periods.

While the fact that uncertainty exists always makes the resource as a whole less valuable ($\lambda^* \leq \overline{\lambda}^*$, see Section 2.5.3) and drives prices down, the specific uncertainty at each time period might drive prices up or down. In particular, $p_t^* < \overline{p}_t^*$ if and only if (i) $p_t^* > x^*$ or (ii) $p_t^* < x^*$ and $\widehat{\delta}_t < b_t(\overline{\lambda}^* - \lambda^*)$. In other words, it is optimal to *increase* prices when prices are low and uncertainty is high. As in the general case, this allows the decision-maker to re-allocate some of the resource to a more profitable time period.

Remark: If the $(a_t - \widehat{\delta}_t)/b_t$ and $(a_t + \widehat{\delta}_t)/b_t$ are increasing with time, then there exist time periods t_1 and t_2 with $t_1 \leq t_2$ such that the robust optimization approach protects the system against more demand than expected early on (for $t < t_1$), which correspond to time periods with low prices, and against less demand than expected later (for $t > t_2$), which correspond to time periods with high prices. A similar property holds when the $(a_t - \widehat{\delta}_t)/b_t$ and $(a_t + \widehat{\delta}_t)/b_t$ are decreasing with time. Hence, the system goes through three sequential stages: (i) a stage where the demand at each time period is equal to its highest value, (ii) a stage where the price is equal to the reference price, and (iii) a stage where the demand at each time period is equal to its lowest value. (The sequence is reversed when the $(a_t - \widehat{\delta}_t)/b_t$ and $(a_t + \widehat{\delta}_t)/b_t$ are decreasing with time.)

2.5.3 Optimal Lagrange multiplier

The marginal value of the resource as a function of the uncertainty is given by:

$$\lambda^* = \frac{\sum_{t=0}^{T-1}[a_t + \widehat{\delta}_t \, sgn(p_t^* - x^*)] - (2\,C + \Delta)}{\sum_{t=0}^{T-1} b_t}. \tag{39}$$

This defines the function F_2 in Equation (18). In particular:

$$\lambda^* = \overline{\lambda}^* + \frac{\sum_{t=0}^{T-1} \widehat{\delta}_t \, sgn(p_t^* - x^*) - \Delta}{\sum_{t=0}^{T-1} b_t}. \tag{40}$$

From Theorem 2.3, we know that $\sum_{t=0}^{T-1} \widehat{\delta}_t \, sgn(p_t^* - x^*) \leq \Delta$ at optimality, so that $\lambda^* \leq \overline{\lambda}^*$, which satisfies Assumption 2.6 (i). We also note that:

$$\frac{\partial \lambda^*}{\partial \Delta} = -\frac{1}{\sum_{t=0}^{T-1} b_t}, \tag{41}$$

which is negative and thus satisfies Assumption 2.6 (ii), and:

$$\frac{\partial \lambda^*}{\partial \widehat{\delta}_t} = \frac{1}{\sum_{s=0}^{T-1} b_s} \cdot sgn(p_t^* - x^*), \tag{42}$$

so that λ^* indeed increases, resp. decreases, in the maximum uncertainty $\widehat{\delta}_t$ at time t when the optimal price at time t strictly exceeds, resp. falls strictly below, the reference price x^*.

2.5.4 Optimal prices as a function of uncertainty

The optimal prices are piecewise linear in the uncertainty, measured by $\widehat{\delta}_t$ for all t. Specifically:

$$p_t^* = \frac{1}{2} \frac{\sum_{s=0}^{T-1}[a_s + \widehat{\delta}_s \, sgn(p_s^* - x^*)] - (2\,C + \Delta)}{\sum_{s=0}^{T-1} b_s} + \begin{cases} \dfrac{a_t - \widehat{\delta}_t \, sgn(p_t^* - x^*)}{2\, b_t}, & \text{if } t \notin \mathcal{T}, \\ \dfrac{\sum_{s \in \mathcal{T}} a_s - \Delta}{2 \sum_{s \in \mathcal{T}} b_s}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{43}$$

It follows that:

$$\frac{\partial p_t^*}{\partial \Delta} = \begin{cases} -\dfrac{1}{2\, \sum_{s=0}^{T-1} b_s}, & \text{if } t \notin \mathcal{T}, \\ -\dfrac{1}{\sum_{s=0}^{T-1} b_s}, & \text{if } t \in \mathcal{T}\ . \end{cases} \tag{44}$$

and:

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_s} = \begin{cases} \dfrac{1}{2} \cdot \left[-\dfrac{1}{b_t} + \dfrac{1}{\sum_{s=0}^{T-1} b_s} \right] \cdot sgn(p_t^* - x^*), & \text{if } t \notin \mathcal{T} \text{ and } s = t, \\ \dfrac{1}{2} \cdot \dfrac{1}{\sum_{s=0}^{T-1} b_s} \cdot sgn(p_s^* - x^*), & \text{otherwise.} \end{cases} \tag{45}$$

The optimal price at time t decreases when the decision-maker's degree of conservatism increases, and converges further towards the reference price x^* when the uncertainty at that time period increases. Moreover, we see here that the optimal price at time t increases, resp. decreases, when the uncertainty at a time period $s \neq t$ for which $p_s^* > x^*$, resp. $p_s^* < x^*$, increases.

We have thus illustrated on an example the role of the reference price in understanding the impact of uncertainty.

3 Single-Product Pricing with Multiplicative Uncertainty

3.1 Description of Uncertainty

We now present the robust optimization framework in the case of *multiplicative* uncertainty. As before, we assume that the average demand, resp. revenue, at time t is convex, resp. concave in the prices. The random demand is modeled by:

$$d_t(p_t) = \overline{d}_t(p_t) \cdot (1 + \delta_t), \tag{46}$$

where δ_t is again a random variable with zero mean and symmetric support $[-\widehat{\delta}_t, \widehat{\delta}_t]$ independent of the price p_t ($\widehat{\delta}_t < 1$). The half-length of the uncertainty interval in the robust formulation will hence be equal to $\overline{d}_t(p_t) \cdot \widehat{\delta}_t$ for all t, yielding the following model for the demand:

$$d_t(p_t) = \overline{d}_t(p_t) + \widehat{\delta}_t \, \overline{d}_t(p_t) \, z_t, \ \ |z_t| \leq 1, \tag{47}$$

where $\overline{d}_t$ verifies Equation (1).

As in Section 2.2, we consider a box uncertainty set with a budget limiting the consumption of the resource by the uncertainty. Here, the uncertainty set depends on the prices:

$$\mathcal{Z}(\mathbf{p}) = \left\{ \left| \sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p_t) \, z_t \right| \leq \Delta, \ |z_t| \leq 1, \forall t \right\}. \tag{48}$$

Since $p_t^{\min} \leq p_t \leq p_t^{\max}$ for all t, we choose $\Delta \leq \sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p_t^{\min})$.

3.2 The Robust Optimization Approach

Similarly to the approach developed in Section 2.3, we seek to maximize the worst-case revenue, where the worst case is taken over the set of allowable scaled deviations, and define the robust problem in the presence of multiplicative uncertainty as:

$$\begin{aligned} \max \quad & \left[\sum_{t=0}^{T-1} p_t \, \overline{d}_t(p_t) + \min \quad \sum_{t=0}^{T-1} p_t \, \widehat{\delta}_t \, \overline{d}_t(p_t) \, z_t \right] \\ & \qquad\qquad \text{s.t.} \quad -\Delta \leq \sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p_t) \, z_t \leq \min\left(\Delta, C - \sum_{t=0}^{T-1} \overline{d}_t(p_t) \right), \\ & \qquad\qquad\qquad\quad |z_t| \leq 1, \ \forall t, \\ \text{s.t.} \quad & p_t^{\min} \leq p_t \leq p_t^{\max}, \ \forall t. \end{aligned} \tag{49}$$

Theorem 3.1 provides a simpler nonlinear formulation of the robust problem and describes how it can be solved efficiently.

Theorem 3.1 (Robust Counterpart and Algorithm)
(a) The robust problem in the case of multiplicative uncertainty can be formulated as a nonlinear problem over a convex feasible set:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t\, \overline{d}_t(p_t)\, |p_t - x| \right] \\ s.t. \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta, \\ & p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t. \end{aligned} \tag{50}$$

(b) If the uncertainty at each time period satisfies:

$$\widehat{\delta}_t \leq 1 + \min_{p_t^{\min} \leq p_t \leq p_t^{\max}} \frac{p_t\, \overline{d}_t^{''}(p_t)}{2\, \overline{d}_t^{'}(p_t)},\ \forall t, \tag{51}$$

Problem (50) for any given x *is convex, and hence can be solved efficiently as a function of the reference price.*
(c) Let the function F be such that, for all x, $F(x)$ is equal to the optimal objective of Problem (50) solved at x given. There exists x^* such that F is nondecreasing on $(-\infty, x^*]$ and nonincreasing on $[x^*, \infty)$. Hence, the optimal reference price in Problem (50) is equal to x^* and can be found efficiently using gradient-ascent methods.

Proof: (a) is a direct extension of Theorem 2.2 to the case with multiplicative uncertainty.
(b) At x given, the part of the objective function that depends on p_t is equal to $p_t\, \overline{d}_t(p_t) - \widehat{\delta}_t\, \overline{d}_t(p_t)\, |p_t - x|$, for each t. The second derivative of this function is equal to:

- $(1 - \widehat{\delta}_t)\, [p_t\, \overline{d}_t^{''}(p_t) + 2\, \overline{d}_t^{'}(p_t)] + \widehat{\delta}_t\, \overline{d}_t^{''}(p_t)\, x$ when $p_t > x$, which is smaller than or equal to $p_t\, \overline{d}_t^{''}(p_t) + 2\, \overline{d}_t^{'}(p_t) - 2\, \widehat{\delta}_t\, \overline{d}_t^{'}(p_t)$ (because $p_t > x$ and $\overline{d}_t$ is convex),
- $(1 + \widehat{\delta}_t)\, [p_t\, \overline{d}_t^{''}(p_t) + 2\, \overline{d}_t^{'}(p_t)] - \widehat{\delta}_t\, \overline{d}_t^{''}(p_t)\, x$ when $p_t \leq x$, which is nonpositive (because $x > 0$, the nominal demand is convex and the revenue is concave).

Moreover, the average demand decreases in the prices, and straightforward calculations show that the slope at the breakpoint x decreases. Therefore, for the objective function at x given to be concave, it suffices that, for all t and for all p_t such that $p_t^{\min} \leq p_t \leq p_t^{\max}$, the condition:

$$p_t\, \overline{d}_t^{''}(p_t) + 2\, \overline{d}_t^{'}(p_t) - 2\, \widehat{\delta}_t\, \overline{d}_t^{'}(p_t) \leq 0 \tag{52}$$

holds. This yields Condition (51) immediately.
(c) The proof is rather technical and hence has been put in the appendix. □

Remark: As in the case with additive uncertainty, the objective in the robust problem (50) has two components: (i) the nominal revenue, and (ii) a penalty term, which penalizes the deviations (both upside and downside) of the decision variables from a **reference price** x for the item, common to all time periods. The unit penalty is equal to the maximum amount of demand uncertainty faced in that time period, measured by the half-length of the uncertainty interval $\widehat{\delta}_t\,\overline{d}_t(p_t)$. The constraints are the same as in the deterministic problem where the quantity of resource available is $C+\Delta$.

3.3 Theoretical Insights

3.3.1 Optimal reference price

Let $p^*_{(t)}$, $t = 0, \ldots, T-1$, be the optimal prices ranked in increasing order ($p^*_{(0)} \leq \ldots \leq p^*_{(T-1)}$).

Theorem 3.2 (Optimal reference price) *The optimal price satisfies:* $x^* = p^*_{(s)}$, *where* s *is the smallest integer such that:*

$$\sum_{t|p_t^*\leq p^*_{(s)}} \widehat{\delta}_t\,\overline{d}_t(p_t^*) > \frac{1}{2}\left[\sum_{t=0}^{T-1}\widehat{\delta}_t\,\overline{d}_t(p_t^*) - \Delta\right]. \tag{53}$$

Proof: When the prices p_t are set to their optimal value p_t^*, $t = 0, \ldots, T-1$, the objective in Problem (50) is piecewise linear in the reference price x, with slope $\sum_{t=0}^{T-1}\widehat{\delta}_t\,\overline{d}_t(p_t^*)\,sgn(p_t^*-x^*)-\Delta$ or equivalently: $\sum_{t=0}^{T-1}\widehat{\delta}_t\,\overline{d}_t(p_t^*) - \Delta - 2\sum_{t|x>p_t^*}\widehat{\delta}_t\,\overline{d}_t(p_t^*)$ (for $x^* \neq p_t^*$, $t = 0, \ldots, T-1$.) The value of the slope decreases as x^* increases, and optimality is reached at x^* for which the slope changes sign, which yields Equation (53). □

3.3.2 Preliminary results

As in Section 2.4.2, let $\mathcal{T}$ be the set of time periods for which $p_t^* = x^*$ (from Theorem 3.2, the set $\mathcal{T}$ is nonempty.) Let λ^* be the optimal Lagrange multiplier associated with the capacity constraint. We only consider changes which do not affect the set $\mathcal{T}$ and assume $p_t^{\min} < p_t^* < p_t^{\max}$ for all t at optimality.

Lemma 3.3
(a) For $t \notin \mathcal{T}$, p_t^* *satisfies:*

$$(p_t^*-\lambda^*)\,\overline{d}_t'(p_t^*) + \overline{d}_t(p_t^*) = \widehat{\delta}_t\,sgn(p_t^*-x^*)\,\left[(p_t^*-x^*)\,\overline{d}_t'(p_t^*) + \overline{d}_t(p_t^*)\right], \tag{54}$$

or equivalently:

$$p_t^* + \frac{\overline{d}_t(p_t^*)}{\overline{d}_t'(p_t^*)} = \frac{\lambda^* - x^*\,\widehat{\delta}_t\,sgn(p_t^*-x^*)}{1-\widehat{\delta}_t\,sgn(p_t^*-x^*)}. \tag{55}$$

Furthermore, x^* *satisfies:*

$$(x^*-\lambda^*)\sum_{t\in\mathcal{T}}\overline{d}_t'(x^*) + \sum_{t\in\mathcal{T}}\overline{d}_t(x^*) = \Delta, \tag{56}$$

and for any $t \in \mathcal{T}$:

$$-\widehat{\delta}_t\,\overline{d}_t(x^*) \leq (x^* - \lambda^*)\,\overline{d}_t'(x^*) + \overline{d}_t(x^*) \leq \widehat{\delta}_t\,\overline{d}_t(x^*). \tag{57}$$

(b) At optimality, all prices strictly exceed the marginal value of the resource λ^*.

(c) Let $\phi_t(p_t, \lambda) = (p_t - \lambda)\,\overline{d}_t'(p_t) + \overline{d}_t(p_t)$ and $\psi(x, \lambda) = (x - \lambda)\sum_{t\in\mathcal{T}} \overline{d}_t'(x) + \sum_{t\in\mathcal{T}} d_t(x)$. Then:

(c-i) $\phi_t(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ decrease at $\lambda \geq 0$ given.

(c-ii) $\phi_t(p_t, \cdot)$, resp. $\psi(x, \cdot)$, increases at p_t, resp. x, given.

Proof: The proof of (a) and (c) is similar to Lemma 2.4 and we omit it here. For (b), we use Condition (57) and note that worst-case demand is nonnegative, so that $(x^* - \lambda^*)\,\overline{d}_t'(p_t^*) \leq -(1 - \widehat{\delta}_t)\,\overline{d}_t(p_t^*)$ requires $x^* \geq \lambda^*$. It follows that $p_t^* \geq \lambda^*$ for all p_t^* greater than x^*. For $p_t^* < x^*$, Condition (55) yields: $p_t^* + \overline{d}_t(p_t^*)/\overline{d}_t'(p_t^*) = (\lambda^* + x^*\,\widehat{\delta}_t)/(1 + \widehat{\delta}_t)$ or equivalently:

$$p_t^* - \lambda^* = \frac{(x^* - \lambda^*)\,\widehat{\delta}_t}{1 + \widehat{\delta}_t} - \frac{\overline{d}_t(p_t^*)}{\overline{d}_t'(p_t^*)}. \tag{58}$$

Since $x^* \geq \lambda^*$, the right-hand side is nonnegative and $p_t^* \geq \lambda^*$. Therefore, all prices are greater than or equal to the marginal value of the resource. If at least one price was equal to λ^*, then in particular $p_t^* = \lambda^*$ for the time period t corresponding to the smallest product price. If $p_t^* < x^*$ for that t, then from Equation (58) we have $x^* = \lambda^*$ (and $\overline{d}_t(p_t^*) = 0$), and if x^* is indeed the smallest price then the condition $x^* = \lambda^*$ is trivial. Reinjecting into Condition (57) yields $\overline{d}_t(x^*) = 0$ for all $t \in \mathcal{T}$ (since $\widehat{\delta}_t < 1$ for all t), which would violate Equation (56). Hence, $p_t^* > \lambda^*$ for all t. □

We now provide a general characterization of the optimal prices. The key difference with the additive case is that the prices as a function of the marginal value of the resource explicitly depend on the reference price.

Lemma 3.4

(a) The optimal reference and product prices are a function of the uncertainty as follows:

(i) For each $t \notin \mathcal{T}$, there exists a function G_{1t} such that:

$$p_t^* = G_{1t}(\widehat{\delta}_t\, sgn(p_t^* - x^*), \lambda^*, x^*). \tag{59}$$

Moreover, there exists a function G_1 such that:

$$x^* = G_1(\Delta, \lambda^*). \tag{60}$$

(ii) There exists a function G_2 such that:

$$\lambda^* = G_2((\widehat{\delta}_t\, sgn(p_s^* - x^*))_{s=0,\ldots,T-1}, \Delta). \tag{61}$$

(iii) For each t, there exists a function G_{3t} such that:

$$p_t^* = G_{3t}((\widehat{\delta}_s\, sgn(p_s^* - x^*))_{s=0,\ldots,T-1}, \Delta). \tag{62}$$

(b) The function G_1 is nonincreasing in its first argument and nondecreasing in its second. Specifically:

$$\frac{\partial G_1}{\partial u_1}(u_1,u_2) = \left(2\sum_{t\in\mathcal{T}} \overline{d}_t'[G_1(u_1,u_2)] + (G_1(u_1,u_2)-u_2)\sum_{t\in\mathcal{T}} \overline{d}_t''[G_1(u_1,u_2)]\right)^{-1}, \tag{63}$$

$$\frac{\partial G_1}{\partial u_2}(u_1,u_2) = \left(2\sum_{t\in\mathcal{T}} \overline{d}_t'[G_1(u_1,u_2)] + (G_1(u_1,u_2)-u_2)\sum_{t\in\mathcal{T}} \overline{d}_t''[G_1(u_1,u_2)]\right)^{-1} \sum_{t\in\mathcal{T}} \overline{d}_t'[G_1(u_1,u_2)]. \tag{64}$$

Proof: Is a straightforward extension of Lemma 2.5. □

We now briefly comment on an assumption that allows for more powerful results.

Assumption 3.5 *The function $p_t \mapsto p_t + \dfrac{\overline{d}_t(p_t)}{\overline{d}_t'(p_t)}$ increases in p_t, for all t.*

This assumption enforces that the ratio of the marginal revenue over the absolute value of the marginal demand decreases in the price (since $\overline{d}_t'(p_t) < 0$). This is equivalent to limiting the curvature of the nominal demand at each time period:

$$\overline{d}_t''(p_t) \le 2\,\frac{[\overline{d}_t'(p_t)]^2}{\overline{d}_t(p_t)},\ \forall t,\ p_t. \tag{65}$$

Note that the curvature is already limited by the concavity of the revenue, so that this assumption simply limits it further. Nominal demands that are linear in the prices obviously satisfy Condition (65).

Lemma 3.6 *Under Assumption 3.5, there exists a nondecreasing function $\widetilde{G}_{1t}$, $t \notin \mathcal{T}$, such that:*

$$p_t^* = \widetilde{G}_{1t}\left(\frac{\lambda^* - x^*\,\widehat{\delta}_t\, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}\right). \tag{66}$$

Proof: Is a direct consequence of Equation (55) combined with Assumption 3.5. □

Remark: Since $\widetilde{G}_{1t}$ only depends on the average demand, the impact of the uncertainty on the prices p_t^* (with $p_t^* \neq x^*$) is captured *in its entirety* by the parameter $\dfrac{\lambda^* - x^*\,\widehat{\delta}_t\, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}$. This depends on the time period only through $\widehat{\delta}_t\, sgn(p_t^* - x^*)$. If $p_t^* < x^*$, this parameter is equal to $\dfrac{\lambda^* + \widehat{\delta}_t\, x^*}{1+\widehat{\delta}_t}$, which is a convex combination of λ^* and x^*, with x^* receiving more weight if there is more uncertainty. If $p_t^* > x^*$, this parameter is equal to $\dfrac{\lambda^* - \widehat{\delta}_t\, x^*}{1-\widehat{\delta}_t}$ and remains strictly smaller than λ^*, since $x^* > \lambda^*$ from Theorem 3.3, and the coefficient of x^* becomes more negative as the uncertainty increases.

Section 3.5 illustrates the insights that the decision-maker can derive from the robust optimization approach in the case of average demand linear in the prices.

3.3.3 Impact of uncertainty

The results in this section require that Assumption 2.6 holds (the resource is always more valuable in the nominal model, and its marginal value decreases as the budget of resource consumption by the uncertainty increases). We start by comparing the optimal prices with those obtained in the deterministic model.

Theorem 3.7 (Comparison with the nominal model)
(a) If $x^* \leq \overline{\lambda}^*$, then all the optimal prices in the robust model have decreased from their nominal values.
(b) If $x^* > \overline{\lambda}^*$, then the robust prices have always decreased from their nominal values when they exceed the reference price in the robust model, and have decreased from their nominal values when they fall below the reference price if and only if the uncertainty is below a threshold:

$$\widehat{\delta}_t < \frac{\overline{\lambda}^* - \lambda^*}{x^* - \overline{\lambda}^*}. \tag{67}$$

Proof: We prove (a) and (b) simultaneously. We know that $\phi_t(\cdot, \overline{\lambda}^*)$ decreases for all t, with $\phi_t(\overline{p}_t^*, \overline{\lambda}^*) = 0$ and $\phi_t(p_t^*, \overline{\lambda}^*) = \widehat{\delta}_t\, sgn(p_t^* - x^*) \left[(p_t^* - x^*)\, \overline{d}_t'(p_t^*) + \overline{d}_t(p_t^*)\right] + (\lambda^* - \overline{\lambda}^*)\, \overline{d}_t'(p_t^*)$ from Lemma 3.3 for $t \notin \mathcal{T}$. Hence, $p_t^* < \overline{p}_t^*$ ($t \notin \mathcal{T}$) if and only if the condition:

$$\widehat{\delta}_t\, sgn(p_t^* - x^*) \left[(p_t^* - x^*)\, \overline{d}_t'(p_t^*) + \overline{d}_t(p_t^*)\right] + (\lambda^* - \overline{\lambda}^*)\, \overline{d}_t'(p_t^*) > 0 \tag{68}$$

holds. Upon dividing by $\overline{d}_t'(p_t^*) < 0$ and injecting Equation (55), this is equivalent to:

$$\widehat{\delta}_t\, sgn(p_t^* - x^*) \left[\frac{\lambda^* - x^*\, \widehat{\delta}_t\, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)} - x^*\right] + \lambda^* - \overline{\lambda}^* < 0. \tag{69}$$

Rearranging the terms yields:

$$\widehat{\delta}_t\, sgn(p_t^* - x^*) \left[\overline{\lambda}^* - x^*\right] < \overline{\lambda}^* - \lambda^*. \tag{70}$$

From Assumption 2.6, this condition is always satisfied if $x^* = \overline{\lambda}^*$. If $x^* < \overline{\lambda}^*$, Equation (70) becomes:

$$\widehat{\delta}_t\, sgn(p_t^* - x^*) < \frac{\overline{\lambda}^* - \lambda^*}{x^* - \overline{\lambda}^*}, \tag{71}$$

which is always satisfied because the right-hand side is greater than 1 ($\lambda^* < x^* < \overline{\lambda}^*$) and $\widehat{\delta}_t < 1$ for all t. If $x^* > \overline{\lambda}^*$, Equation (70) becomes:

$$\widehat{\delta}_t\, sgn(p_t^* - x^*) > -\frac{\overline{\lambda}^* - \lambda^*}{\overline{\lambda}^* - x^*}. \tag{72}$$

We conclude by distinguishing between $p_t^* < x^*$ and $p_t^* > x^*$. □

Remark: In contrast with the case of additive uncertainty, the threshold in Condition (67) does not depend on the time period considered.

To analyze the impact of the parameters Δ and $\widehat{\delta}_t$ on the optimal solution, we restrict ourselves to demand functions for which Assumption 3.5 holds, which include the linear model. The budget of resource consumption by the uncertainty Δ affects the optimal prices as follows.

Theorem 3.8 (Impact of the maximum use of the resource by the uncertainty) *(a) The optimal product prices below or equal to the reference price, the reference price and the marginal value of the resource are all piecewise nonincreasing in the budget of uncertainty impact.* *(b) Let t be a time period for which the optimal price exceeds the reference price. p_t^* decreases with Δ if and only if:*

$$\widehat{\delta}_t \leq \frac{\partial \lambda^*}{\partial \Delta} \left(\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \right), \tag{73}$$

which is equivalent to the following two conditions holding simultaneously:

$$\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \leq 0, \tag{74}$$

and

$$\frac{\partial \lambda^*}{\partial \Delta} \leq \widehat{\delta}_t \left[\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \right]^{-1}. \tag{75}$$

Proof: (a) $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$ is a direct consequence of Assumption 2.6. Differentiating Equation (56) with respect to Δ yields:

$$\frac{\partial x^*}{\partial \Delta} = \left(1 + \frac{\partial \lambda^*}{\partial \Delta} \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) \right) \left[2 \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \right]^{-1}. \tag{76}$$

Since $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$, it follows that $\frac{\partial x^*}{\partial \Delta} \leq 0$ by concavity of the revenue, convexity and monotonicity of the demand. Moreover, differentiating Equation (55) with respect to Δ yields, for $t \notin \mathcal{T}$:

$$\frac{\partial p_t^*}{\partial \Delta} = \frac{1}{1 - \widehat{\delta}_t \, sgn(p_t^* - x^*)} \left[2 - \frac{\overline{d}_t(p_t^*) \, \overline{d}_t''(p_t^*)}{[\overline{d}_t'(p_t^*)]^2} \right]^{-1} \left(\frac{\partial \lambda^*}{\partial \Delta} - \frac{\partial x^*}{\partial \Delta} \widehat{\delta}_t \, sgn(p_t^* - x^*) \right). \tag{77}$$

$\frac{\partial p_t^*}{\partial \Delta} \leq 0$ for $p_t^* < x^*$ (i.e., $sgn(p_t^* - x^*) = -1$) follows immediately from Assumption 3.5.
For $p_t^* > x^*$, $\frac{\partial p_t^*}{\partial \Delta} \leq 0$ is equivalent to:

$$\frac{\partial \lambda^*}{\partial \Delta} \leq \frac{\partial x^*}{\partial \Delta} \widehat{\delta}_t. \tag{78}$$

We obtain Condition (73) by injecting Equation (76) into Equation (78). □

Remark: High prices (those above the reference price) decrease as the decision-maker's risk aversion Δ increases if the uncertainty $\widehat{\delta}_t$ is "small enough", i.e., below a threshold. If the uncertainty is too large, then it is optimal to increase those prices in order to decrease the nominal demand and allocate the resource to more valuable time periods.

We now investigate the impact of the maximum uncertainty at each time period, $\widehat{\delta}_t$, on the optimal prices, for $t \notin \mathcal{T}$.

Theorem 3.9 (Impact of the uncertainty at each time period) *The optimal prices converge further towards the reference price as the uncertainty increases. Specifically:*

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_t} = \frac{(\lambda^* - x^*)\, sgn(p_t^* - x^*)}{\widetilde{G}'_{1t}(p_t^*)\, [1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)]^2}. \tag{79}$$

Proof: Follows by differentiating Equation (55) as a function of $\widehat{\delta}_t$ and invoking Lemma 3.3 (b) and Lemma 3.6. □

3.4 Comparison with the additive case

In this section, we summarize our results in the presence of additive and multiplicative uncertainty and highlight the common points as well as the main differences. To compare our insights in both cases, we need the following assumptions to hold, besides convexity of nominal demand and concavity of nominal revenue: (i) the function $p_t \mapsto p_t + \overline{d}_t(p_t)/\overline{d}_t'(p_t)$ increases in p_t for all t, (ii) $\widehat{\delta}_t \leq 1 + \min_{p_t^{\min} \leq p_t \leq p_t^{\max}} p_t\, \overline{d}_t''(p_t)/[2\, \overline{d}_t'(p_t)]$, (iii) the uncertainty decreases the value of the resource, in the sense that the resource is always more valuable in the nominal model than in the robust one, and its marginal value decreases with the budget of resource consumption by the uncertainty, (iv) except for the mathematical programming formulation, the prices remain strictly within their bounds at optimality.

The common points are:

1. The robust optimization approach introduces one new decision variable, called the reference price of the product, and no new constraint. The robust formulation for a given reference price is a convex programming problem.

2. All product prices (distinct from the reference) converge further toward the reference price as the maximal uncertainty in their time period increases.

3. The prices above the reference at optimality are always smaller than their nominal counterparts, i.e., incorporating uncertainty decreases high prices.

4. The prices below the reference at optimality are smaller than their nominal counterparts when the maximal uncertainty at that time period does not exceed a threshold, i.e., incorporating uncertainty decreases low prices when the uncertainty is small enough.

5. This threshold is proportional to the difference in the marginal value of the resource in the nominal and robust models.

6. Low prices are increased, with respect to their nominal counterparts, when the demand at that time period is too volatile and it is preferable to allocate the resource to more profitable time periods.

7. The prices below or at the reference level always decrease as the budget of resource consumption by the uncertainty, i.e., the decision-maker's degree of risk aversion, increases, and the uncertainty levels at each time period are kept constant.

The main differences are:

1. In the additive case, the objective function of the robust problem is convex in the reference price. In the multiplicative model, the objective function increases then decreases in the reference price, so that its optimal value can be found by gradient-ascent methods.

2. The statement in point 4. above is a necessary and sufficient condition when the uncertainty is additive. In the multiplicative case, the prices are smaller than their nominal counterparts if and only if either the reference price is smaller than the marginal value of the resource in the nominal model, or (when it does not) the statement in point 4. holds.

3. The threshold in point 4. above depends on time and (explicitly) on the demand function in the additive model, but does not in the multiplicative model.

4. In the additive model, prices above the reference always decrease with the decision-maker's degree of risk aversion as the uncertainty levels at each time period are kept constant, but in the multiplicative model this only holds when the uncertainty at these time periods remain below a threshold.

Hence, it appears that additive and multiplicative models of uncertainty affect the single-product pricing problem in many similar ways and differ only on rather technical points. The robust optimization approach provides valuable insights into the impact of uncertainty in general on the optimal solution.

3.5 Example

In this section, we apply the robust optimization framework to the single-product pricing problem with multiplicative uncertainty when the average demand is linear in the prices, i.e.:

$$\overline{d}_t(p_t) = a_t - b_t\, p_t, \text{with } a_t,\ b_t > 0,\ \forall t. \tag{80}$$

Linear functions satisfy Assumption 3.5. The results follow immediately from injecting Equation (80) into the approach developed in Sections 3.2 and 3.3. Therefore, we state them without proof.

3.5.1 Robust formulation

The robust problem when the average demand is linear in the prices and uncertainty is multiplicative can be formulated as:

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} p_t\,(a_t - b_t\,p_t) - \left[\Delta\,x + \sum_{t=0}^{T-1} \widehat{\delta}_t\,(a_t - b_t\,p_t)\,|p_t - x|\right] \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} b_t\,p_t \geq \sum_{t=0}^{T-1} a_t - (C + \Delta), \\
& p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t.
\end{aligned}
\tag{81}
$$

3.5.2 Optimal prices

As in Section 2.5.2, let x^*, $\mathbf{p}^*$ be the optimal decision variables of Problem (50), and λ^* the optimal Lagrange multiplier for the capacity constraint $\sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta$. Let also $\overline{\mathbf{p}}^*$, $\overline{\lambda}^*$ be the optimal prices and Lagrange multiplier in the nominal problem. To simplify the mathematical expressions, we assume that the capacity constraint is tight both in the nominal and the robust models and that $p_t^{\min} < p_t^* < p_t^{\max}$ for all t. We also assume that $\mathcal{T} = \{t | p_t^* = x^*\}$ does not change with infinitesimal changes in the parameters. As before, we denote by $\mathcal{T}$ the set $\{t | p_t^* = x^*\}$.

The optimal price at time t satisfies:

$$
p_t^* = \begin{cases} \dfrac{a_t}{2\,b_t} + \dfrac{\lambda^* - x^*\,\widehat{\delta}_t\,sgn(p_t^* - x^*)}{2\,[1 - \widehat{\delta}_t\,sgn(p_t^* - x^*)]}, & \text{if } t \notin \mathcal{T} \\ \dfrac{\sum_{t\in\mathcal{T}} a_t - \Delta}{2\sum_{t\in\mathcal{T}} b_t} + \dfrac{\lambda^*}{2}, & \text{if } t \in \mathcal{T}. \end{cases}
\tag{82}
$$

As expected, $p_t^* < \overline{p}_t^*$ if and only if $(x^* - \overline{\lambda}^*)\,\widehat{\delta}_t\,sgn(p_t^* - x^*) > \lambda^* - \overline{\lambda}^*$. The robust prices p_t^* with $t \notin \mathcal{T}$ differ at optimality from their nominal counterparts as follows:

$$
p_t^* = \overline{p}_t^* + \frac{\lambda^* - \overline{\lambda}^*}{2} + \frac{(\lambda^* - x^*)\,\widehat{\delta}_t\,sgn(p_t^* - x^*)}{2\,(1 - \widehat{\delta}_t\,sgn(p_t^* - x^*))}.
\tag{83}
$$

Hence, the impact of uncertainty on the optimal prices at each time period has two components:

- a term that depends on the change in the marginal value of the resource $\lambda^* - \overline{\lambda}^*$ and is common to all time periods,
- a term that depends on the time period through $\widehat{\delta}_t\,sgn(p_t^* - x^*)$ and is proportional to the difference between the reference price x^* and the marginal value of the resource λ^*.

Remark: If the a_t/b_t are monotonic and the $\widehat{\delta}_t$ small enough so that the $a_t/b_t + [\lambda^* - x^*\,\widehat{\delta}_t\,sgn(p_t^* - x^*)]\,[1 - \widehat{\delta}_t\,sgn(p_t^* - x^*)]^{-1}$ are also monotonic in time, the time horizon is partitioned into three stages at optimality: (i) a stage where the decision-maker protects the system against more demand than expected, (ii) a stage where the prices are constant, equal to the reference price, and (iii) a stage where the decision-maker protects the system against less demand than expected. Depending on whether the a_t/b_t are increasing or decreasing over time, we observe the stages in this order (i-ii-iii) as time goes by or in the reverse order (iii-ii-i).

3.5.3 Optimal Lagrange multiplier

The marginal value of the resource as a function of the uncertainty, obtained by injecting Equation (82) into $\sum_{t\notin\mathcal{T}} \overline{d}_t(p_t^*) + \sum_{t\in\mathcal{T}} \overline{d}_t(x^*) = C + \Delta$, is given by:

$$\lambda^* = \frac{\sum_{t=0}^{T-1} a_t + (B/2) - (2\,C + \Delta)}{A/2}. \tag{84}$$

where:

$$\begin{cases} A = \displaystyle\sum_{t=0}^{T-1} b_t + \sum_{t=0}^{T-1} \frac{b_t}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}, \\ B = \left(\displaystyle\sum_{t=0}^{T-1} \frac{\widehat{\delta}_t\, b_t\, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}\right) \cdot \left(\frac{\sum_{t\in\mathcal{T}} a_t - \Delta}{\sum_{t\in\mathcal{T}} b_t}\right). \end{cases} \tag{85}$$

Hence, the marginal value of the resource decreases as the decision-maker's risk aversion, i.e., the maximum allowable use of the resource by the uncertainty, increases:

$$\frac{\partial\lambda^*}{\partial\Delta} = -\frac{2}{A}. \tag{86}$$

The marginal value of the resource increases, resp. decreases, in the maximum uncertainty $\widehat{\delta}_t$ when the optimal price at time t strictly exceeds, resp. falls strictly below, the reference price x^*. Specifically:

$$\frac{\partial\lambda^*}{\partial\widehat{\delta}_t} = \frac{2\,b_t\,(x^* - \lambda^*)\,sgn(p_t^* - x^*)}{A\,(1 - \widehat{\delta}_t sgn(p_t^* - x^*))^2}. \tag{87}$$

3.5.4 Optimal prices as a function of uncertainty

The key result of this section is that, in the presence of multiplicative uncertainty, the optimal prices are piecewise rational functions of the $\widehat{\delta}_t$ (piecewise because of the dependence in $\mathcal{T}$):

$$p_t^* = \begin{cases} \dfrac{\sum_{t\in\mathcal{T}} a_t - \Delta}{2\sum_{t\in\mathcal{T}} b_t} + \dfrac{\lambda^*}{2}, & t\in\mathcal{T}, \\ \dfrac{a_t}{2\,b_t} - \dfrac{\widehat{\delta}_t\, sgn(p_t^* - x^*)}{4(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))} \cdot \left(\dfrac{\sum_{s\in\mathcal{T}} a_s - \Delta}{\sum_{s\in\mathcal{T}} b_s}\right) + \lambda^* \cdot \dfrac{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)/2}{2\,(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))}, & t\notin\mathcal{T}. \end{cases} \tag{88}$$

where λ^* is given by Equation (84). It follows that:

$$\frac{\partial p_t^*}{\partial\Delta} = \begin{cases} -\left(\dfrac{1}{2\sum_{s\in\mathcal{T}} b_s} + \dfrac{1}{A}\right), & t\in\mathcal{T}, \\ \dfrac{1}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}\left[\dfrac{1}{2}\,\widehat{\delta}_t\, sgn(p_t^* - x^*)\left(\dfrac{1}{2\sum_{s\in\mathcal{T}} b_s} + \dfrac{1}{A}\right) - \dfrac{1}{A}\right], & t\notin\mathcal{T}. \end{cases} \tag{89}$$

As expected, we find that p_t^* at t such that $p_t^* < x^*$ and x^* decrease with Δ, and that p_t^* at t such that $p_t^* > x^*$ decreases with Δ if and only if the uncertainty at that time period falls below a threshold, here:

$$\widehat{\delta}_t \leq 2 \left[1 + \frac{A}{2\sum_{s\in\mathcal{T}} b_s}\right]^{-1}. \tag{90}$$

Since $2\sum_{s\in\mathcal{T}} b_s < A$ by definition of A, the right-hand side is less than 1. Prices increasing with Δ are therefore prices that are already high (above the reference level) and correspond to time periods with high uncertainty. This allows the capacitated resource to be reallocated to more profitable time periods.

In this example, we can also explicitly formulate the dependence of the prices on the uncertainty at each time period:

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_s} = \begin{cases} -\dfrac{(x^* - \lambda^*)\, sgn(p_t^* - x^*)}{2\,(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))^2} \left[1 - \dfrac{b_t}{A}\left(1 + \dfrac{1}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}\right)\right], & \text{if } t \notin \mathcal{T} \text{ and } s = t, \\ \dfrac{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)/2}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)} \cdot \dfrac{b_s\,(x^* - \lambda^*)\, sgn(p_s^* - x^*)}{A\,(1 - \widehat{\delta}_s\, sgn(p_s^* - x^*))^2}, & \text{if } t \notin \mathcal{T} \text{ and } s \neq t, \\ \dfrac{b_s\,(x^* - \lambda^*)\, sgn(p_s^* - x^*)}{A\,(1 - \widehat{\delta}_s\, sgn(p_s^* - x^*))^2}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{91}$$

As established for general demand functions, if $t \notin \mathcal{T}$ and $s = t$, the price converges further the reference price, since $b_t(1 + 1/(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))) < A$ by definition of A.

Therefore, whether a price is above or below the reference level plays a key role in understanding the impact of an increase in the uncertainty at that time period on all optimal prices.

4 Computational Results

4.1 Additive uncertainty

In this section, we apply the robust optimization approach to an example with 20 time periods, market size at time t a_t decreasing linearly from 500 to 400, and price sensitivity at time t b_t linearly increasing from 4 to 8. This is for instance applicable to a seasonal product. We assume an initial inventory of $C = 3000$ items, select $\mathbf{p^{min}}$ and $\mathbf{p^{max}}$ to enforce nonnegativity of the prices and the demands, and consider the following four functions to describe the uncertainty $\widehat{\delta}_t$ at time t:

(i) $\widehat{\delta}_t$ linearly increasing from 20 to 80,

(ii) $\widehat{\delta}_t$ linearly increasing from 20 to 40 during the first 10 time periods, and linearly increasing from 40 to 80 during the last 10,

(iii) $\widehat{\delta}_t$ quadratically increasing from 1/5 at time 1, to 20 at time 10, to 80 at time 20,

(iv) $\widehat{\delta}_t$ constant at 50.

The functions in cases (i)-(iii) increase with time to model the fact that demand forecasts become less accurate. The budget of resource consumption by the uncertainty Δ is chosen using historical observations of the cumulative residual error $|\sum_{t=0}^{T-1} \widehat{\delta}_t \, z_t|$. Specifically, we observe this quantity 10 times and set Δ to the highest observed value. The realizations of the scaled deviations z_t, $t = 0, \ldots, T-1$ are generated using the same class of distributions at all time periods: either normal with mean 0 and standard deviation 0.25, or uniform over $[-1, 1]$. Table 1 provides the corresponding sample values of Δ in those eight cases. These are the values we will use when we solve the robust

	Uniform	Normal
Case (i)	137	88
Case (ii)	286	93
Case (iii)	135	75
Case (iv)	256	121

Table 1: Values for the budget of resource consumption.

optimization model and compare the robust strategy with the deterministic one.

Impact of Δ on the optimal prices

Figures 1 and 2 show the optimal prices for $\Delta = 0$, 50, 150 and 300 and the four models of maximal uncertainty $\widehat{\delta}_t$, $t = 1, \ldots, 20$.

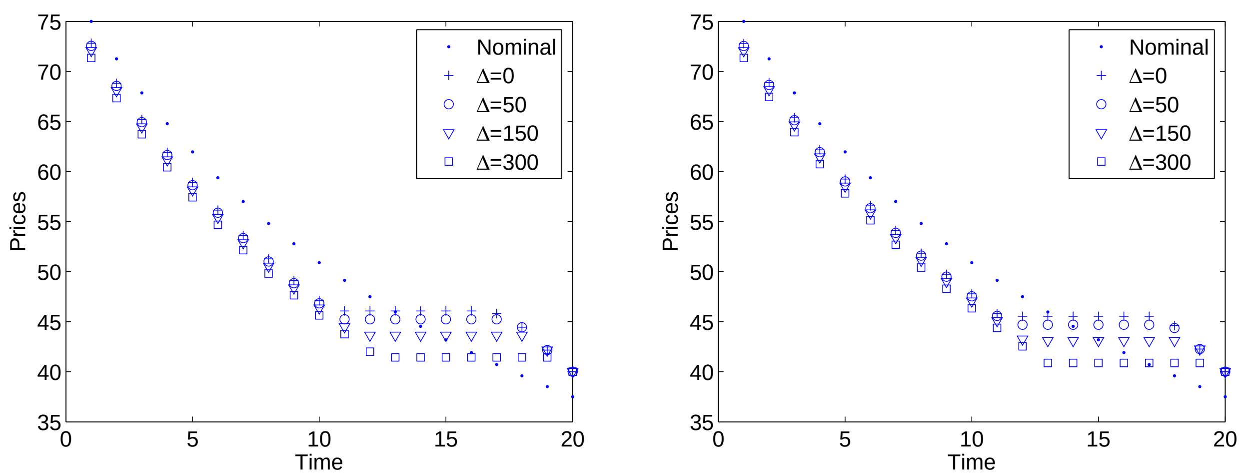

Figure 1: Optimal prices for uncertainty models (i) (left) and (ii) (right).

The actual choice of Δ mostly affects the value of the reference price (which varies by up to 10%) and the set $\mathcal{T}$, i.e., the set of the time periods for which the optimal price is equal to the reference price. The impact of the value of Δ on prices that differ from the reference price is of the order of 1 to 2%. The very fact that a robust optimization approach is implemented accounts for the most

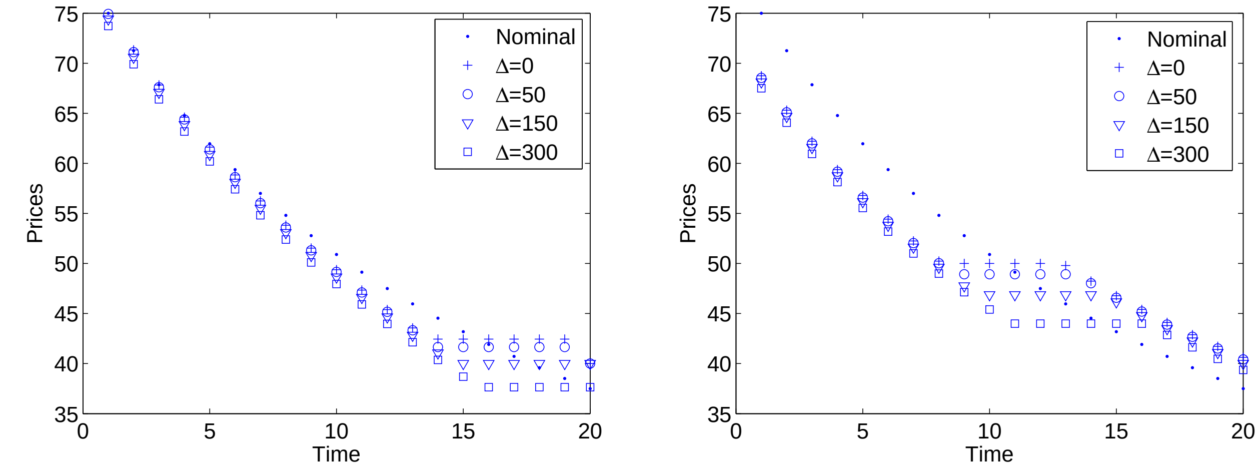

Figure 2: Optimal prices for uncertainty models (iii) (left) and (iv) (right).

significant changes in prices, which suggests that the specific value of Δ does not matter as much as incorporating uncertainty instead of using deterministic models.

Comparison of the nominal and robust strategies

Figures 3-6 represent the histogram, with 20 bins and 5,000 samples, of the actual revenue realized when the scaled deviations follow either a uniform or a normal distribution. The prices implemented are those obtained in the nominal and robust approaches, where the values of Δ are given in Table 1. The decision-maker incurs a unit penalty of \$10 at the last time period for violating the capacity constraint.

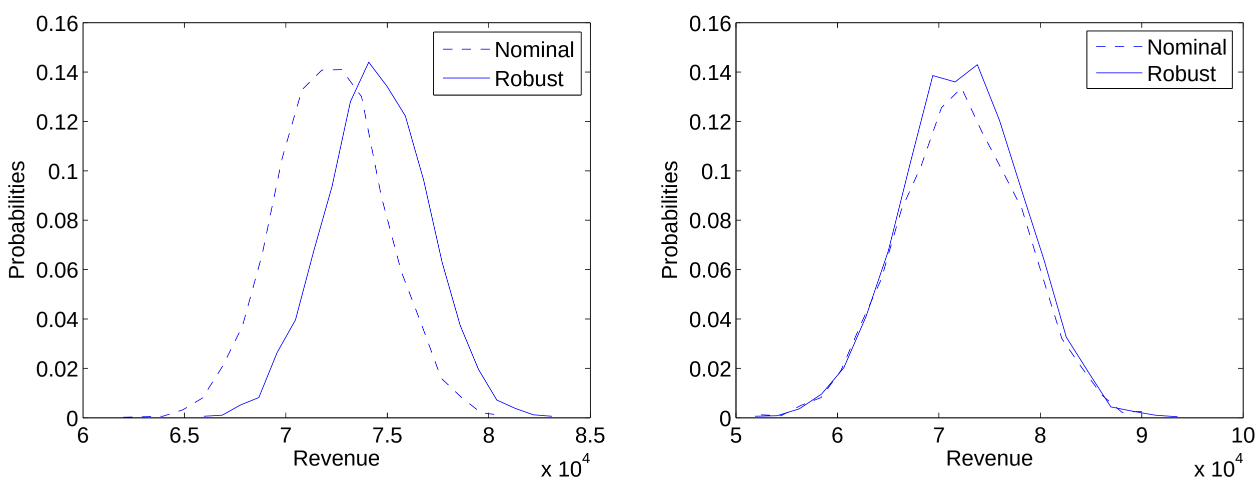

Figure 3: Histogram for uncertainty model (i): uniform (left) and normal (right).

The characteristics of the histograms are summarized in Table 2. The robust optimization approach consistently outperforms the nominal strategy, with the mean revenue increased by up to 12% (case

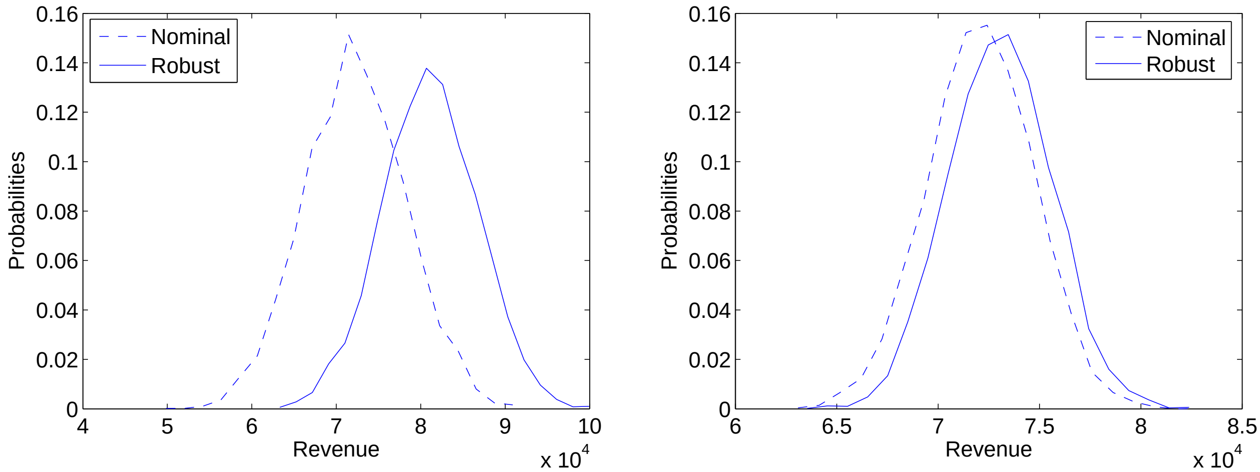

Figure 4: Histogram for uncertainty model (ii): uniform (left) and normal (right).

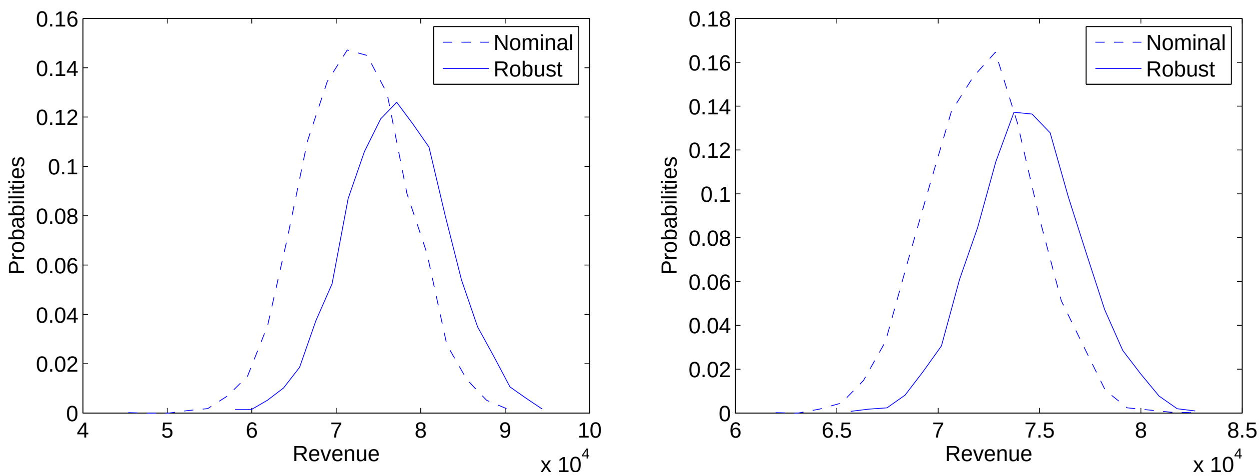

Figure 5: Histogram for uncertainty model (iii): uniform (left) and normal (right).

(ii) with uniform distribution) and the standard deviation decreased by up to 9% (case (iv) with uniform distribution), while the amount of uncertainty faced by the system remains moderate.

	Uniform				Normal			
	Mean ($\times 10^4$)		St. dev. ($\times 10^3$)		Mean ($\times 10^4$)		St. dev. ($\times 10^3$)	
	Nominal	Robust	Nominal	Robust	Nominal	Robust	Nominal	Robust
(i)	7.22	7.45	2.61	2.51	7.22	7.24	6.01	5.89
(ii)	7.22	8.11	6.00	5.64	7.21	7.31	2.61	2.55
(iii)	7.21	7.71	5.99	5.94	7.21	7.45	2.62	2.58
(iv)	7.22	7.56	6.08	5.54	7.21	7.22	2.59	2.48

Table 2: Histogram characteristics.

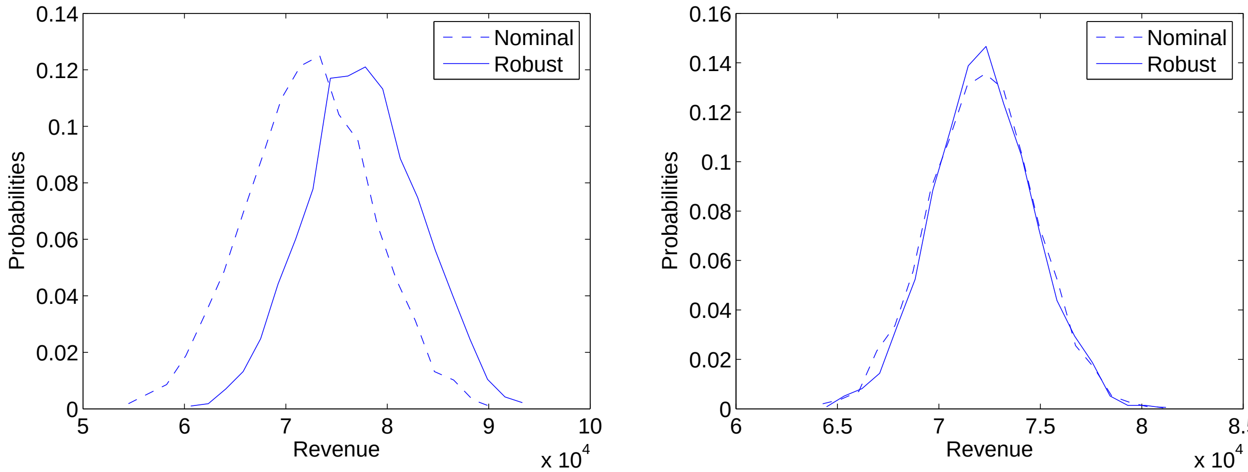

Figure 6: Histogram for uncertainty model (iv): uniform (left) and normal (right).

4.2 Multiplicative uncertainty

We now repeat the experiments performed in Section 4.1 in the case of multiplicative uncertainty. Here the deviations $\widehat{\delta}_t$ cannot exceed 1; we scale the four functions defined above by two times their maximal values. The cumulative scaled deviation $|\sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p_t) \, z_t|$ now depends on the prices; hence, we evaluate this expression for $p_t = a_t/(2\, b_t)$, which would be optimal if there were no uncertainty and no capacity. (The numerical results below indicate that the optimal prices are not very sensitive to the actual value of Δ.) Table 3 presents the values of Δ obtained using the same sampling procedure as in the additive case. Since the expressions now depend on prices and the deviations have been scaled, the numbers cannot be compared with those in Table 1; however, we note that the budgets of uncertainty impact obtained with the uniform and normal distribution still differ by an order of magnitude of 2 in cases (i), (iii) and (iv), and 3 in case (ii).

	Uniform	Normal
Case (i)	50	31
Case (ii)	66	20
Case (iii)	27	15
Case (iv)	99	45

Table 3: Values for the budget of resource consumption.

Figures 7 and 8 show the optimal prices as a function of time. The reference price can be identified by the value where the function of the prices in the robust model becomes flat. As expected, the prices above the reference price have decreased with respect to their nominal values, and the prices below the reference price have increased when the uncertainty exceeds a threshold. For instance in case (iv), $\widehat{\delta}_t = 0.5$ for all t, while the threshold (with $\overline{\lambda}^* = 25$, $\lambda^* = 22.5$, $x^* = 52.8$) is 0.09. As in the additive case, cases (i) and (ii) yields optimal robust prices that are almost identical, although the optimal budgets of uncertainty differ significantly, and the optimal nominal and robust prices in case (iii) are very close, in part because there is little uncertainty in the early time periods.

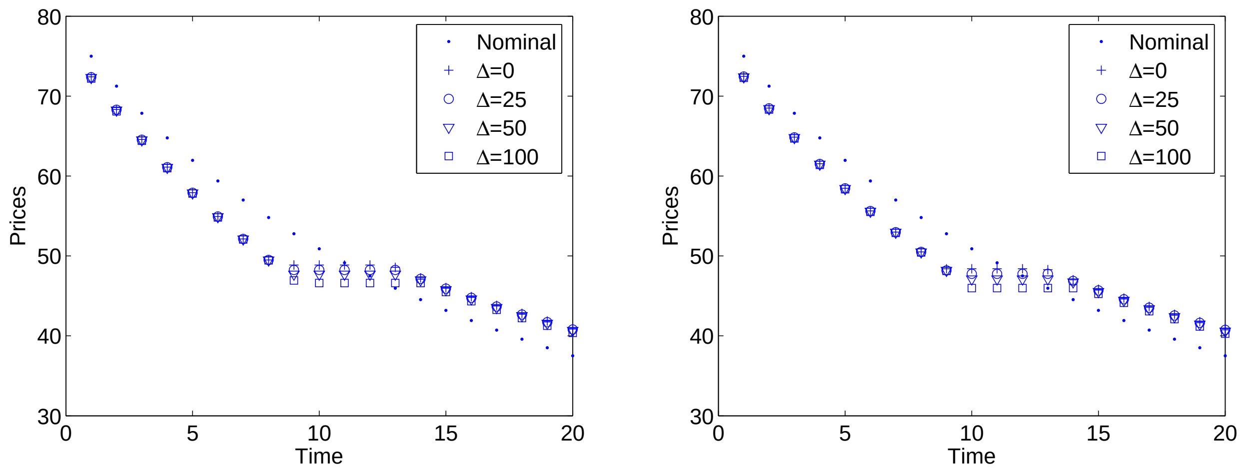

Figure 7: Optimal prices for uncertainty models (i) (left) and (ii) (right).

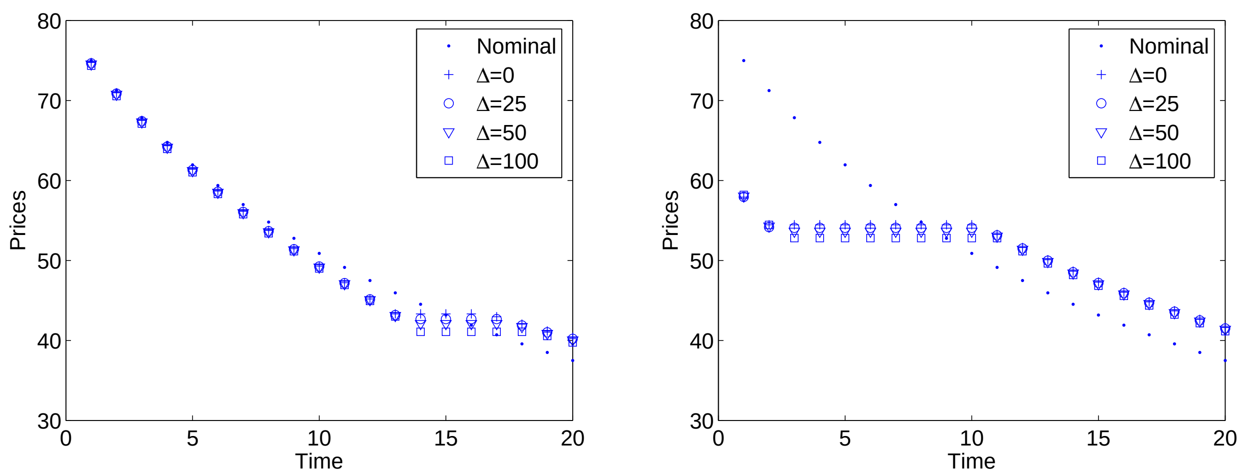

Figure 8: Optimal prices for uncertainty models (iii) (left) and (iv) (right).

Finally, Figures 9-12 present the histograms of the actual revenues, realized with 5,000 samples, while Table 4 summarizes their characteristics in terms of mean and standard deviation. We note that in the multiplicative case, the robust optimization approach decreases the standard deviation of the revenue by up to 5%, but has little impact on the mean, except in case (iv).

These results suggest that the proposed approach is particularly well-suited to the additive model of uncertainty, as it increases the mean of the revenue *and* decreases its standard deviation. When the uncertainty is multiplicative, the performance of the robust optimization approach might be improved by, for instance, introducing a budget of uncertainty impact that would be a function of prices rather than a given parameter.

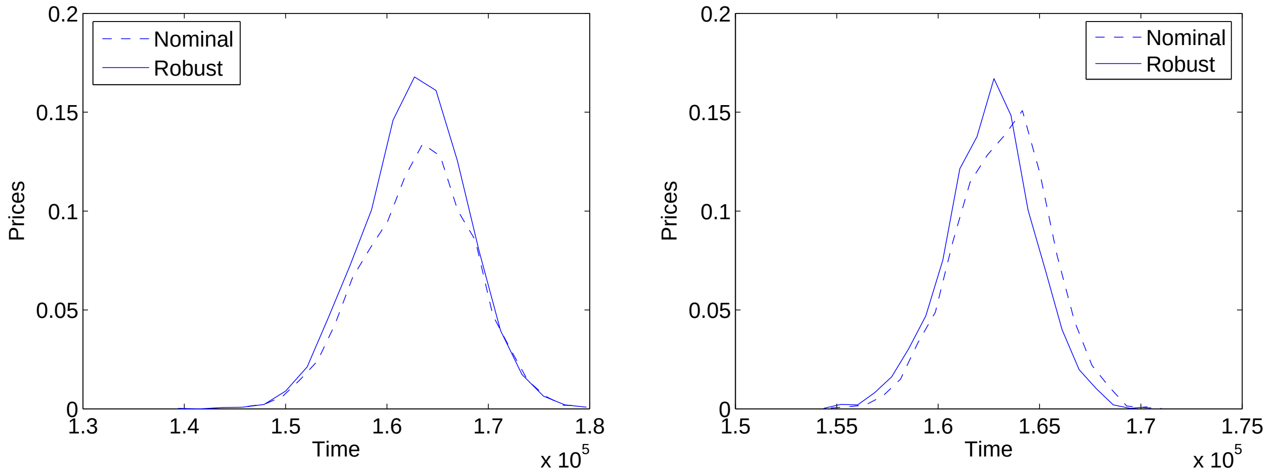

Figure 9: Histogram for uncertainty model (i): uniform (left) and normal (right).

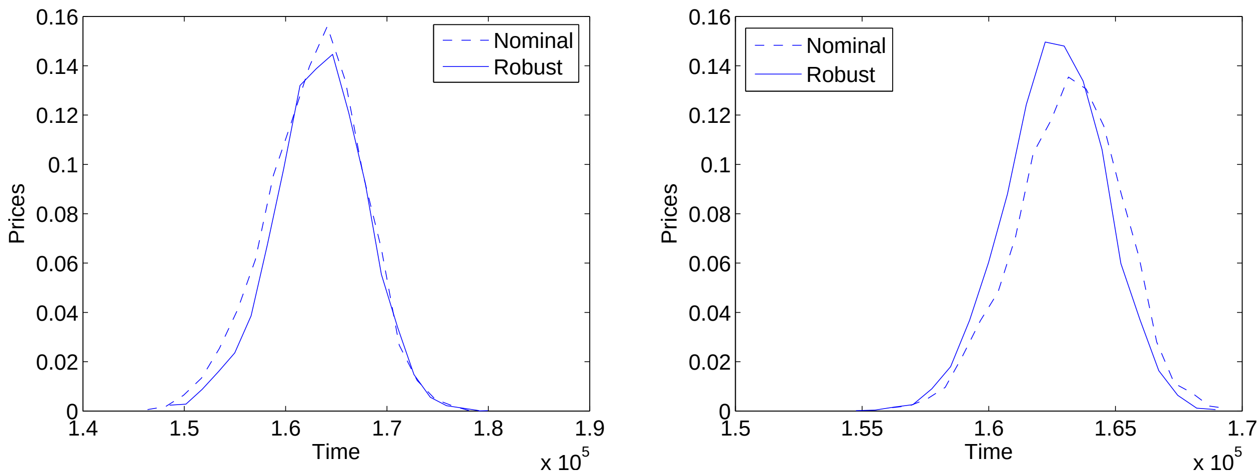

Figure 10: Histogram for uncertainty model (ii): uniform (left) and normal (right).

	Uniform				Normal			
	Mean ($\times 10^5$)		St. dev. ($\times 10^3$)		Mean ($\times 10^5$)		St. dev. ($\times 10^3$)	
	Nominal	Robust	Nominal	Robust	Nominal	Robust	Nominal	Robust
(i)	1.63	1.63	5.30	5.07	1.63	1.63	2.30	2.20
(ii)	1.63	1.63	4.78	4.50	1.63	1.63	2.06	1.96
(iii)	1.63	1.63	2.74	2.65	1.63	1.63	1.19	1.11
(iv)	1.63	1.59	1.06	1.04	1.63	1.58	4.67	4.53

Table 4: Histogram characteristics.

5 Conclusions

We have presented a robust optimization approach to the problem of pricing a single product over time which: (i) does not require the exact knowledge of the demand distributions, and instead

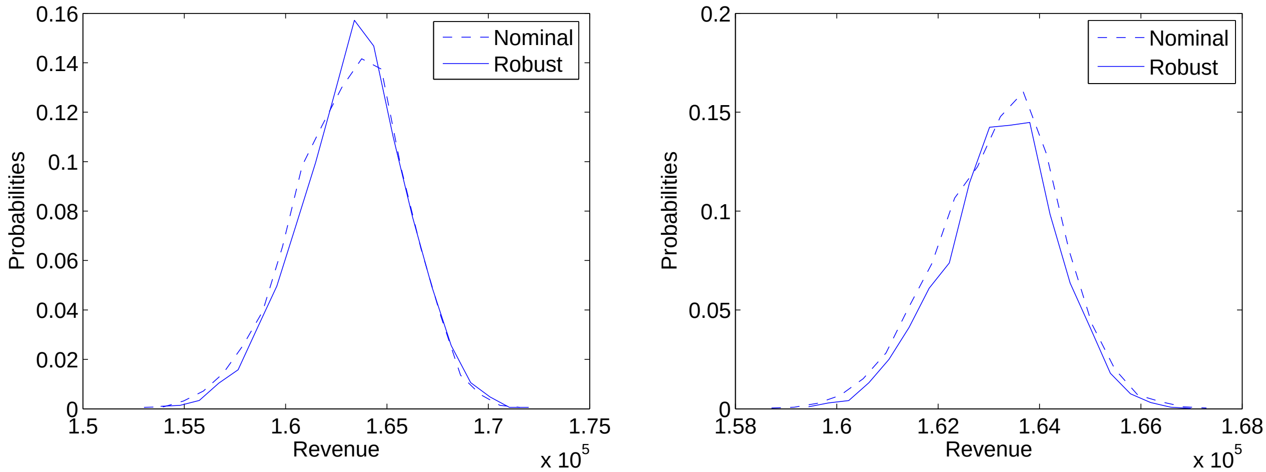

Figure 11: Histogram for uncertainty model (iii): uniform (left) and normal (right).

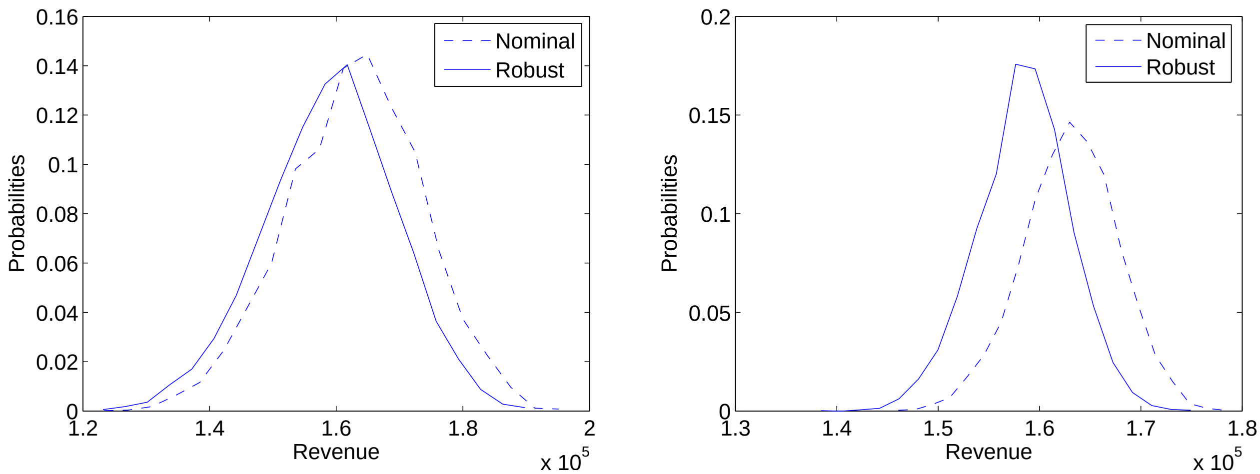

Figure 12: Histogram for uncertainty model (iv): uniform (left) and normal (right).

models random variables through uncertainty sets, (ii) yields tractable mathematical programming problems with only one new decision variable, called the reference price, and no new constraint, and (iii) allows for theoretical insights into the impact of the uncertainty on the solution, for the additive as well as multiplicative models of randomness. In particular, under mild conditions, the optimal prices converge towards the reference price as the uncertainty at each time period increases, and it is not optimal to decrease the prices from their nominal values if prices are low and uncertainty is high. This approach allows the decision-maker to gain a deeper understanding of the impact of demand uncertainty on the optimal prices.

A Appendix: Proof of Theorem 3.1 (c)

Let $(\mathbf{p}, x) \mapsto \widetilde{F}(\mathbf{p}, x)$ be the objective function of Problem (50) and let $x \mapsto F(x)$ be its maximum value over the set of feasible prices. We know that the function F can be computed efficiently since $\widetilde{F}(\cdot, x)$ is convex for any given x (from Theorem 3.1 (b)). We now show that F has a unique local and hence global maximum. Specifically, we show that there exists x^* such that F is nondecreasing over $(-\infty, x^*]$ and nonincreasing over $[x^*, \infty)$, so that gradient-ascent methods always converge to the optimal reference price x^*.

Let x and x' be any two numbers such that $x < x'$, and let the vectors $\mathbf{p}$, resp. $\mathbf{p}'$ be the optimal prices at x, resp. x' given, that is:

$$F(x) = \widetilde{F}(\boldsymbol{p}, x) = \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t\, \overline{d}_t(p_t)\, |p_t - x| \right], \tag{92}$$

and

$$F(x') = \widetilde{F}(\boldsymbol{p'}, x') = \sum_{t=0}^{T-1} p'_t\, \overline{d}_t(p'_t) - \left[\Delta\, x' + \sum_{t=0}^{T-1} \widehat{\delta}_t\, \overline{d}_t(p'_t)\, |p'_t - x'| \right]. \tag{93}$$

It is straightforward to check that:

$$|p'_t - x'| = \begin{cases} x' - x + |x - p'_t|, & \text{if } p'_t < x, \\ x' - x - |p'_t - x|, & \text{if } x \le p'_t \le x', \\ |p'_t - x| + x - x', & \text{otherwise.} \end{cases} \tag{94}$$

This yields:

$$F(x') = \widetilde{F}(\boldsymbol{p'}, x) + (x' - x) \left[-\Delta + \sum_{t|p'_t > x'} \widehat{\delta}_t\, \overline{d}_t(p'_t) - \sum_{t|p'_t \le x'} \widehat{\delta}_t\, \overline{d}_t(p'_t) \right] + 2 \sum_{t|x \le p'_t \le x'} \widehat{\delta}_t\, \overline{d}_t(p'_t) |p'_t - x|. \tag{95}$$

Since $\widetilde{F}(\boldsymbol{p'}, x) \le F(x)$ by definition of F and $\sum_{t|x \le p'_t \le x'} \widehat{\delta}_t\, \overline{d}_t(p'_t)|p'_t - x| \le (x' - x) \sum_{t|x \le p'_t \le x'} \widehat{\delta}_t\, \overline{d}_t(p'_t)$, we obtain after re-arranging the terms:

$$F(x') \le F(x) + (x' - x) \left[-\Delta + \sum_{t|p'_t \ge x} \widehat{\delta}_t\, \overline{d}_t(p'_t) - \sum_{t|p'_t < x} \widehat{\delta}_t\, \overline{d}_t(p'_t) \right]. \tag{96}$$

Similarly, we derive a lower bound on $F(x')$ by expressing $|p_t - x'|$ as a function of $|p_t - x|$ and using that $F(x') \ge \widetilde{F}(\boldsymbol{p}, x')$ and $\sum_{t|x \le p'_t \le x'} \widehat{\delta}_t\, \overline{d}_t(p_t)|p_t - x| \ge 0$. Combining these results leads to:

$$-\Delta + \sum_{t|p_t > x'} \widehat{\delta}_t\, \overline{d}_t(p_t) - \sum_{t|p_t \le x'} \widehat{\delta}_t\, \overline{d}_t(p_t) \le \frac{F(x') - F(x)}{x' - x} \le -\Delta + \sum_{t|p'_t \ge x} \widehat{\delta}_t\, \overline{d}_t(p'_t) - \sum_{t|p'_t < x} \widehat{\delta}_t\, \overline{d}_t(p'_t). \tag{97}$$

From the right-hand side of Equation (97), we conclude that $F(x') < F(x)$ for all $x < x'$ such that $-\Delta + \sum_{t|p'_t \ge x} \widehat{\delta}_t\, \overline{d}_t(p'_t) - \sum_{t|p'_t < x} \widehat{\delta}_t\, \overline{d}_t(p'_t) < 0$. Since the vector $\mathbf{p}'$ depends on x' but not on x,

$-\Delta + \sum_{t|p'_t \geq x} \widehat{\delta}_t \, \overline{d}_t(p'_t) - \sum_{t|p'_t < x} \widehat{\delta}_t \, \overline{d}_t(p'_t)$ is piecewise constant, nonincreasing in x. Therefore, there exists a threshold value which depends on x' and is noted $x^+(x')$ such that, for all x', F decreases on $[x^+(x'), x']$. $x^+(x')$ is defined as:

$$x^+(x') = \min \left\{ y | -\Delta + \sum_{t|p'_t \geq y} \widehat{\delta}_t \, \overline{d}_t(p'_t) - \sum_{t|p'_t < y} \widehat{\delta}_t \, \overline{d}_t(p'_t) < 0 \right\}. \tag{98}$$

It follows that F decreases on $[\min_{x'} x^+(x'), \infty)$.

Similarly, F is nondecreasing on $(-\infty, \max_x x^-(x)]$, where $x^-(x)$ is defined for all x as:

$$x^-(x) = \max \left\{ y | -\Delta + \sum_{t|p'_t \geq y} \widehat{\delta}_t \, \overline{d}_t(p'_t) - \sum_{t|p'_t < y} \widehat{\delta}_t \, \overline{d}_t(p'_t) \geq 0 \right\}. \tag{99}$$

Therefore, the global maximum of F, noted x^*, belongs to $[\max_x x^-(x), \min_{x'} x^+(x')]$. Let $\mathbf{p}^*$ be such that $F(x^*) = \widetilde{F}(\mathbf{p}^*, x^*)$, that is, $\mathbf{p}^* = \arg\max \widetilde{F}(\cdot, x^*)$. Because x^* is the global maximum of F, we also have: $x^* = \arg\max \widetilde{F}(\mathbf{p}^*, \cdot)$. Hence, x^* is the point where the slope of $\widetilde{F}(\mathbf{p}^*, \cdot)$ changes sign. It follows that $x^+(x^*) = x^-(x^*) = x^*$, and F is nondecreasing on $(-\infty, x^*]$ and decreasing on $[x^*, \infty)$.

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