Extreme inequalities for infinite group problems

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Abstract

In this paper we derive new properties of extreme inequalities for infinite group problems. We develop tools to prove that given valid inequalities for the infinite group problem are extreme. These results show that integer infinite group problems have discontinuous extreme inequalities. These inequalities are strong when compared to related classes of continuous extreme inequalities. This gives further insight that these cuts are be computationally important. Furthermore, the methods we develop also yield the first tools to generate extreme inequalities for the infinite group problem from extreme inequalities of finite group problems. Finally, we study the generalization of these results to the mixed integer infinite group problem and prove that extreme inequalities for mixed integer programs are always continuous.

1 Introduction

Generating strong cutting planes for unstructured problems is one of the most fundamental problem in mixed integer programming (MIP). The group problem introduced by Gomory [5] provides an elegant framework for the derivation of such inequalities. It proceeds from the observation that integer programs can be written in the form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & A_B x_B + A_N x_N = b \\ & x_B \in \mathbb{Z}^m_+, x_N \in \mathbb{Z}^{n-m}_+, \end{aligned} \tag{1}$$

where the set of variable indices is partitioned into two groups, B and N, and where $A_B \in \mathbb{R}^{m\times m}$ is invertible, $A_N \in \mathbb{R}^{m\times(n-m)}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. In particular, B can be interpreted as the set of basic variables in a basic solution of the LP relaxation of (1). The corner polyhedron, introduced by Gomory in [5], is the relaxation of (1) obtained by removing the nonnegativity constraints of the basic variables x_B. The feasible region of the corner polyhedron can be further relaxed into the following finite group problem $P(C_{|\mathbb{G}|,r})$

$$\begin{aligned} \sum_{u\in\mathbb{G}} u t_u = r \\ t_u \in \mathbb{Z}_+ \end{aligned} \tag{2}$$

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where $\mathbb{G}$ is a finite cyclic group and $r \in \mathbb{G} \setminus \{0\}$. The typical representation of the cyclic group of cardinality n, C_n, is the set $\{0, 1, 2..., n-1\}$ with the addition of elements taken modulo n. For ease of notation here, we denote the cyclic group C_n as $\{\frac{0}{n}, \frac{1}{n}, ..., \frac{n-1}{n}\}$ with addition taken modulo 1. Thus $P(C_{n,r})$ is the finite group problem of cardinality n with a right hand side r. Extreme inequalities of the finite group problem are defined in [6],

Definition 1 ([6])**.** *Let T be the set of all feasible solutions of (2). A valid inequality for $P(C_{|\mathbb{G}|,r})$ is a vector $(f, f(r)) \in (\mathbb{R}_+^{|G|-1}, \mathbb{R}_+)$ such that $\sum_{u \in \mathbb{G} \setminus \{0\}} f(u)t_u \geq f(r)$ $\forall t \in T$. A valid inequality $(f, f(r))$ is called extreme if it cannot be written as a convex combination of two valid inequalities $(f_1, f(r))$ and $(f_2, f(r))$, where $f_1 \neq f_2$.* □

Extreme inequalities of the finite group problem were obtained in Gomory [5], Gomory, Johnson and Evans [8], Aráoz, Evans, Gomory and Johnson [2], Dash and Gülük [3], Richard, Li and Miller [13] and Miller, Li and Richard [12]. Finite groups yield large families of strong inequalities for (2) and consequently for (1). However they can be difficult to use in practice when the determinant of A_B is large or when A is not an integer matrix. To circumvent this difficulty, Gomory and Johnson [6, 7] introduced a variant of this approach that works with the infinite group I^m of real m-dimensional vectors with the addition modulo 1 componentwise, i.e., $I^m = \{(x_1, x_2...x_m) \mid 0 \leq x_i < 1 \; \forall 1 \leq i \leq m\}$. In the remainder of this paper, we use the symbol $+$ to represent group addition as well as the addition in euclidian space, we refer to the origin $(0, 0..., 0)$ of I^m as o and, for any $u \in I^m$, we let $\hat{u}$ be the element of $\mathbb{R}^m$ that has the same numerical value as u. For any $p \in \mathbb{R}^m$ such that $0 \leq p_i < 1 \; \forall 1 \leq i \leq m$, we denote $\dot{p}$ to be the element of I^m such that $\hat{\dot{p}} = p$. Now we define a metric on I^m.

Definition 2. *For $u, v \in I^m$, let $d(u, v) = \sqrt{\sum_{i=1}^{m}(d'(u_i, v_i))^2}$ where $d'(u_i, v_i) = min\ \{|\hat{u}_i - \hat{v}_i|, 1 - |\hat{u}_i - \hat{v}_i|\}$.*

Proposition 3. *d is a metric.*

Proof. It suffices to prove that d' is a metric on I^1. It is clear that

1. d' is nonnegative,
2. $d'(u, v) = d'(v, u)$ $\forall u, v \in I^1$ and
3. $d'(u, v) = 0$ iff $u = v$ $\forall u, v \in I^1$.

Suppose $u, v, w \in I^1$ such that $\hat{u} \leq \hat{v} \leq \hat{w}$. Since $d'(u, v) = d'(u + x, v + x)$ for any $x \in I^1$, to check that the triangular property holds it is sufficient to check the following conditions:

$$d'(u, v) + d'(v, w) \geq d'(u, w) \tag{3}$$
$$d'(u, v) + d'(u, w) \geq d'(v, w). \tag{4}$$

There are five cases: (i) $|\hat{u} - \hat{w}| < 0.5$, (ii) $|\hat{u} - \hat{w}| \geq 0.5$, $|\hat{u} - \hat{v}| < 0.5$ and $|\hat{v} - \hat{w}| < 0.5$, (iii) $|\hat{u} - \hat{w}| \geq 0.5$, $|\hat{u} - \hat{v}| \geq 0.5$ and $|\hat{v} - \hat{w}| < 0.5$, (iv) $|\hat{u} - \hat{w}| \geq 0.5$, $|\hat{u} - \hat{v}| < 0.5$ and $|\hat{v} - \hat{w}| \geq 0.5$, (v) $|\hat{u} - \hat{w}| \geq 0.5$, $|\hat{u} - \hat{v}| = 0.5$ and $|\hat{v} - \hat{w}| = 0.5$. We prove (ii) since the other cases are trivial or similar. Under the conditions, $d'(u, v) = |\hat{u} - \hat{v}|$, $d'(v, w) = |\hat{v} - \hat{w}|$ and $d'(u, w) = 1 - |\hat{u} - \hat{w}|$. Thus, (3) and (4) are satisfied. □

In this paper we consider the metric topology induced by d on I^m and use it to prove results involving continuity of real valued functions defined on I^m with respect to this topology. We now give the formal definition of the integer infinite group problem $PI(r,m)$.

Definition 4 ([10])**.** *For $r \neq o$, $PI(r,m)$ is the set of functions $t : I^m \to \mathbb{R}$ such that*

1. $\sum_{u \in I^m} ut(u) = r$, $r \in I^m$,
2. $t(u)$ *is a non-negative integer for* $u \in I^m$,
3. t *has a finite support, i.e.,* $t(u) > 0$ *for only a finite subset of* I^m. □

Consider now $(t', 1)$ to be a valid solution to (2). It is easily seen that t defined as

$$t(u) = \begin{cases} t'_u & \text{if } u \in \mathbb{G} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

belongs to $PI(r,m)$. Thus $PI(r,m)$ is a relaxation of (1). Hence, valid inequalities for its convex hull can be used as cutting planes for (1); see Gomory and Johnson [6]. Next we formally present the concept of valid inequalities for $PI(r,m)$.

Definition 5 ([10])**.** *A function $\phi : I^m \to \mathbb{R}_+$ is a valid inequality for PI(r,m) if $\sum_{u \in I^m} \phi(u)t(u) \geq 1$, $\forall\ t \in PI(r,m)$, $\phi(o) = 0$ and $\phi(r) = 1$.* □

In the remainder of this paper, we will use the terms valid function and valid inequality interchangeably.

Definition 6 ([6])**.** *A valid inequality ϕ for $PI(r,m)$ is said to be a subadditive valid inequality if $\phi(u) + \phi(v) \geq \phi(u+v)$, $\forall u, v \in I^m$.* □

Gomory and Johnson [6] prove that all valid functions of $PI(r,m)$ that are not subadditive are dominated by valid subadditive functions of $PI(r,m)$. This result is important since it establishes that it is sufficient to study the valid subadditive functions of $PI(r,m)$. Furthermore, we are interested in deriving only the strongest subadditive valid inequalities for the group problem since they correspond to strong cuts for (1). The following property is necessary for an inequality to be strong.

Definition 7 ([10])**.** *A valid inequality ϕ is minimal for $PI(r,m)$ if there does not exist a valid function ϕ^* for $PI(r,m)$ different from ϕ such that $\phi^*(u) \leq \phi(u)$ $\forall u \in I^m$.* □

The following theorem is adapted from Gomory and Johnson [6] and Johnson [10].

Theorem 8 ([6, 10])**.** *ϕ is minimal for $PI(r,m)$ iff ϕ is subadditive and $\phi(u) + \phi(r-u) = 1$ $\forall u \in I^m$.* □

Minimal inequalities for $PI(r,m)$ are strong because they are not dominated by any single valid inequality. However, they can sometimes be expressed as convex combinations of other valid inequalities. Therefore, Gomory and Johnson introduced the notion of extreme inequalities for $PI(r,m)$ to describe the strongest valid inequalities for $PI(r,m)$.

Definition 9 ([10])**.** *A valid inequality ϕ is extreme for $PI(r,m)$ if whenever $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ for some valid inequalities ϕ_1 and ϕ_2 of $PI(r,m)$ then $\phi = \phi_1 = \phi_2$.* □

All the families of functions that have been derived and proven to be extreme for $PI(r, m)$ are continuous; see Gomory and Johnson [6, 9], Dey and Richard [4]. In Section 2 we introduce tools to show that given discontinuous functions are extreme for $PI(r, 1)$. In particular, we generalize Gomory and Johnson's Interval Lemma [9] in a way that does not assume continuity of the underlying functions. Then we show how extreme inequalities for finite groups can be used to generate extreme inequalities for the infinite group problem. In Section 3, we present three families of discontinuous valid and subadditive functions for $PI(r, 1)$. We then prove that these functions are extreme for $PI(r, 1)$ by using the tools developed in Section 2. We show that these extreme inequalities are strong when compared to related classes of continuous inequalities. In Section 4, we show that, unlike the pure integer case, extreme inequalities of the mixed integer group problem are continuous. We conclude this paper with final remarks in Section 5.

2 Proving that valid functions of $PI(r, 1)$ are extreme

In this section we derive a set of tools that can be used to prove that given valid inequalities for the one dimensional infinite group problem are extreme. In particular Proposition 13 is an extension of Gomory and Johnson's Interval Lemma [9] that can be applied for discontinuous inequalities. Propositions 14 and 17 are tool to prove that given inequalities are extreme.

For $a, b \in I^1$, we use the notation $[a, b]$ to denote the interval $\{x \in I^1 | \hat{a} \leq \hat{x} \leq \hat{b}\}$. Similarly, we use the notations (a, b), $[a, b)$ and $(a, b]$ to denote open or half-open intervals, respectively. Now we present Gomory and Johnson's Interval Lemma.

Proposition 10 ([9]). *(Interval Lemma) Let $U \equiv [u_1, u_2] \subset I^1$, $V \equiv [v_1, v_2] \subset I^1$ and $U + V \equiv [u_1 + v_1, u_2 + v_2]$ such that $u_1 \neq u_2$ and $v_1 \neq v_2$. If there exists a continuous real-valued function g defined over U, V and $U + V$ such that $g(u) + g(v) = g(u + v)$ $\forall u \in U, v \in V$. Then g must be a straight line with constant slope s over U, V and $U + V$.* □

Observe that Proposition 10 assumes that the underlying function g is continuous. We will give in Proposition 13 an analogous result that does not assume continuity. We first present a proposition from Aczél [1] that is used in the proof of Proposition 13.

Proposition 11 ([1]). *Let K be the closed interval $[0, \epsilon] \subset \mathbb{R}$ for $\epsilon > 0$. If $g : K \rightarrow \mathbb{R}$ is such that $g(x) + g(y) = g(x + y)$ $\forall x, y \in K$ satisfying $x + y \in K$ and $g(x) \geq 0$ for arbitrarily small $x \in K$, then $g(x) = cx$ $\forall x \in K$, where $c \in \mathbb{R}$.* □

Proposition 12. *Let $U = [u_1, u_2] \subset \mathbb{R}$, $V = [v_1, v_2] \subset \mathbb{R}$ and $U + V \equiv [u_1 + v_1, u_2 + v_2]$ be two closed set such that $u_1 \neq u_2$. Let g be a real-valued function defined over U, V and $U + V$ such that $g(x) + g(y) = g(x + y)$ $\forall x \in U$ and $\forall y \in V$ and $g(x) = cx + d$ $\forall x \in U$, then $g(x) = cx + d'$ $\forall x \in V$ and $g(x) = cx + d + d'$ $\forall x \in U + V$ for some d, $d' \in \mathbb{R}$.*

Proof. We first consider any $x \in [v_1, v_2)$. We prove that the right derivative of $g(x)$ exists and is equal to c. There exists $\epsilon > 0$, such that $u_1 + \epsilon \in U$ and $x + \epsilon \in V$. Since $g(u_1 + \epsilon) + g(x) = g(u_1 + \epsilon + x) = g(x + \epsilon) + g(u_1)$ we obtain $\frac{g(\epsilon + x) - g(x)}{\epsilon} = c$ for all sufficiently small ϵ. Hence the right derivative of g at x is c. Similarly, it can be shown that for $x \in (v_1, v_2]$ the left derivative of g at x exists and equals c. Thus $g(x) = cx + d'$ $\forall x \in V$. Finally, because $g(x) + g(y) = g(x + y)$ $\forall x \in U$ and $y \in V$ and $g(x) = cx + d$ $\forall x \in U$, we obtain that $g(x) = cx + d + d'$ $\forall x \in U + V$. □

Although Propositions 11 and 12 are proven for functions over $\mathbb{R}$, they can also be verified for functions over I^1. We next present a corollary of Propositions 11 and 12 for functions defined on I^1 that is used throughout the paper. This corollary can easily be verified by constructing from the function g satisfying the properties of Corollary 13, a new function $g' : [0, 1) \to \mathbb{R}_+$ as $g'(x) = g(\dot{x})$, that satisfies the conditions of Propositions 12 and 11. This is a variant of the Interval Lemma that does not require continuity.

Corollary 13. *Let $U \equiv [0, u_2] \subset I^1$, $V \equiv [v_1, v_2] \subset I^1$ and $U + V \equiv [v_1, u_2 + v_2]$ such that $0 < \hat{u}_2 < \hat{u}_2 + \hat{v}_2 < 1$. If there exists a function g defined over U, V and $U + V$, such that*

(i) $g(u) \geq 0\ \forall u \in U$,

(ii) $g(u) + g(v) = g(u + v)\ \forall u, v \in U$ such that $\hat{u} + \hat{v} \leq \hat{u}_2$, and

(iii) $g(u) + g(v) = g(u + v)\ \forall u \in U, v \in V$,

then $g(x) = cx\ \forall x \in U$ and $g(x) = cx + \alpha\ \forall x \in U + V$, for some $\alpha, c \in \mathbb{R}$. □

We next show in Proposition 14 that, under mild assumptions, if a function ϕ is not extreme in the space of valid subadditive functions, i.e., it can be obtained as a convex combination of valid subadditive functions, then these functions are continuous at all points at which ϕ is continuous. This result implies that Gomory and Johnson's Interval Lemma can be used to verify that a certain function is extreme even if the assumption that valid inequalities are continuous is removed.

Proposition 14. *Let $\phi : I^1 \to \mathbb{R}_+$ be a piecewise linear, subadditive and valid function such that $\phi(u) = cu\ \forall u \in U$, where $c > 0$, U is the interval $\{u \in I^1 | 0 \leq \hat{u} \leq w\}$ and w is a non-zero real number less than 1. Assume that $\phi = (1 - \lambda)\phi_1 + \lambda\phi_2$, where $0 < \lambda < 1$ and ϕ_1 and ϕ_2 are some subadditive valid inequalities. Then ϕ_1 and ϕ_2 are continuous at all points at which ϕ is continuous.*

Proof. Let $\frac{U}{2}$ be the interval $\{u \in I^1 | 0 \leq \hat{u} \leq \frac{w}{2}\}$. For $u, v \in \frac{U}{2}$, $\phi(u) + \phi(v) = \phi(u + v)$. By substituting ϕ in terms of ϕ_1 and ϕ_2 we obtain

$$(\phi_2(u) + \phi_2(v) - \phi_2(u + v)) + (\phi_1(u) + \phi_1(v) - \phi_1(u + v)) = 0. \tag{6}$$

It follows from the subadditivity of ϕ_2 and ϕ_1 that $\phi_2(u) + \phi_2(v) = \phi_2(u + v)$ and $\phi_1(u) + \phi_1(v) = \phi_1(u + v)$. Thus the same equalities are satisfied by ϕ_1 and ϕ_2. Therefore by Corollary 13 we obtain that $\phi_1(u) = c_1 u$ and $\phi_2(u) = c_2 u\ \forall 0 \leq \hat{u} \leq U_1$ where $c_1, c_2 \in \mathbb{R}_+$. We assume without loss of generality that $c_1 \geq c_2$.

Note that since ϕ is piecewise linear, for any $u \in I^m$, if ϕ is right continuous at u, then ϕ is right differentiable at u. Similarly if ϕ is left continuous at u then ϕ is left differentiable at u. We first consider the case when ϕ is right continuous at u. Assume by contradiction that there exists u such that ϕ is right continuous at u and ϕ_1 is not right continuous at u. Hence ϕ_2 cannot be right continuous at u as $\phi_2 = \frac{1}{\lambda}\phi - \frac{1-\lambda}{\lambda}\phi_1$. Denote the right derivative of ϕ at u by $\bar{c}$. Since ϕ is right continuous and ϕ is piecewise linear there exists $l > 0$ such that the right derivative of ϕ is $\bar{c}$ in (u, v), where $\hat{v} = \hat{u} + l$. Since ϕ_1 is discontinuous at u, there exists $\epsilon > 0$ and points u' arbitrarily close to u with $|\phi_1(u') - \phi_1(u)| \geq \epsilon$. In particular $\exists x \in I^1$ such that $\delta = d(x, u) < \min\{\frac{\epsilon}{\alpha|\bar{c}| + \beta c_1}, \frac{w}{2}, l\}$ and $|\phi_1(x) - \phi_1(u)| \geq \epsilon$ where $\alpha = \frac{1}{1-\lambda} + 1$ and $\beta = \max\{2, \frac{\lambda}{1-\lambda} + 1\}$. This implies that $|\phi_1(x) - \phi_1(u)| = \alpha|\bar{c}|\delta + \beta c_1 \delta + k$ where $k \in \mathbb{R}_+$. As $\hat{x} > \hat{u}$ and ϕ_1 is subadditive, $\phi_1(u) + \phi_1(\delta) \geq \phi_1(x)$ or $\phi_1(u) - \phi_1(x) \geq -c_1\delta$. Since, $|\phi_1(x) - \phi_1(u)| = \alpha|\bar{c}|\delta + \beta c_1\delta + k$ and $\beta > 1$, we obtain that

$\phi_1(u) = \phi_1(x) + \alpha|\bar{c}|\delta + \beta c_1\delta + k$. Since $\phi_2(x) - \phi_2(u) = \frac{1}{\lambda}(\phi(x) - \phi(u)) - \frac{1-\lambda}{\lambda}(\phi_1(x) - \phi_1(u))$, we obtain $\phi_2(x)-\phi_2(u) = \frac{\bar{c}\delta}{\lambda} + \frac{1-\lambda}{\lambda}\alpha|\bar{c}|\delta + \frac{1-\lambda}{\lambda}\beta c_1\delta + \frac{1-\lambda}{\lambda}k > c_2\delta$ as $\frac{1-\lambda}{\lambda}\alpha > \frac{1}{\lambda}$ and $\frac{1-\lambda}{\lambda}\beta > 1$. Since ϕ_2 is subadditive, $\phi_2(x)+\phi_2(\delta) \geq \phi_2(u)$, i.e., $\phi_2(x) - \phi_2(u) \geq -c_2\delta$. This is the required contradiction.

Now assume by contradiction that there exists u such that ϕ is left continuous at u and ϕ_1 is not left continuous at u. So ϕ_2 cannot be left continuous at u as $\phi_2 = \frac{1}{\lambda}\phi - \frac{1-\lambda}{\lambda}\phi_1$. Denote the left derivative of ϕ at u by c. Since ϕ is left continuous at u, there exists $l > 0$ such that the left derivative of ϕ is $\bar{c}$ in (v, u), where $\hat{v} = \hat{u} - l$. Again choose x such that $|\phi_1(x) - \phi_1(u)| \geq \epsilon$ and $\delta = d(x, u) < \min\{\frac{\epsilon}{\alpha|\bar{c}|+\beta c_1}, \frac{w}{2}, l\}$ where $\alpha = \frac{1}{1-\lambda} + 1$ and $\beta = \max\{2, \frac{\lambda}{1-\lambda} + 1\}$. As $\hat{x} < \hat{u}$ and ϕ_1 is subadditive $\phi_1(x)+\phi_1(\delta) \geq \phi_1(u)$, i.e., $\phi_1(x)-\phi_1(u) \geq -c_1\delta$. Since $|\phi_1(x)-\phi_1(u)| = \alpha|\bar{c}|\delta+\beta c_1\delta+k$, where $k > 0$ and $\beta > 1$ we obtain that $\phi_1(x) = \phi_1(u)+\alpha|\bar{c}|\delta+\beta c_1\delta+k$. Since $\phi_2(x)-\phi_2(u) = \frac{1}{\lambda}(\phi(x)-\phi(u)) - \frac{1-\lambda}{\lambda}(\phi_1(x) - \phi_1(u))$, we obtain $\phi_2(x) - \phi_2(u) = -\frac{\bar{c}\delta}{\lambda} - \frac{1-\lambda}{\lambda}\alpha|\bar{c}|\delta - \frac{1-\lambda}{\lambda}\beta c_1\delta - \frac{1-\lambda}{\lambda}k < -c_2\delta$ as $\frac{1-\lambda}{\lambda}\alpha > \frac{1}{\lambda}$ and $\frac{1-\lambda}{\lambda}\beta > 1$. Since ϕ_2 is subadditive, $\phi_2(x)+\phi_2(\delta) \geq \phi_2(u)$, i.e., $\phi_2(x) - \phi_2(u) \geq -c_2\delta$. This is the required contradiction. □

We now use the above result to study relations between extreme inequalities of finite and infinite group problems. Note that for all $n \in \mathbb{Z}_+$, C_n is a subgroup of I^1. It has been proven that, if $\phi : I^1 \to \mathbb{R}_+$ is an extreme function for $PI(r, 1)$ that is continuous and piecewise linear, then for any finite subgroup $\mathbb{G}$ of I^1 containing all the points at which ϕ is non-differentiable, the valid inequality obtained by restricting ϕ to $\mathbb{G}$ is extreme for the finite group problem defined on $\mathbb{G}$; see Aráoz et al, Evans, Gomory and Johnson [2]. Proposition 14 can be used to prove a weak converse to this result. This weak converse is proven in Proposition 17. Its proof uses the following result.

Theorem 15 ([10])**.** *If ϕ is extreme in the space of subadditive valid inequalities and if ϕ is minimal, then ϕ is extreme in the space of valid inequalities.* □

Although Theorem 15 is proven in Johnson [10] in the context of the mixed integer group problem, it can be easily verified to be true for $PI(r, m)$. Theorem 15 states that if a valid minimal inequality cannot be written as a convex combination of valid subadditive inequalities of $PI(r, m)$, then it is extreme for $PI(r, m)$. Another result used in proving Proposition 17 is from Gomory and Johnson [6]. This result states that every extreme inequality for the finite group problem and $PI(r, 1)$ is also minimal. We next introduce the notation that is used in the statement of Proposition 17.

Definition 16. *Let $\mathbb{G}$ be any finite subgroup of I^1. Then we define $2\mathbb{G}$ to be the subgroup of I^1 with the elements $\{y \in I^1 | y + y \in \mathbb{G}\}$.* □

Thus if $\mathbb{G}$ has a cardinality n, $2\mathbb{G}$ is the subgroup of I^1 of cardinality $2n$ containing $\mathbb{G}$. Inductively, $2^k\mathbb{G}$ as a subgroup of I^1 can be defined. Given a point u in I^1 where $u \notin \mathbb{G}$, we denote u_1 and u_2 to be the closest points of $\mathbb{G}$ to u such that $\hat{u}_1 < \hat{u} < \hat{u}_2$.

Proposition 17. *Let $(\hat{\phi}, 1)$ be a valid subadditive extreme inequality for $\mathbb{G}$. Consider the linear interpolation of $\hat{\phi}$, $\phi : I^1 \to \mathbb{R}_+$, defined as*

$$\phi(u) = \begin{cases} \hat{\phi}(u) & u \in \mathbb{G} \\ \frac{(\hat{u}_2-\hat{u})\hat{\phi}(u_1)+(\hat{u}-\hat{u}_1)\hat{\phi}(u_2)}{\hat{u}_2-\hat{u}_1} & u \notin \mathbb{G}. \end{cases} \quad (7)$$

Suppose that ϕ restricted to $2^k\mathbb{G}$ is an extreme valid inequality for $P(C_{|2^k\mathbb{G}|,r})$ for all $k \in \mathbb{Z}_+$, then ϕ is extreme for the infinite group problem.

Proof. We prove first that ϕ is valid, subadditive and minimal inequality. First note that $\phi(0) = 0$, $\phi(r) = 1$ and $\phi(x) \geq 0$ $\forall x \in I^1$. Now consider any $x \in I^1$. If $x \in \mathbb{G}$ then $\phi(x) + \phi(r - x) = 1$, since $\hat{\phi}$ is minimal. If $x \in I^1 \setminus \mathbb{G}$ then

$$\begin{aligned}
\phi(x) + \phi(r - x) &= \frac{(\hat{x}_2 - \hat{x})\hat{\phi}(x_1) + (\hat{x} - \hat{x}_1)\hat{\phi}(x_2)}{\hat{x}_2 - \hat{x}_1} \\
&+ \frac{((r \hat{-} x)_2 - (r \hat{-} x))\hat{\phi}((r - x)_1)}{(r \hat{-} x)_2 - (r \hat{-} x)_1} \\
&+ \frac{((r \hat{-} x) - (r \hat{-} x)_1)\hat{\phi}((r - x)_2)}{(r \hat{-} x)_2 - (r \hat{-} x)_1} \\
&= 1.
\end{aligned}$$

To check subadditivity, we use the subadditivity checking theorem, of Gomory and Johnson [9]. This states that if a continuous function satisfies the symmetry conditions, i.e., $\phi(x) + \phi(r - x) = 1$ $\forall x \in I^1$ and convex breakpoints satisfy subadditivity, the function is subadditive. Since the break points belong to $\mathbb{G}$, the break points satisfy subadditivity, proving that ϕ is subadditive. Finally, using Theorem 8 we conclude that ϕ is minimal.

Now we prove that ϕ is extreme. Assume by contradiction that ϕ is not extreme. Then, using Theorem 15, we obtain that $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ where $\phi_1 \neq \phi_2$ and ϕ_1, ϕ_2 are valid subadditive inequalities. It is easily proven that the restriction of ϕ_1 to $2^k\mathbb{G}$ is still a valid inequality for $P(C_{|2^k\mathbb{G}|,r})$. Thus if $\phi_1(a) \neq \phi_2(a)$ for $a \in 2^k\mathbb{G}$ then we have a contradiction to the fact that ϕ restricted to $2^k\mathbb{G}$ is extreme. Thus $\phi_1(a) = \phi_2(a)$ $\forall a \in 2^k\mathbb{G}$ $\forall k \in \mathbb{Z}_+$.

Note now that if x_0 is the smallest element of the group, i.e., $\hat{x}_0 < \hat{y}$ $\forall y \in \mathbb{G} \setminus \{0\}$. Then we claim that $\hat{\phi}(x_0) > 0$. Assume by contradiction that this is not true. Let $r = kx_0$ where k is an integer. Then $1 = \hat{\phi}(r) = \hat{\phi}(kx_0) \leq k\hat{\phi}(x_0) = 0$ a contradiction. Also $\hat{\phi}(o) = 0$. Therefore, ϕ is a linear function with positive slope in the interval $[0, x_0]$.

Since ϕ satisfies all the conditions of Proposition 14, ϕ_1 and ϕ_2 are continuous. Therefore $\phi_1(a) = \phi_2(a)$ for a set of points that is dense in I^1 and by continuity, $\phi_1 = \phi_2$, which is the required contradiction. ☐

Although Proposition 17 requires ϕ to be extreme over the sequence of subgroups of the form $2^k\mathbb{G}$, any sequence of subgroup $\{G_k\}_{k=1}^{\infty}$ such that G_k become dense in I^1 as k tends to infinity is sufficient for the result to hold. Many families of valid inequalities as well as conditions under which they are extreme for finite group problems have been derived; see Aráoz, Evans, Gomory and Johnson [2]. The result in Proposition 17 is a useful tool since it allows the use of extreme inequalities of finite group problems to obtain extreme inequalities for $PI(r, 1)$. It is important to observe that the shape of the extreme function in Proposition 17 are not altered when going from the group $2^k\mathbb{G}$ to the larger group $2^{(k+1)}\mathbb{G}$. As an example of the applicability of Proposition 17 we give a proof that the $three - slope$ inequalities introduced in Aráoz, Evans, Gomory and Johnson [2] are extreme for $PI(r, 1)$. We first present the three-slope inequalities as well as conditions under which they are extreme for finite group problems. The result is from Aráoz et al, Evans, Gomory and Johnson [2] although its statement has been adapted to confirm to our notation.

Theorem 18 ([2]). *For positive integers n, r, and d, with* $r+1 \le d \le \lfloor \frac{n+r}{4} \rfloor$, $(\gamma, \gamma(r))$ *defined as follows gives an extreme inequality of* $P(C_{n,\frac{r}{n}})$:

$$\gamma(i) = \begin{cases} i\alpha & i \in \{\frac{1}{n}, ..., \frac{r}{n}\} \\ 1 + (\frac{i}{n} - \frac{r}{n})\beta & i \in \{\frac{r+1}{n}, ..., \frac{d-1}{n}\} \\ i\delta & i \in \{\frac{d}{n}, ..., \frac{n+r-d}{n}\} \\ (i-1)\beta & i \in \{\frac{n+r-d+1}{n}, ..., \frac{n-1}{n}\}, \end{cases} \tag{8}$$

where $\alpha = \frac{n}{r}$, $\delta = \frac{n}{n+r}$, $\beta = \frac{n(n+r-d)}{(n+r)(r-d)}$. □

Proposition 19. *Let r and d be rational numbers such that* $r < d < \frac{1+r}{4} < 1$. *Then the inequality* γ *defined as follows:*

$$\gamma^d(x) = \begin{cases} x\alpha & x \in [o, \dot{r}] \\ 1 + (x-r)\beta & x \in [\dot{r}, \dot{d}] \\ i\delta & x \in [\dot{d}, (1 + \dot{r} - d)] \\ (x-1)\beta & x \in [(1 + \dot{r} - d), \dot{1}), \end{cases} \tag{9}$$

is extreme for $PI(\dot{r}, 1)$ *where* $\alpha = \frac{1}{r}$, $\delta = \frac{1}{1+r}$, $\beta = \frac{1+r-d}{(1+r)(r-d)}$.

Proof. Since r and d are rational such that $r < d < \frac{1+r}{4}$, there exists $n \in \mathbb{Z}$ such that $nr, nd \in \mathbb{Z}$ and $nr + 1 \le nd < \frac{nr+n}{4}$. Denote $r' = nr$ and $d' = dn$. It is easily verified from Theorem 18 that γ restricted to the cyclic subgroup C_n is extreme for the corresponding finite group problem $P(C_{n,r'})$. Also the inequality γ restricted to $C_{2^k n}$ is extreme for $P(C_{2^k n,r})$ since $r' + 1 \le d' \le \left\lfloor \frac{n+r'}{4} \right\rfloor$ implies that $2^k r' + 1 \le 2^k d' \le \left\lfloor \frac{2^k n + 2^k r'}{4} \right\rfloor$. Thus, γ is extreme for $PI(r, 1)$. □

As a final remark, we note that the condition $r < d < \frac{1+r}{4}$ implies that r must satisfy $0 < r < \frac{1}{3}$.

3 Discontinuous extreme inequalities of infinite integer group problem

In this section we show that there exist discontinuous extreme inequalities of $PI(r, 1)$ by proving that three families of discontinuous inequalities are extreme for $PI(r, 1)$. These results are not only theoretically interesting, but also are of practical interest because some of the cuts presented have better performance measures than the Gomory Mixed Integer Cut (GMIC), which has proven to be computationally valuable in commercial software implementations. We also show in these examples how the results developed in Section 2 can be used to show that a given function is extreme. To simplify the notation, we replace $\hat{u}$ with u in this section whenever it is clear from context to which one we refer.

3.1 One-slope inequality

First, we consider the discontinuous function π discovered by Letchford and Lodi [11]. Alternate derivations of the function were later obtained by Richard, Li and Miller [13] and Dash and Günlük [3].

Definition 20. *The function $\pi : I \rightarrow \mathbb{R}_+$ is defined for a right-hand-side r with $r \geq 0.5$ as*

$$\pi(x) = \begin{cases} \frac{x}{r} & 0 \leq x \leq r \\ \frac{x}{r} - \frac{1}{2r} & r < x < 1. \end{cases} \tag{10}$$

It is clear that π is discontinuous at r and the origin o. The function π is illustrated in Figure 1 for a $r = 0.6$.

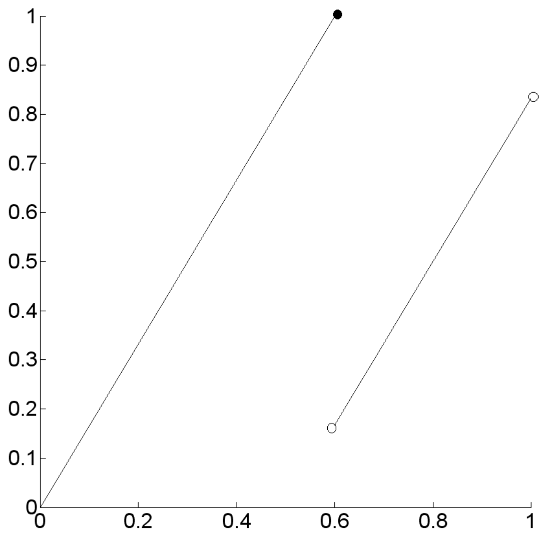

Figure 1: π with $r = 0.6$

It is interesting to observe that π can be obtained as the limiting function of the following family of functions as p tends to zero

$$\pi^p(x) = \begin{cases} \frac{x}{r} & x \leq r \\ 1 + \frac{2p-1}{2pr}(x - r) & r < x \leq r + p \\ \frac{x}{r} - \frac{1}{2r} & r + p < x \leq 1 - p \\ \frac{2p-1}{2pr}(x - 1) & 1 - p < x < 1 \end{cases} \tag{11}$$

where $r \geq 0.5$ and $0 < p \leq \frac{1-r}{2}$. Gomory and Johnson [9] showed that all the functions π^p, which are two-slope functions, are extreme for $PI(r, m)$. In Figure 2, we represent the functions π^p for some values of p and for $r = 0.6$.

First we show that π is minimal. This result was proven in Richard, Li and Miller [13] in a more general setting. For the sake of completeness we next give a simpler but less general proof of this result.

Proposition 21. *π is subadditive and minimal.*

Proof. For subadditivity we consider the following six cases:

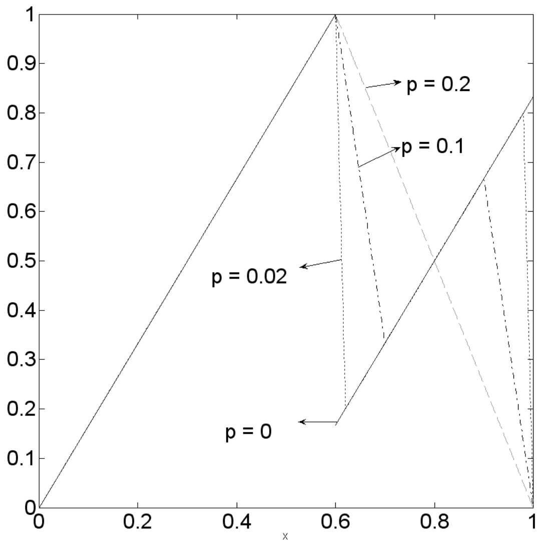

Figure 2: π_p with $r = 0.6$

1. $0 \leq u \leq r$, $0 \leq v \leq r$ and $0 \leq u+v \leq r$. Then $\pi(u) + \pi(v) = \pi(u+v)$.
2. $0 \leq u \leq r$, $0 \leq v \leq r$ and $r < u+v < 1$. Then $\pi(u) + \pi(v) = \frac{u+v}{r} > \frac{(u+v)}{r} - \frac{1}{2r} = \pi(u+v)$.
3. $0 \leq u \leq r$, $r < v < 1$ and $0 \leq u+v \leq r$. Then $\pi(u)+\pi(v) = \frac{u}{r} + \frac{v}{r} - \frac{1}{2r} > \frac{(u+v-1)}{r} = \pi(u+v)$.
4. $0 \leq u \leq r$, $r < v < 1$ and $r < u+v < 1$. Then $\pi(u) + \pi(v) = \pi(u+v)$.
5. $r < u < 1$ $r < v < 1$ and $0 \leq u+v \leq r$. Then $\pi(u) + \pi(v) = \frac{u+v}{r} - \frac{1}{r} = \pi(u+v)$.
6. $r < u < 1$ $r < v < 1$ and $1 < u+v < r$. Then $\pi(u)+\pi(v) = \frac{u+v}{r} - \frac{1}{r} > \frac{u+v-1}{r} - \frac{1}{2r} = \pi(u+v)$.

As π has been proven to be subadditive, it is sufficient to verify that $\pi(u) + \pi(r-u) = 1$ $\forall u \in I^1$ to prove it is minimal. We have two cases:

1. $0 \leq x \leq r$. Then $0 \leq r-x \leq r$. Thus, $\pi(x) + \pi(r-x) = \frac{x+r-x}{r} = 1$.
2. $r < x < 1$. Then $r < r-x < 1$. Thus $\pi(x) + \pi(r-x) = \frac{x}{r} - \frac{1}{2r} + \frac{r-x+1}{r} - \frac{1}{2r} = 1$.

□

Next, we show that π is extreme for $PI(r, 1)$.

Proposition 22. *π is extreme for $PI(r, 1)$.*

Proof. Assume by contradiction that π is not extreme for $PI(r, 1)$. By Theorem 15, π is not extreme in the space of valid subadditive inequalities. Thus, there exist subadditive valid inequalities

π_1 and π_2 such that $\pi_1 \neq \pi_2$ and $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Now take any points $u, v \in I$, such that $\pi(u) + \pi(v) = \pi(u + v)$. By substituting π in terms of π_1 and π_2 we obtain

$$(\pi_2(u) + \pi_2(v) - \pi_2(u + v)) + (\pi_1(u) + \pi_1(v) - \pi_1(u + v)) = 0. \tag{12}$$

It follows from subadditivity of π_2 and π_1 that $\pi_2(u)+\pi_2(v) = \pi_2(u+v)$ and $\pi_1(u)+\pi_1(v) = \pi_1(u+v)$.

Let $U = V = [0, r/2]$. It is clear that $\pi(u_1 + u_2) = \pi(u_1) + \pi(u_2)\ \forall u_1, u_2 \in U$. Thus, π_2 and π_1 also satisfy these equations. Since $U + V = [0, r]$, we have that π_1 and π_2 have the same slope π_1' and π_2' respectively at each point in the interior of $[0, r]$. Moveover, since $\pi_2(r) = \pi_1(r) = 1$, we have that $\pi_2' = \pi_1' = 1/r$. This implies that $\pi_2 = \pi_1$ on $[0, r]$.

Define $a := \pi_2\left(\frac{r+1}{2}\right)$. For any $u \in [\frac{r+1}{2}, 1)$, observe that $\pi(u) = \pi(u - \frac{r+1}{2}) + \pi(\frac{r+1}{2})$. Therefore π_2 must satisfy this equation, i.e., $\pi_2(u) = a+\pi_2(u-\frac{r+1}{2})$ Because $r \geq 0.5$, $\phi_2(u-\frac{r+1}{2}) = (u-\frac{r+1}{2})/r$ and $\phi_2(u) = a+(u-\frac{r+1}{2})/r$. Similarly for any $u \in (r, \frac{r+1}{2}]$, we can obtain that $\pi_2(u) = a+(u-\frac{r+1}{2})/r$. Thus, the value of π_2 on $(r, 1)$ is completely determined by its value at $\frac{r+1}{2}$. Similarly, we can prove that the value of π_1 over $(r, 1)$ is completely determined by its value at $\frac{r+1}{2}$. Note also that both π_1 and π_2 are linear and have a slope of $1/r$ in $(r, 1)$.

Since $\pi_2 \neq \pi_1$ and $\pi\left(\frac{r+1}{2}\right) = \frac{1}{2}$, we may assume that $\pi_2\left(\frac{r+1}{2}\right) = \frac{1}{2} - \delta$ and $\pi_1\left(\frac{r+1}{2}\right) = \frac{1}{2} + \delta$ where $\delta > 0$. Then, $2\pi_2(\frac{r+1}{2}) = 2(1/2 - \delta) = 1 - 2\delta < 1 = \pi_2(2\frac{r+1}{2})$. This shows that π_2 is not subadditive, which is the desired contradiction. □

Discontinuous extreme inequalities cannot be studied using the shooting experiment as they cannot be replicated exactly on a finite grid. One way to estimate the strength of these inequalities is to evaluate their merit index. The merit index was introduced by Gomory and Johnson [9] as a simple way of evaluating the strength of a cut. It was empirically shown to be strongly correlated to the results of the shooting experiment; see Gomory and Johnson [9] for details.

Definition 23 ([9])**.** *Let C_2 be the unit square in two dimensions. The merit index $\mathbb{M}(\phi)$ of a given inequality ϕ is equal to twice the area of the set of points $q \equiv (u_1, u_2) \in C_2$ such that $\phi(u_1)+\phi(u_2) = \phi(u_1 + u_2)$.*

The following proposition gives the merit index of π.

Proposition 24. *For π with a right-hand side r, $\mathbb{M}(\pi) = 4r^2 - 6r + 3$.* □

It is easy to verify that the merit index for Gomory Mixed Integer Cut (GMIC) is $2r^2 - 2r + 1$. Obviously $\mathbb{M}(\pi) > \mathbb{M}(GMIC)\ \forall 0.5 \leq r < 1$. This suggests that the inequality π is strong. To analyze this observation more thoroughly, we compute the merit index of the function π^p for all admissible values of p. It can be easily verified that

$$\mathbb{M}(\pi^p) = \begin{cases} r^2 + 3(1 - r - 2p)^2 + 3p^2 & p \in [0, \frac{1-r}{2}) \\ 2r^2 - 2r + 1 & p = \frac{1-r}{2} \end{cases} \tag{13}$$

We present in Figure 4 the corresponding function for $\mathbb{M}(\pi^p)$ for $r = 0.6$ and $p \in \left[0, \frac{1-r}{2}\right]$. We observe that for a given right-hand side, the merit index is a convex function of p. Note also that the function is discontinuous at $p = \frac{1-r}{2}$. This indicates that there is a significant increase in merit index value when converging from the two-slope family of functions to GMIC. Since the function is convex in p, it seems sensible to assume that the better inequalities are found at the end points of $[0, \frac{1-r}{2}]$, i.e., either $p = 0$ when it is the function π or $p = \frac{1-r}{2}$ when it is the GMIC.

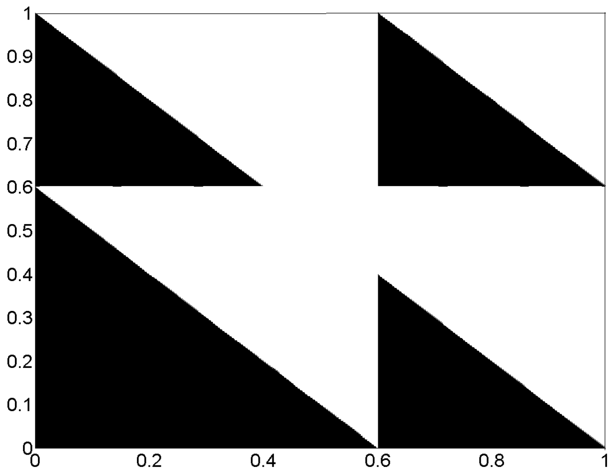

Figure 3: Merit index computation for π with $r = 0.6$

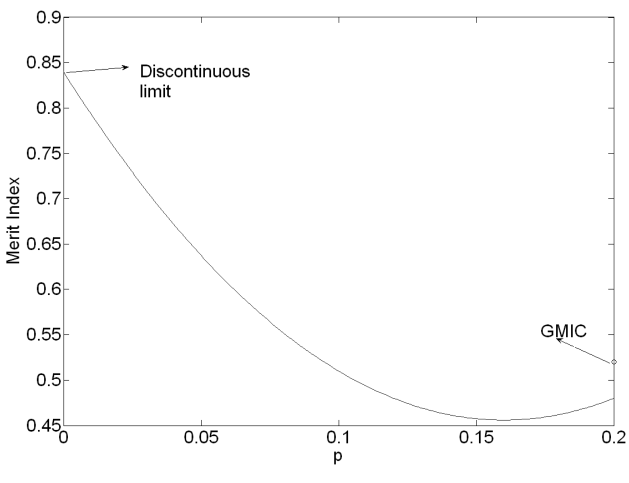

Figure 4: Merit index of π^p as a function of p for $r = 0.6$

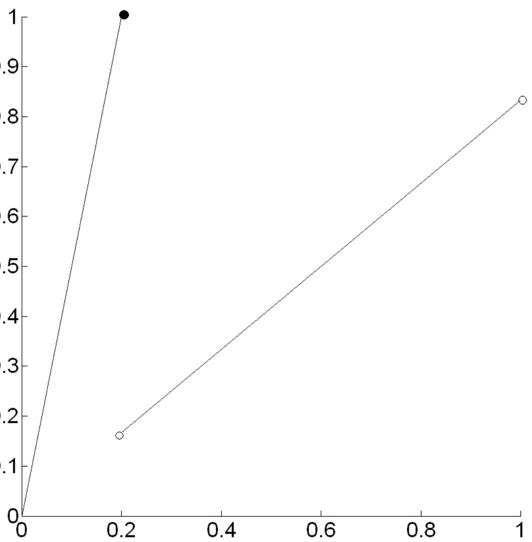

Figure 5: An example κ with $r = 0.2$

3.2 Two-slope inequality - Type I

In this section, we show that the discontinuous function κ derived by Richard, Li and Miller [13] is extreme for $PI(r, 1)$. The proof illustrates the use of Proposition 14 as a tool for showing that functions are extreme for the infinite group problem.

Definition 25. *The function $\kappa : I \to \mathbb{R}_+$ is defined for $r < 0.5$ as*

$$\kappa(u) = \begin{cases} \frac{u}{r} & u \in [0, r] \\ \frac{u}{r+1} & u \in (r, 1) \end{cases} \tag{14}$$

□

The function κ is illustrated in Figure 5 for $r = 0.2$. It is interesting to note that κ can be viewed as the limit of a sequence of the three-slope extreme inequalities when d converges to r, described in Proposition 19. It has been proven in Richard, Li, Miller [13] that κ is subadditive and minimal. For completeness, we give a simpler but less general proof of this fact in Proposition 26.

Proposition 26. *κ is subadditive and minimal.*

Proof. For subadditivity we consider the following six cases:

1. $0 \le u \le r$, $0 \le v \le r$ and $0 \le u + v \le r$. Then $\kappa(u) + \kappa(v) = \kappa(u + v)$.
2. $0 \le u \le r$, $0 \le v \le r$ and $r < u + v < 1$. Then $\kappa(u) + \kappa(v) = \frac{u+v}{r} > \frac{(u+v)}{1+r} = \kappa(u + v)$.
3. $0 \le u \le r$, $r < v < 1$ and $0 \le u + v \le r$. Then $\kappa(u) + \kappa(v) = \frac{(u+v)r+u}{r(1+r)} > \frac{(u+v)r-r+u+v-1}{r(r+1)} = \kappa(u + v)$.

4. $0 \le u \le r$, $r < v < 1$ and $r < u+v < 1$. Then $\kappa(u)+\kappa(v) = \frac{(u+v)r+u}{r(1+r)} > \frac{u+v}{1+r} = \kappa(u+v)$.

5. $r < u < 1$ $r < v < 1$ and $0 \le u+v \le r$. Then $\kappa(u)+\kappa(v) = \frac{u+v}{1+r} > \frac{u+v-1}{r} = \kappa(u+v)$.

6. $r < u < 1$ $r < v < 1$ and $1 < u+v < r$. Then either $\kappa(u)+\kappa(v) = \kappa(u+v)$ or $\kappa(u)+\kappa(v) = \frac{u+v}{1+r} > \frac{u+v-1}{1+r} = \kappa(u+v)$.

As κ has been proven to be subadditive, it is sufficient to verify that $\kappa(u)+\kappa(r-u) = 1$ $\forall u \in I^1$ to prove it is minimal.

1. $0 \le x \le r$. Then $0 \le r-x \le r$. Thus, $\kappa(x)+\kappa(r-x) = \frac{x+r-x}{r} = 1$.
2. $r < x < 1$. Then $r < r-x < 1$. Thus $\kappa(x)+\kappa(r-x) = \frac{x}{1+r} + \frac{r-x+1}{1+r} = 1$.

□

We show next in Proposition 27 that κ is extreme for $PI(r,m)$.

Proposition 27. *κ is extreme for $PI(r,1)$ for $r < 1/3$.*

Proof. Assume by contradiction that κ is not extreme. Since κ is minimal and subadditive, we obtain that $\kappa = \frac{1}{2}\kappa_1 + \frac{1}{2}\kappa_2$ for two different valid subadditive and minimal functions κ_1 and κ_2. Setting $U = V = [0, r/2]$ and using Proposition 13 we obtain that κ_1 and κ_2 are linear on $[0,r]$. Since $\kappa_1(r) = \kappa_2(r) = 1$, we conclude that $\kappa(u) = \kappa_1(u) = \kappa_2(u)$ $\forall u \in [0,r]$.

By Proposition 14 we know that κ_1 and κ_2 are continuous over the interval $(r,1)$. Now consider any point $u_0 \in (r,1)$. We can set up a closed interval $[\frac{1+r}{2} - \Delta, \frac{!+r}{2} + \Delta]$ such that the point u_0 belongs to it and $\frac{1+r}{4} < \Delta < \frac{1-r}{2}$. This is possible because $r < \frac{1}{3}$. Also we know $\frac{1+r}{2} - \Delta < \Delta$. Setting $U = [\frac{1+r}{2} - \Delta, \Delta]$ and $V = [\Delta, \frac{1+r}{2}]$ we, obtain that $U + V = [\frac{1+r}{2}, \frac{1+r}{2} + \Delta]$. Also, if $u \in U$, $v \in V$, then $\kappa(u)+\kappa(v) = \kappa(u+v)$. Thus, by Proposition 10, the function κ_1 (resp. κ_2) has the same slope over $U, V, U+V$. Hence the slope of κ_1 (resp. κ_2) at u_0 is the same as the slope of κ_1 (resp. κ_2) at $\frac{1+r}{2}$. Also, since U and V intersect, the constant term is the same at each point for U and V. Thus for $i \in \{1,2\}$, $\kappa_i(x) = \alpha_i x + \beta_i$ $\forall x \in (r,1)$. Since there exist $u, v, u+v \in (r,1)$ such that $\kappa(u)+\kappa(v) = \kappa(u+v)$, we have $\kappa_i(u)+\kappa_i(v) = \kappa_i(u+v)$ and therefore $\beta_i = 0$. Thus $\kappa_i(x) = (\kappa_i\left(\frac{1+r}{2}\right) / \left(\frac{1+r}{2}\right))x$. The values of κ_1 and κ_2 on the interval $(r,1)$ are determined by their value at $\frac{1+r}{2}$.

Assume $\kappa_1\left(\frac{1+r}{2}\right) > \kappa_2\left(\frac{1+r}{2}\right)$. This implies that $\kappa_2\left(\frac{1+r}{2}\right) < \frac{1}{2}$, and consequently that κ_2 is not subadditive. The case $\kappa_1\left(\frac{1+r}{2}\right) < \kappa_2\left(\frac{1+r}{2}\right)$ can be argued similarly. Thus $\kappa_1\left(\frac{1+r}{2}\right) = \kappa_2\left(\frac{1+r}{2}\right) = \kappa\left(\frac{1+r}{2}\right)$ showing that $\kappa_1 = \kappa_2$. □

In the following proposition, we determine the merit index of κ.

Proposition 28. *For κ with a right-hand side r, $\mathbb{M}(\kappa) = 5r^2 - 4r + 1$.* □

The extreme inequality κ is the limit of the functions γ^d described in Proposition 19. Their merit indices are given by

$$\mathbb{M}(\gamma^d) = r^2 + 3(d-r)^2 + (1+r-3d)^2 \ \forall d \in [r, \frac{1+r}{4}]. \tag{15}$$

For a given right-hand side r, this function is convex in d. This function has a maximum at $d = r$, thus suggesting the use of the discontinuous function κ for pure integer programs over other members of this family.

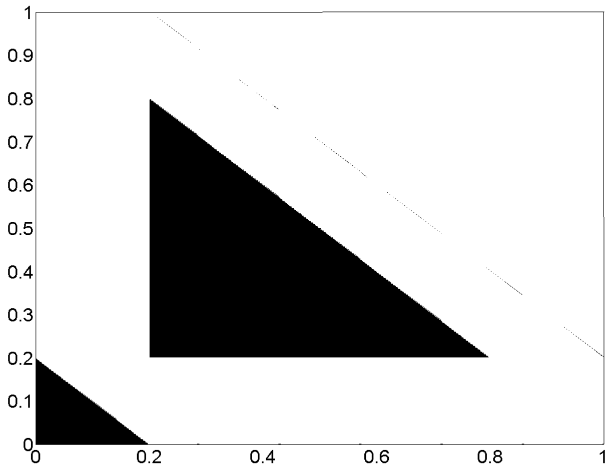

Figure 6: Merit index computation for κ with $r = 0.2$

3.3 Two-slope inequality - Type II

In Sections 3.1 and 3.2, the functions π and κ could be derived respectively as the limits of families of two-slope and three-slope extreme inequalities for the infinite group problem. We prove next that the pointwise limit of a sequence of valid, subadditive and minimal functions for $PI(r, m)$ is also valid, subadditive and minimal. This is a generalization of a result in Dash and Günlük [3], who present a similar result for a specific family of inequalities.

Proposition 29. *Let $f_i : I^m \rightarrow \mathbb{R}_+$ be a valid, subadditive and minimal inequality. If the sequence of inequalities $\{f_i\}_{i=1}^{\infty}$ converges to ϕ pointwise on I^m, then ϕ is valid, subadditive and minimal.*

Proof. By Definition 5, valid functions f_i satisfy $f_i(r) = 1$ and $f_i(o) = 0$ $\forall i \geq 1$. Thus $\phi(r) = 1$ and $\phi(o) = 0$. Also since $f_i(x) \geq 0$ $\forall x \in I^1$ and $\forall i \in \mathbb{Z}_+$, we conclude that $\phi(x) \geq 0$ $\forall x \in I^m$. Assume now by contradiction that ϕ is not subadditive. Therefore there exist u and v such that $\phi(u) + \phi(v) = \phi(u + v) - \delta$, for some $\delta > 0$. Since $\{f_i\}_{i=1}^{\infty}$ converges to ϕ pointwise there exists N such that $|f_i(u) - \phi(u)| < \frac{\delta}{6}$, $|f_i(v) - \phi(v)| < \frac{\delta}{6}$, and $|f_i(u + v) - \phi(u + v)| < \frac{\delta}{6}$ $\forall i > N$. Then $-\phi(u) + f_i(u) - \phi(v) + f_i(v) + \phi(u + v) - f_i(u + v) < \frac{\delta}{2}$, so $f_i(u) + f_i(v) - f_i(u + v) < -\frac{\delta}{2}$ which is the required contradiction since f_i is subadditive.

Next we prove that ϕ is minimal. Since ϕ is already proven to be subadditive, it remains to show that $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$ to prove ϕ is minimal. Assume by contradiction that there exists $v \in I^m$ such that $\phi(v) + \phi(r - v) = 1 + \delta$ for $\delta \neq 0$. Note that $\delta > 0$ since ϕ is subadditive. Again, since $\{f_i\}_{i=1}^{\infty}$ converges to ϕ pointwise there exists N such that $|f_i(v) - \phi(v)| < \frac{\delta}{4}$ and $|f_i(r - v) - \phi(r - v)| < \frac{\delta}{4}$. Thus, $f_i(v) - \phi(v) + f_i(r - v) - \phi(r - v) > -\frac{\delta}{2}$, so $f_i(v) + f_i(r - v) > 1 + \frac{\delta}{2}$, which is a contradiction since f_i is minimal. □

Note that in the proof of Proposition 29 the subadditivity of the limiting function ϕ was established using only the subadditivity of the functions f_i. It is difficult to prove that the limit of a family of extreme inequalities is also extreme. However, we establish the following sufficient conditions for the result to hold.

Proposition 30. *Let $f_i : I^1 \to \mathbb{R}_+$ be piecewise linear, continuous extreme functions of $PI(r,1)$ for $i \geq 1$. Assume that the sequence of functions $\{f_i\}_{i=1}^{\infty}$ converges to ϕ pointwise on I^1 and that ϕ satisfies the conditions of Proposition 14. Let $\mathbb{G}$ be a finite subgroup of I^1 such that if ϕ is discontinuous at u then $u \in \mathbb{G}$. Assume that for every $i \in \mathbb{Z}_+$, there is $k(i) \in \mathbb{Z}_+$, such that the non-differentiable points of f_i belong to $2^k\mathbb{G}$ and $f_i(u) = \phi(u)$ $\forall u \in 2^k\mathbb{G}$. Then ϕ is an extreme function for I^1.*

Proof. Assume by contradiction that ϕ is not extreme. Since it is proven that ϕ is minimal, this implies $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ where $\phi_2 \neq \phi_1$ are subadditive valid inequalities for $PI(r,1)$. Let $i \in \mathbb{Z}_+$ and $k := k(i)$. Consider the restrictions of ϕ, ϕ_1 and ϕ_2 to $2^k\mathbb{G}$. The restrictions of ϕ_1 and ϕ_2 are valid functions over $P(C_{2^k\mathbb{G},r})$. Since $f_i(u) = \phi(u)$ $\forall u \in 2^k\mathbb{G}$ and f_i is non-differentiable only on $2^k\mathbb{G}$, by the result of Aráoz, Evans, Gomory and Johnson [2], the restriction of ϕ to $2^k\mathbb{G}$ is extreme. Thus $\phi(u) = \phi_1(u) = \phi_2(u)$ $\forall u \in 2^k\mathbb{G}$. As the condition holds for all i we conclude that $\phi_1(u) = \phi_2(u)$ for a dense subset of I^1. Then by Proposition 14 we obtain that $\phi_1(u) = \phi_2(u)$ $\forall u \in I^1$ which is a contradiction. □

We next use the result of Proposition 30 to prove that the discontinuous limit of Gomory and Johnson's [9] three-slope function ζ^θ is extreme for $PI(r,1)$. To the best of our knowledge, ζ^θ has not been described in previous literature.

Definition 31. *Given $r \in (0,1)$, the function $\zeta^\theta : I^1 \to \mathbb{R}_+$ is defined for $\hat{\theta} \leq min \left\{\frac{\hat{r}}{2}, \frac{1-\hat{r}}{4}\right\}$ as*

$$\zeta^\theta(x) = \begin{cases} \frac{x}{r} & 0 \leq x \leq r \\ \frac{1-r-\theta}{1-r} - \frac{r+\theta-x}{r} & r < x \leq r+\theta \\ \frac{1-x}{1-r} & r+\theta \leq x \leq 1-\theta \\ \frac{\theta}{1-r} + \frac{x-1+\theta}{r} & 1-\theta \leq x < 1 \end{cases} \tag{16}$$

□

The function ζ^θ is illustrated in Figure 7 for $r = 0.2$ and $\theta = 0.1$. It is the limit of a family of three-slope extreme inequalities presented in Gomory and Johnson [9]. We represent this family as $\zeta^{\theta,q}$, which is given by

$$\zeta^{\theta,q}(x) = \begin{cases} \frac{x}{r} & x \in [0,r] \\ 1-\beta(x-r) & x \in [r, r+q) \\ \frac{1-r-\theta}{1-r} - \frac{(r+\theta-x)}{r} & x \in [r+q, r+\theta] \\ \frac{1-x}{1-r} & x \in [r+\theta, 1-\theta] \\ \frac{\theta}{1-r} + \frac{(x+\theta-1)}{r} & x \in [1-\theta, 1-q) \\ \beta(1-x) & x \in [1-q, 1) \end{cases} , \tag{17}$$

where $\beta = \frac{\theta+qr-q}{(1-r)rq}$ and $q \leq \theta$. The function ζ^θ is obtained by taking the limit of $\zeta^{\theta,q}$ as q tends to 0. It is clear that if $\hat{\theta}$ and $\hat{r}$ are rational then Proposition 30 implies that ζ^θ is extreme for $PI(r,1)$. In the general case when $\hat{\theta}$ and $\hat{r}$ are allowed to be irrational, a proof can be written by combining the proof given by Gomory and Johnson [9] with the result of Proposition 14.

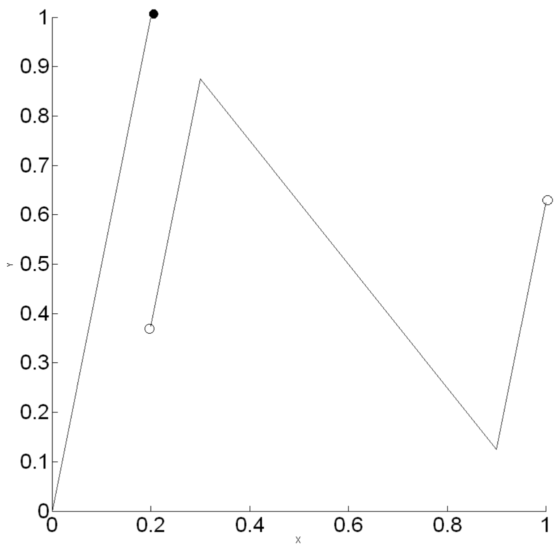

Figure 7: An example ζ^θ with $r = 0.2$ and $\theta = 0.1$

Proposition 32. *ζ^θ is extreme for $PI(r, 1)$.*

Proof. Since ζ^θ is a sequence of extreme inequalities, using Proposition 29 we obtain that ζ^θ is subadditive and minimal. Now assume by contradiction that ζ^θ is not extreme. Since ζ^θ is minimal, we obtain that $\zeta^\theta = \frac{1}{2}\zeta_1 + \frac{1}{2}\zeta_2$ for two different valid subadditive and minimal functions ζ_1 and ζ_2. Setting $U = V = [0, r/2]$ and using Corollary 13, we obtain that ζ_1 and ζ_2 are linear on $[0, r]$. Since $\zeta_1(r) = \zeta_2(r) = 1$, we conclude that $\zeta^\theta(u) = \zeta_1(u) = \zeta_2(u)\ \forall u \in [0, r]$.

Now consider any $u_0 \in (r, r + \theta)$. There exists Δ, such that $2\Delta < \hat{r}$ and $[u_0 - \Delta, u_0 + 3\Delta] \in (r, r + \theta)$. Let $U = [0, 2\Delta]$ and $V = [u_0 - \Delta, u_0 + \Delta]$. Clearly $U + V = [u_0 - \Delta, u_0 + 3\Delta]$. Since all the conditions of Corollary 13 are satisfied, we have that ζ_1 and ζ_2 have the same slope on U, V and $U + V$. In particular, the slope at u_0 for ζ_1 and ζ_2 is $\frac{1}{\hat{r}}$ in the interval $(r, r + \theta)$. Similarly it can be proven that the slope for ζ_1 and ζ_2 is $\frac{1}{\hat{r}}$ for any u_0 belonging to the interval $(1 - \theta, 1)$. By Proposition 14, ζ_1 are ζ_2 are continuous over $(r, 1)$. Thus, the values of ζ_1 and ζ_2 on the intervals $(r, r + \theta]$ and $[1 - \theta, 1)$ depend only on the values of these functions at $r + \theta$ and $1 - \theta$ respectively.

Consider the points $y_1, y_2, y_3 \in I^1$ such that $\hat{y}_1 = \frac{1-\hat{r}}{2} + \hat{r}$, $\hat{y}_2 = \frac{1-\hat{r}}{2} + \hat{r} + \theta$ and $\hat{y}_3 = 1 - \theta$. Let $U = [y_1, y_2]$ and $V = [y_2, y_3]$. It can be verified that $U + V = [r + \theta, y_1]$. Also, as described earlier, ζ_1 and ζ_2 are continuous over U, V and $U + V$. Since $\zeta^\theta(u) + \zeta^\theta(v) = \zeta^\theta(u + v)$ for $u \in U$ and $v \in V$, ζ_1 and ζ_2 satisfy the same equalities. Thus, by using Proposition 10, ζ_1 and ζ_2 have constant slope over U, V and $U + V$. Since U and V intersect and V and $U + V$ intersect, the constant term is same at all points. Thus ζ_1 and ζ_2 are linear functions over U, V and $U + V$. Finally, since $\zeta^\theta(\frac{1+r}{2}) + \zeta^\theta(\frac{1+r}{2}) = \zeta^\theta(r)$, we have $\zeta_1(\frac{1+r}{2}) + \zeta_1(\frac{1+r}{2}) = \zeta_1(r) = 1$ or $\zeta_1(\frac{1+r}{2}) = \frac{1}{2}$. Similarly, it can be shown that $\zeta_2(\frac{1+r}{2}) = \frac{1}{2}$. Thus, $\zeta_1(x) = \zeta_2(x)\ \forall x \in [r + \theta, 1 - \theta]$. This implies, $\zeta_1(x) = \zeta_2(x)$ on the intervals $(r, r + \theta]$ and $[1 - \theta, 1)$ as $\zeta_1(r + \theta) = \zeta_2(r + \theta)$ and $\zeta_1(1 - \theta) = \zeta_2(1 - \theta)$. Hence $\zeta_1 = \zeta_2$, which is the required contradiction. □

The following proposition gives the merit index of ζ^θ.

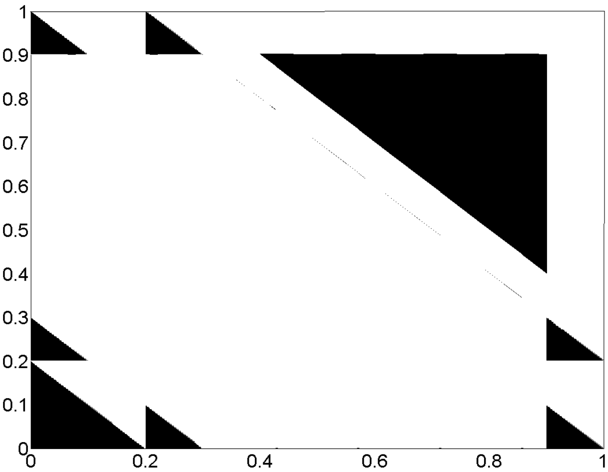

Figure 8: Merit index computation for ζ^θ with $r = 0.2$ and $\theta = 0.1$

Proposition 33. *For ζ^θ with a right-hand side r, $\mathbb{M}(\pi) = r^2 + 6\theta^2 + (1 - r - 3\theta)^2$.* □

The merit indices of the function of family $\zeta^{\theta,q}$ can be computed as

$$\mathbb{M}(\zeta^{\theta,q}) = r^2 + 6(\theta - q)^2 + (1 - r - 3\theta)^2 + 3q^2. \tag{18}$$

It can be easily verified that the merit index of this family is best for $q = 0$, i.e., ζ^θ has the largest merit index in its family.

This trend in merit index can be intuitively explained. There are linear pieces of the function that disappear when the limit is taken. These linear pieces typically have a negative impact on the merit index. Therefore, the merit index progressively improves as these pieces become smaller, and the merit index of the discontinuous function is generally better than those of the nearly discontinuous functions.

4 Continuity of Extreme Functions of Mixed Integer Infinite Group Problem

In this section, we will prove that extreme functions of mixed integer infinite group problem are always continuous.

As an extension of their work, Gomory and Johnson [6, 7] and Johnson [10] introduced a mixed integer variant of $PI(r, m)$ that we call the mixed integer infinite group problem and denote as $MI(r, m)$. We first give a definition of $MI(r, m)$. We let S^m to represent the set of real m-dimensional vectors $w = (w_1, w_2 ... w_m)$, such that $max\{|w_i|| 1 \leq i \leq m\} = 1$, i.e., S^m is the boundary of an m-dimensional hypercube. Let F be the function that componentwise gives the fractional part of a real vector, i.e., $F(x) = (x_1(mod\ 1),\ x_2(mod\ 1),\ ..., x_m(mod\ 1))$.

Definition 34 ([10])**.** *The mixed integer infinite group problem, $MI(r, m)$, is defined as a set of functions $t : I^m \to \mathbb{R}$ and $s : S^m \to \mathbb{R}$ that satisfy*

1. $\sum_{u\in I^m} ut(u) + F(\sum_{v\in S^m} vs(v)) = r$, $r \in I^m$,

2. $t(u)$ *is a non-negative integer for* $u \in I^m$, $s(v)$ *is a non-negative real number for* $v \in S^m$,

3. t *and* s *have finite supports.* □

In the following definitions,we give the concepts of valid inequality, minimal inequality and extreme inequality for $MI(r,m)$.

Definition 35 ([10])**.** *A valid inequality for MI(r,m) is defined as a pair of functions,* $\phi : I^m \to \mathbb{R}_+$ *and* $\mu_\phi : S^m \to \mathbb{R}_+$*, such that* $\sum_{u\in I^m} \phi(u)t(u) + \sum_{v\in S^m} \mu_\phi(v)s(v) \geq 1$, $\forall\ (t,s) \in MI(r,m)$*, where* $\phi(o) = 0$ *and* $\phi(r) = 1$*.* □

Definition 36 ([10])**.** *A valid inequality* (ϕ, μ_ϕ) *is minimal for* $MI(r,m)$ *if* $(\rho_1, \mu_1) = (\rho_2, \mu_2) = (\phi, \mu_\phi)$ *for any valid inequalities* (ρ_1, μ_1) *and* (ρ_2, μ_2) *that satisfy* $(\phi, \mu_\phi) = \frac{1}{2}(\rho_1, \mu_1) + \frac{1}{2}(\rho_2, \mu_2)$*.* □

Definition 37 ([10])**.** *A valid inequality* (ϕ, μ_ϕ) *is extreme for* $MI(r,m)$ *if* $(\phi, \mu_\phi) = \frac{1}{2}(\rho_1, \mu_1) + \frac{1}{2}(\rho_2, \mu_2)$ *for valid functions* (ρ_1, μ_1) *and* (ρ_2, μ_2) *implies* $(\rho_1, \mu_1) = (\rho_2, \mu_2) = (\phi, \mu_\phi)$*.* □

We first give two theorems by Johnson [10].

Theorem 38 ([10])**.** *Let* $\phi : I^m \to \mathbb{R}_+$ *and let* $\tau_\phi : S^m \to \mathbb{R}_+$*. For any* $r \in I \backslash \{0\}$ *and any* $W \subseteq S^m$*,* (ϕ, τ_ϕ) *is a minimal valid inequality if and only if*

$$\begin{array}{rcrl} \phi(u) + \phi(v) & \geq & \phi(u+v) & \forall u, v \in I^m \\ \tau_\phi(w) & = & lim_{h\downarrow 0} \frac{\phi(F(hw))}{h} & \forall w \in W \\ \phi(u) + \phi(u_0 - u) & = & 1 & \forall u \in I^m. \end{array} \tag{19}$$

□

Theorem 39 ([10])**.** *All extreme inequalities of* $MI(r,m)$ *are minimal valid inequalities.* □

Note that the existence of the directional derivatives of ϕ is not sufficient for the continuity of ϕ, as the following well-known example shows.

Example 40. *Let*

$$f(x,y) = \begin{cases} 0 & x = 0, y = 0 \\ \frac{xy^2}{x^2+y^4} & otherwise. \end{cases} \tag{20}$$

The function $f(x,y)$ is not continuous at the origin. To prove this claim, consider the sequence of points $(x_i, y_i) = (\frac{1}{i}, \frac{1}{i^2})$. Then $lim_{i\to\infty}(x_i, y_i) = (0,0)$, but $f(x_i, y_i) = \frac{1}{2}\ \forall i \geq 1$. So $lim_{i\to\infty} f(x_i, y_i) = \frac{1}{2} \neq f(0,0)$. However, restricted to any line on $\mathbb{R}^2$, the function f is continuous and has a derivative. □

We now use Theorem 38 to prove that the function ϕ is continuous in a valid inequality (ϕ, μ_ϕ) of $MI(r,1)$. In order to prove this result, we first show in Proposition 41 that valid subadditive functions are continuous over I^m if and only if they are continuous at o.

Proposition 41. *Let* $\phi : I^m \to \mathbb{R}_+$ *be a valid and subadditive inequality for* $PI(r,m)$*. Then* ϕ *is continuous over* I^m *if and only if it is continuous at* o*.*

Proof. The direct implication is straightforward. For the reverse implication consider, $u \in I^m$ and any sequence of points $\{q_i\}_{i=1}^{\infty}$ that converges to u. We need to show that the sequence $\{\phi(q_i)\}_{i=1}^{\infty}$ converges to $\phi(u)$. By subadditivity, $-\phi(u-q_i)+\phi(u) \leq \phi(q_i) \leq \phi(q_i-u)+\phi(u)$. Since the sequences $\{u-q_i\}_{i=1}^{\infty}$ and $\{q_i-u\}_{i=1}^{\infty}$ converge to o and ϕ is continuous at o, we conclude that $\{\phi(u-q_i)\}_{i=1}^{\infty}$ and $\{\phi(q_i-u)\}_{i=1}^{\infty}$ converge to 0. This proves that $\{\phi(q_i)\}_{i=1}^{\infty}$ converges to $\phi(u)$. □

Proposition 42. *Let (ϕ, μ_ϕ) be a valid inequality for $MI(r,m)$. If ϕ satisfies the first two conditions in (19), then ϕ is continuous.*

Proof. Let e^i be the i^{th} unit vector in $\mathbb{R}^m$. First we claim that given $\gamma > 0$ there exists $h^\gamma > 0$ such that $\phi(F(he^i)) < \gamma$ and $\phi(F(-he^i)) < \gamma$ $\forall 1 \leq i \leq m$ if $h \leq h^\gamma$. To prove this claim, choose any $\rho > 0$. Denote $lim_{h\downarrow 0}\frac{\phi(F(he^i))}{h} = c^i \geq 0$. There exists $h^i > 0$ such that $c^i - \rho < \phi(F(he^i))/h < c^i + \rho$, $\forall h \leq h^i$. Set $h_i^+ = min\{h^i, \frac{\gamma}{(c^i+\rho)}\}$. Thus, if $h \leq h_i^+$, then $\phi(F(he^i)) < h(c^i+\rho) \leq h_i^+(c^i+\rho) \leq \gamma$. Similarly, let h_i^- be such that if $h \leq h_i^-$, then $\phi(F(h(-e^i))) < \gamma$. Finally set $h^\gamma = min\{h_1^+, h_1^-, h_2^+, ..., h_m^-\}$. Thus, if $h \leq h^\gamma$, then $\phi(F(he^i)) < \gamma$ and $\phi(F(h(-e^i))) < \gamma$ $\forall 1 \leq i \leq m$.

We now prove that ϕ is continuous by proving that ϕ is continuous at o and by applying Proposition 41. We show that for $\epsilon > 0$ there is a neighborhood N_ϵ of o such that $\phi(u) < \epsilon$ $\forall u \in N_\epsilon$. Consider the set of points $E = \{u \in I^m | d(o,u) < h^{\epsilon/m}\}$. Since $d(o,u) < h^{\epsilon/m}$, we have that $d'(0,u_i) < h^{\epsilon/m}$. Note here that if $|\hat{u}_i - 1| < h^{\epsilon/m}$, then $\phi(u_i e^i) = \phi(F((|\hat{u}_i - 1|)(-e^i))) \leq \frac{\epsilon}{m}$. By (19), $\phi(u) \leq \sum_{1\leq i\leq m}\phi(u_i e^i) < \epsilon$. Setting N_ϵ to E completes the proof. □

5 Conclusion

In this paper, we show that discontinuous functions are important in the study of infinite group problems by proving they are extreme and by giving indications that these inequalities might be strong for IPs.

We first introduce a variant of Gomory and Johnson's Interval Lemma that is useful for showing that discontinuous functions are extreme for $PI(r,m)$. We also provide conditions under which the Interval Lemma introduced by Gomory and Johnson [9] may be applied when proving that an inequality is extreme over the set of subadditive valid functions that are not necessarily continuous. These tools lead to a new method for constructing extreme inequalities for infinite group problem from extreme inequalities of finite group problems. We then prove that there exist discontinuous functions that are extreme for the integer infinite group problem. In particular we show that three different families of discontinuous valid inequalities are extreme for $PI(r,1)$ and do so by using the different tools we introduced. We generalized these results, to prove that under certain sufficient conditions, the limiting inequality of a sequence of continuous piecewise linear extreme inequalities is extreme. We finally prove that extreme functions for mixed integer infinite group problem are always continuous and therefore discontinuous extreme inequalities of $PI(r,m)$ do not translate into extreme inequalities of $MI(r,m)$.

The results in this paper are important both from a theoretical and practical point of view. First, they show that strong cutting planes can be fundamentally different for pure integer and mixed integer problems. In particular, discontinuous inequalities can be extreme for infinite group problems. This sheds some light on the variety of extreme inequalities for group problems, a variety that is often underestimated. Second the results of this paper suggest that there is a potential computational benefit in studying discontinuous functions as it appears that discontinuous functions always have better merit indices than their continuous counterparts. In particular, the merit index of the function π was shown to be better than that of GMIC, an inequality that is known to be important in computational implementations. This suggests that discontinuous functions should not be omitted when studying infinite group problems, as they may provide interesting and potentially strong cuts for IPs.

References

[1] Aczél, J.: Lectures on functional equations and their applications. New York: Academic Press (1966)

[2] Aráoz, J., Evans, L., Gomory, R. E, Johnson, E.L.: Cyclic groups and knapsack facets. Math. Program. **96**, 377-408 (2003)

[3] Dash, S., Günlük, O.: Valid inequalities based on simple mixed integer set. Math. Program. **106**, 29-53 (2006)

[4] Dey, S.S., Richard, J-P. P.: Two families of facets for the two-dimensional mixed integer infinite group problem. submitted for publication. (2006)

[5] Gomory, R. E.: Some polyhedra related to combinatorial problems. J. Linear Algebra and Applications. **2**, 341- 375 (1969)

[6] Gomory, R. E., Johnson, E. L.: Some continuous functions related to corner polyhedra, Part I. Math. Program. **3**, 23-85 (1972)

[7] Gomory, R. E., Johnson, E. L.: Some continuous functions related to corner polyhedra, Part II. Math. Program **3**, 359-389 (1972)

[8] Gomory, R. E., Johnson, E. L., Evans, L.: Corner polyhedra and their connection with cutting planes. Math. Program. **96**, 321-339 (2003)

[9] Gomory, R. E., Johnson, E. L.: T-space and cutting planes. Math. Program. **96**, 341- 375 (2003)

[10] Johnson, E. L.: On the Group Problem for mixed integer programming. Math. Program. Study **2**, 137- 179 (1974)

[11] Letchford, A.N., Lodi A.: Strengthening Chvátal-Gomory cuts and Gomory fractional cuts. Oper. Res. Lett., **30(2)**, 74-82 (2002)

[12] Miller, L., Li, Y., Richard, J-P. P.: New facets for finite and infinite group problems from approximate lifting. Technical report MN-ISYE-TR-06-004. University of Minnesota Graduate Program in Industrial and Systems Engineering. (2006)

[13] Richard, J-P. P., Li, Y., Miller L.A.: Valid inequalities for MIPs and group polyhedra from approximate liftings. Technical report MN-ISYE-TR-06-003. University of Minnesota Graduate Program in Industrial and Systems Engineering. (2006)