# Clique-based facets for the precedence constrained knapsack problem

Natashia Boland[*], Christopher Fricke[*,†], Gary Froyland[‡], Renata Sotirov [*,§]

19 December 2005

**Abstract**

We consider a knapsack problem with precedence constraints imposed on pairs of items, known as the precedence constrained knapsack problem (PCKP). This problem has applications in management and machine scheduling, and also appears as a subproblem in decomposition techniques for network design and other related problems. We present a new approach for determining facets of the PCKP polyhedron based on clique inequalities. A comparison with existing techniques, that lift knapsack cover inequalities for the PCKP, is also presented. It is shown that the clique-based approach generates facets that cannot be found through the existing cover-based approaches, and that the addition of clique-based inequalities for the PCKP can be computationally beneficial.

*Keywords:* Precedence constrained knapsack problem; Clique inequalities; Integer programming.

## 1 Introduction

In this paper, we consider the polyhedral structure of the *precedence constrained knapsack problem* (PCKP), also known as the *partially ordered knapsack problem.* A set of items $\mathcal{N}$ is given, along with a partial order, or set of precedence relationships, on the items, denoted by $\mathcal{S} \subseteq \mathcal{N} \times \mathcal{N}$. A precedence relationship $(i, j) \in \mathcal{S}$ exists if item $i$ can be placed in the knapsack only if item $j$ is in the knapsack. Each item $i \in \mathcal{N}$ has a value $c_i \in \mathbb{Z}$ and a weight $a_i \in \mathbb{Z}^+$, and the knapsack has a capacity $b \in \mathbb{Z}^+$. The PCKP is the problem of finding a maximum value subset of $\mathcal{N}$ whose total weight does not exceed the knapsack capacity, and that also satisfies the precedence relationships.

The precedence constraints can be represented by the directed acyclic graph $G = (\mathcal{N}, \mathcal{S})$, where the node set is the set of all items $\mathcal{N}$, and each precedence constraint in $\mathcal{S}$ is represented by a directed arc. Note that the precedence constraints are transitive, so without loss of generality we assume that $\mathcal{S}$ does not contain any redundant relationships, that is, $\mathcal{S}$ is the set of all immediate predecessor arcs. If $G$ contains a cycle, all nodes within the cycle must either all be included in, or all be excluded from, the knapsack. Hence the cycle can be contracted into a single node, with cumulative value and weight coefficients, and the resulting directed graph is acyclic.

An integer programming formulation of the PCKP is as follows. Let

$$x_i = \begin{cases} 1, & \text{if item } i \text{ is included in the knapsack} \\ 0, & \text{otherwise} \end{cases} \qquad \text{for all } i \in \mathcal{N}.$$

Then the PCKP may be written as:

[*]Department of Mathematics and Statistics, University of Melbourne, Victoria, Australia, 3010
[†]Corresponding author. Email address: fricke@ms.unimelb.edu.au
[‡]School of Mathematics, University of New South Wales, New South Wales, Australia, 2052
[§]Preprint submitted to Mathematical Programming 19 December 2005

$$
\text{(PCKP)} \quad \begin{aligned}
\max \quad & \sum_{i \in \mathcal{N}} c_i \, x_i & & & (1) \\
\text{s.t.} \quad & \sum_{i \in \mathcal{N}} a_i \, x_i & \leq b & & (2) \\
& x_i & \leq x_j & \quad \text{for all } (i,j) \in \mathcal{S} & (3) \\
& x_i & \in \{0,1\} & \quad \text{for all } i \in \mathcal{N}. & (4)
\end{aligned}
$$

The PCKP appears in a wide range of applications. These include investment problems (Ibarra and Kim [3]), tool management problems (Stecke and Kim [8]), strip mining (Johnson and Niemi [4]) and local access telecommunication network design (Shaw et. al. [7]). In these cases the PCKP has generally been solved using dynamic programming algorithms, when the underlying precedence graph has a special structure such as a tree, or heuristics.

Johnson and Niemi [4] showed that the PCKP is NP-complete. The polyhedral structure of the problem was first investigated by Boyd [2], who extended the concept of a cover inequality for the standard 0-1 knapsack polyhedron to the PCKP polyhedron. Further investigation of the PCKP polyhedron is presented by both Park and Park [6] and van de Leensel et. al. [9], where lifting orders and general sequential lifting procedures are derived to lift valid knapsack cover-based inequalities from lower dimensional polyhedrons into facets of the PCKP polyhedron.

In this paper, we determine facet-defining inequalities for the convex hull $conv(P)$ of the PCKP feasible set $P$ defined by (2)-(4). Unlike previous work [2, 6, 9], we do not take knapsack covers as our starting point, but instead investigate clique inequalities derived from a graph representing pairwise conflict relationships between variables.

We begin in Section 2 by introducing the notation and definitions used throughout the paper. We also derive properties of the precedence relationships that will be useful in our investigation, and present the concept of a conflict graph. In Section 3 we introduce clique inequalities for the PCKP, and derive necessary and sufficient conditions under which they represent facets of $conv(P)$. A comparison of clique inequalities and the results of Boyd [2], Park and Park [6], and van de Leensel et. al. [9], which, as already mentioned, are all based on knapsack cover-like inequalities, is presented in Section 4. We provide a more complete classification of these knapsack cover-like inequalities than has previously been given. The differences, similarities, and computational strength of the various classes of constraints are illustrated in examples in Section 5. We demonstrate that our clique-based approach can generate facet-defining inequalities for $conv(P)$, without the need for the computationally expensive lifting procedures that are used in existing cover-based approaches.

# 2 Notation and Properties of the Precedence Relationships

## 2.1 Summary of Notation

A summary of the notation used throughout this paper is given in Table 1.

## 2.2 Properties of the Precedence Relationships and Feasible Packings

For each $(i,j) \in \mathcal{S}$, item $i$ is an *immediate predecessor* of item $j$ and item $j$ is an *immediate successor* of item $i$. Let $S_i$ be the set of immediate predecessors of item $i$, that is let $S_i = \{j \in \mathcal{N} : (i,j) \in \mathcal{S}\}$. It follows that the set of all precedence relationships $\mathcal{A}$ is the transitive closure of $\mathcal{S}$, and $(i,j) \in \mathcal{A}$ if and only if there exists a path from node $i$ to node $j$ in the directed acyclic graph $G = (\mathcal{N}, \mathcal{S})$. Let $A_i$ be the *minimal set of items*, including item $i$, that must be included in the knapsack for item $i$ to be included, that is $A_i = \{j \in \mathcal{N} : (i,j) \in \mathcal{A}\} \cup \{i\}$. Note that inclusion in the set $A_i$ is also transitive, so if $j \in A_k$ and $k \in A_i$ then $j \in A_i$. Property 1 follows directly.

**Property 1.** *Let $i \in \mathcal{N}$. For all $j \in A_i$ it must be that $A_j \subseteq A_i$.*

| Notation | Definition |
|---|---|
| $\mathcal{N}$ | the set of items available for inclusion in the knapsack. |
| $\mathcal{S}$ | the set of all immediate precedence relationships in the problem instance. |
| $G = (\mathcal{N}, \mathcal{S})$ | the directed graph representing the immediate precedence relationships in the problem instance. |
| $\mathcal{A}$ | the set of all precedence relationships in the problem instance. |
| $c_i$ | the value of item $i \in \mathcal{N}$, $c_i \in \mathbb{Z}$. |
| $a_i$ | the weight of item $i \in \mathcal{N}$, $a_i \in \mathbb{Z}^+$. |
| $b$ | the capacity of the knapsack, $b \in \mathbb{Z}^+$. |
| $S_i$ | the set of immediate predecessors of item $i \in \mathcal{N}$. |
| $A_i$ | the entire precedence set of item $i \in \mathcal{N}$ (including item $i$). |
| $B$ | a set of items, $B \subseteq \mathcal{N}$. |
| $A(B)$ | the union of the entire precedence sets for the items in the set $B$, $A(B) = \cup_{i \in B} A_i$. |
| $H_i$ | the capacity required for item $i$ to be included in the knapsack, $H_i = \sum_{j \in A_i} a_j$. |
| $H(B)$ | the total capacity required to include all items in the set $B$, $H(B) = \sum_{j \in A(B)} a_j$. |
| $D_i$ | the entire successor set of item $i$ (including item $i$). |
| $e_i$ | the $i^{th}$ standard basis vector in $\mathbb{R}^{\vert\mathcal{N}\vert}$. |
| $x(B)$ | the characteristic vector of the set $B$, $x(B) = \sum_{i \in B} e_i$. |
| $\hat{J}_B(k)$ | the descendent set of $k$ in the set $B$, $\hat{J}_B(k) = \{j \in B : k \in A_j\}$ for each $k \in A(B) \setminus B$. |
| $P$ | the PCKP feasible set defined by (2)-(4). |
| $conv(P)$ | the convex hull of the PCKP feasible set $P$. |
| $CG = (\mathcal{N}, E)$ | a conflict graph with edge $\{i, j\} \in E$ if and only if $H(\{i, j\}) > b$. |
| $E$ | the set of edges in the conflict graph $CG$. |
| $\mathcal{C}$ | a set of items that is a clique in the conflict graph $CG$, $\mathcal{C} \subseteq \mathcal{N}$. |
| $\mathcal{P}(\mathcal{C})$ | the set of all items in the intersection of the entire precedence sets of all the items in the clique $\mathcal{C}$, $\mathcal{P}(\mathcal{C}) = \cap_{j \in \mathcal{C}} A_j$. |
| $\mathcal{Q}(\mathcal{C})$ | the set of all items in the intersection of the entire precedence sets of all the items in the clique $\mathcal{C}$, with no items in their entire successor sets $D_i$ that satisfy the same property, $\mathcal{Q}(\mathcal{C}) = \{i \in \mathcal{P}(\mathcal{C}) : \mathcal{C} \nsubseteq D_k \text{ for all } k \in D_i \setminus \{i\}\}$. |
| $C$ | a set of items that is a cover for an instance of the PCKP, $C \subseteq \mathcal{N}$. |
| ($K$-)BMC | a ($K$-)Boyd minimal cover. |
| ($K$-)MIC | a ($K$-)minimal induced cover. |
| $P(B)$ | the convex hull of feasible solutions to (PCKP) restricted to those variables in $A(B)$, $P(B) = conv(proj_{A(B)}\{x(D) \in P : D \subseteq A(B)\})$. |

Table 1: Summary of Notation

In all diagrams throughout this paper, we show the set of immediate predecessors $S_i$ for all $i \in \mathcal{N}$. The $A_i$ sets can be deduced by finding the transitive closure of the $S_i$ sets.

Consider a set of items $B \subseteq \mathcal{N}$. Let $A(B) = \cup_{i \in B} A_i$ be the union of the $A_i$ sets for the items in the set $B$. Then $A(B)$ gives the minimal set of items that must be included for all items in the set $B$ to be included in the knapsack. Now consider the set of items that cannot be included unless item $i$ has been included in the knapsack, and include item $i$ in this set. This is the set of all successors of item $i$, which we denote as $D_i$, hence $D_i = \{j \in \mathcal{N} : i \in A_j\}$. By the transitivity of inclusion in the $A_i$ sets, it follows directly that for any item $j \in A_i$, it must be that $i \in D_j$. Hence, given a set of items $\mathcal{N}$ and the immediate predecessor set $S_i$ for each $i \in \mathcal{N}$, the corresponding entire precedence sets $A_i$ and entire successor sets $D_i$ can be deduced for each $i \in \mathcal{N}$. Note that item $i$ is included in both the entire precedence set $A_i$ and the entire successor set $D_i$. For a given set $B \subseteq \mathcal{N}$ we also require the concept of a subset of $B$ that contains successors of an item $k \in A(B) \setminus B$.

**Definition 1.** *For each set $B \subseteq \mathcal{N}$ with $A(B) \setminus B \neq \emptyset$, for each $k \in A(B) \setminus B$ there exists $j \in B$ such*

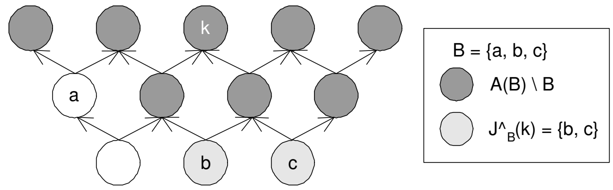


Figure 1: Illustration of a descendent set $\hat{J}_B(k)$.

*that* $k \in A_j$*. Let* $\hat{J}_B(k)$ *denote these* $j$*, that is* $\hat{J}_B(k) = \{j \in B : k \in A_j\}$ *for each* $k \in A(B) \setminus B$*. We refer to* $\hat{J}_B(k)$ *as the* ***descendent set of*** $k$ ***in*** $B$*.*

See Figure 1 for an illustration of a descendent set $\hat{J}_B(k)$.

We now combine the precedence sets with the knapsack constraint (2) to determine the minimum capacity required to include each item in the knapsack. Let $H(B) = \sum_{j \in A(B)} a_j$ be the total capacity required to include the items in the set $B$ in the knapsack. It follows that $H(\{i\}) = \sum_{j \in A_i} a_j$ is the capacity required to include item $i$ in the knapsack. For ease of notation let $H_i = H(\{i\})$. We assume that for every item, there exists a feasible solution in which it is included in the knapsack. Otherwise, the item can be deleted from the problem instance.

**Assumption 1.** *Each item in the set* $\mathcal{N}$ *could be included in the knapsack, that is* $H_i \leq b$ *for all* $i \in \mathcal{N}$*.*

It follows directly from Assumption 1 that the PCKP polyhedron is full-dimensional. We now determine when the inclusion of a given set of items $B \subseteq \mathcal{N}$ in the knapsack is feasible. In what follows, $e_i$ is the $i^{th}$ standard basis vector in $\mathbb{R}^{|\mathcal{N}|}$.

**Definition 2.** *For any set* $B \subseteq \mathcal{N}$*, let* $x(B) \in \{0,1\}^{|\mathcal{N}|}$ *induce* $B$*, that is* $x(B) = \sum_{i \in B} e_i$*.*

**Definition 3.** *Let* $B \subseteq \mathcal{N}$ *such that*

*(i) for all* $j \in B$*,* $A_j \subseteq B$*, and*

*(ii)* $H(B) = \sum_{j \in A(B)} a_j \leq b$*.*

*Then we say that the set of items* $B$ *is a* ***feasible packing*** *of the knapsack.*

We now provide a series of technical results regarding feasible packings and precedence sets. These are all straightforward, but help to simplify the proofs of our main results in Section 3.

**Lemma 1.** *Let* $B_1, \ldots, B_m$*,* $m \in \mathbb{Z}^+$ *be a collection of feasible packings such that* $H(B_1 \cup \ldots \cup B_m) \leq b$*. Then* $B_1 \cup \ldots \cup B_m$ *is a feasible packing.*

*Proof.* Let $B_1, \ldots, B_m$, $m \in \mathbb{Z}^+$ be a collection of feasible packings such that $H(B_1 \cup \ldots \cup B_m) \leq b$. Then for each $i \in B_1 \cup \ldots \cup B_m$ there exists $j$ such that $i \in B_j$. It follows from Property 1 that $A_i \subseteq B_j$. Hence from Definition 3 it follows that $B_1 \cup \ldots \cup B_m$ is a feasible packing. □ □

Hence a union of feasible packings is itself a feasible packing.

**Lemma 2.** *Let* $i$*,* $j \in \mathcal{N}$*. If* $i \in A_j$ *and* $j \in A_i$*, then* $i = j$*.*

*Proof.* Let $i \in A_j \subseteq \mathcal{N}$. From Property 1, it follows that $A_i \subseteq A_j$. Similarly, if $j \in A_i \subseteq \mathcal{N}$ then $A_j \subseteq A_i$. So we have $A_i \subseteq A_j \subseteq A_i$ and hence $A_i = A_j$. It follows directly that $i = j$. □ □

**Lemma 3.** *Let* $B \subseteq \mathcal{N}$*. If* $i \in A(B)$ *then* $A_i \subseteq A(B)$*.*

*Proof.* Let $i \in A(B) = \cup_{j \in B} A_j$. Then there exists $j \in B$ such that $i \in A_j$. It follows from Property 1 that $A_i \subseteq A_j$ and hence $A_i \subseteq A(B)$. □ □

**Lemma 4.** *Let $j \in \mathcal{N} \setminus D_i$ for some $i \in \mathcal{N}$. Then $A_j \subseteq \mathcal{N} \setminus D_i$.*

*Proof.* Let $j \in \mathcal{N} \setminus D_i$ for some $i \in \mathcal{N}$ and suppose that $A_j \not\subseteq \mathcal{N} \setminus D_i$. Then there must exist a $k \in D_i \cap A_j$. By Property 1 we have that $A_k \subseteq A_j$ and since $k \in D_i$ it follows that $i \in A_k$, and from Property 1 we have that $A_i \subseteq A_k$. So $A_i \subseteq A_k \subseteq A_j$, from which it follows that $i \in A_j$. From the definition of $D_i$ we then have that $j \in D_i$, which is a contradiction. Hence $A_j \subseteq \mathcal{N} \setminus D_i$. □ □

**Lemma 5.** *If $B \subseteq \mathcal{N}$ with $H(B) \leq b$, then $A(B)$ is a feasible packing.*

*Proof.* Let $B \subseteq \mathcal{N}$ with $H(B) \leq b$, but assume that $A(B)$ is not a feasible packing. Then it must be that for some $i \in A(B)$, $A_i \not\subseteq A(B)$. This is a contradiction of Lemma 3. Hence $A(B)$ is a feasible packing. □ □

If $B \subseteq \mathcal{N}$ with $H(B) \leq b$ then we say that $B$ **induces** a feasible packing.

**Corollary 1.** *For any item $i \in \mathcal{N}$ it must be that $\{i\}$ induces a feasible packing, that is, $A_i$ is a feasible packing.*

*Proof.* Set $B = \{i\}$ in Lemma 5. Then $H(B) = H_i$ and the result follows directly from Assumption 1. □ □

**Lemma 6.** *Let $B \subseteq \mathcal{N}$ such that*

1. *$B$ is minimal in the sense that $A(B \setminus \{i\}) \subsetneq A(B)$ for all $i \in B$, and*
2. *$H(B) \leq b$.*

*Then $A(B) \setminus \{i\}$ is a feasible packing for any $i \in B$.*

*Proof.* Let $i \in B \subseteq \mathcal{N}$. We now show Conditions (i) and (ii) from Definition 3 of a feasible packing.

(i) Let $j \in A(B) \setminus \{i\}$ and suppose $A_j \not\subseteq A(B) \setminus \{i\}$. Now $j \in A(B)$ so $A_j \subseteq A(B)$ by Lemma 3, and since $A_j \not\subseteq A(B) \setminus \{i\}$ it must be that $i \in A_j$. Hence by Property 1 $A_i \subseteq A_j$. Also, it must be that $j \in A(B \setminus \{i\})$, since otherwise $j \in A_i$, and by Lemma 2 it would be that $j = i$, contradicting $j \in A(B) \setminus \{i\}$. Hence $A_j \subseteq A(B \setminus \{i\})$ by Lemma 3. Then

$$\begin{aligned} A(B) &= \cup_{k \in B} A_k = \cup_{k \in B \setminus \{i\}} A_k \cup A_i \subseteq A(B \setminus \{i\}) \cup A_j && \text{since } A_i \subseteq A_j \\ &= A(B \setminus \{i\}) && \text{since } A_j \subseteq A(B \setminus \{i\}), \end{aligned}$$

so $A(B \setminus \{i\}) \subseteq A(B)$, which is a contradiction to the minimality of $B$ (Condition 1 of Lemma 6). Therefore it must be that $A_j \subseteq A(B) \setminus \{i\}$.

(ii) Since $H(B) \leq b$ and $A(B) \setminus \{i\} \subseteq A(B)$, it follows that $H(A(B) \setminus \{i\}) \leq b$.

Hence $A(B) \setminus \{i\}$ satisfies the conditions for a feasible packing. □ □

**Corollary 2.** *Let $k \in \mathcal{N}$. Then $A_k \setminus \{k\}$ is a feasible packing.*

*Proof.* Let $k \in \mathcal{N}$, and take $B = \{k\}$ in Lemma 6. Then $k \in A(B)$, so $A(B) \neq \emptyset$, and $B$ satisfies Condition 1 of Lemma 6. Also $H(B) = H_k \leq b$ by Assumption 1, so $B$ also satisfies Condition 2 of Lemma 6. Hence $A(B) \setminus \{k\} = A_k \setminus \{k\}$ is a feasible packing. □ □

### 2.3 Conflict Graphs and their Properties

In order to identify potential facet-defining inequalities for $conv(P)$, we require the following definition of a conflict graph for the instance of the PCKP under consideration.

**Definition 4.** *A **conflict graph** $CG = (\mathcal{N}, E)$ contains the edge $\{i, j\} \in E$ if and only if the pair of items $i$ and $j$ **cannot** be included in the knapsack together, that is if and only if $H(\{i, j\}) > b$.*

In all illustrations of a conflict graph throughout this paper, we show only nodes that are not singletons. A **clique** $\mathcal{C} \subseteq \mathcal{N}$ in the conflict graph $CG$ is a set of nodes such that every pair of nodes in $\mathcal{C}$ is joined by an edge. Hence each pair of items in $\mathcal{C}$ cannot be included in the knapsack simultaneously, and it follows that at most one item in $\mathcal{C}$ can be included in the knapsack. A **maximal clique** is a clique that cannot be enlarged by adding any additional node. We now derive technical properties of cliques in the conflict graph, useful in our main results in Section 3.

**Lemma 7.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$. Then for each $i \in \mathcal{C}$, $(A_i \setminus \{i\}) \cap \mathcal{C} = \emptyset$.*

*Proof.* Let $i \in \mathcal{C}$ where $\mathcal{C} \subseteq \mathcal{N}$ is a clique in the conflict graph $CG$. Suppose that there exists $j \in A_i \cap \mathcal{C}$, with $j \neq i$. By Property 1, $A_j \subseteq A_i$ so $A_i \cup A_j = A_i$. By Definition 4, and since $\mathcal{C}$ is a clique, it must be that $H_i = H(\{i, j\}) > b$, which contradicts Assumption 1. Hence $(A_i \setminus \{i\}) \cap \mathcal{C} = \emptyset$. □ □

**Corollary 3.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$. Then for each $i \in \mathcal{C}$, $A_i \cap \mathcal{C} = \{i\}$.*

*Proof.* From Lemma 7 we have that for each $i \in \mathcal{C}$, $(A_i \setminus \{i\}) \cap \mathcal{C} = \emptyset$. Since $i \in \mathcal{C}$, it follows directly that $A_i \cap \mathcal{C} = \{i\}$. □ □

**Lemma 8.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$. Then for each $i \in \mathcal{C}$, $A_i \setminus \{i\} \subseteq A(\mathcal{C}) \setminus \mathcal{C}$.*

*Proof.* Let $i \in \mathcal{C}$ where $\mathcal{C} \subseteq \mathcal{N}$ is a clique in the conflict graph $CG$. We have that $A_i \subseteq A(\mathcal{C})$, so it follows that $A_i \setminus \{i\} \subseteq A(\mathcal{C})$. But $(A_i \setminus \{i\}) \cap \mathcal{C} = \emptyset$ by Lemma 7, and hence $A_i \setminus \{i\} \subseteq A(\mathcal{C}) \setminus \mathcal{C}$. □ □

**Lemma 9.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$. Then for each $k \in A(\mathcal{C}) \setminus \mathcal{C}$, $A_k \subseteq A(\mathcal{C}) \setminus \mathcal{C}$.*

*Proof.* Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$, and let $k \in A(\mathcal{C}) \setminus \mathcal{C}$. Then $k \in A(\mathcal{C})$, and there exists $i \in \mathcal{C}$ such that $k \in A_i$. It follows from Property 1 that $A_k \subseteq A_i$, and since $k \neq i$, we have that $A_k \subseteq A_i \setminus \{i\}$. Suppose that $A_k \nsubseteq A(\mathcal{C}) \setminus \mathcal{C}$. Then there must exist $j \neq i$ such that $j \in A_k \cap \mathcal{C}$. By Property 1 it follows that $A_j \subseteq A_k \subseteq A_i$, so $A_j \cup A_i = A_i$. But $i, j \in \mathcal{C}$, and thus by the definition of $\mathcal{C}$, $H(\{i, j\}) = H_i > b$. This contradicts Assumption 1. Hence $A_k \subseteq A(\mathcal{C}) \setminus \mathcal{C}$. □ □

Along with these properties of the PCKP, we also require general results from polyhedral theory. In particular, we require the following lemma, which is straightforward to prove.

**Lemma 10.** *Let $F$ and $\bar{F}$ be two faces of a non-empty polyhedron $Q$, and let $F \subsetneq \bar{F} \subsetneq Q$. Then $F$ cannot represent a facet of $Q$.*

# 3 Clique-Based Facets for the PCKP Polyhedron

The properties derived in Section 2 are now used to identify facets of $conv(P)$, where $CG = (\mathcal{N}, E)$ is a conflict graph determined according to Definition 4. The following result is obvious.

**Lemma 11.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$. Then the clique inequality*

$$\sum_{j \in \mathcal{C}} x_j \leq 1 \tag{5}$$

*is valid for $P$.*

**Definition 5.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$. Let $\mathcal{P}(\mathcal{C})$ be the set of items in the intersection of the entire precedence sets of all the items in the clique $\mathcal{C}$, that is $\mathcal{P}(\mathcal{C}) = \{k : k \in \cap_{j \in \mathcal{C}} A_j\}$.*

The set $\mathcal{P}(\mathcal{C})$ may or may not be empty. We consider these two cases separately.

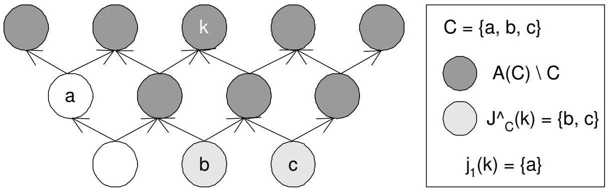


Figure 2: Illustration of the definition of $j_1(k)$ from Condition 1 for Case 1, $\mathcal{P}(\mathcal{C}) = \emptyset$, on the precedence graph $(\mathcal{N}, \mathcal{S})$. Take $a_i = 1$ for all $i \in \mathcal{N}$ and $b = 7$.

### 3.1 Case 1: Empty intersection set, $\mathcal{P}(\mathcal{C}) = \emptyset$

In this case, we are able to determine necessary and sufficient conditions under which (5) is facet-defining for $conv(P)$. We also give a straightforward procedure that, given any maximal clique $\mathcal{C} \subseteq \mathcal{N}$ with $\mathcal{P}(\mathcal{C}) = \emptyset$, can generate a maximal clique satisfying these conditions.

**Definition 6.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$. Let* $F_{\mathcal{C}} = \left\{ x \in conv(P) : \sum_{j \in \mathcal{C}} x_j = 1 \right\}$, *that is, $F_{\mathcal{C}}$ represents the face of $conv(P)$ determined from the valid clique inequality (5).*

The necessary and sufficient conditions on $\mathcal{C}$ so that $F_{\mathcal{C}}$ is facet-defining for $conv(P)$ are given by Condition 1, where $\hat{J}_{\mathcal{C}}(k)$ is defined for all $k \in A(\mathcal{C}) \setminus \mathcal{C}$ according to Definition 1.

**Condition 1.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$. Either $A(\mathcal{C}) \setminus \mathcal{C} = \emptyset$, or for every $k \in A(\mathcal{C}) \setminus \mathcal{C}$ there exists $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$ such that $H(\{j, k\}) \leq b$. Let $j_1(k)$ denote the set of all such $j$.*

See Figure 2 for an illustration of $j_1(k)$ when $A(\mathcal{C}) \setminus \mathcal{C} \neq \emptyset$ in Case 1.

Suppose a maximal clique $\mathcal{C} \subseteq \mathcal{N}$ in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$ is given, and Condition 1 does not hold. The following lemma shows that in this case (5) is redundant in the description of $conv(P)$. It also provides a way to construct another maximal clique $\mathcal{C}'$ from $\mathcal{C}$ for which Condition 1 holds.

**Lemma 12.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$. Suppose Condition 1 does not hold, that is $A(\mathcal{C}) \setminus \mathcal{C} \neq \emptyset$ and for some $k \in A(\mathcal{C}) \setminus \mathcal{C}$, for all $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$, $H(\{j, k\}) > b$. Then $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in $CG$, $\mathcal{P}(\mathcal{C}') = \emptyset$, and the clique inequality (5) for $\mathcal{C}$ is redundant in the description of $conv(P)$.*

*Proof.* Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$. Suppose Condition 1 does not hold, that is $A(\mathcal{C}) \setminus \mathcal{C} \neq \emptyset$ and there exists $k \in A(\mathcal{C}) \setminus \mathcal{C}$ such that for all $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$, $H(\{j, k\}) > b$. Since $\mathcal{P}(\mathcal{C}) = \emptyset$, we have that $\hat{J}_{\mathcal{C}}(k) \subsetneq \mathcal{C}$. Let $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$. Then $|\mathcal{C}'| \geq 2$ and $\mathcal{C}'$ is also a clique in $CG = (\mathcal{N}, E)$. Suppose that $\mathcal{C}'$ is not a maximal clique in $CG$, so there exists $i \notin \mathcal{C}'$ such that $\{i, j\} \in E$ for all $j \in \mathcal{C}'$. Note that $i \notin \hat{J}_{\mathcal{C}}(k)$ since otherwise $k \in A_i$ and $\{i, k\} \notin E$ by Assumption 1. Since $\{i, k\} \in E$ and by Definition 1 we have that $A_k \subseteq A_h$ for all $h \in \hat{J}_{\mathcal{C}}(k)$, it follows that $\{i, h\} \in E$ for all $h \in \hat{J}_{\mathcal{C}}(k)$. So we have $\{i, j\} \in E$ for all $j \in \mathcal{C}' \cup \hat{J}_{\mathcal{C}}(k) = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\} \cup \hat{J}_{\mathcal{C}}(k)$. In particular, $\{i, j\} \in E$ for all $j \in \mathcal{C}$, which contradicts the maximality of $\mathcal{C}$. Hence $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in $CG$.

By definition we have that $k \in A_j$ for all $j \in \hat{J}_{\mathcal{C}}(k)$, and hence by Lemma 3 $A_k \subseteq A_j$ for all $j \in \hat{J}_{\mathcal{C}}(k)$. It follows that $A_k \subseteq \cap_{j \in \hat{J}_{\mathcal{C}}(k)} A_j$. Hence $\mathcal{P}(\mathcal{C}') = \cap_{j \in (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}} A_j = (\cap_{j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)} A_j) \cap A_k \subseteq (\cap_{j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)} A_j) \cap (\cap_{j \in \hat{J}_{\mathcal{C}}(k)} A_j) = \cap_{j \in \mathcal{C}} A_j = \mathcal{P}(\mathcal{C}) = \emptyset$.

Let $H_{\mathcal{C}} = \{x \in \mathbb{R}^{|\mathcal{N}|} : \sum_{j \in \mathcal{C}} x_j = 1\}$ and $H_{\mathcal{C}'} = \{x \in \mathbb{R}^{|\mathcal{N}|} : \sum_{j \in \mathcal{C}'} x_j = 1\}$. Let $I_{\mathcal{C}} = P \cap H_{\mathcal{C}}$ and $I_{\mathcal{C}'} = P \cap H_{\mathcal{C}'}$. We will show that $I_{\mathcal{C}} \subsetneq I_{\mathcal{C}'}$. Let $x \in I_{\mathcal{C}}$. Then there is exactly one $j \in \mathcal{C}$ such that $x_j = 1$. If $j \in \hat{J}_{\mathcal{C}}(k)$ then $k \in A_j$ and $x_k = 1$, implying that $x \in I_{\mathcal{C}'}$. If $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$ then obviously $x \in I_{\mathcal{C}'}$. Hence $I_{\mathcal{C}} \subseteq I_{\mathcal{C}'}$. Now consider the feasible packing $A_k$. Then $x(A_k) \in I_{\mathcal{C}'}$. However, since

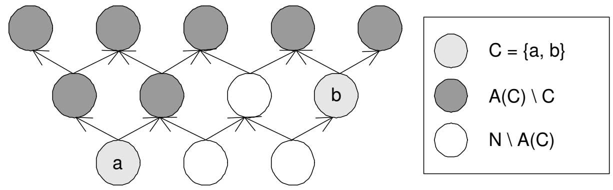


Figure 3: Illustration of the situations for consideration in Case 1, $\mathcal{P}(\mathcal{C}) = \emptyset$.

$k \in A(\mathcal{C}) \setminus \mathcal{C}$, by Lemma 9 we have that $A_k \subseteq A(\mathcal{C}) \setminus \mathcal{C}$, and hence $A_k \cap \mathcal{C} = \emptyset$. Thus $x(A_k) \notin I_\mathcal{C}$. Hence $I_\mathcal{C} \subsetneqq I_{\mathcal{C}'}$. Since $I_\mathcal{C}$ and $I_{\mathcal{C}'}$ are sets of binary vectors, it follows that $conv(I_\mathcal{C}) \subsetneqq conv(I_{\mathcal{C}'})$.

Now observe that $F_\mathcal{C} = conv(P) \cap H_\mathcal{C}$ and $F_{\mathcal{C}'} = conv(P) \cap H_{\mathcal{C}'}$, and furthermore, by Lemma 11, the hyperplanes $H_\mathcal{C}$ and $H_{\mathcal{C}'}$ are defined by *valid* inequalities for $P$. Thus by Lemma 6.1.1 of Balas [1] we have that $conv(I_\mathcal{C}) = conv(P \cap H_\mathcal{C}) = conv(P) \cap H_\mathcal{C} = F_\mathcal{C}$, and that $conv(I_{\mathcal{C}'}) = conv(P \cap H_{\mathcal{C}'}) = conv(P) \cap H_{\mathcal{C}'} = F_{\mathcal{C}'}$, and hence $F_\mathcal{C} \subsetneqq F_{\mathcal{C}'}$. Note also that $0 \in P$ but $0 \notin F_{\mathcal{C}'}$, hence $F_{\mathcal{C}'} \subsetneqq P$. It follows from Lemma 10 that $F_\mathcal{C}$ cannot represent a facet of $conv(P)$, and hence the clique inequality (5) for $\mathcal{C}$ is redundant in the description of $conv(P)$. □ □

We now prove that Condition 1 is necessary and sufficient on $\mathcal{C}$ so that $F_\mathcal{C}$ is facet-defining for $conv(P)$, where $\mathcal{C} \subseteq \mathcal{N}$ is a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$.

**Theorem 1.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$. Then $F_\mathcal{C}$ from Definition 6 is a facet of $conv(P)$ if and only if Condition 1 holds.*

*Proof.* ($\Leftarrow$)

Consider $\mathcal{C} \subseteq \mathcal{N}$ such that $\mathcal{C}$ is a maximal clique in the conflict graph $CG$, $\mathcal{P}(\mathcal{C}) = \emptyset$, and Condition 1 holds. Suppose $\lambda\, x = \lambda_0$ for all $x \in F_\mathcal{C}$ holds for some $(\lambda, \lambda_0)$, where $F_\mathcal{C}$ is given by Definition 6. If it can be shown that $\lambda_k = \lambda_0$ for all $k \in \mathcal{C}$, and $\lambda_k = 0$ otherwise, then by Theorem 3.6 of Nemhauser and Wolsey [5] we will have proved that $F_\mathcal{C}$ represents a facet of $conv(P)$. There are three cases to consider, described in detail below and illustrated in Figure 3.

1. **Case 1(a)**: Let $k \in \mathcal{N} \setminus A(\mathcal{C})$.

   Note that in this case $\hat{J}_\mathcal{C}(k) = \emptyset$. Since $\mathcal{C}$ is maximal there must exist at least one $h \in \mathcal{C}$ such that $H(\{h, k\}) \leq b$. Since $h \in \mathcal{C}$ and $k \in \mathcal{N} \setminus A(\mathcal{C})$, it follows that $h \neq k$. In what follows, we will show that $x(A_h \cup A_k) \in F_\mathcal{C}$ and $x((A_h \cup A_k) \setminus \{k\}) \in F_\mathcal{C}$, and hence deduce that $\lambda_k = 0$.

   We begin by considering $x(A_h \cup A_k)$. By Assumption 1, Corollary 1 and Lemma 1, $A_h \cup A_k$ is a feasible packing and we have

$$x(A_h \cup A_k) = \sum_{j \in A_h \cup A_k} e_j \ \in P.$$

   By Corollary 3, $A_h \cap \mathcal{C} = \{h\}$, and since $k \in \mathcal{N} \setminus A(\mathcal{C})$ it follows that either $A_k \cap \mathcal{C} = \emptyset$, or by Assumption 1 and the definition of the conflict graph $CG$, $A_k \cap \mathcal{C} = \{h\}$. Hence

$$(A_h \cup A_k) \cap \mathcal{C} \ = \ (A_h \cap \mathcal{C}) \cup (A_k \cap \mathcal{C}) \ = \ \{h\}.$$

   Thus $|(A_h \cup A_k) \cap \mathcal{C}| = 1$, and it follows that $x(A_h \cup A_k) \in F_\mathcal{C}$.

   We now consider $x((A_h \cup A_k) \setminus \{k\})$. Since $h \neq k$, we have that $(A_h \cup A_k) \setminus \{k\} = A_h \cup (A_k \setminus \{k\})$. Also $H((A_h \cup A_k) \setminus \{k\}) \leq H(\{h, k\}) \leq b$. By Assumption 1 and Corollary 2, both $A_h$ and $A_k \setminus \{k\}$ are feasible packings, and hence by Lemma 1, $A_h \cup (A_k \setminus \{k\})$ is a feasible packing. So we have

$$x((A_h \cup A_k) \setminus \{k\}) = \sum_{j \in (A_h \cup A_k) \setminus \{k\}} e_j \ \in P.$$

We have shown above that $(A_h \cup A_k) \cap \mathcal{C} = \{h\}$, and since $k \neq h$, it follows that $((A_h \cup A_k) \setminus \{k\}) \cap \mathcal{C} = \{h\}$. So $|((A_h \cup A_k) \setminus \{k\}) \cap \mathcal{C}| = 1$, and we have

$$x((A_h \cup A_k) \setminus \{k\}) \;=\; \sum_{j \in (A_h \cup A_k) \setminus \{k\}} e_j \in F_{\mathcal{C}} \quad \Rightarrow \quad \sum_{j \in (A_h \cup A_k) \setminus \{k\}} \lambda_j \;=\; \lambda_0.$$

Recall that $x(A_h \cup A_k) \in F_{\mathcal{C}}$, so $\sum_{j \in A_h \cup A_k} \lambda_j = \lambda_0$, and hence $\lambda_k = 0$. Since $k \in \mathcal{N} \setminus A(\mathcal{C})$ was chosen arbitrarily, it follows that $\lambda_k = 0$ for all $k \in \mathcal{N} \setminus A(\mathcal{C})$.

2. **Case 1(b)**: Suppose $A(\mathcal{C}) \setminus \mathcal{C} \neq \emptyset$ and let $k \in A(\mathcal{C}) \setminus \mathcal{C}$.

   By Condition 1 there exists $j_1(k) \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$ such that $H(\{j_1(k), k\}) \leq b$. In what follows, we will show that $x(A_{j_1(k)} \cup A_k) \in F_{\mathcal{C}}$ and $x((A_{j_1(k)} \cup A_k) \setminus \{k\}) \in F_{\mathcal{C}}$, and hence deduce that $\lambda_k = 0$.

   We begin by considering $x(A_{j_1(k)} \cup A_k)$. By Assumption 1, Corollary 1 and Lemma 1, $A_{j_1(k)} \cup A_k$ is a feasible packing. Thus $x(A_{j_1(k)} \cup A_k) \in P$. By Corollary 3, $A_{j_1(k)} \cap \mathcal{C} = \{j_1(k)\}$, and since $k \in A(\mathcal{C}) \setminus \mathcal{C}$, by Lemma 9 we have that $A_k \subseteq A(\mathcal{C}) \setminus \mathcal{C}$. Hence $A_k \cap \mathcal{C} = \emptyset$, and it follows that

$$(A_{j_1(k)} \cup A_k) \cap \mathcal{C} \;=\; (A_{j_1(k)} \cap \mathcal{C}) \cup (A_k \cap \mathcal{C}) \;=\; \{j_1(k)\} \cup \emptyset \;=\; \{j_1(k)\}.$$

   Thus $|(A_{j_1(k)} \cup A_k) \cap \mathcal{C}| = 1$ and we have

$$x(A_{j_1(k)} \cup A_k) \quad = \quad \sum_{j \in A_{j_1(k)} \cup A_k} e_j \in F_{\mathcal{C}}.$$

   We now consider $x((A_{j_1(k)} \cup A_k) \setminus \{k\})$. By the definition of $j_1(k)$, we have that $H((A_{j_1(k)} \cup A_k) \setminus \{k\}) \leq H(\{j_1(k), k\}) \leq b$. Note that by Definition 1, $j_1(k) \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$, so $j_1(k) \notin A_k$. Hence $(A_{j_1(k)} \cup A_k) \setminus \{k\} = A_{j_1(k)} \cup (A_k \setminus \{k\})$. By Assumption 1 and Corollary 2, both $A_{j_1(k)}$ and $A_k \setminus \{k\}$ are feasible packings, and by Lemma 1, $A_{j_1(k)} \cup (A_k \setminus \{k\})$ is a feasible packing. So we have

$$x((A_{j_1(k)} \cup A_k) \setminus \{k\}) \quad = \quad \sum_{j \in (A_{j_1(k)} \cup A_k) \setminus \{k\}} e_j \;\in P.$$

   From above we have $(A_{j_1(k)} \cup A_k) \cap \mathcal{C} = \{j_1(k)\}$, and since by definition $j_1(k) \neq k$, it follows that $((A_{j_1(k)} \cup A_k) \setminus \{k\}) \cap \mathcal{C} = \{j_1(k)\}$. Hence $|((A_{j_1(k)} \cup A_k) \setminus \{k\}) \cap \mathcal{C}| = 1$, and it follows that

$$x((A_{j_1(k)} \cup A_k) \setminus \{k\}) \;=\; \sum_{j \in (A_{j_1(k)} \cup A_k) \setminus \{k\}} e_j \in F_{\mathcal{C}} \quad \Rightarrow \quad \sum_{j \in (A_{j_1(k)} \cup A_k) \setminus \{k\}} \lambda_j \;=\; \lambda_0.$$

   Recall that $x(A_{j_1(k)} \cup A_k) \in F_{\mathcal{C}}$, so $\sum_{j \in A_{j_1(k)} \cup A_k} \lambda_j = \lambda_0$, and hence $\lambda_k = 0$. Since $k \in A(\mathcal{C}) \setminus \mathcal{C}$ was chosen arbitrarily, it follows that $\lambda_k = 0$ for all $k \in A(\mathcal{C}) \setminus \mathcal{C}$.

3. **Case 1(c)**: Let $k \in \mathcal{C}$.

   In what follows, we will show that $x(A_k) \in F_{\mathcal{C}}$ and $\lambda_j = 0$ for all $j \in A_k \setminus \{k\}$, and hence deduce that $\lambda_k = \lambda_0$. By Corollary 1, $A_k$ induces a feasible packing and we have

$$x(A_k) = \sum_{j \in A_k} e_j \;\; \in P.$$

Since $k \in \mathcal{C}$, by Corollary 3 $A_k \cap \mathcal{C} = \{k\}$. Hence $|A_k \cap \mathcal{C}| = 1$, and it follows that

$$\begin{aligned} x(A_k) = \sum_{j \in A_k} e_j \in F_{\mathcal{C}} \quad & \Rightarrow \sum_{j \in A_k} \lambda_j \;=\; \lambda_0 \\ & \Rightarrow \sum_{j \in A_k \setminus \{k\}} \lambda_j + \lambda_k \;=\; \lambda_0. \end{aligned} \tag{6}$$

By Lemma 8 we have $A_k \setminus \{k\} \subseteq A(\mathcal{C}) \setminus \mathcal{C}$, and from Case 1(b) we have

$$\lambda_j = 0 \text{ for all } j \in A(\mathcal{C}) \setminus \mathcal{C} \;\Rightarrow\; \lambda_j = 0 \text{ for all } j \in A_k \setminus \{k\}.$$

Hence (6) reduces to $\lambda_k = \lambda_0$. Since $k \in \mathcal{C}$ was chosen arbitrarily, it follows that $\lambda_k = \lambda_0$ for all $k \in \mathcal{C}$.

It has been shown that $\lambda_k = 0$ for all $k \in \mathcal{N} \setminus \mathcal{C}$ and $\lambda_k = \lambda_0$ for all $k \in \mathcal{C}$. Since we assumed that for some $(\lambda, \lambda_0)$, $\lambda\, x = \lambda_0$ for all $x \in F_{\mathcal{C}}$, we have shown that $F_{\mathcal{C}}$ represents a facet of $conv(P)$.

($\Rightarrow$)

Suppose that $\mathcal{C} \subseteq \mathcal{N}$ is a maximal clique in the conflict graph $CG$, $\mathcal{P}(\mathcal{C}) = \emptyset$ and $F_{\mathcal{C}} = \{x \in conv(P) : \sum_{j \in \mathcal{C}} x_j = 1\}$ is a facet of $conv(P)$. Suppose Condition 1 does not hold. Then by Lemma 12 the clique inequality (5) for $\mathcal{C}$ is redundant in the description of $conv(P)$, and hence $F_{\mathcal{C}}$ cannot represent a facet of $conv(P)$, which is a contradiction. □ □

We have shown that Theorem 1 gives necessary and sufficient conditions on $\mathcal{C}$ for $F_{\mathcal{C}}$ from Definition 6 to represent a facet of $conv(P)$ when $\mathcal{C} \subseteq \mathcal{N}$ is a maximal clique in the conflict graph $CG$ and $\mathcal{P}(\mathcal{C}) = \emptyset$.

We now use Lemma 12 to show how, by the application of the following simple procedure, we can generate a maximal clique $\mathcal{C}' \subseteq \mathcal{N}$ that *does* satisfy Condition 1 from a maximal clique $\mathcal{C} \subseteq \mathcal{N}$ with $\mathcal{P}(\mathcal{C}) = \emptyset$ that does not satisfy Condition 1, and hence derive a facet-defining inequality for $conv(P)$.

**Procedure 1.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$, and suppose Condition 1 does not hold. From Lemma 12 it follows that for some $k \in A(\mathcal{C}) \setminus \mathcal{C}$, $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in $CG$ with $\mathcal{P}(\mathcal{C}') = \emptyset$. Replace $\mathcal{C}$ by $\mathcal{C}'$. Repeat this procedure until $\mathcal{C}$ satisfies Condition 1.*

We now prove that Procedure 1 will always terminate with a maximal clique that satisfies Condition 1, and hence yield a clique inequality of the form (5) that defines a facet of $conv(P)$.

**Lemma 13.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$, and suppose Condition 1 does not hold. Then application of Procedure 1 will terminate with a maximal clique $\mathcal{C} \subseteq \mathcal{N}$ with $\mathcal{P}(\mathcal{C}) = \emptyset$ for which Condition 1 does hold.*

*Proof.* Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) = \emptyset$, and suppose Condition 1 does not hold. From Lemma 12 it follows that for some $k \in A(\mathcal{C}) \setminus \mathcal{C}$, $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in $CG$ with $\mathcal{P}(\mathcal{C}') = \emptyset$, and that $F_{\mathcal{C}} \subsetneqq F_{\mathcal{C}'}$. By Proposition 3.1 of Nemhauser and Wolsey [5], we have that for any polyhedron $Q$, the number of distinct faces of $Q$ is finite. Hence it will only be possible to replace $\mathcal{C}$ by $\mathcal{C}'$ a finite number of times before $\mathcal{C}$ satisfies Condition 1. □ □

### 3.2 Case 2: Non-empty intersection set, $\mathcal{P}(\mathcal{C}) \neq \emptyset$

In this case, we are able to determine necessary and sufficient conditions under which a strengthened form of (5) is facet-defining for $conv(P)$. We also give a straightforward procedure that, given any maximal clique $\mathcal{C} \subseteq \mathcal{N}$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, can generate a maximal clique satisfying these conditions. As shown by Lemma 11, for each clique $\mathcal{C} \subseteq \mathcal{N}$ in the conflict graph $CG$, the corresponding clique inequality (5) is valid for $P$. However, in the case where $\mathcal{P}(\mathcal{C}) \neq \emptyset$, it is possible to strengthen this clique inequality as follows.

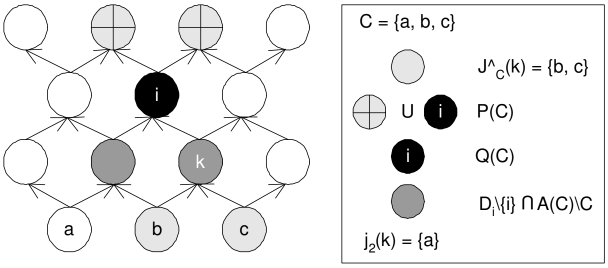


Figure 4: Illustration of the definition of $\mathcal{Q}(\mathcal{C})$ and $j_2(k)$ from Condition 2 for Case 2, $\mathcal{P}(\mathcal{C}) \neq \emptyset$, on the precedence graph $(\mathcal{N}, \mathcal{S})$. Take $a_i = 1$ for all $i \in \mathcal{N}$ and $b = 11$.

**Lemma 14.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph CG with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, and let $i \in \mathcal{P}(\mathcal{C})$. Then the inequality*

$$\sum_{j \in \mathcal{C}} x_j \ \leq \ x_i \tag{7}$$

*is valid for $P$.*

*Proof.* Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, and let $i \in \mathcal{P}(\mathcal{C})$. From Definition 5 we have that $\mathcal{C} \subseteq D_i$. Hence $i \in A_j$ for each $j \in \mathcal{C}$. It follows from the transitivity of the precedence constraints (3) that for all $j \in \mathcal{C}$, $x_j \leq x_i$. Hence if $x_i = 0$ it must be that $x_j = 0$ for all $j \in \mathcal{C}$, and (7) holds. Otherwise, $x_i = 1$ and (7) is equivalent to the clique inequality (5), which is valid for $P$ by Lemma 11. So we have that the strengthened clique inequality (7) is valid for $P$ when $\mathcal{P}(\mathcal{C}) \neq \emptyset$. □ □

**Definition 7.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph CG with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, and let $i \in \mathcal{P}(\mathcal{C})$. Let $F^i_{\mathcal{C}} = \left\{ x \in conv(P) : \sum_{j \in \mathcal{C}} x_j \ = \ x_i \right\}$, that is, $F^i_{\mathcal{C}}$ represents the face of $conv(P)$ determined from the valid inequality (7).*

We now define the set $\mathcal{Q}(\mathcal{C})$ which will be used throughout this section.

**Definition 8.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a clique in the conflict graph CG, and let $\mathcal{P}(\mathcal{C})$ be determined according to Definition 5. Let $\mathcal{Q}(\mathcal{C}) = \{i \in \mathcal{P}(\mathcal{C}) : \mathcal{C} \not\subseteq D_k$ for all $k \in D_i \setminus \{i\}\}$. That is, $\mathcal{Q}(\mathcal{C})$ is the set of items that lie in the intersection of the entire precedence sets of all the items in the clique $\mathcal{C}$, with no items in their successor sets $D_i$ that satisfy the same property.*

See Figure 4 for an illustration of a set $\mathcal{Q}(\mathcal{C})$. Note that if $|\mathcal{P}(\mathcal{C})| = 1$, then $\mathcal{Q}(\mathcal{C}) = \mathcal{P}(\mathcal{C})$. As we will show, the following condition is necessary and sufficient on $\mathcal{C}$ so that $F^i_{\mathcal{C}}$ is facet-defining for $conv(P)$, where $\hat{J}_{\mathcal{C}}(k)$ is defined for all $k \in A(\mathcal{C}) \setminus \mathcal{C}$ according to Definition 1.

**Condition 2.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph CG with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $\mathcal{Q}(\mathcal{C})$ be determined according to Definition 8, and let $i \in \mathcal{Q}(\mathcal{C})$. Either $(D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C}) = \emptyset$, or for every $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$ there exists $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$ such that $H(\{j, k\}) \leq b$. Let $j_2(k)$ denote the set of all such $j$.*

See Figure 4 for an illustration of $j_2(k)$ when $(D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C}) \neq \emptyset$.

Suppose a maximal clique $\mathcal{C} \subseteq \mathcal{N}$ in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$ is given, with $\mathcal{Q}(\mathcal{C})$ determined according to Definition 8, $i \in \mathcal{Q}(\mathcal{C})$, and suppose Condition 2 does not hold. The following lemma shows that in this case (7) is redundant in the description of $conv(P)$. It also provides a way to construct another maximal clique $\mathcal{C}'$ from $\mathcal{C}$ for which Condition 2 holds.

**Lemma 15.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph CG with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $\mathcal{Q}(\mathcal{C})$ be determined according to Definition 8, and let $i \in \mathcal{Q}(\mathcal{C})$. Suppose Condition 2 does not hold. Then*

*$(D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C}) \neq \emptyset$ and for some $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$, for all $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$, $H(\{j,k\}) > b$. Then $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in $CG$. Furthermore, $i \in \mathcal{Q}(\mathcal{C}')$, and the strengthened clique inequality (7) for $\mathcal{C}$ and item $i$ is redundant in the description of $conv(P)$.*

*Proof.* Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $\mathcal{Q}(\mathcal{C})$ be determined according to Definition 8, and let $i \in \mathcal{Q}(\mathcal{C})$. Suppose Condition 2 does not hold. Then $(D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C}) \neq \emptyset$ and there exists $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$ such that for all $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$, $H(\{j,k\}) > b$. From the definition of $\mathcal{Q}(\mathcal{C})$ we have that $\mathcal{C} \not\subseteq D_k$ since $k \in D_i \setminus \{i\}$. Hence $\hat{J}_{\mathcal{C}}(k) \subsetneqq \mathcal{C}$. Let $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$. Then $|\mathcal{C}'| \geq 2$ and $\mathcal{C}'$ is also a clique in $CG = (\mathcal{N}, E)$. Suppose that $\mathcal{C}'$ is not a maximal clique in $CG$, so there exists $m \notin \mathcal{C}'$ such that $\{m, l\} \in E$ for all $l \in \mathcal{C}'$. Note that $m \notin \hat{J}_{\mathcal{C}}(k)$ since otherwise $k \in A_m$ and $\{m, k\} \notin E$ by Assumption 1. Since $\{m, k\} \in E$ and by Definition 1 we have that $A_k \subseteq A_h$ for all $h \in \hat{J}_{\mathcal{C}}(k)$, it follows that $\{m, h\} \in E$ for all $h \in \hat{J}_{\mathcal{C}}(k)$. So we have $\{m, l\} \in E$ for all $l \in \mathcal{C}' \cup \hat{J}_{\mathcal{C}}(k) = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\} \cup \hat{J}_{\mathcal{C}}(k)$. In particular $\{m, l\} \in E$ for all $l \in \mathcal{C}$, which contradicts the maximality of $\mathcal{C}$. Hence $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in $CG$.

We now show that $i \in \mathcal{Q}(\mathcal{C}')$. First, we show that $i \in P(\mathcal{C}')$. To begin, $i \in P(\mathcal{C})$ so $i \in A_j$ for all $j \in \mathcal{C}$, and hence for all $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$. Furthermore $k \in D_i$ so $i \in A_k$ by the definition of descendent sets. Thus $i \in A_j$ for all $j \in \mathcal{C}'$, i.e. $i \in P(\mathcal{C}')$. Second, we suppose that $i \notin \mathcal{Q}(\mathcal{C}')$, and deduce a contradiction. If $i \notin \mathcal{Q}(\mathcal{C}')$ then there must exist $h \in D_i \setminus \{i\}$ such that $D_h \supseteq \mathcal{C}'$. But since $i \in \mathcal{Q}(\mathcal{C})$, we know that $D_h \not\supseteq \mathcal{C}$, so it must be that $D_h \not\supseteq \hat{J}_{\mathcal{C}}(k)$. Now $k \in \mathcal{C}' \subseteq D_h$ so $D_k \subseteq D_h$ by Property 1 and the definition of descendent sets. But $\hat{J}_{\mathcal{C}}(k) \subseteq D_k$, which contradicts $D_h \not\supseteq \hat{J}_{\mathcal{C}}(k)$. Thus it must be that $i \in \mathcal{Q}(\mathcal{C}')$ as required.

Let $H^i_{\mathcal{C}} = \{x \in \mathbb{R}^{|\mathcal{N}|} : \sum_{j \in \mathcal{C}} x_j = x_i\}$ and $H^i_{\mathcal{C}'} = \{x \in \mathbb{R}^{|\mathcal{N}|} : \sum_{j \in \mathcal{C}'} x_j = x_i\}$. Let $I^i_{\mathcal{C}} = P \cap H^i_{\mathcal{C}}$ and $I^i_{\mathcal{C}'} = P \cap H^i_{\mathcal{C}'}$. We will show that $I^i_{\mathcal{C}} \subsetneqq I^i_{\mathcal{C}'}$. Let $x \in I^i_{\mathcal{C}}$. Then either $x_i = 0$ or $x_i = 1$. Consider first $x_i = 0$. By the validity of (7) $x_j = 0$ for all $j \in \mathcal{C}$. Similarly, since $k \in D_i$ it follows that $i \in A_k$ and hence $x_k = 0$. Thus $x \in I^i_{\mathcal{C}'}$. Now consider the case $x_i = 1$. Then there is exactly one $j \in \mathcal{C}$ such that $x_j = 1$. If $j \in \hat{J}_{\mathcal{C}}(k)$ then $k \in A_j$ and $x_k = 1$, implying that $x \in I^i_{\mathcal{C}'}$. If $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$ then obviously $x \in I^i_{\mathcal{C}'}$. Hence $I^i_{\mathcal{C}} \subseteq I^i_{\mathcal{C}'}$. Now consider the feasible packing $A_k$. Since $k \in D_i$, we have that $i \in A_k$ and hence $x(A_k) \in I^i_{\mathcal{C}'}$. However, since $k \in A(\mathcal{C}) \setminus \mathcal{C}$, by Lemma 9 $A_k \subseteq A(\mathcal{C}) \setminus \mathcal{C}$, and hence $A_k \cap \mathcal{C} = \emptyset$. Thus $x(A_k) \notin I^i_{\mathcal{C}}$. Hence $I^i_{\mathcal{C}} \subsetneqq I^i_{\mathcal{C}'}$. Since $I^i_{\mathcal{C}}$ and $I^i_{\mathcal{C}'}$ are sets of binary vectors, it follows that $conv(I^i_{\mathcal{C}}) \subsetneqq conv(I^i_{\mathcal{C}'})$.

Now observe that $F^i_{\mathcal{C}} = conv(P) \cap H^i_{\mathcal{C}}$ and $F^i_{\mathcal{C}'} = conv(P) \cap H^i_{\mathcal{C}'}$, and furthermore by Lemma 14 the hyperplanes $H^i_{\mathcal{C}}$ and $H^i_{\mathcal{C}'}$ are defined by *valid* inequalities for $P$. Thus by Lemma 6.1.1 of Balas [1] we have that $conv(I^i_{\mathcal{C}}) = conv(P \cap H^i_{\mathcal{C}}) = conv(P) \cap H^i_{\mathcal{C}} = F^i_{\mathcal{C}}$, and that $conv(I^i_{\mathcal{C}'}) = conv(P \cap H^i_{\mathcal{C}'}) = conv(P) \cap H^i_{\mathcal{C}'} = F^i_{\mathcal{C}'}$, and hence $F^i_{\mathcal{C}} \subsetneqq F^i_{\mathcal{C}'}$. Now consider the feasible packing $A_i$. Then $x(A_i) \in P$ but since $j \in D_i$ for all $j \in \mathcal{C}'$, $x(A_i) \notin F^i_{\mathcal{C}'}$, and hence $F^i_{\mathcal{C}'} \subsetneqq P$. It follows from Lemma 10 that $F^i_{\mathcal{C}}$ cannot represent a facet of $conv(P)$, and hence the clique inequality (7) for $\mathcal{C}$ and item $i$ is redundant in the description of $conv(P)$. □ □

We now prove that Condition 2 is necessary and sufficient on $\mathcal{C}$ so that $F^i_{\mathcal{C}}$ is facet-defining for $conv(P)$ where $\mathcal{C} \subseteq \mathcal{N}$ is a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$.

**Theorem 2.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $\mathcal{Q}(\mathcal{C})$ be determined according to Definition 8, and let $i \in \mathcal{Q}(\mathcal{C})$. Then $F^i_{\mathcal{C}}$ from Definition 7 is a facet of $conv(P)$ if and only if Condition 2 holds.*

*Proof.* ($\Leftarrow$)

Consider $\mathcal{C} \subseteq \mathcal{N}$ such that $\mathcal{C}$ is a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, $\mathcal{Q}(\mathcal{C})$ is defined as in Definition 8, $i \in \mathcal{Q}(\mathcal{C})$, and Condition 2 holds. Suppose that $\lambda\, x = \lambda_0$ for all $x \in F^i_{\mathcal{C}}$, where $F^i_{\mathcal{C}}$ is given by Definition 7. Note that the zero vector induces a feasible packing, since it is always feasible to have an empty knapsack. Hence $0 \in P$. In this case, $0 \in F^i_{\mathcal{C}}$ as well, and hence $\lambda_0 = 0$. Thus if it can be shown that $\lambda_k = -\lambda_i$ for all $k \in \mathcal{C}$, and $\lambda_k = 0$ for all $k \in \mathcal{N} \setminus (\mathcal{C} \cup \{i\})$, then by Theorem 3.6 of Nemhauser and Wolsey [5] we will have proved that $F^i_{\mathcal{C}}$ represents a facet of $conv(P)$. There are four cases to consider, described in detail below and illustrated in Figure 5.

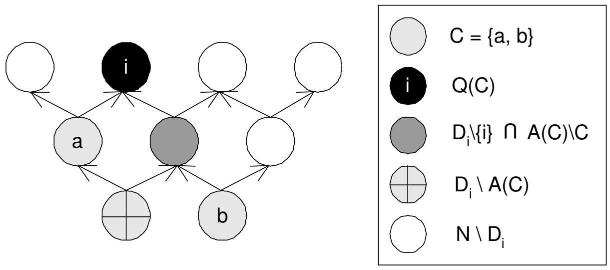


Figure 5: Illustration of the situations for consideration in Case 2, $\mathcal{P}(\mathcal{C}) \neq \emptyset$.

1. **Case 2(a)**: Let $k \in \mathcal{N} \setminus D_i$.

   In what follows, we will show that $x(A_k) \in F^i_{\mathcal{C}}$ and $x(A_k \setminus \{k\}) \in F^i_{\mathcal{C}}$, and hence deduce that $\lambda_k = 0$. We begin by considering $x(A_k)$. By Corollary 1 $A_k$ induces a feasible packing and we have

   $$x(A_k) = \sum_{j \in A_k} e_j \quad \in P.$$

   Since $k \notin D_i$, by Lemma 4 we have that $A_k \subseteq \mathcal{N} \setminus D_i$. From the definition of $\mathcal{Q}(\mathcal{C})$ we also have that $\mathcal{C} \subseteq D_i$, and thus $A_k \cap \mathcal{C} = \emptyset$. From the definition of $D_i$ it follows that since $k \notin D_i$, $i \notin A_k$, and we have $x(A_k) \in F^i_{\mathcal{C}}$.

   We now consider $x(A_k \setminus \{k\})$. By Corollary 2 we have that $A_k \setminus \{k\}$ is a feasible packing. Hence

   $$x(A_k \setminus \{k\}) \;=\; \sum_{j \in A_k \setminus \{k\}} e_j \quad \in P.$$

   From above we have that $A_k \cap \mathcal{C} = \emptyset$, and it follows directly that $(A_k \setminus \{k\}) \cap \mathcal{C} = \emptyset$. Since $i \notin A_k$ we also have that $i \notin A_k \setminus \{k\}$. Hence

   $$x(A_k \setminus \{k\}) \;=\; \sum_{j \in A_k \setminus \{k\}} e_j \in F^i_{\mathcal{C}} \qquad \Rightarrow \sum_{j \in A_k \setminus \{k\}} \lambda_j \;=\; \lambda_0 \;=\; 0.$$

   Recall that $x(A_k) \in F^i_{\mathcal{C}}$, so $\sum_{j \in A_k} \lambda_j = \lambda_0 = 0$, and hence $\lambda_k = 0$. Since $k \in \mathcal{N} \setminus D_i$ was chosen arbitrarily, it follows that $\lambda_k = 0$ for all $k \in \mathcal{N} \setminus D_i$.

2. **Case 2(b)**: Suppose $(D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C}) \neq \emptyset$ and let $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$.

   By Condition 2 there exists $j_2(k) \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$ such that $H(\{j_2(k), k\}) \leq b$. In what follows, we will show that $x(A_{j_2(k)} \cup A_k) \in F^i_{\mathcal{C}}$ and $x((A_{j_2(k)} \cup A_k) \setminus \{k\}) \in F^i_{\mathcal{C}}$, and hence deduce that $\lambda_k = 0$.

   We begin by considering $x(A_{j_2(k)} \cup A_k)$. By Assumption 1, Corollary 1 and Lemma 1, we have that $A_{j_2(k)} \cup A_k$ is a feasible packing. Hence $x(A_{j_2(k)} \cup A_k) \in P$. By Corollary 3, $A_{j_2(k)} \cap \mathcal{C} = \{j_2(k)\}$, and since $k \in A(\mathcal{C}) \setminus \mathcal{C}$, by Lemma 9 we have that $A_k \subseteq A(\mathcal{C}) \setminus \mathcal{C}$. Hence $A_k \cap \mathcal{C} = \emptyset$, and it follows that

   $$(A_{j_2(k)} \cup A_k) \cap \mathcal{C} \;=\; (A_{j_2(k)} \cap \mathcal{C}) \cup (A_k \cap \mathcal{C}) \;=\; \{j_2(k)\} \cup \emptyset \;=\; \{j_2(k)\}.$$

   By the definition of $\mathcal{Q}(\mathcal{C})$, $\mathcal{C} \subseteq D_i$ and thus $j_2(k) \in D_i$. From the definition of $D_i$ it follows that $i \in A_{j_2(k)}$. Hence $|(A_{j_2(k)} \cup A_k) \cap \mathcal{C}| = 1$ and $i \in A_{j_2(k)}$, and we have

   $$x(A_{j_2(k)} \cup A_k) \;=\; \sum_{j \in A_{j_2(k)} \cup A_k} e_j \in F^i_{\mathcal{C}}.$$

We now consider $x((A_{j_2(k)} \cup A_k) \setminus \{k\})$. By the definition of $j_2(k)$, we have that $H((A_{j_2(k)} \cup A_k) \setminus \{k\}) \leq H(\{j_2(k), k\}) \leq b$. Note that by Condition 2 $j_2(k) \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$, so it follows from Definition 1 that $k \notin A_{j_2(k)}$. Hence $(A_{j_2(k)} \cup A_k) \setminus \{k\} = A_{j_2(k)} \cup (A_k \setminus \{k\})$. By Assumption 1 and Corollary 2, both $A_{j_2(k)}$ and $A_k \setminus \{k\}$ are feasible packings, and by Lemma 1, $A_{j_2(k)} \cup (A_k \setminus \{k\})$ is a feasible packing. So we have

$$x((A_{j_2(k)} \cup A_k) \setminus \{k\}) \;=\; \sum_{j \in (A_{j_2(k)} \cup A_k) \setminus \{k\}} e_j \;\in P.$$

From above we have that $(A_{j_2(k)} \cup A_k) \cap \mathcal{C} = \{j_2(k)\}$, and since by definition $j_2(k) \neq k$, it follows that $((A_{j_2(k)} \cup A_k) \setminus \{k\}) \cap \mathcal{C} = \{j_2(k)\}$. Hence $|((A_{j_2(k)} \cup A_k) \setminus \{k\}) \cap \mathcal{C}| = 1$, and since $i \in A_{j_2(k)}$, it follows that

$$x((A_{j_2(k)} \cup A_k) \setminus \{k\}) \;=\; \sum_{j \in (A_{j_2(k)} \cup A_k) \setminus \{k\}} e_j \in F^i_{\mathcal{C}} \quad \Rightarrow \quad \sum_{j \in (A_{j_2(k)} \cup A_k) \setminus \{k\}} \lambda_j \;=\; \lambda_0 \;=\; 0.$$

Recall that $x(A_{j_2(k)} \cup A_k) \in F^i_{\mathcal{C}}$, so $\sum_{j \in A_{j_2(k)} \cup A_k} \lambda_j = \lambda_0 = 0$, and hence $\lambda_k = 0$. Since $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$ was chosen arbitrarily, it follows that $\lambda_k = 0$ for all $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$.

3. **Case 2(c)**: Let $k \in D_i \setminus A(\mathcal{C})$.

   Note that since $i \in A(\mathcal{C})$ we have that $k \neq i$. Since $\mathcal{C}$ is maximal there must exist at least one $h \in \mathcal{C}$ such that $H(\{h, k\}) \leq b$. In what follows, we will show that $x(A_h \cup A_k) \in F^i_{\mathcal{C}}$ and $x((A_h \cup A_k) \setminus \{k\}) \in F^i_{\mathcal{C}}$, and hence deduce that $\lambda_k = 0$.

   We begin by considering $x(A_h \cup A_k)$. By Corollary 1 both $A_h$ and $A_k$ induce a feasible packing, and it follows from Lemma 1 that $A_h \cup A_k$ is a feasible packing. So we have

$$x(A_h \cup A_k) = \sum_{j \in A_h \cup A_k} e_j \;\in P.$$

   Since $h \in \mathcal{C}$, $i \in A_h$ and $k \notin A(\mathcal{C})$, it follows directly from Assumption 1 and the definition of the conflict graph $CG$ that $x(A_h \cup A_k) \in F^i_{\mathcal{C}}$.

   We now consider $x((A_h \cup A_k) \setminus \{k\})$. Since $k \notin A(\mathcal{C})$ and $h \in \mathcal{C}$ we have that $k \notin A_h$, and hence $(A_h \cup A_k) \setminus \{k\} = A_h \cup (A_k \setminus \{k\})$. Also $H((A_h \cup A_k) \setminus \{k\}) \leq H(\{h, k\}) \leq b$. By Assumption 1 and Corollary 2, both $A_h$ and $A_k \setminus \{k\}$ are feasible packings, and by Lemma 1 $A_h \cup (A_k \setminus \{k\})$ is a feasible packing. So we have

$$x((A_h \cup A_k) \setminus \{k\}) = \sum_{j \in (A_h \cup A_k) \setminus \{k\}} e_j \;\in P.$$

   Since $h \in \mathcal{C}$, $i \in A_h$ and $k \neq h$ it follows that

$$x((A_h \cup A_k) \setminus \{k\}) \;\in F^i_{\mathcal{C}} \quad \Rightarrow \quad \sum_{j \in (A_h \cup A_k) \setminus \{k\}} \lambda_j \;=\; \lambda_0 \;= 0.$$

   Recall that $x(A_h \cup A_k) \in F^i_{\mathcal{C}}$, so $\sum_{j \in A_h \cup A_k} \lambda_j = \lambda_0 = 0$, and hence $\lambda_k = 0$. Since $k \in D_i \setminus A(\mathcal{C})$ was chosen arbitrarily, it follows that $\lambda_k = 0$ for all $k \in D_i \setminus A(\mathcal{C})$.

4. **Case 2(d)**: Let $k \in \mathcal{C}$.

In what follows, we will show that $x(A_k) \in F^i_{\mathcal{C}}$ and $\lambda_j = 0$ for all $j \in A(\mathcal{C}) \setminus (\mathcal{C} \cup \{i\})$, and hence deduce that $\lambda_k = -\lambda_i$. By Corollary 1, $A_k$ induces a feasible packing and we have

$$x(A_k) = \sum_{j \in A_k} e_j \quad \in P.$$

Since $k \in \mathcal{C}$, from Corollary 3 $A_k \cap \mathcal{C} = \{k\}$, and $|A_k \cap \mathcal{C}| = 1$. By the definition of $\mathcal{Q}(\mathcal{C})$, $\mathcal{C} \subseteq D_i$ and thus $k \in D_i$. From the definition of $D_i$ it follows that $i \in A_k$, and we have

$$x(A_k) = \sum_{j \in A_k} e_j \in F^i_{\mathcal{C}} \qquad \Rightarrow \sum_{j \in A_k} \lambda_j \ = \ \lambda_0 \ = \ 0. \tag{8}$$

By Lemma 8 we have $A_k \setminus \{k\} \subseteq A(\mathcal{C}) \setminus \mathcal{C}$. From Case 2(a) we have that $\lambda_j = 0$ for all $j \in \mathcal{N} \setminus D_i$ and from Case 2(b) we have that $\lambda_j = 0$ for all $j \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$. Hence $\lambda_j = 0$ for all $j \in A(\mathcal{C}) \setminus (\mathcal{C} \cup \{i\})$, and (8) reduces to

$$\lambda_k + \lambda_i \ = \ \lambda_0 \ = \ 0 \quad \Rightarrow \quad \lambda_k \ = \ -\lambda_i.$$

Since $k \in \mathcal{C}$ was chosen arbitrarily, it follows that $\lambda_k = -\lambda_i$ for all $k \in \mathcal{C}$.

It has been shown that $\lambda_k = 0$ for all $k \in \mathcal{N} \setminus (\mathcal{C} \cup \{i\})$ and $\lambda_k = -\lambda_i$ for all $k \in \mathcal{C}$. Since we assumed that for some $(\lambda, \lambda_0)$, $\lambda\, x = \lambda_0$ for all $x \in F^i_{\mathcal{C}}$, we have shown that $F^i_{\mathcal{C}}$ represents a facet of $conv(P)$.

($\Rightarrow$)

Suppose that $\mathcal{C} \subseteq \mathcal{N}$ is a maximal clique in the conflict graph $CG$, $\mathcal{P}(\mathcal{C}) \neq \emptyset$, $\mathcal{Q}(\mathcal{C})$ has been determined according to Definition 8, $i \in \mathcal{Q}(\mathcal{C})$, and $F^i_{\mathcal{C}} = \{x \in conv(P) : \sum_{j \in \mathcal{C}} x_j = x_i\}$ is a facet of $conv(P)$. Suppose Condition 2 does not hold. Then by Lemma 15 the inequality (7) for $\mathcal{C}$ and item $i$ is redundant in the description of $conv(P)$, and hence $F^i_{\mathcal{C}}$ cannot represent a facet of $conv(P)$, which is a contradiction. □ □

We have shown that Theorem 2 gives necessary and sufficient conditions on $\mathcal{C}$ for $F^i_{\mathcal{C}}$ given by Definition 7 to represent a facet of $conv(P)$ when $\mathcal{C} \subseteq \mathcal{N}$ is a maximal clique in the conflict graph $CG$, $\mathcal{P}(\mathcal{C}) \neq \emptyset$, $\mathcal{Q}(\mathcal{C})$ has been determined according to Definition 8, and $i \in \mathcal{Q}(\mathcal{C})$.

We now use Lemma 15 to show how, by the application of the following simple procedure, we can generate a maximal clique $\mathcal{C}' \subseteq \mathcal{N}$ that *does* satisfy Condition 2 from a maximal clique $\mathcal{C} \subseteq \mathcal{N}$ that does not satisfy Condition 2, and hence derive a facet-defining inequality for $conv(P)$.

**Procedure 2.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $\mathcal{Q}(\mathcal{C})$ be determined according to Definition 8, and let $i \in \mathcal{Q}(\mathcal{C})$. Suppose Condition 2 does not hold. From Lemma 15 it follows that for some $k \in (A(\mathcal{C}) \setminus \mathcal{C}) \cap (D_i \setminus \{i\})$, $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in $CG$. Replace $\mathcal{C}$ by $\mathcal{C}'$. Repeat this procedure until $\mathcal{C}$ satisfies Condition 2.*

We now prove that Procedure 2 will always terminate with a maximal clique that satisfies Condition 2, and hence yield a clique inequality of the form (7) that defines a facet of $conv(P)$.

**Lemma 16.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $\mathcal{Q}(\mathcal{C})$ be determined according to Definition 8, and let $i \in \mathcal{Q}(\mathcal{C})$. Suppose Condition 2 does not hold. Then application of Procedure 2 will terminate with a maximal clique $\mathcal{C} \subseteq \mathcal{N}$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$ for which Condition 2 does hold.*

*Proof.* Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$ with $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $\mathcal{Q}(\mathcal{C})$ be determined according to Definition 8, and let $i \in \mathcal{Q}(\mathcal{C})$. Suppose Condition 2 does not hold. From Lemma 15 it follows that for some $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$, $\mathcal{C}' = (\mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)) \cup \{k\}$ is also a maximal clique in the conflict graph $CG$, $i \in \mathcal{Q}(\mathcal{C}')$, and that $F^i_{\mathcal{C}} \subsetneq F^i_{\mathcal{C}'}$. By Proposition 3.1 of Nemhauser and Wolsey [5] we have that for any polyhedron $Q$, the number of distinct faces of $Q$ is finite. Hence it will only be possible to replace $\mathcal{C}$ by $\mathcal{C}'$ a finite number of times before $\mathcal{C}$ satisfies Condition 2. □ □

We now consider the polyhedral investigations of the PCKP carried out by other authors, and give a comparison of the different approaches.

## 4 Cover-Based Polyhedral Approaches to the PCKP

Investigation of the polyhedral structure of the PCKP has previously been carried out by Boyd [2], Park and Park [6] and van de Leensel et. al. [9]. All of these authors consider the derivation of strong inequalities for $P$ by applying lifting procedures to valid knapsack cover-based inequalities. We now consider some of the knapsack cover-based inequalities studied by these authors, and compare them to the clique inequalities introduced in Section 3. We begin by giving a summary of the relevant PCKP terminology and the main cover-based results.

### 4.1 PCKP Terminology and Cover-Based Results

The basic terminology for the PCKP used by Boyd [2], Park and Park [6] and van de Leensel et. al. [9] is as follows. Two items $i,\ j \in \mathcal{N}$ are called **incomparable** if $i \notin A_j$ and $j \notin A_i$. A set $B \subseteq \mathcal{N}$ is **incomparable** if the elements of $B$ are pairwise incomparable. A set $C \subseteq \mathcal{N}$ is a **cover** if $H(C) > b$. A cover is a **minimal cover** if no proper subset of it is a cover. It is obvious that for any cover $C \subseteq \mathcal{N}$ the inequality

$$\sum_{j \in C} x_j \ \leq \ |C| - 1 \tag{9}$$

is valid for $P$.

Park and Park [6] define a cover $C \subseteq \mathcal{N}$ to be an **induced cover (IC)** if $C$ is incomparable. An induced cover is a **minimal induced cover (MIC)** if $H(C \setminus \{i\}) \leq b$ for all $i \in C$, in which case the associated cover inequality (9) is called an **MIC inequality**. This definition differs from that used by Boyd [2] and van de Leensel et. al. [9]. Boyd [2] defines a cover $C \subseteq \mathcal{N}$ to be minimal if $C$ is incomparable and $H(C) - a_i \leq b$ for all $i \in \mathcal{C}$. We shall call such a cover a **Boyd minimal cover (BMC)**, and the associated cover inequality a **BMC inequality**. As Park and Park [6] note, in general, a BMC is also an MIC, but the converse does not hold.

Boyd [2] and van de Leensel et. al. [9] also consider a generalized version of a cover. We define a **$K$-cover** to be a cover $C \subseteq \mathcal{N}$ with the property that for all $B \subseteq C$ with $|B| = K$, $B$ is a cover. It is obvious that if $C$ is a $K$-cover then

$$\sum_{j \in C} x_j \ \leq \ K - 1 \tag{10}$$

is valid for $P$. In fact, both Boyd and van de Leensel et. al. fail to observe this explicitly; their definition requires $C$ to be incomparable and $B$ to be a BMC for all $B \subseteq C$ with $|B| = K$. We will refer to such a $C$ as a **$K$-Boyd minimal cover ($K$-BMC)**. Boyd and van de Leensel et. al. do not seek results for $K$-covers under looser conditions. Note that $C$ a $|C|$-cover is simply a cover, and $C$ a $|C|$-BMC is a BMC.

Van de Leensel et. al. [9] are the only authors known to us who show how to derive facet-defining inequalities for $conv(P)$. They redefine the term minimal induced cover to coincide with the definition of a Boyd minimal cover, and consider BMCs (which they call MICs) in their study of the PCKP polyhedron $P$. Throughout this paper we will distinguish between MICs and BMCs. Van de Leensel et. al. [9] consider BMCs and $K$-BMCs separately, and for both cases develop a general sequential lifting procedure to lift the ($K$-)cover inequality to a facet of $conv(P)$. The resulting facet-defining inequality takes the form

$$\sum_{i \in C} x_i \ + \sum_{i \in A(C) \setminus C} \alpha_i (1 - x_i) \ + \sum_{i \in \mathcal{N} \setminus A(C)} \alpha_i x_i \ \leq \ K - 1, \tag{11}$$

where of course $K = |C|$ in the case of a BMC.

They note that in general, calculating each lifting coefficient $\alpha_i$ requires the solution of a separate PCKP. Further, they show that determining the maximal lifting coefficients for the items in the set $\mathcal{N} \setminus A(C)$ is NP-complete in the strong sense. Finally, a polynomial time algorithm for determining the maximal lifting coefficients for items in the set $A(C) \setminus C$ is presented. This algorithm applies only in the special case $K = |C|$. We will use it to determine strengthened BMC inequalities in examples in Section 5.

Park and Park [6] consider MICs $C \subseteq \mathcal{N}$, and present a heuristic for determining lifting coefficients for items in the set $A(C) \setminus C$ to strengthen the MIC inequality (9). They show that under certain conditions, this lifted inequality is facet-defining for the lower-dimensional polyhedron $P(C)$, defined as $P(C) = conv(proj_{A(C)}\{x(D) \in P : D \subseteq A(C)\})$ for any incomparable set $C \subseteq \mathcal{N}$. That is, $P(C)$ is the convex hull of $P$ restricted to those variables in $A(C)$. No further significant results for MICs have been developed.

For our investigation, we extend the concept of an MIC in a similar manner to that used by Boyd [2] and van de Leensel et. al. [9] for BMCs. A set $C \subseteq \mathcal{N}$ is a $K$**-MIC** if $C$ is incomparable, and for all $B \subseteq C$ with $|B| = K$, $B$ is an MIC, in which case we call the corresponding $K$-cover inequality (10) a $K$**-MIC inequality**. As in the case of BMCs, MICs are special cases of $K$-MICs. Hence we consider only $K$-BMCs and $K$-MICs from here forward. Note that the concept of a $K$-MIC has not been investigated by any previous author.

### 4.2 Comparison of Covers and Cliques in the Conflict Graph

Consider an instance of (PCKP) and let $CG = (\mathcal{N}, E)$ be a conflict graph determined according to Definition 4. We now compare the covers investigated by Boyd [2], Park and Park [6] and van de Leensel et. al. [9] with cliques in the conflict graph $CG$. Of particular interest in our investigation are 2-BMCs and 2-MICs: there is a precise correspondence between 2-MICs and cliques in the conflict graph $CG$. The following results are straightforward to prove.

**Lemma 17.** *Let $\mathcal{C} \subseteq \mathcal{N}$.*

1. *$\mathcal{C}$ is a 2-MIC if and only if $\mathcal{C}$ is a clique in the conflict graph.*
2. *If $\mathcal{C}$ is a $K$-BMC then $\mathcal{C}$ is a $K$-MIC.*
3. *If $\mathcal{C}$ is a $K$-MIC then $\mathcal{C}$ is not a $k$-MIC for any $k \neq K$.*

**Corollary 4.** *Let $\mathcal{C} \subseteq \mathcal{N}$. If $\mathcal{C}$ is a clique in the conflict graph then*

*(i) $\mathcal{C}$ **cannot** be a $K$-MIC where $K > 2$; and*

*(ii) $\mathcal{C}$ **cannot** be a $K$-BMC where $K > 2$.*

We use Lemma 17 and Corollary 4 to demonstrate where cliques determined from the conflict graph $CG$ fit into the set of all $K$-covers; these results justify everything in Figure 6 except for the placement of the set of maximal cliques that satisfy Conditions 1 or 2. In fact, this, too, is justified, as we will now show.

**Lemma 18.** *Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$. If $\mathcal{C}$ is also a 2-BMC, then either $\mathcal{P}(\mathcal{C}) = \emptyset$ and Condition 1 is satisfied, or $\mathcal{P}(\mathcal{C}) \neq \emptyset$ and Condition 2 is satisfied.*

*Proof.* Let $\mathcal{C} \subseteq \mathcal{N}$ be a maximal clique in the conflict graph $CG$, and suppose $\mathcal{C}$ is also a 2-BMC.

If $\mathcal{P}(\mathcal{C}) = \emptyset$, then suppose $A(\mathcal{C}) \setminus \mathcal{C} \neq \emptyset$ (otherwise Condition 1 holds), and let $k \in A(\mathcal{C}) \setminus \mathcal{C}$. It must be that $\hat{J}_{\mathcal{C}}(k) \neq \mathcal{C}$, otherwise $k \in \mathcal{P}(\mathcal{C})$, which is a contradiction.

If $\mathcal{P}(\mathcal{C}) \neq \emptyset$, let $i \in \mathcal{Q}(\mathcal{C})$, suppose $(D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C}) \neq \emptyset$ (otherwise Condition 2 holds), and let $k \in (D_i \setminus \{i\}) \cap (A(\mathcal{C}) \setminus \mathcal{C})$. It must be that $\hat{J}_{\mathcal{C}}(k) \neq \mathcal{C}$, otherwise $k \in \mathcal{P}(\mathcal{C})$ and $\mathcal{C} \subseteq D_k$, which is a contradiction of the assumption that $i \in \mathcal{Q}(\mathcal{C})$.

In either case, choose any $j \in \mathcal{C} \setminus \hat{J}_{\mathcal{C}}(k)$, and also choose any $m \in \hat{J}_{\mathcal{C}}(k)$. Now $k \in A_m$, so by Property 1, $A_k \subseteq A_m$, and hence $A(\{j,k\}) \subseteq A(\{j,m\})$. Also $m \notin A_k$, since $A_k \subseteq A(\mathcal{C}) \setminus \mathcal{C}$ by Lemma 3, and $m \in \hat{J}_{\mathcal{C}}(k) \subseteq \mathcal{C}$. Furthermore, $m, j \in \mathcal{C}$ so $m \notin A_j$. Thus $m \notin A_j \cup A_k = A(\{j,k\})$. It follows that $A(\{j,k\}) \subseteq A(\{j,m\}) \setminus \{m\}$. Since $\mathcal{C}$ is a 2-BMC and $j, m \in \mathcal{C}$ we have $H(A(\{j,m\}) \setminus \{m\}) = H(\{j,m\}) - a_m \leq b$ and hence $H(\{j,k\}) \leq b$.

In the case $\mathcal{P}(\mathcal{C}) = \emptyset$, this shows that Condition 1 is satisfied; otherwise it shows that Condition 2 is satisfied. □ □

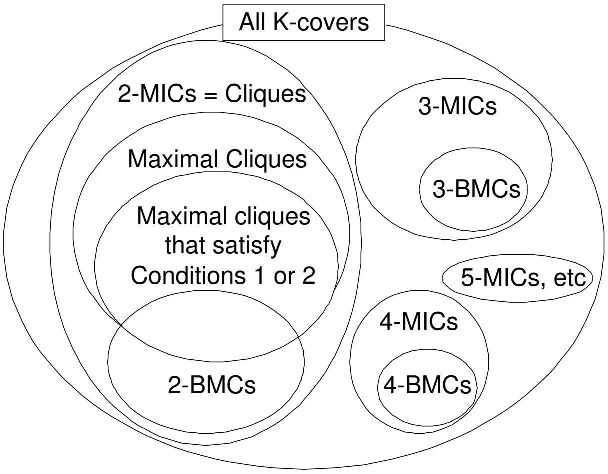


Figure 6: Diagram of the set of all $K$-covers

To complete the classification given in Figure 6, and highlight the new contribution of clique-based inequalities, we note that there *do* exist examples of maximal cliques satisfying Conditions 1 or 2 that are *not* 2-BMCs: Example 1 in the following section has *no* 2-BMCs, but has four maximal cliques in the conflict graph satisfying either Condition 1 or Condition 2.

Of course, it is certainly possible that our facet-defining clique-based inequalities could be derived by maximal lifting of 2-MIC inequalities. (It is not hard to see from the form of (11) that it would be impossible to arrive at an inequality of the form of either (5) or (7) by maximal lifting, unless $K = 2$.) However, as van de Leensel et. al. [9] show, maximal lifting may require solution of NP-complete subproblems; our clique-based inequalities require no such effort to yield facet-defining constraints.

Furthermore, we note that van de Leensel et. al. [9] do not discuss the facet-defining status of (10) in the case that $C$ is a 2-MIC rather than 2-BMC. Furthermore, their polynomial time lifting algorithm only applies to the special case that $K = |C|$ and only applies to coefficients of variables in $A(C) \setminus C$, not to the whole of $\mathcal{N}$.

## 5 Application of Clique-Based Inequalities to PCKP Examples

We now demonstrate that our clique-based approach to determining facets of $conv(P)$ can find facets that would not be found using the cover-based approaches of previous authors, and demonstrate their relative strengthening effect on the LP relaxation. To be fair, we restrict our attention to polynomial time approaches, and do not attempt to any lifting except that for which polynomial time algorithms have been developed.

We give two PCKP examples. For each, we find all $K$-BMCs, and where applicable, use the polynomial time algorithm of van de Leensel et. al. [9] to lift these. We also find all maximal cliques in the conflict graph, and apply Procedures 1 or 2 as appropriate to derive all facet-defining clique-based inequalities.

Consider Example 1 given in Figure 7. There are eight $K$-BMCs in this example, all of which are 3-BMCs. Note in particular that there are no 2-BMCs in this example. For the 3-BMCs with $|C| = 3$, for which $A(C) \setminus C \neq \emptyset$, we apply the polynomial time lifting algorithm of van de Leensel et. al. [9] for the items in the predecessor sets $A(C) \setminus C$ to strengthen the 3-BMC inequalities of the form (10). The 3-BMCs and corresponding inequalities are given in Table 2.

Of course if lifting for $K$-BMCs with $|C| > K$ were available, it would also be possible to strengthen the inequality for the cover $\{4, 5, 6, 7\}$: this could be lifted to either $x_4 + x_5 + x_6 + x_7 \leq x_{10} + 1$, or $x_4 + x_5 + x_6 + x_7 \leq x_{11} + 1$. For interest, we tried adding these to the LP relaxation; we found doing so did not change the value reported in Table 6.

Applying the approach for deriving facets of $conv(P)$ from clique inequalities to this example, we

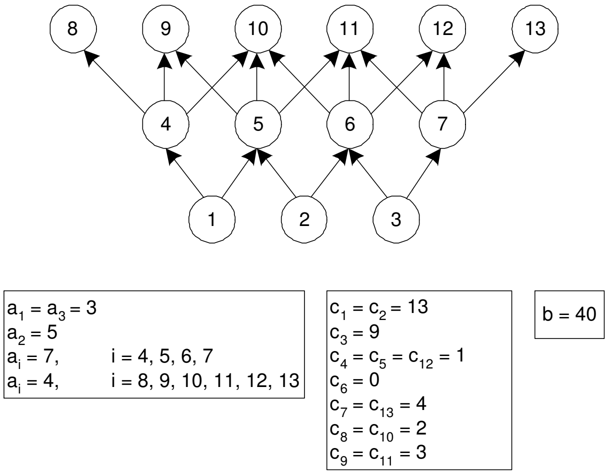


Figure 7: PCKP Example 1.

| $K$-BMC | Corresponding $K$-BMC inequality (lifted when $A(C) \setminus C \neq \emptyset$ and $K = \|C\|$) |
|---|---|
| $\{4, 5, 6\}$ | $x_4 + x_5 + x_6 \leq x_9 + x_{10}$ |
| | $x_4 + x_5 + x_6 \leq x_9 + x_{11}$ |
| | $x_4 + x_5 + x_6 \leq x_{10} + x_{11}$ |
| $\{5, 6, 7\}$ | $x_5 + x_6 + x_7 \leq x_{10} + x_{11}$ |
| | $x_5 + x_6 + x_7 \leq x_{10} + x_{12}$ |
| | $x_5 + x_6 + x_7 \leq x_{11} + x_{12}$ |
| $\{4, 5, 7\}$ | $x_4 + x_5 + x_7 \leq x_9 + x_{11}$ |
| | $x_4 + x_5 + x_7 \leq x_{10} + x_{11}$ |
| $\{4, 6, 7\}$ | $x_4 + x_6 + x_7 \leq x_{10} + x_{11}$ |
| | $x_4 + x_6 + x_7 \leq x_{10} + x_{12}$ |
| $\{4, 5, 6, 7\}$ | $x_4 + x_5 + x_6 + x_7 \leq 2$ |
| $\{1, 12, 13\}$ | $x_1 + x_{12} + x_{13} \leq 2$ |
| $\{3, 8, 9\}$ | $x_3 + x_8 + x_9 \leq 2$ |
| $\{2, 8, 13\}$ | $x_2 + x_8 + x_{13} \leq 2$ |

Table 2: $K$-BMCs for PCKP Example 1

obtain the conflict graph given in Figure 8. Note that each clique in this conflict graph represents a 2-MIC that is not a 2-BMC. There are five maximal cliques $\mathcal{C} \subseteq \mathcal{N}$ in this conflict graph as shown in Table 3, all of which are such that $\mathcal{P}(\mathcal{C}) \neq \emptyset$. One of the maximal cliques ($\mathcal{C}^3 = \{1, 2, 3\}$) does not satisfy Condition 2. However, a maximal clique that does satisfy Condition 2 can be derived from $\mathcal{C}^3$ by the application of Procedure 2. For example, by replacing items 1 and 2 in $\mathcal{C}^3$ with their immediate predecessor item 5, we obtain $\mathcal{C}^2 = \{3, 5\}$, which does satisfy Condition 2.

The maximal cliques and the corresponding facet-defining strengthened clique inequalities are given in Table 3. Note that none of the six facet-defining inequalities derived from these maximal cliques appear in Table 2. Hence we see that the clique-based approach has derived facets of *conv*$(P)$ that cannot be found using the polynomial time cover-based approach of previous authors.

Consider now Example 2 given in Figure 9. There are nine $K$-BMCs $C \subseteq \mathcal{N}$ in this example, all of which are 2-BMCs. For the 2-BMCs with $|C| = 2$ for which $A(C) \setminus C \neq \emptyset$, application of the polynomial time lifting algorithm of van de Leensel et. al. [9] for the items in the predecessor sets $A(C) \setminus C$ allows us to strengthen the 2-BMC inequalities of the form (10). The 2-BMCs and corresponding inequalities are given in Table 4.

Applying the approach for deriving facets of *conv*$(P)$ from clique inequalities to this example,

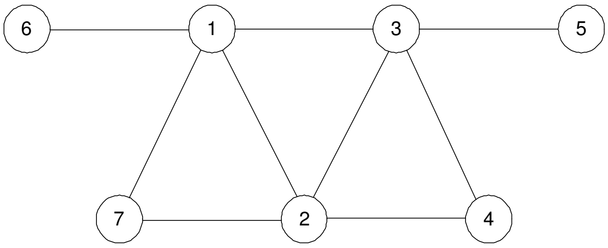


Figure 8: Conflict Graph for PCKP Example 1.

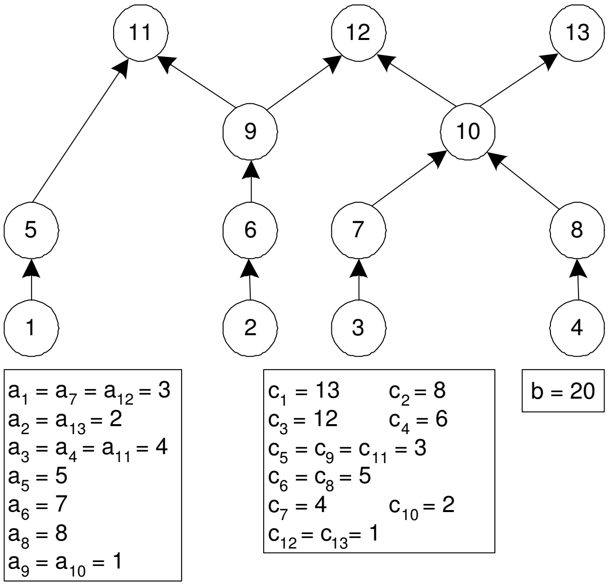


Figure 9: PCKP Example 2.

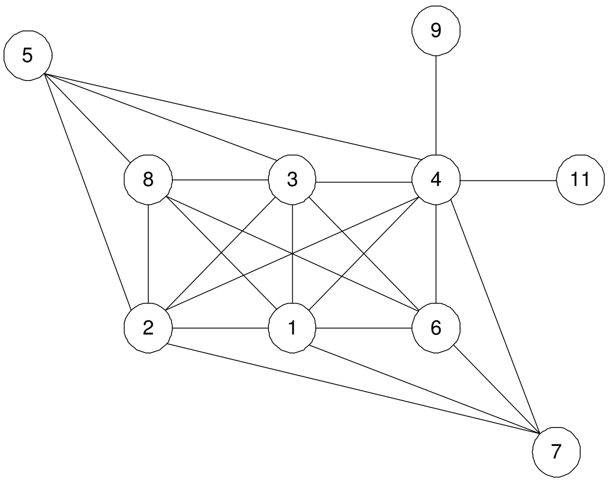


Figure 10: Conflict Graph for PCKP Example 2.

| Maximal Clique | Corresponding facet-defining clique inequality |
|---|---|
| $\{1,6\}$ | $x_1 + x_6 \leq x_{10}$ |
| | $x_1 + x_6 \leq x_{11}$ |
| $\{3,5\}$ | $x_3 + x_5 \leq x_{10}$ |
| | $x_3 + x_5 \leq x_{11}$ |
| $\{1,2,3\}$ | Not facet-defining |
| $\{1,2,7\}$ | $x_1 + x_2 + x_7 \leq x_{11}$ |
| $\{2,3,4\}$ | $x_2 + x_3 + x_4 \leq x_{10}$ |

Table 3: Maximal Cliques for PCKP Example 1

| $K$-BMC | Corresponding $K$-BMC inequality (lifted when $A(C) \setminus C \neq \emptyset$ and $K = \|C\|$) |
|---|---|
| $\{1,7\}$ | $x_1 + x_7 \leq 1$ |
| $\{3,5\}$ | $x_3 + x_5 \leq 1$ |
| $\{3,8\}$ | $x_3 + x_8 \leq x_{10}$ |
| $\{4,7\}$ | $x_4 + x_7 \leq x_{10}$ |
| $\{4,11\}$ | $x_4 + x_{11} \leq 1$ |
| $\{5,8\}$ | $x_5 + x_8 \leq 1$ |
| $\{6,7\}$ | $x_6 + x_7 \leq x_{12}$ |
| $\{6,8\}$ | $x_6 + x_8 \leq x_{12}$ |
| $\{3,5,8\}$ | $x_3 + x_5 + x_8 \leq 1$ |

Table 4: $K$-BMCs for PCKP Example 2

we obtain the conflict graph given in Figure 10. Again note that each clique in this conflict graph represents a 2-MIC; only $\{4,11\}$ is a 2-BMC. There are ten maximal cliques $\mathcal{C} \subseteq \mathcal{N}$ in this conflict graph, as shown in Table 5, all of which are such that $\mathcal{P}(\mathcal{C}) = \emptyset$, and six of which do not satisfy Condition 1. However, a maximal clique that does satisfy Condition 1 can be derived from these maximal cliques in all instances, by the application of Procedure 1. As a result, there are four clique-based inequalities that are facet-defining for this example, as seen in Table 5. In this case, the maximal clique $\{4,11\}$ is also a 2-BMC, and since it satisfies Condition 1, we see that Lemma 18 holds. The remaining maximal cliques in the conflict graph $CG$ all contain 2-BMCs within them, and we see that all 2-BMCs are cliques in the conflict graph, but not necessarily maximal cliques. In this case the facet-defining clique-based inequalities could be reproduced by the lifting approach of van de Leensel et. al. [9], but in all cases some of the lifted variables lie in the set $\mathcal{N} \setminus A(C)$, and so would require solution of a difficult lifting problem. Here using maximal cliques in the conflict graph $CG$ has bypassed the need to solve difficult lifting problems.

A comparison of the LP-relaxations for the PCKP Examples 1 and 2 is presented in Table 6. The cases tested are those of the standard integer programming formulation (PCKP), and this formulation with the addition of the $K$-BMC inequalities (lifted on their predecessor variables where possible), and also with the addition of the facet-defining clique inequalities. It is evident from Table 6 that the addition of the $K$-BMC inequalities, lifted on their predecessor variables where possible, results in a reduction in the root node gap (of approximately 6% in example 1 and 10% in example 2). The addition of the facet-defining clique-based inequalities to the PCKP formulation results in a further reduction in root node gap (of approximately 15% in both cases). In the second example the optimal integer solution is found by solving the LP-relaxation of (PCKP) with the addition of the facet-defining clique inequalities. These results indicate that the addition of facet-defining clique-based inequalities for the PCKP is beneficial in certain instances.

| Maximal Clique | Corresponding facet-defining clique inequality |
|---|---|
| $\{1,2,3,4\}$ | Not facet-defining |
| $\{1,2,3,8\}$ | Not facet-defining |
| $\{1,3,4,6\}$ | Not facet-defining |
| $\{1,3,6,8\}$ | $x_1 + x_3 + x_6 + x_8 \le 1$ |
| $\{1,2,4,7\}$ | Not facet-defining |
| $\{1,4,6,7\}$ | $x_1 + x_4 + x_6 + x_7 \le 1$ |
| $\{2,3,4,5\}$ | Not facet-defining |
| $\{2,3,5,8\}$ | $x_2 + x_3 + x_5 + x_8 \le 1$ |
| $\{4,9\}$ | Not facet-defining |
| $\{4,11\}$ | $x_4 + x_{11} \le 1$ |

Table 5: Maximal Cliques for PCKP Example 2

| Example Number | Formulation | LP relaxation | IP value | Gap (%) |
|---|---|---|---|---|
| 1 | PCKP formulation only | 35.73 | 29.00 | 23.20 |
| | PCKP formulation with 3-BMC inequalities | 34.00 | 29.00 | 17.24 |
| | PCKP formulation and facet-defining clique inequalities | 29.75 | 29.00 | 2.59 |
| 2 | PCKP formulation only | 32.31 | 26.00 | 24.26 |
| | PCKP formulation with 2-BMC inequalities | 29.75 | 26.00 | 14.42 |
| | PCKP formulation and facet-defining clique inequalities | 26.00 | 26.00 | 0.00 |

Table 6: Summary of Results for PCKP Examples

# 6 Conclusions and Future Work

We have presented a new approach for determining facets of the PCKP polyhedron based on clique inequalities. The conditions derived in Section 3 can be checked to determine whether clique inequalities derived from the conflict graph are facet-defining whenever the problem instance contains pairs of items that cannot be included in the knapsack together. A procedure to generate a facet-defining clique inequality from any maximal clique in the conflict graph is also presented in Section 3. A comparison with previous polyhedral approaches to the PCKP based on knapsack cover-like inequalities in Sections 4 and 5 has demonstrated that the clique-based approach can generate facet-defining inequalities that cannot be found through the cover-based approach of previous authors. We note that a relaxation of the conditions under which previous results were obtained could have allowed additional facet-defining inequalities to be determined. We provide a thorough classification of PCKP covers and cliques, showing the relationships between them. We have also shown computationally that the addition of facet-defining clique-based inequalities for the PCKP is beneficial in certain instances.

The derivation of a maximal lifting procedure for 2-MICs is an obvious direction for future research. In addition, computational application of our clique-based facet-defining inequalities, through both *a priori* addition of inequalities and solution of the separation problem for finding inequalities violated by a fractional solution to the PCKP, is an area of interest. This work is in progress.

# References


[1] Balas, E.: Disjunctive Programming. Ann. Discr. Math. **5**, 3-51 (1979)

[2] Boyd, E.A.: Polyhedral results for the precedence-constrained knapsack problem. Discrete Appl. Math. **41**, 185-201 (1993)

[3] Ibarra, O.H., Kim, C.E.: Approximation algorithms for certain scheduling problems. Math. Oper. Res. **4**, 197-204 (1978)

[4] Johnson, D.S., Niemi, K.A.: On knapsacks, partitions, and a new dynamic programming technique for trees. Math. Oper. Res. **8**, 1-14 (1983)

[5] Nemhauser, G.L., Wolsey, L.A.: Integer and Combinatorial Optimization, Wiley-Interscience, New York (1988)

[6] Park K., Park, S.: Lifting cover inequalities for the precedence-constrained knapsack problem. Discrete Appl. Math. **72**, 219-241 (1997)

[7] Shaw, D.X., Cho, G., Chang, H.: A depth-first dynamic programming procedure for the extended tree knapsack problem in local access network design. Telecommunication Systems **7**, 29-43 (1997)

[8] Stecke, K.E., Kim, I.: A study of part type selection approaches for short-term production planning. Int. J. Flexible Manufacturing Systems **1**, 7-29 (1988)

[9] Van de Leensel, R.L.M.J., van Hoesel, C.P.M., van de Klundert, J.J.: Lifting valid inequalities for the precedence constrained knapsack problem. Math. Program. **86**, 161-185 (1999)