SURVIVABLE ENERGY MARKETS

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Abstract. In this paper we present a centralized model for managing, at the same time, the day-ahead energy market and the reserve market in order to price through the market, beside energy, the overall cost of reliability and to assure that the power grid survives the failure of any single components, so to avoid extended blackouts. The model addresses, also, the very important point of long-term market efficiency, giving the right economic signals and incentives to guide investments in transmission as well as in production facilities. The model does not use the locational price scheme but determines a unique energy price (which reflects, though, externalities due to transmission and operational constraints) by the means of an auction scheme. In the proposed market model, energy price, reserve price and transmission charges are kept separate as in decentralized systems but they are jointly determined. The model is based on a mathematical optimization program and we propose a simple heuristic coupled with a *cutting plane algorithm* to solve it very quickly. The *cutting plane algorithm* requires an extended polyhedral study presented in the second part of the paper.

Key words. energy markets design, survivable networks, polyhedral characterization.

AMS subject classifications.

1. Introduction.

Recent liberalization of the energy sector in many countries has brought forth market mechanisms for the efficient allocation of demand/supply. The subsequent utility industry restructuring has led to separate the Generation, Transmission and Distribution functions so that these activities are now owned separately and interact through the activities of independent organizations. The reason for this is that the "power grid" is an incredible complex mechanism and needs to be operated in a centralized way. The activities of this central organization, from now on referred to as the *Network Operator*, span real time operations as well as the planning phase. In real time operations, its essential activity is to "follow the load" (i.e. continuously match energy supply and demand) and to "manage congestion" (that can occur due to scarce transmission capacity): it does so by increasing or decreasing energy production in real time. Because of technical constraints (such as ramp rates, capacity, minimum load, etcetera), not all generators can adjust easily and quickly their output. Hence, the *Network Operator* needs to acquire into the market, before actual operations, *Reserves* that are to be used to keep the system stable. Also, depending on the market structure, some *Network Operator* can rely on real time markets in which generators offer to increase or decrease energy production for a price. In the planning phase the latitude of the *Network Operator* activities varies according to the chosen market design. In any case, though, the *Network Operator* needs to assure that the scheduled energy production meets the transmission system constraints. The *Network Operator* is also responsible for very high system reliability. Its guiding principle is the so called "N minus 1 criterion": "the system must be operated in such a way to remain secure upon failure of the most important component (generator or transmission line)". In general, it is required to ensure a high degree of "survivability" for the many contingencies, more or less likely, that can happen. This implies that the network operator does not only need to real-time operate the grid but also needs to plan for the unexpected. The severe blackouts that were experienced recently in so many countries, though, have brought many to think that energy markets operating procedures need to be revised. This operational

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complexity makes the design of efficient energy markets a real challenge. If we look at the models for energy markets proposed in the literature and implemented worldwide we can make a first big classification: decentralized systems and centralized systems. In the former, implemented, for example, in California and Scandinavia (NordPool), there are usually several markets (a market for energy, a reserve market, a real time imbalance market, a congestion management market, etc.) which are run in sequence and in which prices are set (mostly) as equilibria between supply and demand. In the latter, the *Network Operator* optimizes the allocation of energy subject to operational constraints by solving an appropriate optimization program. Procedural rules varies a lot in practical implementation but, in theory, the *Network Operator* should receive supply and demand curves (possibly constrained: e.g. minimum load, maximum capacity, maximum price, etc.) and, keeping in consideration all system and operative constraints, should find the least cost allocation that optimize each player utility function. Energy prices are set as the shadow prices at each node in the network. This method of determining prices is called "nodal pricing" and it reflects the cost of producing energy as well as transmission charges and possibly other costs related to reliability and security. The nodal price method is deemed necessary because it is not possible to use explicit transmission capacity price curves in the optimization program since transmission capacity usage is dictated only by physics laws. In most practical implementations, though, not all operational constraints are considered at the same time: this is to avoid that the corresponding optimization program becomes intractable or solved too slowly with respect to market needs. In particular, usually, the reservation plan is made after a feasible (with respect to transmission constraints) energy production plan is obtained. Also, the *n-1* criterion reliability analysis is a static analysis that is pursued after a feasible energy production schedule is available. No model that explicitly price, through the market, the cost of system reliability has been presented, although this need was first expressed in [17]. For an in-depth analysis of energy markets design and procedural rules implemented in the various markets worldwide please refer to [2],[10],[13],[17], [31],[39],[48],[49],[50], [52],[59],[60],[63],[64],[65], [66]. For survivable network models the reader is referred to [28], [45], [56]. If we would like to compare the two systems, decentralized and centralized, it is clear that, from a pure optimization standpoint, a centralized system in which all operative constraints are considered at the same time gives the "best" solution. Moreover, it avoids "gaming" that can easily happen when running sequential markets. For example, a supplier with a location advantage will withhold some of its capacity from the day ahead energy market if it can profit more in subsequent markets. This strategic behavior can create artificial system congestion or an energy price sky-rocket increase in the real-time spot market. This is actually what happened in California when its decentralized energy market started operations. To avoid gaming various counter-measures can be taken. For example, in Scandinavia (NordPool) price bid in subsequent market are carried over from the day ahead energy market. The advocates of decentralized systems, though, support the idea that gaming can be suppressed by the right procedural rules and that the optimal dispatch of an optimization program is usually just an approximation to the real optimal solution with no real guarantee that energy prices paid by demanders will be lower than those obtained in a decentralized system. The real strength, though, of decentralized system supporters is in the claim that the price formation in tight power pools is too obscure and doesn't give the right economic signals to market participants, neither in the primary (bidding) nor in the secondary (financial) markets. The separation among the various markets, in fact, makes clear the cost component of each resource in the determination of the final price. Also, a separate energy market allows the determination of a unique energy price that clears the market. The determination of a single price for energy is important if one is to construct a secondary market for forward, futures, hedges, etc. The point made by decentralized system advocates of the need of a single energy price that clears the market is important but this by itself doesn't disqualify centralized systems. In fact, we will shortly present a centralized system model in which a single price is determined. Also, the model addresses the problem of giving the right incentives to market players to reach long-term efficiency, in particular regarding transmission utilization. In the current models, both centralized and decentralized, transmission asset owners receive the "congestion rent" only when the transmission capacity is used up to its limit. This scheme doesn't give any incentive to transmission assets owners to install more capacity. Actually, it gives incentives to remove capacity! Finally, some consideration about price determination. In decentralized systems the energy price is determined as an equilibrium price between supply and demand. Following Hicks, [30], we can say that it is a *flex-price* system. This is the way prices are determined in basically all commodity markets. But, even if electricity can be seen from many point of views as a commodity, it has a very small demand elasticity which becomes practically zero in the day-ahead and hour ahead market. Small consumers (that make up most of the electricity consumption) are barred by directly bidding in energy markets and large consumers (e.g. industrial facilities) can not economically change their production schedules (that consume energy) in the time frame of a day or few hours ahead. Hence, the demand bids in the market are from retailers and distribution companies and they do not reflect the true utility functions of final consumers. Retailers and distributors derived demand is anyway quite inelastic. As the time horizon increases, though, demand can become more price sensitive. In this case, though, long-term bilateral contracts (either privately managed by the parties or sold through an exchange) are best suited to pursue demand interests. But actually they are also best suited to pursue supply interests, especially those of energy producers whose optimal dispatch configuration requires to generate energy continuously at the same rate for consecutive hours. The day-ahead and hour-ahead markets are, hence, best suited for a *fix-price* system in which the price is set by sellers (suppliers) on the basis of their own costs. The appropriate market design is then that of an auction among all suppliers in order to allocate the energy demand (known or predicted). Various type of auction can be used, all of them having has an output the allocation of energy demand at a single price.

2. Survivable energy markets: a comprehensive model. In this section we will present a centralized model for managing, at the same time, the day-ahead (or hour-ahead) energy market and the reserve market in order to optimize, beside energy price, the reliability costs, both hidden and not-hidden. The model assures that the power grid survives failure of any single network component and that a given percentage of the total demand will be shielded by the risk of blackouts without the need of securing reserves. Energy price is not determined, as usual in centralized model, by the shadow prices at each node, but by the means of a constrained auction. This market model has the advantage of providing a unique energy price for the entire electrical grid (no zones) and to give clear economic signals to market player as to price formation and competitors profiles. In the proposed market model, energy price, reserve price and transmission charges are kept separate as in decentralized systems but they are jointly determined. The model allows bilateral contracts which are priced as CFD (Contracts for Differences). Finally, a simple scheme is proposed to reward transmission assets owners consistently, giving them the right incentives to upgrade capacity, if needed. In the proposed market model, demand is considered totally unelastic (*auction*) but one could implement as well a *first best allocation* by allowing demand bidding, so optimizing social welfare. We assume that the market is cleared, sequentially, every hour, independently of bids submitted in different hours. This is customary in basically all the commodity markets and is the design adopted in most energy markets

As said, the proposed market model is a constrained auction, in which suppliers bid in their hourly supply curve as well as their capacity reservation prices and the *Network Operator* selects, among all the bids and among all the bilateral contracts production schedules, the production pattern that will satisfy demand at the minimum overall cost while abiding to operative constraints and will determine an optimal reservation plan. Beside transmission constraints (Kirchoffs laws, capacity limits, etc.) the model explicitly considers the cost of reliability, represented by the "N minus 1 criterion". Most implemented systems do an ex-post, static analysis of the reliability constraints: given an energy flow, they simulate system behavior when a single link fails, i.e. they will check if the flow will safely redistribute over remaining lines upon the failure of any single transmission link. If this feasibility check fails, ad hoc procedures are undertaken to get a feasible, reliable energy flow. Once a feasible energy flow is obtained, the *Network Operator* will still need to reserve capacity in the *Reserve Market* to face the event that a single supplier, scheduled for production, is cut off from the grid. This can happen if, for any reason, all of a sudden, the supplier can not provide all or part of the scheduled energy or if a transmission line cuts off the supplier from the demand locations. The cost of *Reserves* is paid by all market players and, in the end, by the final consumer. As we will show, appropriate production schemes, that redistribute production load among various network locations, can significantly reduces the amount of this type of reserves.

Let us detail the model now. Let be n the number of suppliers bidding both for energy and reserve and let be m the number of suppliers bidding only for reserve. A supplier i bidding for both will submit, hence, an energy unit price function $P_i(q)$ and a reserve (or availability) unit price function $R_i(q)$. Similarly, a supplier bidding only for reserves will submit a reserve unit price function $S_i(q)$. Here *P(q)* indicates the minimum unit price that a supplier is willing to accept for producing an amount q of energy while *R(q)* and *S(q)* indicates the unit price that a supplier needs to be paid for reserving capacity up to the amount q. These availability prices are paid regardless the supplier's actual energy delivery in real time operations. Suppliers are usually constrained by a minimum load m and a maximum capacity C. We may easily embed these constraints in the supply and reserve curves by setting a huge value in the intervals $[0, m[$ and $]C, \infty]$. A supplier i that entered bilateral contracts will submit, also, the total contracted amount, b_i. Bilateral contracts do not have priorities over transmission lines capacity usage: they are treated as bids with zero unit price up to the amount agreed in the contract. Let us indicate with $h_i \geq 0$ the variable amount of energy to be produced by supplier i according to its bid and with $g_i \geq 0$ the variable amount of energy to be produced by supplier i up to the limit b_i contracted in bilateral contracts. For ease, let us introduce the variable $f_i = h_i + g_i$ which represents the total variable amount of energy produced by supplier i. Let us indicate with $x_i \geq 0$ the variable amount of capacity reserved at supplier i in the *day-ahead market* and with $y_i \geq 0$ the variable amount of capacity reserved at supplier

i in the *reserve market*. Usually capacity is reserved in integer amounts so that x_i and y_i are required to be integer. As we know, there are different types of *reserves* (spinning, not spinning, etc.), hence, we should distinguish among them by adding a variable for each type and for each supplier. For the sake of simplicity, we will assume that there is only one type of *reserve*, keeping in mind that all the results hold true (with the obvious modifications) when all the different types of *reserves* are modelled in the optimization program. Demand at each network location, d_j is predicted and decomposed in two components: the part covered by bilateral contracts d_j^{cov} and the part to be covered by bids d_j^{bid}. The total amount of energy demanded across all locations is hence $D = \sum_j d_j$. The total amount of reserve is usually computed by the *Network Operator* as a percentage of total demand. Let be $H = \lceil (1 + \rho\%)D \rceil$ where ρ is the reservation parameter.

The *Network Operator* needs to solve the following optimization program:

$$Min \sum_{i=1}^{m} S_i(y_i)y_i + \sum_{i=1}^{n} (R_i(x_i)x_i + P_i(h_i)h_i)$$

s.t.

$$h_i + g_i = f_i \qquad \forall\ i = \{1, \ldots n\} \tag{2.1}$$

$$g_i \leq b_i \qquad \forall\ i = \{1, \ldots n\} \tag{2.2}$$

$$y_0 = \sum_{i=1}^{m} y_i \tag{2.3}$$

$$\sum_{i=1}^{n} f_i = D \tag{2.4}$$

$$\sum_{i \neq j}^{n} f_i \geq L \qquad \forall j = \{1 \ldots n\} \tag{2.5}$$

$$y_0 + \sum_{i \neq j}^{n} x_i \geq \lceil D \rceil \qquad \forall j = \{1 \ldots n\} \tag{2.6}$$

$$\sum_{i=1}^{n} x_i + y_0 \geq H \tag{2.7}$$

$$f_i \leq x_i \qquad \forall\ i = \{1 \ldots n\} \tag{2.8}$$

$$x_i \leq C_i \qquad \forall\ i = \{1 \ldots n\} \tag{2.9}$$

$$y_i \leq C_i \qquad \forall\ i = \{1 \ldots m\} \tag{2.10}$$

$$F = (f_1, \ldots, f_n) \in A \tag{2.11}$$

$$F = (f_1, \ldots, f_n) \in A^t \qquad \forall\ t \in E \tag{2.12}$$

$$F^j = (y_1, \ldots y_m, x_1, \ldots x_{j-1}, x_{j+1}, \ldots x_n) \in A \qquad \forall j = \{1 \ldots n\} \tag{2.13}$$

$$f_i \geq 0; \;\; g_i \geq 0; \;\; h_i \geq 0; \;\; x_i \in N^+; \qquad \forall\ i = \{1, \ldots n\} \tag{2.14}$$

$$y_i \in N^+ \qquad \forall\ i = \{1, \ldots m\} \tag{2.15}$$

Constraint (2.4) equates supply and demand while constraint (2.7), (under our assumption of one single type of reserve), just says that the total amount of capacity reserved, both in the day-ahead market and in the reserve market, is equal to the total demand plus a given percentage of it. Constraint (2.6) assures that, in the case generator j is cut off from the grid, there is enough reserved capacity to cover the demand D while constraint (2.5) expresses the trade off between load balancing and reserve requirement. This constraint assures that, if any supplier is cut off from the grid, there is still, at least, an amount of flow L circulating in the grid, so that, at most, the amount $\lceil D - L \rceil$ needs to be reserved. The higher the value of L the lower the amount one needs to reserve. There is, hence, a clear trade-off between load balancing across location (even if this may imply scheduling for dispatch more expensive generators) and reserving capacity in the *Reserve Market*. L can be a variable set in the optimization program, by adding, for example, the constraint $L = D - \sum_{i=1}^{m} y_i$ or it may be a parameter derived by physical and reliability constraints. For example, when a generator is cut off from the grid the reserved energy needs to be available basically in real time in order to avoid blackouts. If the total available capacity in the market for this type of reserve is T then it has to be the case that $L \geq D - T$. In general, if L is the amount that *survives* any failure situation then $\frac{L}{D}$ is the percentage of total demand that is automatically shielded by possible blackouts, at least theoretically. Constraint (2.8) is the supplier's availability constraint while (2.9) and (2.10) are the supplier's capacity constraints. We assume here that the minimum load requirement is embedded in the bid functions. We could easily model other supplier's constraints such as ramp rates, sustained duration of the load, etc. For example, if a supplier can't adjust its production schedule in real time then we may add the constraint that $\lceil f_i \rceil = x_i$ or, if he can increase its production only in a given range from its scheduled production level, we may write $x_i \leq \lfloor f_i + r(f_i) \rfloor$ where $r(f_i)$ is a value depending on the flow value f_i. In the proposed model we are taking the view that suppliers with complex production and inter-temporal constraints (such as thermal generators) can optimize their own production schedule by entering in bilateral contracts. We believe that this scheme should give incentive to these suppliers to be more active in the forward market of bilateral contracts and, in general, in the secondary market, contributing, hence, to its depth and liquidity. Finally, constraint (2.11) summarizes transmission constraints, (i.e.) A is the set of production pattern which are feasible with respect to network constraints, particularly the capacity limit at each line. Constraint (2.12) guarantees that the flow safely redistributes over the remaining lines upon failure of a single transmission link t: A^t is the set of feasible, with respect to transmission constraint, production plan in a network in which the link t has been removed. The constraint (2.13) guarantees that the reservation plan is feasible with respect to network constraints. In fact, we indicate by $F^j = (y_1, \ldots y_m, x_1, \ldots x_{j-1}, x_{j+1}, \ldots x_n)$ all the possible flow patterns that can be created by using the reserved capacity in order to satisfy demand D when supplier j is cut-off from the grid. Then, it has to be that at least one of this flow pattern satisfies transmission constraints. For brevity we may simply write $F^j \in A$.

By solving this program the network operator will get a least cost feasible production plan $F = (f_1^*, \ldots, f_n^*)$ as well as a reservation plan $Res = (x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*)$. Let be N^* the set of suppliers scheduled for production in the day ahead market, (i.e.) $f_i^* > 0$ for all $i \in N^*$ and $f_i^* = 0$ for all $i \notin N^*$. Also let be $Q^* \supseteq N^*$ the set of suppliers in the day ahead market included in the reservation plan and let M^* be the set of suppliers in the reserve market included in the reservation plan, (i.e.) $x_i^* > 0$ for all $i \in Q^*$, $y_i^* > 0$ for all $i \in M^*$ and $x_i^* = y_i^* = 0$ otherwise. We can set the market price for energy, p, as $p = Max_{\{i \in N^*\}}\{P_i(h_i^*)\}$ (we remind the reader that $f_i^* = g_i^* + h_i^*$). Similarly, we could use the same pricing scheme for *reserves*, namely,

$r = Max\{Max_{\{i \in Q^*\}} R_i(x_i^*), Max_{\{i \in M^*\}} S_i(y_i^*)\}$ or, we could pay each selected supplier the amount it tendered. Bilateral contracts will be cleared as CFDs (see ([17])). Each demand node will be charged, hence, $p * \hat{d}_i^{bid} + r * (1 + \rho\%)d_i$, where $\hat{d}_i^{bid}$ is the actual demand fulfilled at location i, net of bilateral contracts which will be finalized as CFDs. Each supplier i will receive, accordingly, $r * x_i^*$ or $r * y_i^*$ plus $p * \hat{h}_i^*$ where $\hat{h}_i^*$ is the actual energy delivered by supplier i, net of bilateral contracts. A transmission charge will be collected from each market player (demander and supplier) proportionally to the used transmission capacity and distributed to transmission assets owners proportionally to their effectively used capacity. This transmission charge will be based on a transmission capacity uniform unit price, t_e, determined by keeping in consideration costs related to maintenance and the capital opportunity cost. At operation, each demand node j will be charged the amount $\frac{1}{2} t_e * \hat{d}_j$ where $\hat{d}_j$ is the actual demand realized at operation at location j while each supplier i will be charged $\frac{1}{2} t_e * \hat{f}_i$ where $\hat{f}_i$ is the amount of energy flow actually dispatched by supplier i at operations. The owner of the transmission link e will receive $t_e * \hat{f}_e$ where $\hat{f}_e$ is the actual flow on the transmission link e. Its owner, hence, can receive at most $t_e * c_e$ where c_e is the link capacity. This scheme rewards transmission assets owners proportionally to the amount of the owned capacity that is effectively used to transmit energy and not for its utilization (proportion of capacity used). If a transmission link is congested the owner doesn't get an additional benefit but the link capacity limit becomes an obstacle for additional revenue stream. This reward scheme gives the right economic incentives to transmission assets owners to upgrade capacity if needed. Our proposed optimization model can be used to carry simulations to help investors decide which is the most profitable amount of capacity that needs to be added and where. There are other information that can be collected to give market players the right economic signals to guide behavior and investments. For example, let us run, by using the same energy bids, an unconstrained (with respect to transmission and reliability constraints) auction to fulfil demand D at the minimum price. Let T^* be the optimal set of suppliers. If $T^* = N^*$ then the operative constraints are not hindering competition. The selected suppliers are the most efficient in producing energy and other suppliers need to reduce cost and/or invest in more efficient technologies if they want to compete. If, though, $T^* \neq N^*$ then $N^* - T^*$ will indicate those suppliers that have a competitive advantage due to their position in the network. This information could be used to guide investments in more efficient production facilities in these locations so that in the long run we should have $T^* = N^*$. We may as well run the optimization program in which, though, all transmission capacity limits have been removed. Let be V^* the set of suppliers selected in this case for dispatch. Again, if $V^* \neq N^*$ then transmission constraints are favoring less efficient suppliers. By analyzing the transmission line flow one can easily see how much capacity needs to be added and in which lines in order to have $V^* = N^*$.

To summarize, the proposed model merges together the *day ahead market* and the *reserve market* through the survivability requirements. Basic optimization theory tells that this scheme is more cost efficient than running the two markets in sequence, as it is done in most energy markets (both centralized and decentralized). Also, by merging these two markets, one avoids possible gaming from suppliers and unwanted competitive advantage for some suppliers due to their location. Altogether these elements heavily support the idea that an efficient and economically sound energy market design needs to merge somehow and run at the same time the *day-ahead energy market* and the *reserve market*. The proposed market model does not use the nodal pricing scheme, as existing centralized energy market do, but a constrained auction scheme. The final energy price as well as the reliability cost are deduced by actual bids and this helps market players to better define their competitive profile and bidding strategies. Finally, the simple transmission pricing scheme proposed in the model gives transmission assets owners the right economic incentives to upgrade capacity, if needed. In the long run, the proposed model should guarantee that investments in transmission facilities as well as in more efficient energy production technology will be pursued. As of today, none of the existing models has completely addressed the very important point of the market long term efficiency.

3. Solving the model: a polyhedral approach. The last (but not least) point that needs to be addressed is how to solve the optimization program underlying the proposed market model. Detractors of centralized models often stress the fact that the actual "optimal" solution obtained by solving complex, constrained optimization models, is usually only an "approximation" of the real optimal solution so that the supposed superiority of this solution to the one obtained through the typical market mechanisms in place in decentralized models is all to be proved. Also, the market clearing mechanisms of decentralized models are extremely fast in providing solutions, and speed is a key element in this environment where market is cleared every hour. Hence, we need to assure that we are able to solve the proposed optimization program very quickly and that the obtained solution is optimal or very close to it. First of all, note that, by adding variables, we may approximate the bid functions by linear functions. Also, in basically all real markets, the bid functions are given as step functions, (i.e.) constant over disjoint intervals so that we can solve our optimization program by solving an equivalent problem whose objective function is linear. Hence, from now on, under very mild assumptions, we will assume, for the sake of simplicity, that the bid functions are linear functions. Also, since our purpose here is to find the cheapest allocation of energy supply and reserve, we can linearize the transmission constraints. The problem we want to solve, hence, will be a Mixed-Integer Program i.e. an optimization program with a linear objective function and linear constraints except for the requirement that some variables need to be integer.

$$Min \sum_{i=1}^{m} s_i y_i + \sum_{i=1}^{n} (r_i x_i + p_i h_i)$$

s.t.

$$h_i + g_i = f_i \qquad \forall\ i = \{1, \dots n\} \tag{3.1}$$

$$g_i \leq b_i \qquad \forall\ i = \{1, \dots n\} \tag{3.2}$$

$$y_0 = \sum_{i=1}^{m} y_i \tag{3.3}$$

$$\sum_{i=1}^{n} f_i = D \tag{3.4}$$

$$\sum_{i \neq j}^{n} f_i \geq L \qquad \forall j = \{1 \dots n\} \tag{3.5}$$

$$y_0 + \sum_{i \neq j}^{n} x_i \geq \lceil D \rceil \qquad \forall j = \{1 \dots n\} \tag{3.6}$$

$$\sum_{i=1}^{n} x_i + y_0 \geq H \tag{3.7}$$

$$f_i \leq x_i \qquad \forall\ i = \{1 \dots n\} \tag{3.8}$$

$$x_i \leq C_i \qquad \forall\ i = \{1 \dots n\} \tag{3.9}$$

$$y_i \leq C_i \qquad \forall\ i = \{1 \dots m\} \tag{3.10}$$

$$F = (f_1, \dots, f_n) \in A \tag{3.11}$$

$$F = (f_1, \dots, f_n) \in A^t \qquad \forall\ t \in E \tag{3.12}$$

$$F^j = (y_1, \dots y_m, x_1, \dots x_{j-1}, x_{j+1}, \dots x_n) \in A \qquad \forall j = \{1 \dots n\} \tag{3.13}$$

$$f_i \geq 0;\ \ g_i \geq 0;\ \ h_i \geq 0;\ \ x_i \in N^+; \qquad \forall\ i = \{1 \dots n\} \tag{3.14}$$

$$y_i \in N^+ \qquad \forall\ i = \{1 \dots m\} \tag{3.15}$$

Mixed Integer Programs, or MIP for brevity, are sometimes hard to solve through general purpose algorithms (and, hence, through commercial code for MIP problems) and to get an optimal solution may require more time than is granted in energy markets. Hence, we may think of getting an approximate solution, if, though, this solution is not too far away from the optimal one. Note that even a small difference between the optimal solution and the approximate one may be a significant value in absolute terms. After all, the optimal solution value is the energy and reservation cost of an entire electrical network!!! If we look at the literature, there have been many computational studies on survivable networks, ([6],[18],[42],[51]), basically all based on *cutting plane algorithms* and *branch and cut* method. The efficiency of both approaches relies, more or less, on a good knowledge of the polyhedron that defines the set of feasible points and there has been an extensive study on polyhedra arising from survivable networks, ([5],[11],[19],[20],[26],[27]). Here, in order to quickly solve our problem, we propose a *cutting plane algorithm*, based on an extensive polyhedral study, followed by a heuristic. In our testing, the proposed approach has always yield the optimal solution except one case. The gap, though, from the solution obtained and the true optimal solution was very small $(0,02\%)$. This case had energy costs comparable with reservation costs which is not, anyway, likely to happen in real markets. We have studied the structure of two polyhedra, referred to in the sequel as $F_n(L)$ and $Q_{n+1}(D,H)$. The polyhedron $F_n(L)$ is defined by the following set of inequalities:

$$\begin{aligned} \sum_{i \neq j} f_i \quad &\geq L \qquad \forall\ j = \{1 \dots n\} \\ x_i \quad &\geq f_i \qquad \forall\ i = \{1 \dots n\} \\ x_i \in Z^+,\ \ f_i &\geq 0 \qquad \forall\ i = \{1 \dots n\} \end{aligned}$$

i.e. inequalities (3.5),(3.8), (3.14) of our model; while $Q_{n+1}(D,H)$ is defined by:

$$\begin{aligned} x_0 + \sum_{i=1}^{n} x_i \ &\geq H \\ x_0 + \sum_{i \neq j}^{n} x_i \ &\geq D \quad \forall j = \{1 \dots n\} \\ x_i &\in Z^+ \quad \forall\ i = \{0 \dots n\} \end{aligned}$$

where H and D are supposed integer and $H \geq D$. The $Q_{n+1}(D,H)$ defining inequalities correspond in our model to the inequalities (3.6),(3.7),(3.14) and (3.15). We are able to characterize completely the polyhedral structure of $Q_{n+1}(D,H)$ so that the *cutting plane algorithm* applied to an optimization problem over $Q_{n+1}(D,H)$ always yield the optimal integer solution. The $F_n(L)$ polyhedral structure is, instead, more complex. Nonetheless, we are able to completely describe it via extreme points and exhibit a class of facets which, together with those described in [5], are very effective in closing the gap between the LP relaxation and its MIP optimal value. The following table summarize our findings:

Table 3.1
Testing on $F_n(L)$. LP is the original formulation with the integrality constraints relaxed: LP-cuts is the LP formulation plus the facets derived by our polyhedral study and those in [5]

Average Gap MIP-LP	Min Gap	Max Gap	Average Gap MIP-LPcuts	Min Gap	Max Gap
$1,69\%$	$0,46\%$	$4,25\%$	$0,18\%$	0%	$0,44\%$

The separation procedure on both polyhedra is quite simple and is based on adding the facets which are tight for points with n^* (or n^*+1 and n^*-1) positive components, where n^* is the number of positive components of the LP optimal solution. Going back to our original problem, after running the *cutting plane algorithm* we do reduce the gap between the MIP and the LP problem but we do not get, usually, a feasible solution. Instead of solving the MIP to optimality through the standard methods of *branch and bound* and *branch and cut*, which can be too slow for the needs of the energy markets, we resort to heuristics. We could, for example, use a very simple heuristic, based on rounding each variable required to be integer to its ceiling, (i.e.) the smallest integer greater than its value. This procedure always yield a feasible solution under the assumption, met in all electrical networks planning models, that small variations in a supplier production output are feasible towards real transmission constraints and that each supplier maximum capacity is an integer value in the mathematical model. This rounding procedure, however, most of the time, will not give the optimal MIP solution, as it is confirmed by our testing. We need to find a better heuristic. In the sequel, we will assume that we can increase and/or decrease by a very small amount each supplier's output (and hence reserve) without violating the real transmission constraints. This "assumption" is not really such since it actually reflects the *Network Operator* modus operandi . In fact, when scheduling production, the *Network Operator* will never allow lines to be loaded up to their limit since it knows that, in real time operations, demand (and even supply) can diverge from the planned one and hence it needs to have room to real-time adjust production. The heuristic we propose is the following: let $P^* = (f_1^*, \dots f_n^*, x_1^*, \dots, x_n^*, y_1^*, \dots y_m^*)$ be the optimal solution obtained from the cutting plane algorithm. If x_i and y_i are integer for all i then we have found the MIP solution. If that is not the case, we note that the optimal MIP solution, let us call it $\hat{P} = (\hat{f_1}, \dots \hat{f_n}, \hat{x_1}, \dots, \hat{x_n}, \hat{y_1}, \dots \hat{y_m})$, will have either $f^* = \hat{f}$ or $\sum_{i=1}^n f_i^* = \sum_{i=1}^n \hat{f_i}$ i.e. $\hat{f}$ can be obtained from f^* by redistributing the flow. Suppose that we are in the first case, i.e. $f^* = \hat{f}$. Note that this is going to be the case if energy costs are significantly higher than reservation costs. Hence, from constraint (3.8), it follows that $\hat{x_i} \geq \lceil f_i^* \rceil$. Also, note that, if it is feasible to decrease the flow by ϵ on the x_i variable and increase by the same quantity a variable y_j then it must be the case that $r_i \leq s_j$ since, otherwise, the point P^* wouldn't be optimal. Hence $\hat{y_j} \leq \lceil y_j^* \rceil$. Now, let be y_j such that

$s_j = max\{s_i : m_i < y_i^*\}$ (where m_i is the variable i lower bound) and let be q the number of positive components of x^* which are not integers. Let be $z = min(\lceil x_i \rceil - x_i)$ for all x_i not integer. Let be $t = y_j - \lfloor y_j \rfloor \geq 0$ and $u = \lfloor (q-1)z - t \rfloor$. By setting $w = min(\lfloor y_j \rfloor - m_j, u)$ we can decrease y_j by $w + t$ and increase each not integer variable x_i by $\delta = \frac{t+w}{q-1}$ while remaining feasible. Assuming, without loss of generality, that the q not integer components in x^* are the first q we have that the point $Q^a = (f_1^*, \ldots, f_n^*, \lceil x_1^* + \delta \rceil, \ldots, \lceil x_q^* + \delta \rceil, x_{q+1}^*, \ldots x_n^*, \lceil y_1^* \rceil, \ldots, y_j^* - w - t, \ldots, \lceil y_m^* \rceil)$ is feasible for the MIP and has a better objective value than the point R obtained by the simple rounding heuristic, namely, $R = (f_1^*, \ldots f_n^*, \lceil x_1^* \rceil, \ldots, \lceil x_n^* \rceil, \lceil y_1^* \rceil, \ldots \lceil y_m^* \rceil)$. By our choice of δ, in fact, is $\lceil x_i^* + \delta \rceil = \lceil x_i^* \rceil$. Moreover, if $w = u$, the point Q has, at least, two more coordinate integer, namely, y_j and x_l where $z = \lceil x_l \rceil - x_l$. If, instead, $w < u$ then y_j will drop to its (integer) lower bound. Let us consider now $k = argmax\{r_i : x_i > \lceil f_i \rceil\}$. If $r_k > s_j$ then we need to check the value of another feasible point. Hence, let be $z' = min(\lceil x_i \rceil - x_i)$ for all x_i not integer and $i \neq k$. Let be $t' = x_k - \lfloor x_k \rfloor \geq 0$ and $u' = \lfloor (q-2)z' - t' \rfloor$. By setting $w' = min(\lfloor x_j \rfloor - \lceil f_k \rceil, u')$ we can decrease x_k by $w' + t'$ and increase each not integer variable x_i, $i \neq k$ by $\delta' = \frac{t'+w'}{q-2}$ while remaining feasible. As before, one or two coordinates x_i will become integer. The point $Q^b = ((f_1^*, \ldots, f_n^*, \lceil x_1^* + \delta' \rceil, \ldots, x_k^* - t' - w', \ldots, \lceil x_q^* + \delta' \rceil, x_{q+1}^*, \ldots x_n^*, \lceil y_1^* \rceil, \ldots, \lceil y_m^* \rceil)$ is feasible for the MIP and has a better objective value than the point R obtained by the simple rounding heuristic. We set $Q^1 = min(Q^a, Q^b)$ where the minimum is with respect to the objective function and will reapply the procedure to the point P^1 that is equal to the point Q^1 except that we have removed all the ceiling from the not integer variables. At the end, we will get the point $Q^t = (f_1^*, \ldots, f_n^*, \lceil x_1^t \rceil, \ldots, \lceil x_n^t \rceil, \lceil y_1^t \rceil, \ldots, \lceil y_m^t \rceil)$ where t is the number of iteration of this procedure and where most of the x_i variable will be integer. Let us consider the point $P^t = (f_1^*, \ldots, f_n^*, \lceil x_1^t \rceil, \ldots, \lceil x_n^t \rceil, y_1^t, \ldots y_m^t)$. Let $a = argmin\{s_j : m_j \leq y_j^t < C_j \text{ and } y_j^t \text{ not integer}\}$ and $b = argmax\{s_j : m_j < y_j \leq C_j \text{ and } y_j^t \text{ not integer}\}$ and consider $\delta = min\{(\lceil y_a \rceil - y_a), y_b - \lfloor y_b \rfloor)$. The point P^{t+1} obtained from P^t by increasing y_a by δ and decreasing y_b by δ is feasible, has a no worse objective value than Q^t and has at least one more component y_i integer. By reapplying the procedure, all the y components, except at most one, will be integer. By rounding this one component we will get a point Q^* feasible for the MIP and that will be our approximate solution for the MIP problem. In our testing this heuristic has always yield the optimal MIP solution when $f^* = \hat{f}$. If this is not the case, though, the heuristic still provides a feasible point which is anyway better than the one obtained by simple rounding. In our testing, we came across one case in which $f^* \neq \hat{f}$ and the heuristic provided a solution that was $0,02\%$ away from the optimal MIP solution. We could easily improve the heuristic (by appropriately redistributing the flow among the existing positive variable plus one) to take care of the case in which $f^* \neq \hat{f}$ but, since this is quite a rare event and the improvement is going to be probably quite small, we will not detail here the procedure. The following tables summarize our testing:

TABLE 3.2
Cutting Plane Algorithm: LPcuts is the LP obtained through the cutting plane algorithm, MIP is the problem we want to solve

Average Gap MIP-LPcuts	Min Gap	Max Gap
$0,10\%$	0	$0,37\%$

TABLE 3.3

Heuristics comparison. SRH is the simple rounding heuristic in which the ceiling of each not integer variable is taken, ERH is the enhanced rounding heuristic outlined above

Average Gap MIP-SRH	Min Gap	Max Gap	Average Gap MIP-ERH	Min Gap	Max Gap
0,56%	0,02%	1%	0%	0%	0,02%

A final note. These percentages can be misleading, since they are referred to the whole electrical grid cost, i.e. big numbers, and where usually energy cost is much larger than the reservation cost. Hence, in many cases, the total energy cost in both the LP relaxed solution (and hence the rounded solution) and the MIP solution is the same. Should we consider, instead, the percentage difference between the reservation cost of the rounded solution and the reservation cost of the optimal MIP solution we would get, on the same test bed, an average of 1,60% with a minimum of 0,88% and a maximum of 2,83%.

The proposed scheme, the *cutting plane algorithm* coupled with the *heuristic*, is a polynomial time algorithm versus exact methods for MIPs which are, as known, NP. As our testing proves, we can solve, for all practical means, our model very quickly through this scheme and hence it can be practically used in energy markets. In the next sections we will present the polyhedral study behind the *cutting plane algorithm*.

4. The Polyhedron $F_n(L)$. The polyhedron $F_n(L)$ is defined by the following set of inequalities:

$$\begin{aligned} \sum_{i\neq j} f_i \; &\geq L \quad \forall\, j = \{1 \dots n\} \\ x_i \; &\geq f_i \quad \forall\, i = \{1 \dots n\} \\ x_i \in Z^+, f_i \geq 0 &\quad \forall\, i = \{1 \dots n\} \end{aligned}$$

First, we are going to consider the polyhedron extreme points. This characterization is useful, on one hand, for finding facets by the lifting procedure and, on the other, for proving the validity of inequalities. Next, we will present a class of facets whose structure is quite different than the usual structure found in polyhedra arising from networks and this shed light on the complexity of survivable networks.

The first thing we note is that, where not for the integrality constraints, all $F_n(L)$ extreme points will be of the type:

$$\begin{aligned} x_i = f_i = \frac{L}{|S|-1} &\quad \forall\, i \in S \\ x_i = f_i = 0 &\quad \forall\, i \notin S \end{aligned}$$

for any subset S of indices $\{1 \dots n\}$ with $|S| \geq 2$.

Note also that $F_2(L)$ has a unique extreme point, namely $f_1 = f_2 = L, x_1 = x_2 = \lceil L \rceil$, and that any extreme point of $F_m(L)$ with exactly t positive coordinates naturally correspond to extreme points of $F_{m+k}(L)$ with exactly t positive coordinates.

LEMMA 4.1. *If $W = (x, f)$ is an extreme point of $F_n(L)$ (with at least 3 positive coordinates) such that exists a partition $(S, \tilde{S}, T)$ of indices $\{1 \dots n\}$, with $|S| = m \geq 2$, ($\tilde{S}, T$ eventually empty), for which $x_i > f_i > 0 \;\forall\, i \in S, x_i = f_i > 0 \;\forall\, i \in \tilde{S}$ and $x_i = f_i = 0 \;\forall\, i \in T$, then $\exists \xi \in \Re$, k, μ integers, $\xi > k \geq \mu \geq 0$, such that*

$f_i = \xi,\ \ x_i = \lceil \xi \rceil\ \forall\, i \in S, x_i = f_i = k\ \forall\, i \in \tilde{S} - \{l\}, x_l = f_l = \mu$ *for some* $l \in \tilde{S}$. *If* $|\tilde{S}| \geq 2$ *then* $k + 1 > \xi > k \geq \mu$.

Proof. Let be $W = (x, f)$ an extreme point of $F_n(L)$ and let $(S, \tilde{S}, T)$ be a partition of $\{1 \dots n\}$ as described in the Lemma statement. By contradiction suppose that $\exists\, i, j, k\ \in S$ such that $f_i > f_j \geq f_k > 0$. Then the points $A = (x, g)$ e $B = (x, h)$, with

$$\begin{aligned} g_j =& f_j + \epsilon & h_j &= f_j - \epsilon & \\ g_k =& f_k - \epsilon & h_k &= f_k + \epsilon & \\ g_l =& \ \ f_l & h_l &= f_l & & otherwise \end{aligned}$$

are feasible for $\epsilon > 0$ small enough and $W = \frac{1}{2}A + \frac{1}{2}B$. By a similar argument one can show that $\nexists\, i, j, k \in\ \tilde{S}$ with $f_i > f_j \geq f_k > 0$.

Hence it must be that $f_t = \eta$, $x_t = \lceil \eta \rceil$ for some $t\ \in S$, $f_i = \xi$, $x_i = \lceil \xi \rceil$ for all $i\ \in S - \{t\}$; $f_l = x_l = \mu$ for some $l\ \in \tilde{S}$, $f_j = x_j = k$ for all $j\ \in S - \{l\}$ and $\xi \geq \eta$, $k \geq \mu$. Also, since $|S| \geq 2$, it can not be $k > \xi$. Otherwise, one can, as above, construct points A e B such that $W = \frac{1}{2}A + \frac{1}{2}B$. Similarly, if $|\tilde{S}| \geq 2$ it has to be $k + 1 > \xi \geq k \geq \mu$.

Now let us prove that $\xi = \eta$. By contradiction suppose not. Then we have:

$$\begin{aligned} (m-2)\xi + \eta + (n-m-1)k + \mu &= L \\ (m-1)\xi + (n-m-1)k + \mu &> L \\ (m-1)\xi + \eta + (n-m-2)k + \mu &> L \\ (m-1)\xi + \eta + (n-m-1)k &> L \end{aligned}$$

and it exits $\epsilon > 0$ such that the points $A = (a, f^a)$ and $B = (b, f^b)$

$$\begin{aligned} a_h &= x_h & &\forall\ h \\ f_i^a &= \xi + \frac{\epsilon}{m-2} & &\forall\ i\ \in S - \{t\} \\ f_t^a &= \eta - \epsilon & & \\ f_l^a &= f_l & &otherwise \end{aligned}$$

$$\begin{aligned} b_h &= x_h & &\forall\ h \\ f_i^b &= \xi - \frac{\epsilon}{m-2} & &\forall\ i\ \in S - \{t\} \\ f_t^b &= \eta + \epsilon & & \\ f_l^b &= f_l & &otherwise \end{aligned}$$

are feasible. Let $(\bar{c}, c)$ be a cost vector such that W is the unique solution to the optimization problem:

$$min\ \ (\sum_i \bar{c}_i y_i + \sum_i c_i g_i)\ \ s.t.\ \ (y, g) \in F_n(L)$$

Then

$$\sum_i \overline{c}_i a_i + \sum_i c_i f_i^a > \sum_i \overline{c}_i x_i + \sum_i c_i f_i$$

and

$$\sum_i \overline{c}_i b_i + \sum_i c_i f_i^b > \sum_i \overline{c}_i x_i + \sum_i c_i f_i$$

Hence $\frac{\epsilon}{m-2}\sum_{i\in S-\{t\}} c_i - \epsilon c_t > 0$ and $-\frac{\epsilon}{m-2}\sum_{i\in S-\{t\}} c_i + \epsilon c_t > 0$. Contradiction□

LEMMA 4.2.

If $W=(x,f)$ *is an extreme point of* $F_n(L)$ *(with at least 3 positive coordinates) such that exists a partition* $(S,\tilde{S},T)$ *of indices* $\{1\dots n\}$*, with* $|S|=1$*, for which* $x_i > f_i > 0$ *for* $i\in S, x_i = f_i > 0\ \forall\ i\in\tilde{S}$ *and* $x_i = f_i = 0\ \forall\ i \in T$ *then* $\exists\ \xi\ \in \Re, k, \mu$ *integers such that* $f_i = \xi, x_i = \lceil \xi \rceil$ *for* $i \in S, x_i = f_i = k\ \forall\ i\in \tilde{S}-\{l\}, x_l = f_l = \mu$ *for some* $l\in\tilde{S}$ *and* $k > \xi > k-1 \geq \mu$ *or* $k=\mu > \xi \geq 1$ *or* $k \geq \mu \geq 1 > \xi$.

Proof.

As in the previous proof one can show that $W=(x,f)$ is of the form:

$$\begin{array}{ll} x_i = \lceil f_i \rceil & i\in S \\ f_i = \xi & i \in S \\ f_l = \mu & l \in \tilde{S} \\ f_j = k & \forall\ j\in\tilde{S}-\{l\} \end{array}$$

and $k\geq\mu$. It can not be $\xi > k$ otherwise W will not be an extreme point.

Similarly, if $k > \xi > \mu$ then it must be the case that $k > \xi > k-1$; and if $k \geq \mu \geq \xi \geq 1$ then $k=\mu$. □

LEMMA 4.3.

If $W=(x,f)$ *is an extreme point of* $F_n(L)$ *(with at least 3 positive coordinates) such that exists a partition* $(\tilde{S},T)$ *of indices* $\{1\dots n\}$*, for which* $x_i = f_i > 0\ \forall\ i\in\tilde{S}$ *and* $x_i = f_i = 0\ \forall\ i\in T$*, then exist* k,μ *integers* $k\geq\mu$ *such that* $x_j = f_j = \mu$ *for some* $j\in\tilde{S}, x_i = f_i = k\ \forall\ i \in\tilde{S}-\{j\}, x_l = f_l = 0\ \forall\ l\in T$.

Proof. See the first part of the proof of Lemma 4.1. □

PROPOSITION 4.4. *Note that* $F_n(L)\cap\{x_{i_1}=0\}\cap\ldots\cap\{x_{i_t}=0\}\simeq F_{n-t}(L)$ *up to renaming variables. Hence we need to characterize only the extreme points with exactly* n *positive components.*

Now let us see which are the possible values of ξ, k, μ.

THEOREM 4.5. *Let* $t = L - \lfloor L \rfloor, v = \lfloor L \rfloor - (n-1)\lfloor\frac{\lfloor L\rfloor}{n-1}\rfloor, w = \lfloor L\rfloor - (n-2)\lfloor\frac{\lfloor L \rfloor}{n-2}\rfloor$ *and define*

$$k_1(N) = \lceil \frac{\lfloor L\rfloor - N}{n-1} \rceil$$
$$k_2(N) = \lfloor \frac{\lfloor L\rfloor - N}{n-2} \rfloor$$

If $W=(x,f)$ *is an extreme point of* $F_n(L)$ *with* n *positive components such that* $x_i > f_i\ \forall\ i\in S\subset\{1\dots n\}, 2\leq|S|=m\leq n-1$ *and* $x_i = f_i > 0\ \forall\ i\in\tilde{S}$,

$\tilde{S} \subset \{1 \ldots n\}$, $S \cap \tilde{S} = \emptyset$. *Then* $\lfloor L \rfloor \geq n-2$ *and*

$$\begin{aligned}
x_h &= \lceil f_h \rceil && \forall\ h \\
f_i &= k + \frac{t + N^1}{m-1} && \forall\ i \in S \\
f_j &= \lfloor L \rfloor - N^1 - (n-2)k && for\ some\ j \in \tilde{S} \\
f_l &= k = \lceil \frac{\lfloor L \rfloor - N^1}{n-1} \rceil && \forall\ l \in \tilde{S} - \{j\}
\end{aligned}$$

for some $N^1 \in \{a, b, c, d\}$ *(whenever* $a, b, c,$ *or* d *exist) where*
$a = min\{p : 0 \leq p \leq min(v-1; m-2) \quad and \quad k_1(p) \leq k_2(p)\}$
$b = max\{p : 0 \leq p \leq min(v-1; m-2) \quad and \quad k_1(p) \leq k_2(p)\}$
$c = min\{p : v \leq p \leq m-2 \quad and \quad k_1(p) \leq k_2(p)\}$
$d = max\{p : v \leq p \leq m-2 \quad and \quad k_1(p) \leq k_2(p)\}$
or

$$\begin{aligned}
x_h &= \lceil f_h \rceil && \forall\ h \\
f_i &= k + \frac{t + N^2}{m-1} && \forall\ i \in S \\
f_j &= \lfloor L \rfloor - N^2 - (n-2)k && for\ some\ j \in \tilde{S} \\
f_l &= k = \lfloor \frac{\lfloor L \rfloor - N^2}{n-2} \rfloor && \forall\ l \in \tilde{S} - \{j\}
\end{aligned}$$

for some $N^2 \in \{a^1, b^1, c^1, d^1\}$*,(whenever* a^1, b^1, c^1*or* d^1 *exist), where*
$a^1 = min\{p : 0 \leq p \leq min(w; m-2) \quad and\ k_1(p) \leq k_2(p)\}$
$b^1 = max\{p : 0 \leq p \leq min(w; m-2) \quad and\ k_1(p) \leq k_2(p)\}$
$c^1 = min\{p : w+1 \leq p \leq m-2 \quad and\ k_1(p) \leq k_2(p)\}$
$d^1 = max\{p : w+1 \leq p \leq m-2 \quad and\ k_1(p) \leq k_2(p)\}$

Proof. By Lemma 4.1 we can write:

$$\begin{aligned}
x_i &= \lceil f_i \rceil && \forall\ i \\
f_i &= k + q && \forall\ i \in S \\
f_j &= \mu && for\, some\, j \in \tilde{S} \\
f_l &= k && \forall\ \ l \in \tilde{S} - \{j\}
\end{aligned}$$

where $0 < q < 1$ and $k \geq \mu$.

The survivability constraint becomes:

$$\begin{aligned}
(n-2)k + \mu + mq &= L + u \\
(n-1)k + mq &\geq L + u \\
(n-2)k + \mu + (m-1)q &= L
\end{aligned}$$

with $u > 0$ implying $u = q$.

Hence

$$\begin{aligned}
(n-2)k + \mu - \lfloor L \rfloor &= t - (m-1)q \\
(n-1)k - \lfloor L \rfloor &\geq t - (m-1)q
\end{aligned}$$

Then $t-(m-1)q=-N$ where $0 \le N \le m-2$ and N integer. For fixed N, from the previous inequalities we get: $k \ge \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil$ and $\mu = \lfloor L \rfloor - N - (n-2)k$. Since μ has to be non negative, it must be that $\mu = \lfloor L \rfloor - N - (n-2)k \ge 0$ (i.e) $k \le \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor$. Note that from $k \ge \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil$ we also get $\mu \le k$. Altogether hence, for fixed N it has to be:

$$0 < k_1(N) = \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil \le k \le \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor = k_2(N)$$

A convexity argument can be used to prove that $k = k_1(N)$ or $k = k_2(N)$. Also, it has to be $\lfloor L \rfloor \ \ge n-2$. Otherwise $k_1(N) > k_2(N)$ or $k_2(N) \le 0$ for all $0 \le N \le m-2$.

Since $m \le n-1$ then $m-2 < n-2$ and $\lfloor L \rfloor - N > 0$. Hence $k_1(N) > 0$ for all $0 \le N \le m-2$. Note that:

$$\begin{aligned} \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil &= \lceil \frac{\lfloor L \rfloor}{n-1} \rceil \qquad & 0 \le N \le v-1 \\ \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil &= \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor \qquad & v \le N \le m-2 < n-1 \end{aligned}$$

This is because

$$\lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor + \lceil \frac{v-N}{n-1} \rceil$$

and

$$\lceil \frac{\lfloor v \rfloor - N}{n-1} \rceil = \{ \begin{array}{ll} 0 & if\, v-N \le 0 \\ 1 & if\, v-N > 0 \end{array}$$

Similarly

$$\begin{aligned} \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor &= \lfloor \frac{\lfloor L \rfloor}{n-2} \rfloor \qquad & 0 \le N \le w \\ \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rceil &= \lfloor \frac{\lfloor L \rfloor}{n-2} \rfloor - 1 \qquad & w+1 \le N \le m-2 < n-1 \end{aligned}$$

Define a, b, c, d and a^1, b^1, c^1, d^1 as in the statement of the theorem. A convexity argument can be used to show that the point corresponding to N where, for example, $a < N < b$ can not be an extreme point. □

THEOREM 4.6. *If $W = (x, f)$ is an extreme point of $F_n(L)$ with n positive components such that $x_i > f_i$ for all $i \in \{1 \dots n\}$ then $f_i = \frac{L}{n-1}$, $x_i = \lceil \frac{L}{n-1} \rceil$ for all $i \in \{1 \dots n\}$.*

Proof. Along the same lines as above. □

THEOREM 4.7. *If $W = (x, f)$ is an extreme point of $F_n(L)$ with n positive components such that $x_i > f_i$ for some $i \in \{1 \dots n\}$ and $x_j = f_j \ \forall\, j \in \ \{1 \dots n\} - \{i\}$, then L is not integer and W is one of the following points:*

$$\begin{array}{ll} x_l = \lceil f_l \rceil & \forall\ l \\ f_i = \xi & for\ \ some\ i \\ f_j = \mu & for\ \ some\ j \ne i \\ f_g = k & \forall\ g \ne i, j \end{array}$$

where

$$\xi = k - (\lceil L \rceil - L)$$
$$\mu = \lceil L \rceil - (n-2)k$$
$$k = \begin{cases} k_1 = & \lceil \frac{\lceil L \rceil + (\lceil L \rceil - L)}{n-1} \rceil \\ k_2 = & \lfloor \frac{\lceil L \rceil}{n-2} \rfloor \end{cases}$$

$$\textit{if } k_1 \leq k_2$$

or

$$\xi = L - (n-2)k$$
$$\mu = k$$
$$k = \begin{cases} k_1 = & \lceil \frac{\lceil L \rceil}{n-1} \rceil \\ k_2 = & \lfloor \frac{L-1}{n-2} \rfloor \end{cases}$$

$$\textit{if } k_1 \leq k_2$$

or

$$\xi = L - \lfloor L \rfloor$$
$$\mu = \lfloor L \rfloor - (n-3)k$$
$$k = \begin{cases} k_1 = & \lceil \frac{\lfloor L \rfloor}{n-2} \rceil \\ k_2 = & \lfloor \frac{\lfloor L \rfloor - 1}{n-3} \rfloor \end{cases}$$

$$\textit{if } k_1 \leq k_2$$

Proof. Similar to the proof in the theorem 4.5.□

THEOREM 4.8. *If $W = (x, f)$ is an extreme point of $F_n(L)$ with n positive components such that $x_i = f_i$ for all $i \in \{1 \dots n\}$ then W is one of the points:*

$$\begin{array}{ll} x_i = k & \forall\ i \in \{1 \dots n\} - \{j\} \\ x_j = \lceil L \rceil - (n-2)k & for\ some\ j \\ k = \begin{cases} k_1 = & \lceil \frac{\lceil L \rceil}{n-1} \rceil \\ k_2 = & \lfloor \frac{\lceil L \rceil}{n-2} \rfloor \end{cases} & \end{array}$$

Proof. Along the some lines as the previous proofs.□

So far we have proved that if W is an extreme point of $F_n(L)$ with n positive components, it must be one of those previously described. By varying n we can therefore obtain a set containing all $F_n(L)$ extreme points.

We could use this information in order to find a characterization of all its facets. The description by facets of $F_n(L)$, though, has revealed a much more complex task. We refer the reader to [5] for some of $F_n(L)$ classes of facets. Here we would like to present a class of facets with a quite complex structure that shed a light on the complex polyhedra structure of survivable networks. Before we do that, let us prove the following:

LEMMA 4.9.

$$\frac{m}{v+1} + \frac{v}{m-1} \geq 2 \tag{4.1}$$

for all $m \geq 2$, $v \geq 0$, m and v integers.

Proof. Fix v. Then (4.1) is a function in m that is not increasing for $m \leq v+1$ and not decreasing for $m \geq v+1$ (i.e.) it has a global minimum at $m = v+1$. In fact

$$\frac{m+1}{v+1} + \frac{v}{m} \geq \frac{m}{v+1} + \frac{v}{m-1}$$

iff

$$\frac{1}{v+1} \geq \frac{v}{m(m-1)}$$

iff

$$m(m-1) \geq v(v+1)$$

Hence $\frac{m}{v+1} + \frac{v}{m-1} \geq \frac{v+1}{v+1} + \frac{v}{v} = 2$. □

THEOREM 4.10. *Let $t = L - \lfloor L \rfloor \neq 0$, $v = \lfloor L \rfloor - (n-1)\lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor$. If $\lceil \frac{\lfloor L \rfloor - v}{n-1} \rceil \leq \lfloor \frac{\lfloor L \rfloor - v}{n-2} \rfloor$ then for any $\tilde{m}$ such that $v+2 \leq \tilde{m} \leq n-2$ the following inequality*

$$\sum_{i \neq j} x_i + (2\tilde{m} - 1 - v)x_j + \frac{\tilde{m}(\tilde{m}-1)}{v+t} \sum_{i \neq j} f_i \geq \lceil L \rceil + (2\tilde{m} - 1 - v)\lceil \frac{\lfloor L \rfloor}{n-1} \rceil + \frac{\tilde{m}(\tilde{m}-1)}{v+t} L$$

is a facet of $F_n(L)$.

Proof.

Let us call $\gamma = (2\tilde{m} - 1 - v)$, $\beta = \frac{\tilde{m}(\tilde{m}-1)}{v+t}$ and $\beta' = \frac{\tilde{m}(\tilde{m}-1)}{v+1}$. We will show, first, that the above inequality is valid for $F_n(L)$ and then that is facet defining. In order to prove that this inequality is valid for $F_n(L)$ we will show that it is valid for all its extreme points, whose characterization was given previously. Let $W = (x, f)$ be an extreme point of $F_n(L)$. If $x_j \geq \lceil \frac{\lfloor L \rfloor}{n-1} \rceil$ the point is valid. So suppose that $x_j < \lceil \frac{\lfloor L \rfloor}{n-1} \rceil$. From Lemmas (4.1), (4.2) and (4.3) we know that all extreme points have the form:

$$\begin{aligned}
x_a &= \mu && a \in I \\
x_b &= k && \forall\ b \in I - \{a\} \\
x_c &= \lceil \xi \rceil && \forall\ c \in NI \\
f_a &= \mu && a \in I \\
f_b &= k && \forall\ b \in I - \{a\} \\
f_c &= \xi && \forall\ c \in NI
\end{aligned}$$

where (I, NI) is a partition of indices $\{1 \dots n\}$ and $k \geq \mu\ k, \mu$ integer. One of the set I or NI can be, eventually, empty. Suppose that $\exists\ l \in \{1 \dots n\} - \{j\}$ such that $x_l = f_l = k > x_j$. Then the following inequality holds:

$$\sum_{i \neq j,l} x_i + x_l + \gamma x_j + \beta \sum_{i \neq j,l} f_i + \beta f_l \geq \sum_{i \neq j,l} x_i + x_j + \gamma x_l + \beta \sum_{i \neq j,l} f_i + \beta f_j \tag{4.2}$$

In fact this is true iff

$$\frac{\beta}{\gamma - 1} \geq \frac{x_l - x_j}{f_l - f_j}$$

Note that

$$\frac{x_l - x_j}{f_l - f_j} = \frac{k - x_j}{k - f_j} \leq 1$$

and hence it is enough to show that

$$\frac{\beta}{\gamma - 1} \geq 1$$

Since $\frac{\beta}{\gamma-1} \geq \frac{\beta^{'}}{\gamma-1}$ the result follows from Lemma (4.9). From our characterization of $F_n(L)$ extreme points, it is easy to check that $k \geq \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor$ with equality holding only if $W = (x, f)$ is one of the following points:

$$\begin{aligned}
x_i = f_i = k = \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil & \qquad \forall \quad i \in I - \{j\} \qquad with \quad |I| = n - m - 1 \\
x_j = f_j = \mu = \lfloor L \rfloor - N - (n-2)k & \\
f_i = \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil + \frac{t + N}{m - 1} & \qquad \forall \quad i \in NI \qquad with \quad |NI| = m \\
x_i = \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil + 1 & \qquad \forall \quad i \in NI
\end{aligned}$$

with $v + 2 \leq m \leq n - 1$ and $v \leq N \leq m - 2$. In this case in fact $k = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor$.

Now we will proof that for these points the RHS of (4.2) is greater or equal to the RHS of the inequality in the theorem statement (i.e.):

$$(n - m - 2)k + \lfloor L \rfloor - N - (n-2)k + m(k+1) + \gamma k + \beta((n - m - 2)k +$$

$$\lfloor L \rfloor - N - (n-2)k + m(k + \frac{t + N}{m - 1})) \geq \lceil L \rceil + \gamma \lceil \frac{\lfloor L \rfloor}{n-1} \rceil + \beta L$$

This holds if and only if:

$$\lfloor L \rfloor - N + m + \beta(\lfloor L \rfloor - N + m\frac{t + N}{m - 1}) - \gamma \geq \lceil L \rceil + \beta L$$

iff

$$m-1-N+\beta\frac{t+N}{m-1}-\gamma\geq 0$$

When $N=v$ we have:

$$m-1-v-2\tilde{m}+v+1+\frac{\tilde{m}(\tilde{m}-1)}{m-1}\geq 0$$

iff

$$\frac{m}{\tilde{m}}+\frac{\tilde{m}-1}{m-1}\geq 2$$

which holds true for any m and $\tilde{m}$ as in the theorem statement.
When $N=m-2$ we have:

$$1+\beta\frac{t+m-2}{m-1}-\gamma\geq 0$$

This is true iff

$$1+\beta'(1+\frac{1-t}{v+t})(\frac{t+m-2}{m-1})-\gamma\geq 0$$

iff

$$1+\beta'(1+\frac{1-t}{v+t})(\frac{t}{m-1}+\frac{m-1}{m-1}-\frac{1}{m-1})-\gamma\geq 0$$

iff

$$1+\beta'(1+\frac{1-t}{v+t})(1+\frac{t-1}{m-1})-\gamma\geq 0$$

iff

$$1+\beta'(1+\frac{t-1}{m-1}+\frac{1-t}{v+t}\frac{t-1}{m-1}+\frac{1-t}{v+t})-\gamma\geq 0$$

if

$$\frac{-1}{m-1}+\frac{-(1-t)}{(v+t)(m-1)}+\frac{1}{v+t}\geq 0$$

iff

$$-v-t-1+t+m-1\geq 0$$

iff

$$m - 2 \geq v$$

which is true.

From Theorems (4.5), (4.6), (4.7) and (4.8) we see that the only points for which $\nexists\ l \in \{1 \dots n\}$ such that $x_l = f_l > x_j$ are those with p positive flow coordinates and in which only one of these flow coordinates is integer (i.e.):

$$\begin{aligned} f_j = x_j = \mu = \lfloor L \rfloor - N - (p-2)k & \\ f_i = \xi = k + \frac{t+N}{p-2} & \qquad \forall\ i\ \in I \\ x_i = \lceil \xi \rceil = k + 1 & \qquad \forall\ i \in I \\ f_u = x_u = 0 & \qquad otherwise \end{aligned}$$

where $I \subset \{1 \dots n\} - \{j\}$ such that $|I| = p - 1$; $0 \leq N \leq p - 3$ for any $3 \leq p \leq n$ and k as in theorem (4.5). We know that for these points is $x_i \geq \lceil \frac{\lfloor L \rfloor}{n-1} \rceil\ i \neq j$.

Note that for fixed p and N and $\tilde{k} > k$ is:

$$(p-1)(\tilde{k}+1) + \gamma(\lfloor L \rfloor - N - (p-2)\tilde{k}) + \beta(p-1)(\tilde{k} + \frac{t+N}{p-2}) \geq$$

$$(p-1)(k+1) + \gamma(\lfloor L \rfloor - N - (p-2)k) + \beta(p-1)(k + \frac{t+N}{p-2})$$

iff

$$(p-1) - \gamma(p-1) + \beta(p-1) + \gamma \geq 0$$

which holds true by Lemma (4.9). So we should be considering only those points with $k = \lceil \frac{\lfloor L \rfloor - N}{p-1} \rceil$. Let us consider a feasible point with p positive coordinates.

$$\begin{aligned} x_j = f_j = \mu & \\ f_i = k + q & \qquad \forall\ i \neq j \\ x_i = k + 1 & \qquad \forall\ i \neq j \end{aligned}$$

where $0 < q < 1$. Suppose that $q + \frac{1}{p-2} > 1$. Then the point

$$\begin{aligned} y_j = g_j = \mu - 1 & \\ g_i = k + q + \frac{1}{p-2} & \qquad \forall\ i \neq j \\ y_i = k + 2 & \qquad \forall\ i \neq j \end{aligned}$$

is feasible and

$$\sum_{i\neq j} y_i + \gamma y_j + \beta \sum_{i\neq j} g_i \geq \sum_{i\neq j} x_i + \gamma x_j + \beta \sum_{i\neq j} f_i$$

In fact it is enough to show that:

$$(p-1) - \gamma + \beta(1 + \frac{1}{p-2}) \geq 0$$

which is true by Lemma (4.9). Then we should consider only those points such that:

$$\begin{aligned} x_j = f_j = \mu = \lfloor L \rfloor - N - (p-2)k & \\ f_l = \xi = k + \frac{t+N}{p-2} & \qquad \forall\ l \in I \subset \{1 \dots n\} - \{j\} \\ x_l = \lceil \xi \rceil = k+1 & \qquad \forall\ l \in I \subset \{1 \dots n\} - \{j\} \end{aligned}$$

such that $|I| = p-1$ $v_p \leq N \leq p-3$ for any $3 \leq p \leq n$ and $v_p = \lfloor L \rfloor - (p-1)\lfloor \frac{\lfloor L \rfloor}{p-1} \rfloor$. We know that in this case is $\lceil \frac{\lfloor L \rfloor - N}{p-1} \rceil = \lfloor \frac{\lfloor L \rfloor}{p-1} \rfloor$. Suppose $p = n$.

$$(n-1)(k+1) + \gamma(\lfloor L \rfloor - N - (n-2)k) + \beta((n-1)(k + \frac{t+N}{n-2})) \geq$$

$$(v+1)(k+1) + (n-v-2)k + \gamma(k+1) + \beta((v+1)(k + \frac{v+t}{v+1}) + (n-v-2)k) =$$

$$\lceil L \rceil + \gamma \lceil \frac{\lfloor L \rfloor}{n-1} \rceil + \beta L$$

iff

$$n - v - 2 + \gamma(v - N - 1) + \beta((n-1)\frac{t+N}{n-2} - v - t) \geq 0$$

Suppose $N = n-3$

$$(n-v-2)(1-\gamma+\beta) + \beta(-1 + \frac{t+(n-3)}{n-2}) \geq 0$$

iff

$$(n-v-2)(1-\gamma+\beta) + \beta(\frac{-n+2+t+n-3}{n-2}) \geq 0$$

iff

$$(n-v-2)(1-\gamma+\beta)+\beta\frac{t-1}{n-2}\geq 0$$

since $v \leq n-3$ if

$$(1-\gamma+\beta)+\beta\frac{t-1}{n-2}\geq 0$$

$$1-\gamma+\beta^{'}(1+\frac{1-t}{v+t})+\beta^{'}(1+\frac{1-t}{v+t})\frac{t-1}{n-2}\geq 0$$

if

$$\beta^{'}(\frac{1-t}{v+t}+\frac{t-1}{n-2}+\frac{(1-t)(t-1)}{(v+t)(n-2)})\geq 0$$

iff

$$\frac{1}{v+t}-\frac{1}{n-2}-\frac{1-t}{(v+t)(n-2)}\geq 0$$

iff

$$n-3-v\geq 0$$

When $N=v$ we have that

$$n-v-2+\gamma(v-N-1)+\beta(t+N-v-t+\frac{t+N}{n-2})\geq 0$$

iff

$$n-1-v-1-2\tilde{m}+v+1+\frac{\tilde{m}(\tilde{m}-1)}{n-2}\geq 0$$

iff

$$\frac{n-1}{\tilde{m}}+\frac{\tilde{m}-1}{n-2}\geq 2$$

which holds true because this function achieves its minimum at $n-1=\tilde{m}$ for fixed $\tilde{m}$. Suppose now that $p<n$. Let us consider the case $N=p-3$.

Let $\tilde{k}=\lfloor\frac{\lfloor L\rfloor}{p-1}\rfloor$ and $k=\lfloor\frac{\lfloor L\rfloor}{n-1}\rfloor$. Let $v_p=\lfloor L\rfloor-(p-1)\tilde{k}$.

We will show that

$$(p-1)(\tilde{k}+1)+\gamma(\lfloor L\rfloor-(p-3)-(p-2)\tilde{k})+\beta((p-1)(\tilde{k}+\frac{t+(p-3)}{p-2}))\geq\lceil L\rceil+\gamma(k+1)+\beta L$$

This is true iff

$$\lfloor L\rfloor - v_p + p - 1 + \gamma(\tilde{k} + v_p - (p-3) - k - 1) + \beta(\lfloor L\rfloor - v_p + t + p - 3 + \frac{t+p-3}{p-2}) \geq \lceil L\rceil + \beta L$$

Note that $v_p \leq p-3 < p-2$. Then

$$(p-3) - v_p - \gamma(-\tilde{k} + k + (p-2) - v_p) + \beta((p-3) - v_p + \frac{t+p-3}{p-2}) \geq 0$$

since $p < n$ is $\tilde{k} - k \geq 1$. Hence

$$(p-3) - v_p - \gamma(-\tilde{k} + k + (p-2) - v_p) + \beta((p-3) - v_p + \frac{t+p-3}{p-2} \geq$$

$$(p-3) - v_p - \gamma((p-3) - v_p) + \beta((p-3) - v_p + \frac{t+p-3}{p-2}) \geq 0$$

where the last inequality is true by Lemma (4.9). When $N = v_p$ is $\mu = \lfloor L\rfloor - v_p - (p-2)\tilde{k} = \tilde{k} \geq \lceil \frac{\lfloor L\rfloor}{n-1}\rceil$.

Now let us show that the inequality described in the theorem statement is, indeed, a facet. We note that the points:

A:

$$\begin{aligned}
&x_i^a = k = \lfloor \frac{\lfloor L\rfloor}{n-1} \rfloor && \forall\ i \in I - \{j\} \\
&x_j^a = k && \\
&x_l^a = k+1 && \forall\ l \in NI \\
&f_i^a = k && \forall\ i \in I - \{j\} \\
&f_j^a = k && \\
&f_l^a = k + \frac{t+v}{\tilde{m}-1} && \forall\ l \in NI
\end{aligned}$$

with $|NI| = \tilde{m}$ and (I, NI) a partition of $\{1 \dots n\}$

B:

$$\begin{aligned}
&x_i^b = k = \lfloor \frac{\lfloor L\rfloor}{n-1} \rfloor && \forall\ i \in I \\
&x_j^b = k+1 && \\
&x_l^b = k+1 && \forall\ l \in NI - \{j\} \\
&f_i^b = k && \forall\ i \in I \\
&f_j^b = k + \frac{t+v}{v+1} && \\
&f_l^b = k + \frac{t+v}{v+1} && \forall\ \ l \in NI - \{j\}
\end{aligned}$$

with $|NI| = v+2$ and (I, NI) a partition of $\{1 \dots n\}$

C:

$$
\begin{aligned}
&x_i^c = k = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor && \forall i \in I - \{j\} \\
&x_j^c = k && \\
&x_l^c = k + 1 && \forall l \in NI \\
&f_i^c = k && \forall i \in I - \{j\} \\
&f_j^c = k && \\
&f_l^c = k + \frac{t+v}{\tilde{m}} && \forall l \in NI
\end{aligned}
$$

with $|NI| = \tilde{m} + 1$ and (I, NI) a partition of $\{1 \dots n\}$ are tight for the inequality in the theorem statement. Let us consider an inequality:

$$
\sum_i \alpha_i x_i + \sum_i \beta_i f_i \geq \pi \tag{4.3}
$$

which is tight for all points that are tight for the inequality in the theorem statement. The point D:

$$
\begin{aligned}
&x_i^d = x_i^b && \forall i \neq j \\
&f_j^d = f_j^b + \epsilon && \\
&f_i^d = f_i^b && \forall i \neq j
\end{aligned}
$$

which is tight for the inequality in the theorem statement implies $\beta_j = 0$. Since (I, NI) can be any partition we get from point A that:

$$
\alpha_i k + \alpha_l (k+1) + \beta_i k + \beta_l (k + \frac{t+v}{\tilde{m}-1}) = \alpha_i (k+1) + \alpha_l k + \beta_i (k + \frac{t+v}{\tilde{m}-1}) + \beta_l k
$$

for all $i, l \neq j$ (i.e.)

$$
\alpha_l - \alpha_i = -(\beta_l - \beta_i)\frac{t+v}{\tilde{m}-1} + \beta_l k
$$

while from the point C we get:

$$
\alpha_l - \alpha_i = -(\beta_l - \beta_i)\frac{t+v}{\tilde{m}}
$$

for all $i, l \neq j$. These two inequalities imply that

$$
\beta_i = \beta_l = \beta \qquad \forall\ i, l \neq j
$$

and hence

$$
\alpha_i = \alpha_l = \alpha = 1 \qquad \forall\ i, l \neq j
$$

The inequality (4.3) now is:

$$\sum_{i\neq j} x_i + \gamma x_j + \beta \sum_{i\neq j} f_i \geq \pi$$

and since the points A, B, C are tight for (4.3) and are linearly independent we have that (4.3) is equivalent to the inequality in the theorem statement□

By Proposition (4.4) this characterization of extreme points and facets applies to each of $F_m(L)$ for all $3 \leq m \leq n$. Then, we can lift sequentially any facet of $F_m(L)$ for $m < n$ to get a facet of $F_n(L)$. We can easily compute the lifting coefficients through our characterization of extreme points.

5. The polyhedron $Q_{n+1}(D, H)$. The polyhedron $Q_{n+1}(D, H)$ is defined by:

$$\begin{aligned} x_0 + \sum_{i=1}^{n} x_i &\geq H \\ x_0 + \sum_{i\neq j}^{n} x_i &\geq D \qquad \forall\ j = \{1 \ldots n\} \\ x_i &\in Z^+ \qquad \forall\ i = \{0 \ldots n\} \end{aligned}$$

where H and D are supposed integer and $H \geq D$. The structure of $Q_{n+1}(D, H)$ is, indeed, simpler. By the same type of argument used to find $F_n(L)$ extreme points one can find $Q_{n+1}(D, H)$ extreme points which are as described in the following theorem.

THEOREM 5.1. *The extreme points of $Q_{n+1}(D, H)$ are:*

$$\begin{aligned} x_0 &= 0 \\ x_i &= k \qquad &\forall\ i \in S \\ x_j &= D - (|S| - 1)k \\ x_l &= 0 \qquad &\forall\ l \in T \end{aligned}$$

where

$$k = \begin{cases} k_1 = max(H - D, \lceil \frac{D}{|S|} \rceil) \\ k_2 = \lfloor \frac{D}{|S|-1} \rfloor \end{cases}$$

for any partition $(S, \{j\}, T)$ of indexes $\{1, \ldots, n\}$ such that $1 \leq |S| < n$ and

$$max(H - D, \lceil \frac{D}{|S|} \rceil) \leq \lfloor \frac{D}{|S| - 1} \rfloor$$

or

$$\begin{aligned} x_0 &= D - (|S| - 1)k \\ x_i &= k \qquad &\forall\ i \in S \\ x_l &= 0 \qquad &\forall\ l \in T \end{aligned}$$

where

$$k = \begin{cases} k_1 = & H - D \\ \\ k_2 = & \lfloor \frac{D}{|S|-1} \rfloor \end{cases}$$

for any partition (S, T) *of indexes* $\{1, \ldots, n\}$ *such that* $2 \le |S| \le n$ *and*

$$H - D \le \lfloor \frac{D}{|S| - 1} \rfloor$$

or

$$\begin{aligned} &x_0 = D \\ &x_i = H - D && for \quad some \ i \in \{1, \ldots, n\} \\ &x_j = 0 && \forall \ j \in \{1, \ldots, n\} - \{i\} \end{aligned}$$

or

$$\begin{aligned} &x_0 = H \\ &x_i = 0 \qquad \forall \ i \in \{1, \ldots, n\} \end{aligned}$$

Proof. Similar to the proof of theorem (4.8) □

THEOREM 5.2. *The inequalities defining* $Q_{n+1}(D, H)$, *(i.e.)* $x_0 + \sum_{i=1}^{n} x_i \ge H$, $x_0 + \sum_{i \ne j} x_i \ge D$ *for all* $j = \{1, \ldots n\}$ *and* $x_i \ge 0$ *for all* $i = \{0, \ldots n\}$ *are facets of* $Q_{n+1}(D, H)$.

Proof. Standard□

THEOREM 5.3. *Let be* $I \subset \{1 \ldots n\}$, $|I| = m$, $2 \le m \le n$, $r = D - (m-1)\lfloor \frac{D}{m-1} \rfloor$, *and* $\alpha = \frac{r+1}{r}$. *If* $\lceil \frac{mD}{m-1} \rceil > H$, $\frac{D}{m-1} \notin Z$, *it exists an extreme point of* $Q_{n+1}(D, H)$ *with* $m < n$ *positive components and the* 0^{th} *component equal to* 0 *and* $p > 0$ *is the smallest integer such that it exists an extreme point of* $Q_{n+1}(D, H)$ *with* $m + p \le n$ *positive components and the* 0^{th} *equal to zero then*

$$\alpha x_0 + \sum_{i \in I} x_i + \alpha \sum_{i \in N - I} x_i \ge \lceil \frac{mD}{m-1} \rceil \tag{5.1}$$

is a facet of $Q_{n+1}(D, H)$.

Proof. It is easy to check that this inequality is valid for all $Q_{n+1}(D, H)$ extreme points. By the hypothesis the following points

Type A:

$$\begin{aligned} &x_0 = 0 \\ &x_j = D - (m-2)\lceil \frac{D}{m-1} \rceil && for \ \ some \ j \in I \\ &x_i = \lceil \frac{D}{m-1} \rceil && \forall \ i \in I - \{j\} \\ &x_l = 0 && \forall \ l \in N - I \end{aligned}$$

Type B:

$$\begin{aligned} &x_0 = 0 && \\ &x_i = \lfloor \frac{D}{m+p-2} \rfloor && \forall\ i \in I \\ &x_j = D - (m+p-2)\lfloor \frac{D}{m+p-2} \rfloor && for\ some\ j \in N-I \\ &x_l = \lfloor \frac{D}{m+p-2} \rfloor && \forall\ l \in T \subset N - I - \{j\}\ with\ |T| = p-1 \\ &x_v = 0 && otherwise \end{aligned}$$

Type C:

$$\begin{aligned} &x_0 = D - (m+p-2)\lfloor \frac{D}{m+p-2} \rfloor && \\ &x_i = \lfloor \frac{D}{m+p-2} \rfloor && \forall\ i \in I \\ &x_l = \lfloor \frac{D}{m+p-2} \rfloor && \forall\ l \in T \subset N - I\ with\ |T| = p-1 \\ &x_v = 0 && otherwise \end{aligned}$$

exists and are tight for the inequality (5.1). Type B and C points are tight because it can be proved that, under the hypothesis, it must be $\lceil \frac{D}{m+p-1} \rceil = \lfloor \frac{D}{m+p-2} \rfloor = \lfloor \frac{D}{m-1} \rfloor$. Also, it is easy to find, among type A, B and C points, $n+1$ points which are linearly independent. Hence the inequality (5.1) is a facet of $Q_{n+1}(D,H)$□

Actually, it can be proved that the class of facets described in Theorem (5.2) and Theorem (5.3) are indeed all the facets of $Q_{n+1}(D,H)$ (i.e.) they describe its convex hull. For the proof details the interested reader is referred to the author Ph.D. thesis [45].

REFERENCES

[1] D. Alevras, M. Grötschel and R. Wessaly, *Cost efficient network synthesis from leased lines*, ZIB Preprint, ZIB, Berlin, (1997).

[2] D. Armstrong, S. Cowan, and J. Vickers, *Regulatory reform: economic analysis and British experience*, MIT Press, Chapter 9, (1994).

[3] A. Balakrishnan, T. L. Magnanti, P. Michandani, *Connectivity splitting models for survivable network design* , Networks, 43 (2004), pp. 10–27.

[4] F. Barahona, A. R. Mahjoub, *On two-connected subgraph polytopes* , Discr. Math., 147 (1995), pp. 19–34.

[5] D. Bienstock, G. Muratore, *Strong inequalities for capacitated survivable network design problems* , Mathematical Programming, vol. 89, (2000).

[6] D. Bienstock,S. Mattia, *Using mixed-integer programming to solve power grid blackout problems*, CORC Technical Report, (October 2005).

[7] E. Bohn, C. Caramanis, C. Schweppe, *Optimal pricing in electrical networks over space and time*, Rand Journal of Economics, 15 (1984), pp. 360–376.

[8] F. Bolle, *Supply function equilibria and the danger of tacit collusion: the case of spot markets for electricity*, Energy Economics, 14 (1992), pp. 94–102.

[9] S. Borenstein, J. B. Bushnell,S. E. Stoft, *The competitive effects of transmission capacity in a deregulated electricity industry* , Rand Journal of Economics, 31 (1998), pp. 294–325.

[10] J. Boucher, Y. Smeers,*Alternative models of restructured electricity systems, part 1: No market power*, Operations Research, 49 (2001), pp. 821–838.

[11] S. C. Boyd, T. Hao, *An integer polytope related to the design of survivable communications networks* , SIAM Journal Discr. Math., 6 (1993), pp. 612–630.

[12] G. Brightwell, G. Oriolo ,F. B. Shepherd, *Reserving resilient capacity in a network*, SIAM J. Discrete Math, 14 (4) (2001), pp. 524–539.
[13] G. M. Casolino, *Mercato elettrico liberalizzato: gestione ottimale con un approccio di decomposizione/coordinamento*, PH.D. Thesis, Universit degli Studi di Cassino, (2003).
[14] H. Chao , H. G. Huntington, *Designing competitive electricity markets*, Kluwer Academic Press, (1998).
[15] H. Chao, S. Peck, *An istitutional design for an electricity contract market with central dispatch*, Energy Journal, 18 (1997), pp. 85–110.
[16] H. Chao, S. Peck, S. S. Oren, R. Wilson, *Flow based transmission rights and congestion management*, The Electricity Journal, 13 (2000), pp. 38–58.
[17] H. Chao, R. Wilson, *Design of wholesale electricity markets*, Electric Power Research Institute,(2001).
[18] G. Dahl, M. Stoer, *A cutting plane algorithm for multicommodity survivable network design problems*, INFORMS Journal of Computing, 10 (1998), pp. 1–11.
[19] M. Didi Biha, H. Kerivin, A. R. Mahjoub, *On the polythope of the (1-2) survivable network design problem*, Preprint, (2004).
[20] M. Didi Biha, A. R. Mahjoub, *Steiner k-edge connected subgraph polyhedra*, Journal of Combinatorial Optimization, 4 (2000), pp. 131–144.
[21] M. A. Einhorn,*From regulation to competition: new frontiers in electricity markets*, Kluwer Academic Publishers, (1994).
[22] B. Fortz, M. Labb, *Polyhedral results for two-connected networks with bounded rings*, Mathematical Programming, 93 (2002), pp. 27–54.
[23] B. Fortz, M. Labb, F. Maffioli, *Solving the two-connected network with bounded meshes problem*, Operation Research, 48 (2000), pp. 866–877.
[24] F. D. Galiana, M. Ili'c, *A mathematical framework for the analysis and management of power transactions under open access*, IEEE Transaction on Power Systems, Vol. 13 No. 2 (1998), pp. 681–687.
[25] R. J. Green and D. M. Newbery, *Competition in the British electricity spot market*, Journal of Political Economy, 100(5) (1992), pp. 929–953.
[26] M. Grötschel, C. L. Monma, *Integer polyhedra associated with certain network design problems with connectivity constraints*, SIAM J. Discrete Math., 3 (1990), pp. 502–523.
[27] M. Grötschel, C. Monma and M. Stoer, *Facets for polyhedra arising in the design of communication networks with low-connectivity constraints*, SIAM J. Opt. 2 (1992), pp. 474–504.
[28] M. Grötschel, C. Monma and M. Stoer, *Design of survivable networks* vol. Network Models of Handbooks in Operations research and Management Science, chp. 10 (North-Holland,1995), pp. 617–672.
[29] A. Henney, S. Crisp, *Lessons for the U.S.? Transmission pricing, constraints, and gaming in England and Wales*, Electricity Journal, (January-February 1997), pp. 17-23.
[30] J. Hicks *Methods of Dynamic Analysis*, 25 Economic Essays in honor of Erik Lindahl, (1956)
[31] W. W. Hogan, *Contract networks for electric power transmission*, J. Regulatory Econ., (1992), pp. 211-242.
[32] W. W. Hogan, *Market in real electric networks require reactive prices*, The Energy Journal, Vol. 14 No. 3 (1993).
[33] W. W. Hogan , J. Chandley, S. Harvey, *Congestion management in California*, John F. Kennedy School of Government, Harvard University, Cambridge, MA, (2000).
[34] W. W. Hogan, *Successful market design and failure diagnosis: blackoutss and lamppost in regulating electricity markets* , J.F.K School of Governement, Harvard University, USA (October 2003).
[35] X. Hu, D. Ralph, E. K. Ralph, P. Bardsley, M. C. Ferris, *Electricity generation with looped transmission networks: Bidding to an ISO*, preprint, (2004).
[36] M. Ili'c, E. Allen, Z. Younes, *Providing for transmission in times of scarcity: an ISO cannot do it all*, International Journal of Electrical Power and Energy Systems, Vol. 21 (1999).
[37] P. Joskow, *Restructuring, competition and regulatory reform in the U.S. electricity sector*, Journal of Economic Perspectives, 11(3) (1997), pp. 119–138.
[38] P. Joskow and R. Schmalensee, *Markets for power*, MIT Press,(1983).
[39] P. Joskow, J. Tirole, *Transmission rights and market power on electric power networks*, Rand Journal, 31 (2000), pp. 450–487.
[40] P. Neuhoff, *Integrating transmission and energy markets mitigates market power*, Working Paper, University of Cambridge, (2003).
[41] H. Kerivin, A. R. Majhoub, *On survivable network polyhedra*, Discr. Math, (to appear).
[42] H. Kerivin, A. R. Majhoub, C. Nocq, *(1-2) Survivable networks:facets and branch-and-*

cut. The sharpest Cut , MPS-SIAM Series in Optimization, M. Grötschel (Editor) (2004), pp. 121–152.
[43] T. L. Magnanti, S. Raghavan,*Strong formulations for network design problems with connectivity requirements* , Networks 45 (2005), pp. 61–79.
[44] M. Minoux, *Discrete cost multicommodity network optimization problems and exact solution methods*, Ann. Oper. Res. 106 (2001), pp. 19–46.
[45] G. Muratore, *Polyhedral approaches to survivable network design*, Ph.D. Thesis, Columbia University, New York, (1999).
[46] K. Neuhoff, J. Barquin, M. G. Boots, A. Ehrenmann, B. F. Hobbs, F. A.M. Rijkers, *Network-constrained Cournot models of liberalized electricity markets: the devil is in the details*, Energy Economics, (2005).
[47] D .Newbery, *Power markets and market power*, Energy Journal, 16(3) (1995), pp. 41–66.
[48] R. P. ONeill, P. M. Sotkiewicz, B. F. Hobbs, M. H. Rothkopf, W. R. Stewart Jr, *Efficient market-clearing prices in markets with nonconvexities*, European J. Operational Research, (2004).
[49] S. Oren, *Economic Inefficiency of passive transmission rights in congested electricity systems with competitive generation*, Energy Journal, 18(1)(1997), pp. 63-84.
[50] J. -S.Pang, B. F. Hobbs, C. Day, *Electricity market models with conjectured resource price responses*, Mathematical Programming B., vol.101 (1) (2004), pp. 57-94.
[51] D. Rajan , A. Atamturk, *A directed cycle-based column-and-cut generation method for capacitated survivable network design*, Networks 43 (4)(2004), pp. 201–211.
[52] R. Riley, *SeTrans market design issues*, FERC SMD Technical Conference,(2003 Atlanta).
[53] L .E. Ruff, *Competition in electricity: where do we go from here?*, Lectures on Regulation Series VIII, London Business School, (October 1998).
[54] F. C. Schweppe, M. Caramanis, R. Tabors, R. Bohn, *Spot pricing of electricity*, Kluwer Academic Publisher, Boston, (1988).
[55] Y. Smeers,J. -Y. Wei *Spatial oligopolistic electricity models with Cournot generators and regulated transmission prices*, Operations Research, 47(1) (1999), pp. 102–112.
[56] M. Stoer, *Design of survivable networks* , Lecture Notes in Mathematics, 1531 (1992), Springer-Verlag, Berlin.
[57] J. Vickers and G. K. Yarrow, *The British electricity experiment*, Economic Policy, 12 (1991), pp. 188–232.
[58] N-H. Von der Fehr and D. Harbord, *Spot market competition in the U.K. electricity industry*, Economic Journal, 103 (1993), pp. 531–546.
[59] B. Willems, *Electricity networks and generation market power* , Ph.D. Thesis Faculteit Economische en Toegepaste Economische, Wetenshappen Katholieke Universiteit Leuven, (2004).
[60] R. Wilson, *Architecture of power markets*, Econometrica, 70 (2002), pp. 1299–1340.
[61] F. Wolak, *Market design and price behavior in restructured electricity markets: an international comparison*, Energy Institute, University of California, Berkeley CA, (1997).
[62] C. D. Wolfram, *Strategic bidding in a multi-unit auction: an empirical analysis of bids to supply electricity in England and Wales*, Rand Journal of Economics, 29 (1998), pp. 703–725.
[63] *White paper: wholesale power market platform*, Federal Energy Regulatory Commission- USA, (April 2003).
[64] *Interim report: causes of the August 14th blackout in the United States and Canada* , Us-Canada Power System Outage task Force, (November,2003).
[65] *Planning and operating standards under Betta*, An OFGEM/DTI Conclusion Document, (30 June 2003).
[66] *Determinazione dell'equilibrio del mercato del giorno prima dell'energia con prezzo uniforme per i consumatori e prezi zonali per i generatori*, Analisi Tecnica n. 5/02, GME, (2002).