Representing the space of linear programs as a Grassmannian *

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Abstract: We represent the space of linear programs as the space of projection matrices. Projection matrices of the same dimension and rank comprise a Grassmannian, which has rich geometric and algebraic structures. An ordinary differential equation on the space of projection matrices defines a path for each projection matrix associated with a linear programming instance and the path leads to a projection matrix associated with an optimal basis of the instance. In this way, any point (projection matrix) in the Grassmannian is connected to a stationary point of the differential equation. We will present some basic properties of the stationary points, in particular, the characteristics of eigenvalues and eigenvectors. We will show that there are only a finite number of stable points. Thus, the Grassmannian can be partitioned into a finite number of attraction regions, each associated with a stable point. The structures of the attraction regions will be important for applications which will be discussed at the end of this paper.

Keywords: Linear programming, Space of linear programs, Grassmannian/Grassmann manifold, Projection matrix.

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1 Introduction

Many powerful methods for solving individual instances of linear programming (LP) have been developed. However, relations between instances are seldom explored, except those like sensitivity analysis which are still local in nature. In this paper, we represent the space of linear programs as the space of projection matrices. This representation provides a framework to study the linear programming as a whole.

The study of the space of linear programs provides a new perspective to understand the nature of linear programming. It can lead to new methods for solving groups of LP instances, e.g. parametric LP, stochastic LP and robust LP, and new theories which are based on collections of LP instances, e.g. on probability distribution on the space of linear programs.

There are many possible ways to look at the space of LP. One can simply consider the collection of all coefficients (A, b, c) of LP instances. However, such a collection may not have good geometric and algebraic structures.

The first goal of this paper is to build a model which can represent the space of LP. The model we will propose is the space of projection matrices, i.e. the Grassmannian, which has rich geometric and algebraic structures.

It is known that a projection matrix captures full information of a LP instance. A deep observation is due to a "universal" ordinary differential equation (ODE) of projection matrices, $\frac{dM}{dt} = h(M)$ where h is a mapping from the space of symmetric matrices to itself, which was first presented in [5, 3]. This observation has motivated the study of the space of projection matrices.

The ODE is "universal" in the sense that its coefficients are universal, i.e. independent of LP instances. Thus, its solution only depends on the initial projection matrix. Each LP instance determines a projection matrix. Starting from this projection matrix, the solution of the ODE forms a path. The path converges to a projection matric which can determine an optimal basis of the LP instance. In this way, any LP instance can be represented by a (starting) projection matrix. Therefore, we can use the space of projection matrices to represent the space of linear programming.

Basic structures of the space of projection matrices we study in this paper are stationary points, stable points and their attraction regions of the ODE. We will show that starting from any projection matrix, the solution of the ODE converges to a stationary point. We will present characterizations of stationary points and stable points. We will completely describe the eigenvalues and eigenvectors of the Jacobian of the mapping h at each stationary point. By virtue of the description of eigenvalues and eigenvectors, we show that there are only a finite number of stable points and thus the space of projection matrices can be partitioned into a finite number of attraction regions (each associated with a stable point) with their boundaries consisting of stationary points and points flowing into the stationary points.

The study of linear programming as a space is a new area for research. This paper is intended to introduce a representation of the space of LP and to study some basic structures of the space. We have not covered this whole research area. Indeed, we have not even had a full view of this area yet. We will discuss some possible topics of this research area and possible applications at the end of this paper.

The rest of this paper is organized as follows: In Section 2, we show how the linear programming is related to the space of projection matrices. Section 3 studies stationary points and their relation to optimal solutions of LP. Section 4 is devoted to the study of eigenvalues and eigenvectors. Section 5 uses the results in Section 4 to characterize stable points. Finally, In Section 6, we discuss some possible topics of future research and some questions we have not yet been able to answer.

2 Relating Linear Programming to Grassmannians

Consider the linear program:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2.1}$$

and its dual

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y + s = c \\ & s \geq 0 \end{aligned} \tag{2.2}$$

where $A \in R^{m \times n}$ is of full row rank, and b, c, x and s are vectors of appropriate dimensions.

Definition 2.1 *We will refer to a set of coefficients* (A, b, c) *as a* **strictly feasible instance** *(in short,* **instance***) of linear programming if the primal and dual problems, (2.1) and (2.2),* *have strictly feasible solutions, i.e. both primal and dual feasible regions have relative interior points.*

For any vectors $x, s \in R^n$ and scalar $\alpha \in R$, we denote $x \circ s = (x_1 s_1, \ldots, x_n s_n)^T$ and $x^\alpha = (x_1^\alpha, \ldots, x_n^\alpha)^T$. We use the symbol $\mathbf{1}$ for the vector of all ones regardless of its dimension.

For any strictly feasible instance, the perturbed KKT system:

$$\begin{aligned} x \circ s &= e^{-t}\mathbf{1} \\ Ax &= b \\ A^T y + s &= c \\ x > 0, \quad & \quad s > 0 \end{aligned} \tag{2.3}$$

has unique solution for any $t \in R$. We refer to this solution $(x(t), s(t))$, $t \geq 0$, as the **central path** of the LP instance.

In this paper, we often refer to the following well-known property.

Theorem 2.2 *Any strictly feasible instance* (A, b, c) *possesses a unique central path* $(x(t), s(t))$, $t \geq 0$, *defined by (2.3), which converges to a pair of strictly complementary primal and dual optimal solutions.*

At any point $(x, s) > 0$, the derivative of the path can be computed through the following system

$$\begin{aligned} x \circ s' + s \circ x' = -e^{-t}\mathbf{1} \\ Ax' = 0 \\ A^T y' + s' = 0 \end{aligned} \tag{2.4}$$

We define the scaled derivatives

$$p(t) = -e^t s(t) \circ x'(t), \qquad q(t) = -e^t x(t) \circ s'(t). \tag{2.5}$$

We also define a projection matrix at any point $(x, s) > 0$

$$M = DA^T(AD^2A^T)^{-1}AD, \tag{2.6}$$

where

$$D = [x^{1/2} \circ s^{-1/2}].$$

Throughout this paper we denote by $[x]$ the diagonal matrix of vector x.

Using $x \circ s = e^{-t}\mathbf{1}$, we can write (p, q) and D in various forms, such as,

$$\begin{aligned}(p(t), q(t)) &= -(x^{-1} \circ x', s^{-1} \circ s') = -(e^{t/2} D^{-1} x', e^{t/2} D s') \\ D(t) &= e^{-t/2}[s^{-1}] = e^{t/2}[x].\end{aligned}$$

The projection matrix M plays an important role in the interior point method from both computational and theoretical perspectives. In particular, it is closely related to the derivatives of the central path as shown below, see, e.g. [3],

$$p(t) = (I - M(t))\mathbf{1}, \qquad q(t) = M(t)\mathbf{1}. \tag{2.7}$$

Definition 2.3 *For any instance (A, b, c), we refer to $(x(0), s(0))$, the solution of (2.3) at $t = 0$, as the* **center** *of (A, b, c), and denote by $M(A, b, c)$ the projection matrix defined by (2.6) with $(x, s) = (x(0), s(0))$.*

For any strictly feasible instance, we can define a projection matrix. Conversely,

Lemma 2.4 *For any projection matrix M, we can construct a strictly feasible instance (A, b, c), such that $M = M(A, b, c)$.*

Proof. Since M is positive semidefinite, we can choose an $n \times m$-matrix A of rank m such that $M = A^T A$. Choose $b = A\mathbf{1}$ and $c = \mathbf{1}$. Then $(x, s) = (\mathbf{1}, \mathbf{1})$, $y = 0$ and $t = 0$ satisfy (2.3).

Since $(\mathbf{1}, \mathbf{1})$ is an interior feasible solution, this instance (A, b, c) is strictly feasible. The projection matrix defined with $(x(0), s(0)) = (\mathbf{1}, \mathbf{1})$ by (2.6) is $M(A, b, c) = A^T(AA^T)^{-1}A$. On the other hand, because $M = A^T A$ is a projection matrix, we have $MM = M$, i.e. $A^T A A^T A = A^T A$, which implies $AA^T = I$ since A is of full row rank. Therefore, $M(A, b, c) = M$. □

Now we give a formal definition of the space of projection matrices.

We denote by S^n the space of all real symmetric n by n matrices, and by $G(n, m)$ the set of all $n \times n$ projection matrices of rank m, i.e.,

$$G(n, m) = \{M \in S^n \mid M \text{ is of rank } m \text{ and satisfies } MM = M\}$$

Lemma 2.5 *$G(n,m)$ is a* **Grassmannian**, *i.e. the set of all m-dimensional subspaces in R^n.*

Proof. Any m-dimensional subspace of R^n can be defined by a full row rank matrix $B \in R^{m\times n}$, namely, $\{B^T y \mid y \in R^m\}$. This subspace can be written as $\{Mx \mid x \in R^n\}$, where $M = B^T(BB^T)^{-1}B \in G(n,m)$.

On the other hand, suppose that $M_1, M_2 \in G(n,m)$ define the same subspace, i.e.

$$\{M_1 x \mid x \in R^n\} = \{M_2 x \mid x \in R^n\}.$$

Then, for any $x, y \in R^n$, there exists $z \in R^n$ with $M_1 z = M_2 y$, and the following holds

$$x^T(I - M_1)M_2 y = x^T(I - M_1)M_1 z = x^T(M_1 - M_1 M_1)z = 0$$

This implies $(I - M_1)M_2 = 0$. Similarly, we have $M_1(I - M_2) = 0$. Therefore,

$$M_2 = M_1 M_2 = M_1.$$

This shows the one-to-one correspondence of the two sets. □

Grassmannians have rich geometric and algebraic structures, cf. [1], which can be used in our analysis.

We have seen that the central path $(x(t), s(t))$ defines a path $M(t)$. The following theorem presents another way, namely, an differential equation to determine the path $M(t)$. The following differential equation was first presented in [3]. Here we present a slight modification, and for completeness we include the proof.

Theorem 2.6 *For any strictly feasible instance (A, b, c), the path $M(t)$ defined by (2.6) with $(x, s) = (x(t), s(t))$ is the unique solution of the following differential equation*

$$\frac{dM}{dt} = M[M\mathbf{1}] + [M\mathbf{1}]M - 2M[M\mathbf{1}]M. \tag{2.8}$$

with $M(0) = M(A, b, c)$. Furthermore, $M(t)$ is real-analytic on R.

Proof. The matrix $M(t)$ defined by (2.6) can be written as

$$M = [s^{-1}]A^T(A[s^{-2}]A^T)^{-1}A[s^{-1}].$$

Denote $H = A[s^{-2}]A^T$, We have

$$\frac{dM}{dt} = [\frac{d(s^{-1})}{dt}][s]M + M[s][\frac{d(s^{-1})}{dt}] - [s^{-1}]A^T H^{-1} A[\frac{d(s^{-2})}{dt}]A^T H^{-1} A[s^{-1}].$$

Using

$$[\frac{d(s^{-1})}{dt}] = -[s^{-2} \circ \frac{ds}{dt}] = [q][s^{-1}] = [s^{-1}][q],$$

$$[\frac{d(s^{-2})}{dt}] = 2[s^{-1}][q][s^{-1}],$$

and $q = M\mathbf{1}$, we have

$$\frac{dM}{dt} = [M\mathbf{1}]M + M[M\mathbf{1}] - 2M[M\mathbf{1}]M.$$

Let $h : R^{n\times n} \to R^{n\times n}$ be defined by

$$h(M) = [M\mathbf{1}]M + M[M\mathbf{1}] - 2M[M\mathbf{1}]M. \tag{2.9}$$

Since h is real-analytic, the differential equation $M' = h(M)$ with any initial point $M_0 \in R^{n\times n}$ has a unique real-analytic solution $M(t)$. Since the path $M(t)$ defined by (2.6) on the central path $(x(t), s(t))$ satisfies the equation $M' = h(M)$ and $M(0) = M(A, b, c)$, it is the unique solution of $M' = h(M)$ with $M(0) = M(A, b, c)$. □

Throughout this paper, we will use the notation $h(M)$ defined in (2.9). The differential equation $M' = h(M)$ with $h : S^n \to S^n$ is defined on the linear space S^n. Since our objective is the space of projection matrices, we shall assure that $M' = h(M)$ is indeed defined on $G(n, m)$ in the sense of the following theorem.

Theorem 2.7 *For any $M_0 \in G(n, m)$, the solution $M(t)$ of $M' = h(M)$ with $M(0) = M_0$ is in $G(n, m)$.*

Proof. As shown by Lemma 2.4, there is an strictly feasible instance (A, b, c) with $A \in R^{m\times n}$ of rank m, such that $M_0 = M(A, b, c)$. Then, by Theorem 2.6, the solution $M(t)$ coincides with the matrix defined by (2.6) on the central path $(x(t), s(t))$ of the instance (A, b, c). Thus, $M(t)$ is a projection matrix. Furthermore, since $(x(t), s(t)) > 0$, the rank of $M(t)$ is equal to the rank of A which is m. Therefore, $M(t) \in G(n, m)$. □

Remark 2.8 *The importance of the differential equation $M' = h(M)$ is that it is a "universal" equation and it links any initial point $M_0 \in G(n,m)$ to an "optimal" point $M_\infty \in G(n,m)$. (The existence and optimality of M_∞ will be shown in the next section.) Thus, we can study relations between any LP instance (A,b,c), represented by $M(A,b,c,)$, and its optimal basis, represented by M_∞, completely on $G(n,m)$ through $M' = h(M)$.*

We will show that the differential equation $M' = h(M)$ is symmetric under permutations.

Lemma 2.9 *Let $Q \in R^{n\times n}$ be a permutation matrix. Then*

$$Q[M\mathbf{1}]Q^T = [QMQ^T\mathbf{1}], \qquad \forall M \in G(n,m). \tag{2.10}$$

Proof. Let Q be a permutation matrix and $q = M\mathbf{1}$. Multiplying Q and Q^T on the two sides of $[q]$, we still obtain a diagonal matrix with diagonal elements permuted by Q, i.e.,

$$Q[q]Q^T = [Qq].$$

On the other hand,

$$QMQ^T\mathbf{1} = QM\mathbf{1} = Qq.$$

Thus, $[QMQ^T\mathbf{1}] = [Qq] = Q[M\mathbf{1}]Q^T$. □

Actually, the converse of the theorem is also true: If an orthogonal matrix Q satisfies (2.10) for all $M \in G(n,m)$, then Q must be a permutation matrix. This can be shown as follows. Suppose that there is a row in Q containing two or more nonzero elements, say $Q_{ik} \neq 0$ and $Q_{jk} \neq 0$. WLOG, assume $k = 1$, we can choose $A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in R^{n\times m}$ with $B^TB = I$ and $B^T\mathbf{1} = 0$ (such B exists since $m < n$). Then $M = AA^T$ is a projection matrix and $M\mathbf{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. That is, we can choose a $M \in G(n,m)$ such that $M\mathbf{1} = e_k$ (the k-th unit vector). For this M, the (i,j)-element of $Q[M\mathbf{1}]Q^T$ is

$$\sum_{l=1}^{n} Q_{il}(M\mathbf{1})_l Q_{jl} = Q_{ik}Q_{jk} \neq 0.$$

This shows that $Q[M\mathbf{1}]Q^T$ is not a diagonal matrix which contradicts to (2.10).

Thus, we have shown that every row of Q can have at most one nonzero element. Since Q is an orthogonal matrix, every row of Q must be either positive or negative unit vector. Such matrix can be represented as a product of a permutation matrix and a reflection matrix.

If Q is a reflection matrix, WLOG, let $Q = \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & I \end{pmatrix}$. Consider $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ where $M_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and M_2 is an appropriate projection matrix such that $M \in G(n,m)$. Then $QMQ^T = \begin{pmatrix} \bar{M}_1 & 0 \\ 0 & M_2 \end{pmatrix}$ where $\bar{M}_1 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$. Hence

$$QMQ^T\mathbf{1} = \begin{pmatrix} 0 \\ 0 \\ M_2\mathbf{1} \end{pmatrix}, \qquad M\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ M_2\mathbf{1} \end{pmatrix}.$$

$$[QMQ^T\mathbf{1}] \neq [M\mathbf{1}] = Q[M\mathbf{1}]Q^T.$$

This shows that Q cannot be a reflection matrix. Therefore, Q can only be a permutation matrix.

Theorem 2.10 *For any permutation matrix Q the following hold:*

(i) $M(t)$ is the solution of $M' = h(M)$ iff so is $QM(t)Q^T$;

(ii) $M(t)$ converges to a projection matrix M_∞ iff $QM(t)Q^T$ converges to $QM_\infty Q^T$.

Proof. It follows from $Q[M\mathbf{1}]Q^T = [QMQ^T\mathbf{1}]$ (by Lemma 2.9) and $QQ^T = I$ that $Qh(M)Q^T = h(QMQ^T)$. Hence $M' = h(M)$ iff $(QMQ^T)' = h(QMQ^T)$. This proves (i). Statement (ii) is an immediate consequence of (i). □

By virtue of this theorem, we need only study the ODE in a subset of $G(n,m)$ and results can be extended to whole $G(n,m)$ by the permutations.

3 Stationary points and optimal solutions of LP

For $G(n,m)$ to fully represent the space of LP, we still need to fill up a gap in the representation: We need to define a projection matrix corresponding to the optimal solution of an instance (A,b,c). More precisely, let $(\bar{x},\bar{s}) = \lim_{t\to\infty}(x(t),s(t))$ where $(x(t),s(t))$ is the central path of the instance (A,b,c). What is the projection matrix corresponding to $(\bar{x},\bar{s})$? The catch is that $(\bar{x},\bar{s}) \not> 0$, thus we cannot use (2.6) to define a corresponding projection matrix. Indeed, it is even not obvious if $M(t)$ is convergent as $t \to \infty$.

In this section, we will show that, starting from any point $M_0 \in G(n,m)$, the solution $M(t)$ of $M' = h(M)$ converges to a *stationary point* (defined below) of $M' = h(M)$, and the stationary point will determine an optimal solution of the LP instance (A, b, c).

Definition 3.1 *$M \in G(n,m)$ is said to be a* **stationary point** *if $h(M) = 0$.*

In order to show that $M(t)$ converges to a stationary point, we shall first show the convergence of $M(t)$ as $t \to \infty$. The difficulty lies in that at the limit point

$$(\bar{x}, \bar{s}) = \lim_{t\to\infty} (x(t), s(t))$$

the matrix (AD^2A^T) may be singular, and thus M cannot be defined by (2.6). We will overcome this difficulty by the following lemma.

Lemma 3.2 *Let the projection matrix $M(t)$ be defined by (2.6) on the central path $(x(t), s(t))$ of an instance (A, b, c). Then $M(t)$ converges to a projection matrix as $t \to \infty$.*

Proof. By Theorem 2.2, the central path converges to a strictly complementary solution of (A, b, c). Let

$$(\bar{x}, \bar{s}) = \lim_{t\to\infty} (x(t), s(t)).$$

Suppose that (B, N) is a partition of $\{1, 2, \ldots, n\}$ such that $\bar{x}_B > 0$ and $\bar{x}_N = 0$. By Theorem 2.2, $\bar{x}$ and $\bar{s}$ are strictly complementary, thus we also have $\bar{s}_B = 0$ and $\bar{s}_N > 0$. With this partition, we write

$$A = (A_B, A_N).$$

Let J be an index set such that A_{JB} consists of a maximum set of linearly independent rows in A_B. Let $K = \{1, 2, \ldots, m\} \setminus J$. Let $Q_1 \in R^{m\times m}$ be a nonsingular matrix such that

$$Q_1 A_B = \begin{pmatrix} A_{JB} \\ 0 \end{pmatrix}.$$

With the same partition of rows, we write

$$Q_1 A_N = \begin{pmatrix} A_{JN} \\ A_{KN} \end{pmatrix}.$$

Since $Q_1 A$ is of full row rank, so is A_{KN}. Denote

$$\begin{aligned} Q_2 &= \begin{pmatrix} I & -A_{JN}[x_N]^2 A_{KN}^T (A_{KN}[x_N]^2 A_{KN}^T)^{-1} \\ 0 & I \end{pmatrix} \\ P &= [x_N] A_{KN}^T (A_{KN}[x_N]^2 A_{KN}^T)^{-1} A_{KN}[x_N]. \end{aligned}$$

Then

$$Q_2Q_1A[x] = Q_2\begin{pmatrix} A_{JB}[x_B] & A_{JN}[x_N] \\ 0 & A_{KN}[x_N] \end{pmatrix} = \begin{pmatrix} A_{JB}[x_B] & A_{JN}[x_N](I-P) \\ 0 & A_{KN}[x_N] \end{pmatrix}. \quad (3.1)$$

Denote $Q = Q_2Q_1$, which is nonsingular. Noting that $(I-P)(I-P) = (I-P)$ and $(I-P)[x_N]A_{KN}^T = 0$, we have

$$QA[x]^2A^TQ^T = \begin{pmatrix} A_{JB}[x_B]^2A_{JB}^T + A_{JN}[x_N](I-P)[x_N]A_{JN}^T & 0 \\ 0 & A_{KN}[x_N]^2A_{KN}^T \end{pmatrix}. \quad (3.2)$$

We can partition $M(t) = [x]A^T(A[x]^2A^T)^{-1}A[x]$ according to $A[x] = (A_B[x_B], A_N[x_N])$:

$$\begin{aligned} M(t) &= \begin{pmatrix} [x_B]A_B^T(A[x^2]A^T)^{-1}A_B[x_B] & [x_B]A_B^T(A[x^2]A^T)^{-1}A_N[x_N] \\ [x_N]A_N^T(A[x^2]A^T)^{-1}A_B[x_B] & [x_N]A_N^T(A[x^2]A^T)^{-1}A_N[x_N] \end{pmatrix} \\ &= \begin{pmatrix} [x_B]A_B^TQ^T(QA[x^2]A^TQ^T)^{-1}QA_B[x_B] & [x_B]A_B^TQ^T(QA[x^2]A^TQ^T)^{-1}QA_N[x_N] \\ [x_N]A_N^TQ^T(QA[x^2]A^TQ^T)^{-1}QA_B[x_B] & [x_N]A_N^TQ^T(QA[x^2]A^TQ^T)^{-1}QA_N[x_N] \end{pmatrix} \\ &= \begin{pmatrix} M_1(t) & M_0(t) \\ M_0(t)^T & M_2(t) \end{pmatrix}. \end{aligned}$$

Using (3.1) and (3.2) and because of $x_N = e^{-t}s_N^{-1}$, we can write

$$M_1(t) = [x_B]A_{JB}^T\left(A_{JB}[x_B]^2A_{JB}^T + E\right)^{-1}A_{JB}[x_B].$$

where

$$E := e^{-2t}A_{JN}[s_N]^{-1}(I-P)[s_N]^{-1}A_{JN}^T.$$

Because $\lim_{t\to\infty} s_N(t) = \bar{s}_N > 0$, $E \to 0$ as $t \to \infty$. Thus,

$$M_1(t) \to \bar{M}_1 = [\bar{x}_B]A_{JB}^T\left(A_{JB}[\bar{x}_B]^2A_{JB}^T\right)^{-1}A_{JB}[\bar{x}_B] \qquad \text{as } t \to \infty.$$

Here $\bar{M}_1$ is a projection matrix.

We also have

$$\begin{aligned} M_2(t) &= \begin{pmatrix} A_{JN}[x_N](I-P) \\ A_{KN}[x_N] \end{pmatrix}^T \begin{pmatrix} A_{JB}[x_B]^2A_{JB}^T + E & 0 \\ 0 & A_{KN}[x_N]^2A_{KN}^T \end{pmatrix}^{-1} \begin{pmatrix} A_{JN}[x_N](I-P) \\ A_{KN}[x_N] \end{pmatrix} \\ &= e^{-2t}(I-P)[s_N]^{-1}A_{JN}^T\left(A_{JB}[x_B]^2A_{JB}^T + E\right)^{-1}A_{JN}[s_N]^{-1}(I-P) \\ &\quad + [s_N]^{-1}A_{KN}^T\left(A_{KN}[s_N]^{-2}A_{KN}^T\right)^{-1}A_{KN}[s_N]^{-1} \end{aligned}$$

Because $A_{JB}[\bar{x}_B]^2A_{JB}^T$ is positive definite and $E \to 0$, the inverse $\left(A_{JB}[x_B]^2A_{JB}^T + E\right)^{-1}$ is bounded on $t \in [0,\infty)$. Thus the the first term of the last equation tends to zero as

$t \to \infty$. The second term of the last equation is a projection matrix and the inverse $\left(A_{KN}[s_N]^{-2}A_{KN}^T\right)^{-1}$ is bounded on $t \in [0, \infty)$. Therefore, $M_2(t)$ tends to the projection matrix $\bar{M}_2 = [\bar{s}_N]^{-1}A_{KN}^T \left(A_{KN}[\bar{s}_N]^{-2}A_{KN}^T\right)^{-1} A_{KN}[\bar{s}_N]^{-1}$ as $t \to \infty$.

Finally, we want to show that $M_0(t) \to 0$ as $t \to \infty$. We consider

$$
\begin{aligned}
tr(M_0(t)^T M_0(t)) &= tr\left([x_N]A_N^T(A[x^2]A^T)^{-1}A_B[x_B^2]A_B^T(A[x^2]A^T)^{-1}A_N[x_N]\right) \\
&= tr\left(QA_N[x_N^2]A_N^TQ^T(QA[x^2]A^TQ^T)^{-1}QA_B[x_B^2]A_B^TQ^T(QA[x^2]A^TQ^T)^{-1}\right) \\
&= tr\left(\begin{pmatrix} E & 0 \\ 0 & W \end{pmatrix}\begin{pmatrix} H+E & 0 \\ 0 & W \end{pmatrix}^{-1}\begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} H+E & 0 \\ 0 & W \end{pmatrix}^{-1}\right) \\
&= tr\left(\begin{pmatrix} E(H+E)^{-1}H(H+E)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right)
\end{aligned}
$$

where

$$
H := A_{JB}[x_B^2]A_{JB}^T, \qquad W = A_{KN}[x_N^2]A_{KN}^T.
$$

Since H is nonsingular with a bounded inverse and $E \to 0$, the trace in the last line tends to zero as $t \to \infty$. This implies that the positive semidefinite matrix $M_0(t)^T M_0(t)$ tends to zero, and in turn, $M_0(t)$ tends to zero as $t \to \infty$. □

Lemma 3.3 *Let $M(t)$ be the solution of $M' = h(M)$ with $M(0) \in G(n, m)$. If $M(t) \to \bar{M}$ as $t \to \infty$, then $\bar{M}$ is stationary, i.e., $h(\bar{M}) = 0$.*

Proof. Since h is continuous, $M(t) \to \bar{M}$ implies $h(M(t)) \to h(\bar{M})$.

If $h(\bar{M}) \neq 0$, we may assume that an entry $h_{ij}(\bar{M}) > 0$ (it can be similarly treated if $h_{ij}(\bar{M}) < 0$). Thus, $M'_{ij}(t) = h_{ij}(M(t)) \geq h_{ij}(\bar{M})/2$, $\forall\, t \in [\bar{t}, \infty)$ for some $\bar{t} \geq 0$. This yields $M_{ij}(t) \to \infty$ as $t \to \infty$, which contradicts to the boundedness of projection matrices. □

Theorem 3.4 *For any $M_0 \in G(n, m)$, the solution of $M' = h(M)$ with $M(0) = M_0$ converges to a stationary point.*

Proof. By Lemma 2.4, there is a strictly feasible instance (A, b, c) such that $M_0 = M(A, b, c)$. Let $(x(t), s(t))$ be the central path of (A, b, c) and let $M(t)$ be the projection matrix defined by (2.6) with $(x(t), s(t))$. By Theorem 2.6, $M(t)$ is the unique solution of $M' = h(M)$ with

$M(0) = M_0$. By Lemma 3.2, $M(t)$ converges to a projection matrix $\bar{M}$, which is a stationary point by Lemma 3.3. □

The following theorem gives a characterization of stationary points.

Theorem 3.5 *A projection matrix $M \in G(n, m)$ is a stationary point iff there exists a partition $\{J, K\}$ of the index set $\{1, \ldots, n\}$ such that*

(i) $q_i = \begin{cases} 1 & \text{if } i \in J \\ 0 & \text{if } i \in K \end{cases}$ *, where* $q = M\mathbf{1}$

(ii) $M_{ij} = 0$ *for all* $i \in J$ *and* $j \in K$*.*

We can write it compactly: $M \in G(n, m)$ *is stationary iff* $M = \begin{pmatrix} M_J & 0 \\ 0 & M_K \end{pmatrix}$ *satisfies* $M_J \mathbf{1}_J = \mathbf{1}_J$ *and* $M_K \mathbf{1}_K = 0$*.*

Proof. Let M be a stationary point, i.e., $h(M) = 0$. Then using $Mq = q$, we have

$$0 = h(M)\mathbf{1} = q^2 + Mq - 2Mq^2 = q^2 - q - 2M(q^2 - q).$$

Let $u = q^2 - q$. Then

$$u = 2Mu = 2M(2Mu) = 2^2 Mu = \ldots = 2^l Mu, \qquad \forall\, l = 1, 2, \ldots$$

Thus, $u = Mu = 0$. $q^2 - q = 0$ implies $q_i \in \{0, 1\}$. (i) is proved.

Without loss of generality, let $q = \begin{pmatrix} \mathbf{1}_J \\ 0 \end{pmatrix}$ and denote $M = \begin{pmatrix} M_J & M_0 \\ M_0^T & M_K \end{pmatrix}$. Then

$$\begin{aligned} h(M) &= \begin{pmatrix} I & \\ & 0 \end{pmatrix} M + M \begin{pmatrix} I & \\ & 0 \end{pmatrix} - 2M \begin{pmatrix} I & \\ & 0 \end{pmatrix} M \\ &= \begin{pmatrix} 2M_J & M_0 \\ M_0^T & 0 \end{pmatrix} - 2 \begin{pmatrix} M_J M_J & M_J M_0 \\ M_0^T M_J & M_0^T M_0 \end{pmatrix} \end{aligned} \tag{3.3}$$

It follows from $h(M) = 0$ that $M_0^T M_0 = 0$ and thus $M_0 = 0$. (ii) is proved.

Conversely, suppose that $q = \begin{pmatrix} \mathbf{1}_J \\ 0 \end{pmatrix}$ and $M = \begin{pmatrix} M_J & 0 \\ 0 & M_K \end{pmatrix}$. Because M is a projection matrix, M_J and M_K are projection matrices, too. Using $M_J M_J = M_J$ and $M_0 = 0$, we have $h(M) = 0$ from (3.3). □

Now we will show that the optimal solution of LP can be determined by the limits of the scaled derivatives.

Theorem 3.6 *Let (A, b, c) be a strictly feasible instance. Suppose that $M(t)$ is the solution of*

$$M' = h(M), \quad M(0) = M(A, b, c).$$

Denote $\bar{M} = \lim_{t\to\infty} M(t)$, $\bar{q} = \bar{M}\mathbf{1}$, and the partition $J = \{i : \bar{q}_i = 1\}$ and $K = \{i : \bar{q}_i = 0\}$ as in Theorem 3.5. Then there is an optimal solution $(\bar{x}, \bar{s}, \bar{y})$ (the limit of the central path) of the instance (A, b, c) which satisfies

$$\begin{aligned} \bar{x}_K &= 0 \\ \bar{s}_J &= 0 \\ A_J \bar{x}_J &= b \\ A_J^T \bar{y} &= c_J \\ \bar{s}_K &= c_K - A_K^T \bar{y}. \end{aligned}$$

Proof. Let $(x(t), s(t))$ be the central path and $(\bar{x}, \bar{s}) = \lim_{t\to\infty}(x(t), s(t))$ be an optimal solution of the instance (A, b, c). By (2.7),

$$q(t) = M(t)\mathbf{1} = -s(t)^{-1} \circ s'(t), \qquad p(t) = (I - M(t))\mathbf{1} = -x(t)^{-1} \circ x'(t).$$

Let $r = e^{-t}$ and $(\hat{x}(r), \hat{s}(r)) = (x(t), s(t))$. Then

$$\frac{d}{dt}(x(t), s(t)) = -e^{-t}\frac{d}{dr}(\hat{x}(r), \hat{s}(r)).$$

It is known, cf. [4], that $(\hat{x}(r), \hat{s}(r))$ is analytic for $r > 0$ and analytically extendable to $r = 0$. Thus, $\frac{d}{dr}(\hat{x}(r), \hat{s}(r))$ is bounded on $r \in [0, 1]$, which implies that $\frac{d}{dt}(x(t), s(t)) \to 0$ as $t \to +\infty$. Therefore, if $x_i(t) \to \bar{x}_i > 0$, then $p_i(t) \to 0$ as $t \to \infty$. If $x_i(t) \to \bar{x}_i = 0$, then since the central path always converges to a strictly complementary solution, $s_i(t) \to \bar{s}_i > 0$. This implies $q_i(t) \to 0$ as $t \to \infty$.

Summarizing the above and by Theorem 3.5, we have

$$\bar{x}_i > 0 \Rightarrow \bar{q}_i = 1 \quad \text{and} \quad \bar{x}_i = 0 \Rightarrow \bar{q}_i = 0.$$

Thus, $\bar{x}_J > 0$ and $\bar{x}_K = 0$. By strict complementarity of $\bar{x}$ and $\bar{s}$, we have $\bar{s}_J = 0$ and $\bar{s}_K > 0$. The remaining equations are satisfied because of the feasibility of $(\bar{x}, \bar{s}, \bar{y})$. □

Theorem 3.6 shows that once we find $\bar{q}$ we can determine the basis of an optimal solution to LP, and in turn can determine the optimal solution. This motivates the study of approaches for finding $(p(t), q(t))$ and their limit. We expect that finding $(p(t), q(t))$ is easier than finding

$(x(t), s(t))$ because the former is normalized, see (i) in Theorem 3.5. Furthermore, given any initial projection matrix M_0, we can find the limit $\bar{M}$ by the differential equation, and determine the basis of the optimal solution (x^*, s^*) by $\bar{q} = \bar{M}\mathbf{1}$.

An obvious stationary point is of the form $M = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$. The following example presents some other forms.

Example 1: It is easy to verify that $M_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in G(2,1)$, $M_2 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \in G(2,1)$, and $M_3 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \in G(4,2)$. They satisfy (i) and (ii) in Theorem 3.5, thus they are stationary points.

The following example shows that $M\mathbf{1} = \begin{pmatrix} \mathbf{1}_J \\ 0 \end{pmatrix}$ does not imply $M = \begin{pmatrix} M_J & 0 \\ 0 & M_K \end{pmatrix}$. That is, the condition (i) in Theorem 3.5 is not sufficient for M to be a stationary point.

Example 2: Let

$$M_1 = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \qquad M_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

where x and y are any number satisfy $x + y = 1$ and $x > y > 0$. Let

$$M = \begin{pmatrix} M_1 & \alpha M_2 \\ \alpha M_2 & \beta M_2 \end{pmatrix}.$$

We will show that α and β can be chosen such that $\bar{M}$ is a projection matrix, i.e., $\bar{M}\bar{M} = \bar{M}$.

Since

$$MM = \begin{pmatrix} M_1^2 + \alpha^2 M_2^2 & \alpha M_1 M_2 + \alpha\beta M_2^2 \\ \alpha M_2 M_1 + \alpha\beta M_2^2 & (\alpha^2 + \beta^2) M_2^2 \end{pmatrix},$$

using $M_2^2 = M_2$, we see that $MM = M$ iff

$$M_1 - M_1^2 = \alpha^2 M_2 \tag{3.4}$$

$$\alpha M_1 M_2 = \alpha M_2 - \alpha\beta M_2 \tag{3.5}$$

$$\alpha^2 + \beta^2 = \beta \tag{3.6}$$

From the condition $x + y = 1$ it follows that

$$x + y = x^2 + y^2 + 2xy$$

that is

$$x - (x^2 + y^2) = 2xy - y = y(x - y)$$

The last equality uses $x + y = 1$. Since

$$M_1 - M_1^2 = \begin{pmatrix} x - (x^2 + y^2) & y - 2xy \\ y - 2xy & x - (x^2 + y^2) \end{pmatrix} = y(x - y) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

we see that (3.4) holds iff

$$\alpha^2 = 2y(x - y). \tag{3.7}$$

Since

$$M_1 M_2 = \frac{1}{2} \begin{pmatrix} x - y & -(x - y) \\ -(x - y) & x - y \end{pmatrix} = (x - y) M_2$$

The equation (3.5) holds iff

$$\alpha(x - y) = \alpha(1 - \beta),$$

that is

$$\beta = 1 - (x - y) = 2y. \tag{3.8}$$

It is easy to verify that α and β determined by (3.7) and (3.8) also satisfy (3.6). Therefore, $\bar{M}$ is a projection matrix with this choice of α and β.

It is easy to see that $M\mathbf{1} = (1, 1, 0, 0)^T$. But the condition (ii) in the theorem is not satisfied since $\alpha \neq 0$. □

4 Eigen-values and -vectors

As we will see in the next section, the Grassmannian $G(n, m)$ can be divided into a finite number of regions, called attraction regions, each corresponding to a stable point. Stationary points which are not stable are on the boundaries of these attraction regions. We guess that the boundaries can be fully determined by certain characteristics of the stationary points, such as eigenvalues and eigenvectors of Jacobian of h at the stationary points. In this section, we will find explicit descriptions of these eigenvalues and eigenvectors.

Throughout this paper, we will denote by $\nabla f(x)$ the *gradient* of f if f is a scalar function, $f : R^n \to R$, or the *Jacobian* of f if f is a vector function (mapping), $f : R^n \to R^m$. More precisely,

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}, \qquad \nabla f(x) = \nabla \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \dots & \vdots \\ \frac{\partial f_m(x)}{\partial x_n} & \dots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}.$$

Let G be a differential manifold in the Euclidean space R^n and $f : R^n \times R \to R^n$ a differentiable function. Then the differential equation $x' = f(x,t)$ is well defined on R^n. We say that the differential equation $x' = f(x,t)$ is defined on G if for any initial condition $x(0) = x^0 \in G$ the solution $x(t)$ lies in G.

In order to investigate the behavior of solutions of $x' = f(x,t)$ on G around a point $\bar{x}$, e.g., the stability of $\bar{x}$ (details will be given in the next section), we consider a coordinate neighborhood of G at $\bar{x}$, (ϕ, U), where $U \subset R^k$ is an open set and $\phi : U \to G$ is a differentiable mapping with a full column rank Jacobian $\nabla\phi(u) : R^k \to R^n$. Because ϕ is a diffeomorphism from U to $\phi(U) \subset G$, its Jacobian $\nabla\phi(u)$ is a isomorphism from R^k onto the tangent space of G at $\phi(u)$, cf. (1.4) Corollary in [1]. We denote by $T_x(G)$ the **tangent space** of G at x. Then $T_x(G)$ at $x = \phi(u)$ can be expressed by

$$T_x(G) = \{\nabla\phi(u)\eta \mid \eta \in R^k\},$$

which is a k-dimensional subspace of R^n.

Because the solution $x(t)$ of $x' = f(x,t)$ is on G (for an initial $x(0) \in G$), $u(t) = \phi^{-1}(x(t))$ can be defined on $x(t) \in \phi(U)$ and satisfies

$$\nabla\phi(u)u' = f(\phi(u),t),$$

Since $\nabla\phi(u)$ is of full column rank, the above can be written as

$$u' = g(u,t),$$

where

$$g(u,t) := \left(\nabla\phi(u)^T \nabla\phi(u)\right)^{-1} \nabla\phi(u)^T f(\phi(u),t). \tag{4.1}$$

With g defined in (4.1), $u' = g(u,t)$ is an expression of $x' = f(x,t)$ on G.

Since the differential equation $u' = g(u,t)$ is defined on the Euclidean space R^k, we can use eigenvalues/vectors of $\nabla g(u,t)$ to analyze the local behavior of solutions of $u' = g(u,t)$ around

$\bar{u} = \phi^{-1}(\bar{x})$. Because $u' = g(u,t)$ is a local representation of $x' = f(x,t)$, the local behavior of solutions of $x' = f(x,t)$ around $\bar{x}$ will be hence observed.

For an arbitrary point $x \in G$, the Jacobian $\nabla g(u,t)$ at $u = \phi^{-1}(x)$ can be very complex. We will only consider $\nabla g(\bar{u})$ where $\bar{x} = \phi(\bar{u})$ is a stationary point, i.e. $f(\bar{x}) = 0$, of the autonomous differential equation $x' = f(x)$. Differentiating g in (4.1) and using $f(\phi(\bar{u})) = f(\bar{x}) = 0$, we obtain

$$\nabla g(\bar{u}) = \left(\nabla\phi(\bar{u})^T \nabla\phi(\bar{u})\right)^{-1} \nabla\phi(\bar{u})^T \nabla f(\bar{x}) \nabla\phi(\bar{u}). \tag{4.2}$$

Finding eigenvalues of $\nabla g(\bar{u},t) : R^k \to R^k$ may be difficult because coordinate neighborhoods (ϕ, U) can be involved, in particular, when we consider the Grassmann manifold. Finding eigenvalues of $\nabla f(\bar{x},t) : R^n \to R^n$ is usually much easier. The following shows how to determine eigenvalues of $\nabla g(\bar{u},t)$ via eigenvalues of $\nabla f(\bar{x},t)$.

Lemma 4.1 *Let G be a differential manifold in the Euclidean space R^n and $f : R^n \to R^n$ a differentiable function. Suppose that for any initial point $x(0) = x^0 \in G$ the solution $x(t)$ of $x' = f(x)$ lies in the manifold G. Let $\bar{x}$ be any stationary point, i.e. $f(\bar{x}) = 0$, let (ϕ, U) with $U \subset R^k$ and $\phi : U \to G$ be a differentiable coordinate neighborhood of G at $\bar{x}$, and denote $\bar{u} = \phi^{-1}(\bar{x})$.*

If (λ, ξ) are an eigenvalue and eigenvector of the Jacobian $\nabla f(\bar{x}) : R^n \to R^n$ and ξ is on the tangent space of G at $\bar{x}$, let $\eta \in R^k$ be such that $\xi = \nabla\phi(\bar{u})\eta$, then (λ, η) are an eigenvalue and eigenvector of the Jacobian $\nabla g(\bar{u}) : R^k \to R^k$, where g is defined in (4.1).

The total number of linearly independent eigenvectors of $\nabla f(\bar{x})$ which are in $T_{\bar{x}}(G)$ is less than or equal to k, the dimension of G.

Proof. If $\nabla f(\bar{x})\xi = \lambda\xi$ and $\xi = \nabla\phi(\bar{u})\eta$, then

$$\begin{aligned}
\nabla g(\bar{u})\eta &= \left(\nabla\phi(\bar{u})^T \nabla\phi(\bar{u})\right)^{-1} \nabla\phi(\bar{u})^T \nabla f(\bar{x})\xi \\
&= \lambda \left(\nabla\phi(\bar{u})^T \nabla\phi(\bar{u})\right)^{-1} \nabla\phi(\bar{u})^T \xi \\
&= \lambda \left(\nabla\phi(\bar{u})^T \nabla\phi(\bar{u})\right)^{-1} \nabla\phi(\bar{u})^T \nabla\phi(\bar{u})\eta \\
&= \lambda\eta.
\end{aligned}$$

This shows that (λ, η) are an eigenvalue and eigenvector of $\nabla g(\bar{u})$.

If $\{\xi_1, \ldots, \xi_p\} \subset T_{\bar{x}}(G)$ are linearly independent eigenvectors of $\nabla f(\bar{x})$, and $\xi_i = \nabla\phi(\bar{u})\eta_i$, then $\{\eta_1, \ldots, \eta_p\}$ must also be linearly independent since $\sum_{i=1}^{p} \alpha_i \eta_i = 0$ implies $\sum_{i=1}^{p} \alpha_i \xi_i = \nabla\phi(\bar{u}) \sum_{i=1}^{p} \alpha_i \eta_i = 0$ and implies $\alpha_1 = \ldots = \alpha_p = 0$. Now, because $\{\eta_1, \ldots, \eta_p\} \subset R^k$, we have $p \leq k$. □

For simplicity, we refer to an eigenvector of $\nabla f(\bar{x})$ which lies in $T_{\bar{x}}(G)$ as an **eigenvector of $\nabla f(\bar{x})$ on** G. It is notable that eigenvalues/vectors of $\nabla f(\bar{x})$ on G are independent of choice of coordinate neighborhoods.

In what follows, we will concentrate on finding eigenvalues and eigenvectors of $\nabla h(M)$ on $G(n, m)$. Let us first describe the Grassmannian $G(n, m)$ and the tangent space, $T_M(G(n, m))$, of $G(n, m)$ at any $M \in G(n, m)$.

Lemma 4.2 *(i) Grassmannian $G(n, m)$ can be expressed by*

$$G(n, m) = \{M \in \mathcal{S}^n \mid MM = M, \quad tr(M) = m\}.$$

(ii) The tangent space, $T_M(G(n, m))$, of $G(n, m)$ at M can be represented by

$$T_M(G(n, m)) = \{D \in S^n \mid MD + DM = D\}.$$

Remark 4.3 *Here and below, we regard $D \in S^n$ as a vector in an $\frac{n(n+1)}{2}$-dimensional Euclidean space equipped with the inner product "$\bullet$" defined by $D \bullet \tilde{D} = tr(D\tilde{D})$.*

Proof. Let $M \in G(n, m)$. As shown in Lemma 2.5, Grassmannian $G(n, m)$ is the set of all real symmetric $n \times n$-matrix of rank m satisfying $MM = M$. From $MM = M$ and $M \in S^n$, it follows that each eigenvalue of M is either 0 or 1. Thus, rank$M = m$ iff M has m eigenvalues equal to 1 and $n - m$ eigenvalues equal to 0. This holds iff $tr(M) = m$. This shows (i).

To show (ii), let us denote

$$\tilde{T} = \{D \in S^n \mid MD + DM = D\}.$$

Since $(M + D)(M + D) - (M + D) = MD + DM - D + DD$, the linearization of $(M + D)(M + D) - (M + D)$ is $MD + DM - D$. A tangent D of $G(n, m)$ at M must satisfy the linearization of equations defining $G(n, m)$ in (i), i.e.

$$MD + DM - D = 0, \qquad tr(D) = 0.$$

Thus, $D \in \tilde{T}$. This shows $T_M(G(n,m)) \subseteq \tilde{T}$.

To show $\tilde{T} \subseteq T_M(G(n,m))$, we must show that $tr(D) = 0$ for any $D \in \tilde{T}$, and the dimension of $\tilde{T}$ is $m(n-m)$, (because dimension of $T_M(G(n,m))$ is equal to the dimension of $G(n,m)$ and thus equal to $m(n-m)$, c.f. [1]).

Let $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22} \end{pmatrix} \in \tilde{T}$. If $M = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$, where I_m is the $m \times m$ identity matrix, then

$$MD + DM - D = \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D_{11} & 0 \\ D_{12}^T & 0 \end{pmatrix} - \begin{pmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22} \end{pmatrix} = \begin{pmatrix} D_{11} & 0 \\ 0 & -D_{22} \end{pmatrix}.$$

Thus, $MD + DM - D = 0$ is equivalent to $D_{11} = 0$ and $D_{22} = 0$, which implies $tr(D) = 0$.

For any $M \in G(n,m)$, there exists an orthogonal matrix Q such that

$$QMQ^T = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\tilde{D} = QDQ^T$. Then $MD + DM - D = 0$ iff $QMQ^T\tilde{D} + \tilde{D}QMQ^T - \tilde{D} = 0$, which is equivalent to $\tilde{D}_{11} = 0$ and $\tilde{D}_{22} = 0$. Thus, $tr(\tilde{D}) = 0$, which in turn implies $tr(D) = 0$.

From the above, we have observed that $D \in \tilde{T}$ if and only if $\tilde{D}_{11} = 0$ and $\tilde{D}_{22} = 0$, but $\tilde{D}_{12}$ is free. This means

$$\tilde{T} = \{D \in S^n : QDQ^T = \begin{pmatrix} 0 & \tilde{D}_{12} \\ \tilde{D}_{12}^T & 0 \end{pmatrix}, \tilde{D}_{12} \in R^{m\times(n-m)}\}.$$

Therefore the dimension of $\tilde{T}$ is $m(n-m)$. So, $\tilde{T} \subseteq T_M(G(n,m))$ is shown. □

Let $M = \begin{pmatrix} M_J & 0 \\ 0 & M_K \end{pmatrix} \in G(n,m)$ be a stationary point. Denote

$$q = M\mathbf{1} = \begin{pmatrix} q_J \\ q_K \end{pmatrix}, \qquad d = D\mathbf{1} = \begin{pmatrix} d_J \\ d_K \end{pmatrix}.$$

Note that $q_J = \mathbf{1}_J$ and $q_K = 0$. We can write

$$\begin{aligned} h(M+D) &= (M+D)[q+d] + [q+d](M+D) - 2(M+D)[q+d](M+D) \\ &= M[d] + [d]M - 2M[d]M + D[q](I-2M) + (I-2M)[q]D + H(D), \end{aligned}$$

where

$$H(D) = D[d] + [d]D - 2D[q+d]D - 2M[d]D - 2D[d]M$$

contains quadratic and higher order terms of D. The Jacobian of h at M is the linear operator $\nabla h(M) : S^n \to S^n$ defined by the linear part of the above expansion, i.e.,

$$\nabla h(M)D = M[D\mathbf{1}] + [D\mathbf{1}]M - 2M[D\mathbf{1}]M + D[M\mathbf{1}](I - 2M) + (I - 2M)[M\mathbf{1}]D. \quad (4.3)$$

The following three lemmas will find all eigenvalues/vectors, (λ, D), of $\nabla h(M)$ on $G(n, m)$.

Lemma 4.4 *Let $M \in G(n, m)$ be a stationary point of the form $M = \begin{pmatrix} M_J & 0 \\ 0 & M_K \end{pmatrix}$ with $M_J\mathbf{1}_J = \mathbf{1}_J$ and $M_K\mathbf{1}_K = 0$. Denote*

$$D = M[d](I - M) + (I - M)[d]M \quad (4.4)$$

where $d = \begin{pmatrix} d_J \\ d_K \end{pmatrix} \in R^n$ satisfies $M_J d_J = 0$ and $M_K d_K = d_K$. Then

(i) $D\mathbf{1} = d$.

(ii) D is a tangent of $G(n, m)$ at M.

(iii) $\nabla h(M)D = D$.

Let n_J and n_K be the dimensions and m_J and m_K the ranks of M_J and M_K, respectively. Then $\nabla h(M)$ has $n_J - m_J + m_K$ linearly independent eigenvectors of the form (4.4) on the tangent space of $G(n, m)$ at M associated with eigenvalue $\lambda = 1$.

Proof. (i) is shown below:

$$\begin{aligned} D\mathbf{1} &= M[d](I - M)\mathbf{1} + (I - M)[d]M\mathbf{1} \\ &= M[d]\begin{pmatrix} 0_J \\ \mathbf{1}_K \end{pmatrix} + (I - M)[d]\begin{pmatrix} \mathbf{1}_J \\ 0_K \end{pmatrix} \\ &= \begin{pmatrix} d_J - M_J d_J \\ M_K d_K \end{pmatrix} \\ &= \begin{pmatrix} d_J \\ d_K \end{pmatrix}. \end{aligned}$$

To show (ii), by Lemma 4.2, we need only to show $MD + DM = D$. Since $MM = M$, it follows from (4.4) that $MD = M[d](I - M)$. Thus $MD + DM = M[d](I - M) + (I - M)[d]M = D$.

Now we show (iii).

$$\nabla h(M)D = M[D\mathbf{1}] + [D\mathbf{1}]M - 2M[D\mathbf{1}]M + D[M\mathbf{1}](I-2M) + (I-2M)[M\mathbf{1}]D.$$

Since $M\mathbf{1} = \begin{pmatrix} \mathbf{1}_J \\ 0_K \end{pmatrix}$, we have

$$D[M\mathbf{1}](I-2M) \;=\; \begin{pmatrix} D_J(I_J - 2M_J) & 0 \\ 0 & 0 \end{pmatrix}.$$

Using (ii), we have $M_J D_J + D_J M_J = D_J$. Thus

$$D_J(I_J - 2M_J) + (I_J - 2M_J)D_J = D_J - 2D_J M_J + D_J - 2M_J D_J = 0$$

This shows that

$$D[M\mathbf{1}](I-2M) + (I-2M)[M\mathbf{1}]D = 0.$$

Now, since $D\mathbf{1} = d$, it is easy to see that

$$\nabla h(M)D \;=\; M[d] + [d]M - 2M[d]M = D.$$

Finally, we shall show that there are $n_J - m_J + m_K$ linearly independent eigenvectors. Because M_J is a projection matrix with dimension n_J and rank m_J, it has $n_J - m_J$ linearly independent eigenvectors $\{d_J^1, \ldots, d_J^{n_J-m_J}\}$ associated with eigenvalue 0, i.e. $M_J d_J^i = 0$, $i = 1, \ldots, n_J - m_J$. Similarly, M_K has m_K linearly independent eigenvectors $\{d_K^1, \ldots, d_K^{m_K}\}$ associated with eigenvalue 1, i.e. $M_K d_K^i = d_K^i$, $i = 1, \ldots, m_K$. Denote $D(d) := M[d](I-M) + (I-M)[d]M$.

Since $d_J^i \neq 0$ and $d_K^i \neq 0$, by (i), $D(\begin{pmatrix} d_J^i \\ 0 \end{pmatrix}) \neq 0$ and $D(\begin{pmatrix} 0 \\ d_K^i \end{pmatrix}) \neq 0$, and by (iii), they are eigenvectors of $\nabla h(M)$ associated with eigenvalue 1. (Note that for (iii) to hold, d_J and d_K are not required to be nonzero).

For any numbers $\{\alpha_1, \ldots, \alpha_{n_J-m_J}, \beta_1, \ldots, \beta_{m_K}\}$, suppose

$$\sum_{i=1}^{n_J-m_J} \alpha_i D(\begin{pmatrix} d_J^i \\ 0 \end{pmatrix}) + \sum_{i=1}^{m_K} \beta_i D(\begin{pmatrix} 0 \\ d_K^i \end{pmatrix}) = 0.$$

Then, by linearity of D on d,

$$D(\begin{pmatrix} \sum_{i=1}^{n_J-m_J} \alpha_i d_J^i \\ \sum_{i=1}^{m_K} \beta_i d_K^i \end{pmatrix}) = 0.$$

As shown in (i),

$$\begin{pmatrix} \sum_{i=1}^{n_J-m_J} \alpha_i d_J^i \\ \sum_{i=1}^{m_K} \beta_i d_K^i \end{pmatrix} = D(\begin{pmatrix} \sum_{i=1}^{n_J-m_J} \alpha_i d_J^i \\ \sum_{i=1}^{m_K} \beta_i d_K^i \end{pmatrix})\mathbf{1} = 0.$$

The independence of $\{d_J^1, \ldots, d_J^{n_J-m_J}\}$ and $\{d_K^1, \ldots, d_K^{m_K}\}$ yields $\alpha_1 = \ldots = \alpha_{n_J-m_J} = 0$ and $\beta_1 = \ldots = \beta_{m_K} = 0$. Therefore, the $n_J - m_J + m_K$ eigenvectors

$$\{D(\begin{pmatrix} d_J^i \\ 0 \end{pmatrix}) : i = 1, \ldots, n_J - m_J\} \cup \{D(\begin{pmatrix} 0 \\ d_K^i \end{pmatrix}) : i = 1, \ldots, m_K\}$$

of $\nabla h(M)$ are linearly independent. □

Lemma 4.5 *Let $M \in G(n,m)$ be a stationary point of the form $M = \begin{pmatrix} M_J & 0 \\ 0 & M_K \end{pmatrix}$ with $M_J \mathbf{1}_J = \mathbf{1}_J$ and $M_K \mathbf{1}_K = 0$. Let n_J and n_K be the dimensions and m_J and m_K the ranks of M_J and M_K, respectively. Suppose that $u \in R^{n_J}$ and $v \in R^{n_K}$ satisfy*

$$M_J u = u, \quad M_K v = 0 \tag{4.5}$$

or

$$M_J u = 0, \quad M_K v = v. \tag{4.6}$$

Denote

$$\begin{aligned} Q_J &= -v^T \mathbf{1} \left(M_J[u] + [u]M_J - 2M_J[u]M_J \right) \\ Q_K &= -u^T \mathbf{1} \left(M_K[v] + [v]M_K - 2M_K[v]M_K \right) \end{aligned}$$

Then,

$$D = \begin{pmatrix} Q_J & uv^T \\ vu^T & Q_K \end{pmatrix} \tag{4.7}$$

is an eigenvector of $\nabla h(M)$ on the tangent space of $G(n,m)$ at M. More precisely, there are $m_J(n_K - m_K)$ linearly independent eigenvectors associated with eigenvalue $\lambda = -1$ in the form of (4.7) in which (u,v) satisfies (4.5) and $(n_J - m_J)m_K$ linearly independent eigenvectors associated with eigenvalue $\lambda = 1$ in the form of (4.7) in which (u,v) satisfies (4.6).

Remark 4.6 *For most cases, $Q_J = 0$ and $Q_K = 0$. Note that u and $\mathbf{1}_J$ are eigenvectors of M_J and v and $\mathbf{1}_K$ are eigenvectors of M_K. If $u \neq \mathbf{1}_J$ and $v \neq \mathbf{1}_K$, we can choose u and v such that $u^T \mathbf{1}_J = 0$ and $v^T \mathbf{1}_K = 0$, then $Q_J = 0$ and $Q_K = 0$. If $u = \mathbf{1}_J$ and $v = \mathbf{1}_K$, it is also easy to see that $Q_J = 0$ and $Q_K = 0$. Therefore, Q_J and Q_K can be nonzero only if "$u = \mathbf{1}_J$ and $v \neq \mathbf{1}_K$" or "$v = \mathbf{1}_K$ and $u \neq \mathbf{1}_J$".*

Proof. First we show that D is a tangent of $G(n,m)$ at M.

$$\begin{aligned} M_J Q_J &= -v^T \mathbf{1}_K \left(M_J[u] + M_J[u]M_J - 2M_J[u]M_J\right) \\ &= -v^T \mathbf{1}_K \left(M_J[u] - M_J[u]M_J\right). \end{aligned}$$

Thus,

$$\begin{aligned} M_J Q_J + Q_J M_J &= -v^T \mathbf{1}_K \left(M_J[u] + [u]M_J - 2M_J[u]M_J\right) \\ &= Q_J. \end{aligned}$$

Similarly, $M_K Q_K + Q_K M_K = Q_K$. Now, together with condition (4.5) or (4.6), we have

$$\begin{aligned} MD + DM &= \begin{pmatrix} M_J Q_J + Q_J M_J & M_J u v^T + u v^T M_K \\ M_K v u^T + v u^T M_J & M_K Q_K + Q_K M_K \end{pmatrix} \\ &= \begin{pmatrix} Q_J & u v^T \\ v u^T & Q_K \end{pmatrix} \\ &= D. \end{aligned}$$

By Lemma 4.2 (ii), D is a tangent of $G(n,m)$ at M.

Since $[M\mathbf{1}] = \begin{pmatrix} I_J & 0 \\ 0 & 0 \end{pmatrix}$,

$$\begin{aligned} D[M\mathbf{1}](I-2M) &+ (I-2M)[M\mathbf{1}]D \\ &= \begin{pmatrix} Q_J(I_J - 2M_J) & 0 \\ vu^T(I_J - 2M_J) & 0 \end{pmatrix} + \begin{pmatrix} (I_J - 2M_J)Q_J & (I_J - 2M_J)uv^T \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (I_J - 2M_J)uv^T \\ vu^T(I_J - 2M_J) & 0 \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} 0 & -uv^T \\ -vu^T & 0 \end{pmatrix}, & \text{if } M_J u = u \\ \begin{pmatrix} 0 & uv^T \\ vu^T & 0 \end{pmatrix}, & \text{if } M_J u = 0. \end{cases} \end{aligned} \tag{4.8}$$

Now, $Q_J \mathbf{1}_J = -v^T \mathbf{1}_K (M_J u + u - 2M_J u)$. If (4.5) holds, then $M_J u + u - 2M_J u = 0$, thus $Q_J \mathbf{1}_J = 0$. If (4.6) holds, then v is an eigenvector of M_K associated with eigenvalue 1, ($M_K v = v$). Since $\mathbf{1}_K$ is an eigenvector of M_K associated with eigenvalue 0 ($M_K \mathbf{1}_K = 0$) and M_K is symmetric, we have $v^T \mathbf{1}_K = 0$, thus $Q_J \mathbf{1}_J = 0$. Similarly, one can verify that $Q_K \mathbf{1}_K = 0$. Therefore, $D\mathbf{1} = \begin{pmatrix} (v^T \mathbf{1}_K)u \\ (u^T \mathbf{1}_J)v \end{pmatrix}$. This leads to

$$M[D\mathbf{1}] + [D\mathbf{1}]M - 2M[D\mathbf{1}]M = \begin{pmatrix} -Q_J & 0 \\ 0 & -Q_K \end{pmatrix}.$$

For the case of (4.5), we obtain

$$\begin{aligned}\nabla h(M)D &= M[D\mathbf{1}] + [D\mathbf{1}]M - 2M[D\mathbf{1}]M + D[M\mathbf{1}](I-2M) + (I-2M)[M\mathbf{1}]D \\ &= \begin{pmatrix} -Q_J & -uv^T \\ -vu^T & -Q_K \end{pmatrix} \\ &= -D.\end{aligned}$$

For the case of (4.6), we note that $u^T\mathbf{1}_J = 0$ and $v^T\mathbf{1}_K = 0$ since u and $\mathbf{1}_J$ (v and $\mathbf{1}_K$) are eigenvectors of M_J (M_K) associated with different eigenvalues, thus $Q_J = 0$ and $Q_K = 0$. Therefore, it follows from the second case of (4.8) that

$$\nabla h(M)D = \begin{pmatrix} 0 & uv^T \\ vu^T & 0 \end{pmatrix} = D.$$

Finally, we shall count the number of linearly independent eigenvectors. Let $\mathcal{B}_J$ be the set of all orthogonal eigenvectors of M_J and $\mathcal{B}_K$ be the set of all orthogonal eigenvectors of M_K. Denote

$$D_{uv} = \begin{pmatrix} Q_J & uv^T \\ vu^T & Q_K \end{pmatrix}, \qquad D^0_{uv} = \begin{pmatrix} 0 & uv^T \\ vu^T & 0 \end{pmatrix}.$$

Let

$$\begin{aligned}\mathcal{D} &= \{D_{uv} \mid u \in \mathcal{B}_J, v \in \mathcal{B}_K\} \\ \mathcal{D}^0 &= \{D^0_{uv} \mid u \in \mathcal{B}_J, v \in \mathcal{B}_K\}\end{aligned}$$

For any $D^0_{uv} \neq D^0_{\bar{u}\bar{v}} \in \mathcal{D}^0$, $(u,v) \neq (\bar{u},\bar{v})$. Thus, either $u^T\bar{u} = 0$ or $v^T\bar{v} = 0$, i.e., $u^T\bar{u}v^T\bar{v} = 0$. This leads to

$$\begin{aligned}tr(D^0_{uv}D^0_{\bar{u}\bar{v}}) &= tr(uv^T\bar{v}\bar{u}^T) + tr(vu^T\bar{u}\bar{v}^T) \\ &= 2tr(u^T\bar{u}v^T\bar{v}) \\ &= 0.\end{aligned}$$

This shows that the vectors in $\mathcal{D}^0$ are orthogonal, and thus, linearly independent. Because each D^0_{uv} can be regarded as a subvector of D_{uv} (by fixing the diagonal blocks to zeros), vectors in $\mathcal{D}$ are also linearly independent.

Since there are m_J orthogonal u's and $n_K - m_K$ orthogonal v's satisfying (4.5), there are $m_J(n_K - m_K)$ linearly independent D_{uv}'s associated with eigenvalue -1.

Similarly, because there are $n_J - m_J$ orthogonal u's and m_K orthogonal v's satisfying (4.6), there are $(n_J - m_J)m_K$ linearly independent D_{uv}'s associated with eigenvalue 1. □

Lemma 4.7 *Let $M \in G(n,m)$ be a stationary point of the form $M = \begin{pmatrix} M_J & 0 \\ 0 & M_K \end{pmatrix}$ with $M_J \mathbf{1}_J = \mathbf{1}_J$ and $M_K \mathbf{1}_K = 0$. Let n_J and n_K be the dimensions and m_J and m_K the ranks of M_J and M_K, respectively. Denote*

$$G_s(n,m) = \{\begin{pmatrix} \tilde{M}_J & 0 \\ 0 & \tilde{M}_K \end{pmatrix} : \tilde{M}_J \in G(n_J, m_J), \tilde{M}_K \in G(n_K, m_K), \tilde{M}_J \mathbf{1}_J = \mathbf{1}_J, \tilde{M}_K \mathbf{1}_K = 0\}.$$

Then any tangent D of the manifold $G_s(n,m)$ at M is an eigenvector of $\nabla h(M)$ on $G(n,m)$ associated with the eigenvalue 0. Equivalently, any $D = \begin{pmatrix} D_J & 0 \\ 0 & D_K \end{pmatrix}$ satisfying

$$MD + DM = D, \qquad D\mathbf{1} = 0, \tag{4.9}$$

is an eigenvector of $\nabla h(M)$ on $G(n,m)$ associated with the eigenvalue 0.

There are

$$\begin{cases} (m_J - 1)(n_J - m_J) + (n_K - m_K - 1)m_K & \text{if } n_J n_K \neq 0 \\ (m-1)(n-m) & \text{if } n_K = 0 \\ (n-m-1)m & \text{if } n_J = 0 \end{cases} \tag{4.10}$$

linearly independent eigenvectors in the form of D satisfying (4.9).

Proof. Because $M_J \mathbf{1}_J = \mathbf{1}_J$ and $M_K \mathbf{1}_K = 0$, we see that $M \in G_s(n,m)$. Let D be a tangent of $G_s(n,m)$ at M. Then, D must be block-diagonal as $\tilde{M} \in G_s(n,m)$, i.e., $D = \begin{pmatrix} D_J & 0 \\ 0 & D_K \end{pmatrix}$. Because $G_s(n,m)$ is a submanifold of $G(n,m)$, D is also a tangent of $G(n,m)$ at M. Hence, by Lemma 4.2 (ii), D satisfies $MD + DM = D$. Furthermore, the linearization of $\tilde{M}_J \mathbf{1}_J = \mathbf{1}_J$ and $\tilde{M}_K \mathbf{1}_K = 0$ is $D\mathbf{1} = 0$. Thus, D satisfies conditions (4.9). The converse is also true by Lemma 4.2 (ii). This shows the equivalence:

$$\left\{ D = \begin{pmatrix} D_J & 0 \\ 0 & D_K \end{pmatrix} \mid MD + DM = D, \quad D\mathbf{1} = 0 \right\} \;=\; T_M(G_s(n,m)). \tag{4.11}$$

Next, we will show that D is an eigenvector. It follows from $D\mathbf{1} = 0$ that

$$M[D\mathbf{1}] + [D\mathbf{1}]M - 2M[D\mathbf{1}]M = 0.$$

The condition $MD + DM = D$ implies $M_J D_J + D_J M_J = D_J$. Thus,

$$D_J(I_J - 2M_J) + (I_J - 2M_J)D_J = 2(D_J - D_J M_J - M_J D_J) = 0.$$

Since $[M\mathbf{1}] = \begin{pmatrix} I_J & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$D[M\mathbf{1}](I-2M) + (I-2M)[M\mathbf{1}]D \;=\; 0.$$

Summarizing the above, we have

$$\nabla h(M)D = M[D\mathbf{1}] + [D\mathbf{1}]M - 2M[D\mathbf{1}]M + D[M\mathbf{1}](I-2M) + (I-2M)[M\mathbf{1}]D = 0.$$

This shows that any tangent D of $G_s(n,m)$ at M is an eigenvector of $\nabla h(M)$ on $G(n,m)$ associated with eigenvalue 0.

Now we will determine the number of linearly independent eigenvectors D satisfying the conditions (4.9). By the equivalence (4.11), this number is equal to the dimension of $T_M(G_s(n,m))$, i.e., the dimension of $G_s(n,m)$. Therefore, we need only to show that the dimension of $G_s(n,m)$ is equal to the number in (4.10).

Denote

$$\begin{aligned} G_1(p,k) &= \{\tilde{M} \in G(p,k) \mid \tilde{M}\mathbf{1} = \mathbf{1}\} \\ G_0(p,k) &= \{\tilde{M} \in G(p,k) \mid \tilde{M}\mathbf{1} = 0\}. \end{aligned}$$

For each $M \in G_1(p,k)$, one can construct a unique k-dimensional subspace $\{\tilde{M}x \mid x \in R^p\}$ of R^p. This subspace contains the vector $\mathbf{1}$ because $\tilde{M}\mathbf{1} = \mathbf{1}$. Thus, $G_1(p,k)$ is isomorphic to the set of all k-dimensional subspaces in R^p which contain $\mathbf{1}$. Now, any k-dimensional subspace in R^p which contains $\mathbf{1}$ is uniquely determined by its intersection with the $(p-1)$-dimensional subspace which is perpendicular to $\mathbf{1}$ and this intersection is a $(k-1)$-dimensional subspace. Hence, the set of all k-dimensional subspaces in R^p which contain $\mathbf{1}$ is isomorphic to the set of all $(k-1)$-dimensional subspaces in R^{p-1} (for visualization, one may rotate $\mathbf{1}$ to the p-th unit vector e_p and whereby rotate the $(p-1)$-dimensional subspace which is perpendicular to $\mathbf{1}$ to R^{p-1}), which is $G(p-1,k-1)$. Therefore, the dimension of $G_1(p,k)$ is equal to the dimension of $G(p-1,k-1)$ which is $(k-1)(p-k)$.

For $G_0(p,k)$, we have

$$\begin{aligned} G_0(p,k) &= \{I - \tilde{M} \in G(p,p-k) : (I-\tilde{M})\mathbf{1} = \mathbf{1}\} \\ &= G_1(p,p-k) \\ &\cong G(p-1,p-k-1), \end{aligned}$$

where $\cong$ stands for the relation of isomorphism. Therefore, the dimension of $G_0(p,k)$ is $(p-k-1)k$.

Since

$$G_s(n,m) = \{\begin{pmatrix} \tilde{M}_J & 0 \\ 0 & M_K \end{pmatrix} \mid \tilde{M}_J \in G_1(n_J, m_J), \tilde{M}_K \in G_0(n_K, m_K)\},$$

its dimension is

$$\begin{cases} (m_J - 1)(n_J - m_J) + (n_K - m_K - 1)m_K & \text{if } n_J n_K \neq 0 \\ (m-1)(n-m) & \text{if } n_K = 0 \\ (n-m-1)m & \text{if } n_J = 0 \end{cases}$$

Here we notice that $m_J \geq 1$ if $n_J \geq 1$ because $M_J \mathbf{1}_J = \mathbf{1}_J$, and similarly, $n_K - m_K \geq 1$ if $n_K \geq 1$ because $M_K \mathbf{1}_K = 0$. We also notice that $(n_J, m_J) = (n, m)$ if $n_K = 0$ and $(n_K, m_K) = (n, m)$ if $n_J = 0$. □

Now we summarize Lemmas 4.4, 4.5 and 4.7 in a theorem, and show a complete set of eigenvectors of $\nabla h(M)$ on $G(n,m)$.

Theorem 4.8 *Let $M \in G(n,m)$ be a stationary point of the differential equation $M' = h(M)$. Then the eigenvectors determined in Lemmas 4.4, 4.5 and 4.7 are the complete set of eigenvectors of $\nabla h(M)$ on $G(n,m)$. There are totally $m(n-m)$ linearly independent eigenvectors.*

Proof. It is easy to see that the eigenvectors determined in the three lemmas are linearly independent. As shown in Lemma 4.1, the total number of linearly independent eigenvectors of $\nabla h(M)$ on $G(n,m)$ is less than or equal to the dimension of $G(n,m)$ which is $m(n-m)$. Thus, we need only to show that there are altogether $m(n-m)$ linearly independent eigenvectors of $\nabla h(M)$ on $G(n,m)$.

We refer to the eigenvectors in Lemmas 4.4, 4.5 and 4.7, as types I, II and III. Eigenvectors of different types are obviously linearly independent.

We count the total number of eigenvectors below. If $n_J n_K \neq 0$, then there are

(i) $n_J - m_J + m_K$ eigenvectors of type I associated with $\lambda = 1$;

(ii) $m_J(n_K - m_K)$ eigenvectors of type II associated with $\lambda = -1$ and $(n_J - m_J)m_K$ eigenvectors of type II associated with $\lambda = 1$;

(iii) $(m_J - 1)(n_J - m_J) + (n_K - m_K - 1)m_K$ eigenvectors of type III associated with $\lambda = 0$.

One can see the total number of the eigenvectors is $m(n-m)$.

If $n_K = 0$, then $m_K = 0$, $n_J = n$ and $m_J = m$. If $n_J = 0$, then $m_J = 0$, $n_K = n$ and $m_K = m$. One can similarly verify that the total number of eigenvectors is $m(n-m)$. □

5 Attraction regions

In this section, we will present a complete characterization of stable/unstable points. Through this characterization, the attraction regions can be outlined.

Definition 5.1 *$M_\infty \in G(n,m)$ is said to be a* **stable point** *of $M' = h(M)$ on $G(n,m)$ if there exists a neighborhood $\mathcal{N}(M_\infty) \subset G(n,m)$ of M_∞ such that for any $M_0 \in \mathcal{N}(M_\infty)$, the solution $M(t)$ of $M' = h(M)$ starting from M_0 converges to M_∞ as $t \to +\infty$. The largest neighborhood $\mathcal{N}(M_\infty)$ possessing the above property is called the* **attraction region** *of M_∞.*

Remark 5.2 *The differential equation $M' = h(M)$ is actually defined on S^n. However, the stability in the above definition is restricted to the Grassmannian $G(n,m)$ because we are interested in projection matrices.*

We first consider the general differential equation $x' = f(x)$ as we considered in Lemma 4.1. A stationary point $\bar{x}$ of $x' = f(x)$ is said to be stable in G if there is a neighborhood $N \subset G$ of $\bar{x}$ such that for any $x(0) = x^0 \in N$ the solution $x(t)$ converges to $\bar{x}$. Taking any coordinate neighborhood (ϕ, U) of G at $\bar{x}$, we see that $\bar{x}$ is stable if and only if there is a neighborhood $V \subset U$ of $\bar{u} = \phi^{-1}(\bar{x})$ such that for any $u(0) = u^0 \in V$ the solution $u(t)$ of $u' = g(u)$, defined in (4.1), converges to $\bar{u}$.

With the basic theory of ordinary differential equations, we can show the stability of $u' = g(u)$ at $\bar{u}$ with eigenvalues of $\nabla g(\bar{u})$ because $u' = g(u)$ is defined on the Euclidean space R^k. If all eigenvalues of $\nabla g(\bar{u})$ have negative real parts, then $\bar{u}$ is a stable point of $u' = g(u)$, and as shown above, $\bar{x}$ will then be a stable point of $x' = f(x)$ on G. On the other hand, if there is an eigenvalue of $\nabla g(\bar{u})$ which has a positive real part, then $\bar{u}$ is an unstable point of $u' = g(u)$, and thus $\bar{x}$ is an unstable point of $x' = f(x)$ on G. By Lemma 4.1, eigenvalues/vectors of $\nabla g(\bar{u})$ can be determined by eigenvectors of $\nabla f(\bar{x})$ on G. Thus we have proved the following theorem.

Theorem 5.3 *Let G be a differentiable manifold in the Euclidean space R^n and $f : R^n \to R^n$ be differentiable. Suppose that, for any $x(0) = x^0 \in G$, the solution $x(t)$ of $x' = f(x)$ lies in G. Let $\bar{x}$ is a stationary point of $x' = f(x)$. If $\nabla f(\bar{x}) : R^n \to R^n$ has an eigenvalue $\lambda > 0$ and its corresponding eigenvector $\xi \neq 0$ is in the tangent space of G at $\bar{x}$, then $\bar{x}$ is not a stable point of $x' = f(x)$ on G. Conversely, if each eigenvalue of $\nabla f(\bar{x})$ associated with an eigenvector in the tangent space of G is negative, then $\bar{x}$ is a stable point of $x' = f(x)$ on G.*

Using the eigenvalues/vectors we have found in the last section, we can characterize the stable points.

Theorem 5.4 *$\bar{M} \in G(n,m)$ is a stable point of $M' = h(M)$ on $G(n,m)$ if and only if there exist a partition $\{J, K\}$ of $\{1, 2, \ldots, n\}$ such that $\bar{M} = \begin{pmatrix} \bar{M}_J & 0 \\ 0 & \bar{M}_K \end{pmatrix}$ and $\bar{M}_J = I$ and $\bar{M}_K = 0$.*

Proof. If $\bar{M}_J = I$ and $\bar{M}_K = 0$, then $n_J = m_J = m$, $m_K = 0$ and $n_K = n - m$. There are $m_J(n_K - m_K) = m(n-m)$ eigenvectors of type II (as shown in Lemma 4.5) with $\bar{M}_J u = u$ and $\bar{M}_K v = 0$, which are associated with the eigenvalue -1. This shows that all eigenvalues are negative. Therefore, by Theorem 5.3, $\bar{M}$ is stable.

Conversely, if $\bar{M}_J = I$ and $\bar{M}_K = 0$ do not both hold, either $\bar{M}_J$ has an eigenvalue equal to 0 or $\bar{M}_K$ has an eigenvalue equal to 1. If $\bar{M}_J$ has an eigenvalue equal to 0, let $d_J \neq 0$ be a corresponding eigenvector, i.e., $\bar{M}_J d_J = 0$, otherwise, let $d_J = 0$. If $\bar{M}_K$ has an eigenvalue equal to 1, let $d_K \neq 0$ such that $\bar{M}_K d_K = d_K$, otherwise, let $d_K = 0$. Then, $d = \begin{pmatrix} d_J \\ d_K \end{pmatrix} \neq 0$ satisfies the condition in Lemma 4.4. Hence, by the lemma, the matrix $D = \bar{M}[d](I - \bar{M}) + (I - \bar{M})[d]\bar{M}$ is nonzero because $D\mathbf{1} = d \neq 0$, and D is an eigenvector of $\nabla h(\bar{M})$ on $G(n,m)$ associated with the eigenvalue $\lambda = 1$. Therefore, by Theorem 5.3, $\bar{M}$ is not stable. □

By Theorem 2.10, starting from any point $M_0 \in G(n,m)$ the path $M(t)$ determined by $M' = h(M)$ must converge to a stationary point M_∞. In this sense, we say that M_0 is connected to M_∞. Every point in $G(n,m)$ is either connected to a stable point or an unstable stationary point. By the stable manifold theorem, cf. [2] §2.7, the flow into an unstable point cannot be of full dimension. Thus, the dimension of the set of all unstable stationary points and all points connected to them is less than $m(n-m)$. Therefore, one can imagine that $G(n,m)$ consists of $\binom{n}{m}$ attraction regions, each associated with a stable point, and the boundaries of attraction regions comprise unstable stationary points and points connected to them. We have characterized all stationary points. However, we have not been able yet to determine all points connected to unstable stationary points. The explicit description of eigenvectors of the Jacobian of h at stationary points will be useful to characterize the points connected to the stationary points and thus to characterize the boundaries of attraction regions.

6 Possible topics and questions

6.1 Characterization of attraction regions

We have had some clues, but have not been able yet to determine the boundaries of attraction regions. This will hopefully be done in the near future.

There are many other interesting questions, for instance:

(i) Let S_i denote the attraction region of the stable point M_i, $i = 1, \dots, \binom{n}{m}$. Do any two attraction regions S_i and S_j share a common boundary, i.e., $\bar{S}_i \cap \bar{S}_j \neq \emptyset$?

(ii) Can the attraction regions, which are defined by the differential equation, be defined purely by the geometry of the Grassmannian?

6.2 Algorithms for finding the stable point connected to a given starting point

Given any point $M_0 \in G(n, m)$, there are many possible ways for finding a stable point connected to M_0:

(i) Solve the differential equation $M' = h(M)$, $M(0) = M_0$. Or, follow the path $M(t)$ approximately, as $t \to \infty$.

(ii) Solve the algebraic equation $h(M) = 0$ by the Newton method or other methods, starting from M_0.

(iii) Determine the attraction region in which M_0 is located through conditions characterizing attraction regions.

There are many questions related to the implementation and efficiency of these methods.

How to keep iterates generated by the methods (i) and (ii) on $G(n,m)$?

Can the Newton method (ii) find the stable point which is connected with M_0, but not other stable points?

Are the methods (i) and (ii) essentially equivalent? More precisely, if (i) uses Euler scheme $M^{k+1} = M^k + \Delta t \cdot h(M^k)$ and (ii) uses Newton scheme $M^{k+1} = M^k - \alpha \cdot \nabla h(M^k)^{-1} h(M^k)$ or a first-order scheme, are these two schemes equivalent?

How to analyze the complexity of these algorithm? Can we obtain any new complexity results? What role the curvature of $G(n,m)$ will play in the complexity analysis?

For solving individual LP instance, these methods may not be useful and necessary. However, these methods may be needed, together with other means, for solving sets of LP instances, as will be discussed in the following subsections.

6.3 Solving sets of linear programs

Parametric LP, stochastic LP and robust LP directly or indirectly involve sets of linear programs.

A set of linear programs can be viewed as a set of points in $G(n,m)$. Explicit representation of such sets in $G(n,m)$ shall be studied. Geometric structure of such sets can be helpful for finding efficient algorithms. Algorithms in previous subsection together with characterizations of the boundaries of attraction regions can be useful in solving these problems.

6.4 Complexity of IPM from perspective of probability

If we can estimate the complexity of the interior point method (IPM) for each instance (A,b,c), i.e. for each point $M = M(A,b,c) \in G(n,m)$, and we denote this complexity by $C(M)$, then the average complexity of IPM is

$$\frac{\int_{G(n,m)} C(M) d\mu(M)}{\int_{G(n,m)} d\mu(M)}.$$

where μ is a probability measure on $G(n, m)$. A way to estimate the complexity of IPM for an instance (A, b, c) is to estimate the curvature integral starting from the center of (A, b, c), cf [5, 6].

Since the Grassmannian $G(n, m)$ and the differential equation $M' = h(M)$ are symmetric under permutations, we need only to analyze complexity in one attraction region.

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