CORRECTOR-PREDICTOR METHODS
FOR SUFFICIENT LINEAR COMPLEMENTARITY PROBLEMS
IN A WIDE NEIGHBORHOOD OF THE CENTRAL PATH

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Abstract. A higher order corrector-predictor interior-point method is proposed for solving sufficient linear complementarity problems. The algorithm produces a sequence of iterates in the $\mathcal{N}_\infty^-$ neighborhood of the central path. The algorithm does not depend on the handicap κ of the problem. It has $O((1 + \kappa)\sqrt{n}L)$ iteration complexity and is superlinearly convergent even for degenerate problems.

Key words. linear complementarity, interior-point, path-following, corrector-predictor, wide neighborhood

AMS subject classifications. 90C51, 90C33

1. Introduction. The MTY predictor-corrector algorithm proposed by Mizuno, Todd and Ye [9] is a typical representative of a large class of MTY type predictor-corrector methods, which play a very important role among primal-dual interior point methods. It was the first algorithm for linear programming (LP) that had both polynomial complexity and superlinear convergence. This result was extended to monotone linear complementarity problems that are nondegenerate, in the sense that they have a strictly complementarity solution [6, 23]. It turned out that the nondegeneracy assumption is not restrictive, since according to [10] a large class of interior point methods, which contains MTY, can have only linear convergence if this assumption is violated. However, it is possible to obtain arbitrarily high order of convergence for degenerate problems by using higher order information of the central path [19, 21].

The existence of a central path is crucial for interior-point methods. An important result of the 1991 monograph of Kojima et al. [7] shows that the central path exists for any P_* linear complementarity problem, provided that the relative interior of its feasible set is nonempty. We recall that every P_* linear complementarity problem is a $P_*(\kappa)$ problem for some $\kappa \geq 0$, i.e.

$$P_* = \cup_{\kappa \geq 0} P_*(\kappa).$$

The class of sufficient matrices was introduced by Cottle et al. [3] in connection with the linear complementarity problems. A matrix $M \in R^{n \times n}$ is said to be column sufficient if:

$$z_i(Mz)_i \leq 0 \text{ for all } i \quad \text{implies} \quad z_i(Mz)_i = 0 \text{ for all } i.$$

The matrix M is called row sufficient if its transpose is column sufficient. The matrix M is sufficient if it is both column and row sufficient. It is proved in the same book that the class of sufficient matrices is closely related to the existence of the solution of the linear complementarity problems and the convexity of the solution set. A surprising result given by Väliaho [22] showed that the class of P_* matrices coincides with the class of sufficient matrices. Therefore, every P_* linear complementarity problem

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is a sufficient linear complementarity problem, and vice versa. The class of sufficient linear complementarity problems is a very general framework for studying interior-point methods. In 1995 Miao [8] extended the MTY predictor-corrector algorithm for $P_*(\kappa)$ linear complementarity problems. His algorithm has $O((1+\kappa)\sqrt{n}L)$ iteration complexity, and is quadratically convergent for nondegenerate problems. However, the constant κ is explicitly used in the construction of the algorithm, which implies that the algorithm cannot be used for sufficient linear complementarity problems. Potra and Sheng [17] extended the MTY predictor-corrector algorithm further for sufficient complementarity problems. While the algorithms of [17] do not depend on the constant κ, their computational complexity does: if the problem is a $P_*(\kappa)$ linear complementarity problem they terminate in at most $O((1+\kappa)\sqrt{n}L)$ iterations. Moreover, the algorithms may attain arbitrarily high orders of convergence on nondegenerate problems. Predictor-corrector algorithms with arbitrarily high order of convergence for degenerate sufficient linear complementarity problems were given in [19]. The algorithms, as shown in [18], have $O((1+\kappa)\sqrt{n}L)$ iteration complexity for $P_*(\kappa)$ linear complementarity problems.

All the above algorithms operate in l_2 neighborhoods, also known as the small neighborhoods, of the central path. It is well known however that primal-dual interior-point methods have better practical performances in wide neighborhoods of the central path. Unfortunately, the iteration complexity of the predictor-corrector methods that use wide neighborhoods are worse than the complexity of the corresponding methods for small neighborhoods. Moreover, as shown in [2, 4], it is more difficult to develop and analyze predictor-corrector methods in wide neighborhoods. The best iteration complexity achieved by any known interior-point method for monotone linear complementarity problems in the wide neighborhoods using first order information is $O(nL)$. By using a large neighborhood defined by a suitable self regular proximity measure, Peng et al. [12] have obtained a predictor-corrector method with $O(\log n \sqrt{n}L)$ iteration complexity which is superlinearly convergent on nondegenerate problems. It turns out that the complexity result can be improved by using higher order information. The algorithms described in [11, 5, 24] have $O(\sqrt{n}L)$ iteration complexity. However, these algorithms are not of a predictor-corrector type, and they are not superlinearly convergent. The algorithm described in [20] operates in the δ_∞^- neighborhood, and is superlinear convergent for sufficient linear complementarity problems, but no complexity results have been proved for this algorithm. A predictor-corrector method for monotone linear complementarity problems using wide neighborhoods of the central path was proposed in [15]. The algorithm has $O(\sqrt{n}L)$ iteration complexity by using a higher order predictor, and it is superlinear convergence even for degenerate problems. In a recent paper, Potra and Liu [16] extended the algorithm in [15] to sufficient linear complementarity problems. Two algorithms are analyzed in [16]. Both algorithms are of predictor-corrector type acting in between two wide neighborhoods of the central path. The radii of those neighborhoods have to satisfy an inequality that depends on the handicap κ of the problems. The first algorithm in [16] depends also on κ, while the second does not. The second algorithm uses the first algorithm by assigning $\kappa = 1$ and then doubles κ until a certain criterion is satisfied. Both algorithms have $O((1+\kappa)^{1+1/m}\sqrt{n}L)$ iteration complexity and are superlinearly convergent even for degenerate problems.

The traditional predictor-corrector algorithms operate between two neighborhoods of the central path. The predictor step aims to decrease the duality gap while keeping the point in the outer neighborhood. It is followed by a corrector step, which

brings the point back into the inner neighborhood so that the next predictor-corrector iteration can be performed. As analyzed in a recent paper [14], the centering direction is not as efficient in the wide neighborhoods as in the small neighborhoods, so that a line search on the centering direction is always needed in the corrector step using wide neighborhoods. Moreover, since the pure centering direction is anyhow inefficient in the wide neighborhoods, a corrector-predictor method was proposed in [14] where the corrector is used to improve both optimality and centrality. In the present paper, we generalize this algorithm to sufficient linear complementarity problems. By using higher order information, the algorithm has $O((1 + \kappa)\sqrt{n}L)$ iteration complexity, which matches the best iteration complexity obtained in the small neighborhoods. Moreover, our algorithm is superlinearly covergent even for degenerate problems. More precisely, by using a predictor with order $m_p > 1$, we show that the duality gap converges to zero with Q-order $m_p + 1$ in the nondegenerate case, and with Q-order $(m_p + 1)/2$ in the degenerate case. Our algorithm improves considerably the results of [16]. First, the algorithm is a corrector-predictor interior-point method so that it uses only one wide neighborhood of the central path, whose radius can be any number between 0 and 1, and therefore does not depend on κ. Second its iteration complexity is improved ($O((1 + \kappa)\sqrt{n}L)$ versus $O((1 + \kappa)^{1+1/m}\sqrt{n}L)$). Finally, by contrast with the algorithms of [16] the present algorithm reduces the duality gap both in the corrector and the predictor steps and therefore it is more efficient. In the present paper we work on horizontal linear complementarity problems (HLCP), which is a slight generalization of the standard linear complementarity problem. Equivalence results of different variants of linear complementarity problems can be found in [1]. We choose to work on HLCP because of its symmetry.

We denote by $I\!N$ the set of all nonnegative integers. $I\!R$, $I\!R_+$, $I\!R_{++}$ denote the set of real, nonnegative real, and positive real numbers respectively. For any real number κ, $\lceil \kappa \rceil$ denotes the smallest integer greater or equal to κ. Given a vector x, the corresponding upper case symbol denotes, as usual, the diagonal matrix X defined by the vector. The symbol e represents the vector of all ones, with dimension given by the context.

We denote component-wise operations on vectors by the usual notations for real numbers. Thus, given two vectors u, v of the same dimension, uv, u/v, etc. will denote the vectors with components $u_i v_i$, u_i/v_i, etc. This notation is consistent as long as component-wise operations always have precedence in relation to matrix operations. Note that $uv \equiv Uv$ and if A is a matrix, then $Auv \equiv AUv$, but in general $Auv \neq (Au)v$. Also if f is a scalar function and v is a vector, then $f(v)$ denotes the vector with components $f(v_i)$. For example if $v \in I\!R_+^n$, then $\sqrt{v}$ denotes the vector with components $\sqrt{v_i}$, and $1 - v$ denotes the vector with components $1 - v_i$. Traditionally the vector $1 - v$ is written as $e - v$, where e is the vector of all ones. Inequalities are to be understood in a similar fashion. For example if $v \in I\!R^n$, then $v \geq 3$ means that $v_i \geq 3$, $i = 1, \ldots, n$. Traditionally this is written as $v \geq 3e$. If $\| \cdot \|$ is a vector norm on $I\!R^n$ and A is a matrix, then the operator norm induced by $\| \cdot \|$ is defined by $\| A \| = \max\{\| Ax \| ; \| x \| = 1\}$. As a particular case we note that if U is the diagonal matrix defined by the vector u, then $\| U \|_2 = \| u \|_\infty$.

We use the notations $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, and $o(\cdot)$ in the standard way to express asymptotic relationships between functions. The most common usage will be associated with a sequence $\{x^k\}$ of vectors and a sequence $\{\tau_k\}$ of positive real numbers. In this case $x^k = O(\tau_k)$ means that there is a constant K (dependent on problem data) such that for every $k \in I\!N$, $\| x^k \| \leq K\tau_k$. Similarly, if $x^k > 0$, $x^k = \Omega(\tau_k)$ means that

$(x^k)^{-1} = O(1/\tau_k)$. If we have both $x^k = O(\tau_k)$ and $x^k = \Omega(\tau_k)$, we write $x^k = \Theta(\tau_k)$

If $x, s \in I\!\!R^n$, then the vector $z \in I\!\!R^{2n}$ obtained by concatenating x and s will be denoted by $\lceil x, s \rfloor$, i.e.,

$$(1.1) \qquad z = \lceil x, s \rfloor = \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} x^T, s^T \end{bmatrix}^T .$$

Throughout this paper the mean value of xs will be denoted by

$$(1.2) \qquad \mu(z) = \frac{x^T s}{n} .$$

2. The $P_*(\kappa)$ horizontal linear complementarity problem. Given two matrices $Q, R \in I\!\!R^{n \times n}$, and a vector $b \in I\!\!R^n$, the horizontal linear complementarity problem (HLCP) consists in finding a pair of vectors $z = \lceil x, s \rfloor$ such that

$$(2.1) \qquad \begin{aligned} xs &= 0 \\ Qx + Rs &= b \\ x, s &\geq 0. \end{aligned}$$

The standard (monotone) linear complementarity problem (SLCP or simply LCP) is obtained by taking $R = -I$, and Q positive semidefinite. Let $\kappa \geq 0$ be a given constant. We say that (2.1) is a $P_*(\kappa)$ HLCP if

$$Qu + Rv = 0 \text{ implies } (1 + 4\kappa) \sum_{i \in \mathcal{I}^+} u_i v_i + \sum_{i \in \mathcal{I}^-} u_i v_i \geq 0, \text{ for any } u, v \in I\!\!R^n ,$$

where $\mathcal{I}^+ = \{i : u_i v_i > 0\}$ and $\mathcal{I}^- = \{i : u_i v_i < 0\}$. If the above condition is satisfied we say that (Q, R) is a $P_*(\kappa)$ pair and we write $(Q, R) \in P_*(\kappa)$. In case $R = -I$, $(Q, -I)$ is a $P_*(\kappa)$ pair if and only if Q is a $P_*(\kappa)$ matrix in the sense that:

$$(1 + 4\kappa) \sum_{i \in \hat{\mathcal{I}}^+} x_i [Qx]_i + \sum_{i \in \hat{\mathcal{I}}^-} x_i [Qx]_i \geq 0, \quad \forall x \in I\!\!R^n ,$$

where $\hat{\mathcal{I}}^+ = \{i : x_i [Qx]_i > 0\}$ and $\hat{\mathcal{I}}^- = \{i : x_i [Qx]_i < 0\}$. Problem (2.1) is then called a $P_*(\kappa)$ LCP and it is extensively discussed in [7]. If (Q, R) belongs to the class

$$P_* = \cup_{\kappa \geq 0} P_*(\kappa) ,$$

then we say that (Q, R) is a P_* pair and (2.1) is a P_* HLCP.

The class of sufficient matrices was defined by Cottle et al. in [3]. The appropriate generalization to sufficient pair [18, 19] is in terms of the null space of the matrix $[Q \ R] \in I\!\!R^{n \times 2n}$

$$(2.2) \qquad \Phi := \mathcal{N}([Q \ R]) = \{\lceil u, v \rfloor | Qu + Rv = 0\}$$

and its orthogonal space

$$(2.3) \qquad \Phi^\perp = \left\{\lceil u, v \rfloor | u = Q^T x, v = R^T x, \text{ for some } x \in I\!\!R^n \right\} .$$

(Q, R) is called column sufficient if

$$\lceil u, v \rfloor \in \Phi, \quad uv \leq 0 \text{ implies } uv = 0 ,$$

and row sufficient if

$$\lceil u,\, v \rceil \in \Phi^{\perp}, \quad uv \geq 0 \text{ implies } uv = 0\,.$$

(Q, R) is a sufficient pair if it is both column and row sufficient. The corresponding results of row and column sufficient matrices in [3] can be extended to row and column sufficient pairs (see, for example, [20]): (Q, R) is a sufficient pair if and only if for any b, the HLCP (2.1) has a convex (perhaps empty) solution set and every KKT point of

$$\begin{array}{rl} \min_{x,s} & x^T s \\ \text{s.t.} & Qx + Rs = b \\ & x, s \geq 0 \end{array}$$

is a solution of (2.1).

Väliaho's result [22] states that a matrix is sufficient if and only if it is a $P_*(\kappa)$ matrix for some $\kappa \geq 0$. The result can be extended to sufficient pairs by using the equivalence results from [1] (see also [20]): (Q, R) is a sufficient pair if and only if there is a finite $\kappa \geq 0$ so that (Q, R) is a $P_*(\kappa)$ pair. By extension, a P_* HLCP will be called a sufficient HLCP and a P_* pair will be called a sufficient pair.

Let us note that if (Q, R) is a sufficient pair, then the matrix $[Q\ R]$ is full rank. In fact, we have the following slightly stronger result.

THEOREM 2.1. *Given two matrices $Q, R \in \mathbb{R}^{n \times n}$, if the pair (Q, R) is column sufficient, the matrix $[Q\ R]$ is full rank.*

Proof. Let r be the rank of Q, the LU factorization of Q can be written as

$$PQ = L \begin{bmatrix} F_1 & F_2 \\ 0 & 0 \end{bmatrix},$$

where P is a permutation matrix, $L \in \mathbb{R}^{n \times n}$ is an invertible lower triangular matrix, $F_1 \in \mathbb{R}^{n-r \times n-r}$ is an invertible upper triangular matrix, and F_2 and the zeros are matrices with the correct dimensions.
Let us denote by G

$$G = L^{-1} P R = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

where $G_{11} \in \mathbb{R}^{n-r \times n-r}$, and G_{12}, G_{21}, G_{22} are with the correct dimensions.
Since permutation matrices are invertible, and L is invertible, we have

$$rank([Q\ R]) = rank(L^{-1} P [Q\ R]) = rank \left(\begin{bmatrix} F_1 & F_2 & G_{11} & G_{12} \\ 0 & 0 & G_{21} & G_{22} \end{bmatrix} \right).$$

We denote by u_1 and u_2 the components of u in the first r and last $n - r$ indices, respectively, and similarly for v. Therefore, $\lceil u,\, v \rceil \in \Phi$ is equivalent to

$$\begin{cases} F_1 u_1 + F_2 u_2 & + & G_{11} v_1 & + & G_{12} v_2 & = & 0 \\ & & G_{21} v_1 & + & G_{22} v_2 & = & 0 \end{cases}.$$

For any $v_2 \in Ker(G_{22})$, we construct a pair of vectors u and v such that

$$v_1 = 0, \quad u_2 = -v_2, \quad u_1 = -F_1^{-1}(F_2 u_2 + G_{12} v_2)\,.$$

Clearly we have $\lceil u, v \rceil \in \Phi$. Moreover, we also obtain

$$u_1 v_1 \leq 0, \quad \text{and} \quad u_2 v_2 = -v_2^2 \leq 0.$$

Because (Q, R) is column sufficient, we have

$$u_1 v_1 = 0, \quad \text{and} \quad u_2 v_2 = 0.$$

We thus have $v_2 \in Ker(G_{22})$ implies $v_2 = 0$. Therefore G_{22} is invertible, and

$$rank([Q\ R]) = rank\left(\begin{bmatrix} F_1 & F_2 & G_{11} & G_{12} \\ 0 & 0 & G_{21} & G_{22} \end{bmatrix}\right) = n.$$

∎

It is interesting to remark that row sufficiency alone does not imply the full rank property. For example, take

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

it is easily seen that (Q, R) is row sufficient, but $rank([Q\ R]) = 1$. We also note that in [19, 18, 20], the full rank property was given as an assumption, which in fact always hold because of the above theorem.

We denote the set of all feasible points of HLCP by

$$\mathcal{F} = \{z = \lceil x, s \rceil \in I\!\!R_+^{2n} : \quad Qx + Rs = b\},$$

and its solution set by

$$\mathcal{F}^* = \{z^* = \lceil x^*, s^* \rceil \in \mathcal{F} : \quad x^* s^* = 0\}.$$

The relative interior of $\mathcal{F}$, which is also known as the set of strictly feasible points or the set of interior points, is given by

$$\mathcal{F}^0 = \mathcal{F} \bigcap I\!\!R_{++}^{2n}.$$

It is known (see, for example, [7]) that if $\mathcal{F}^0$ is nonempty, then the nonlinear system,

$$\begin{aligned} xs &= \tau e \\ Qx + Rs &= b \end{aligned}$$

has a unique positive solution for any $\tau > 0$. The set of all such solutions defines the central path $\mathcal{C}$ of the HLCP, that is,

$$\mathcal{C} = \{z \in I\!\!R_{++}^{2n} : F_\tau(z) = 0, \tau > 0\},$$

where

$$F_\tau(z) = \begin{bmatrix} xs - \tau e \\ Qx + Rs - b \end{bmatrix}.$$

If $F_\tau(z) = 0$, then it is easy to see that $\tau = \mu(z)$, where $\mu(z)$ is given by (1.2). The wide neighborhood $\mathcal{N}_\infty^-(\alpha)$ is defined as

$$\mathcal{N}_\infty^-(\alpha) = \{z \in \mathcal{F}^0 : \delta_\infty^-(z) \leq \alpha\},$$

where $0 < \alpha < 1$ is a given parameter and

$$\delta_\infty^-(z) := \left\| \left[\frac{xs}{\mu(z)} - e \right]^- \right\|_\infty$$

is a proximity measure of z to the central path. Alternatively, if we denote

$$\mathcal{D}(\beta) = \{z \in \mathcal{F}^0 : xs \geq \beta\mu(z)\},$$

then the neighborhood $\mathcal{N}_\infty^-(\alpha)$ can also be written as

$$\mathcal{N}_\infty^-(\alpha) = \mathcal{D}(1 - \alpha).$$

It is well known (see, for example, the proof in [14]) that

$$\lim_{\alpha\downarrow 0} \mathcal{N}_\infty^-(\alpha) = \lim_{\beta\uparrow 1} \mathcal{D}(\beta) = \mathcal{C}, \quad \lim_{\alpha\uparrow 1} \mathcal{N}_\infty^-(\alpha) = \lim_{\beta\downarrow 0} \mathcal{D}(\beta) = \mathcal{F}.$$

3. A higher order corrector-predictor algorithm. The higher order corrector and predictor use higher derivatives of the central path. Given a point $z = \lceil x, s \rfloor \in \mathcal{D}(\beta)$, we consider the curve given by an mth order vector valued polynomial of the form

$$(3.1) \qquad\qquad z(\theta) = z + \sum_{i=1}^m w^i \theta^i,$$

where the vectors $w^i = \lceil u^i, v^i \rfloor$ are obtained as solutions of the following linear systems

$$(3.2) \qquad
\begin{cases} su^1 + xv^1 &= \gamma\mu e - (1+\epsilon)xs \\ Qu^1 + Rv^1 &= 0 \end{cases},
$$
$$
\begin{cases} su^2 + xv^2 &= \epsilon xs - u^1 v^1 \\ Qu^2 + Rv^2 &= 0 \end{cases},
$$
$$
\begin{cases} su^i + xv^i &= -\sum_{j=1}^{i-1} u^j v^{i-j} \\ Qu^i + Rv^i &= 0 \end{cases}, \quad i = 3,\ldots,m.
$$

In a corrector step we choose $\epsilon = 0$ and $\gamma \in [\underline{\gamma}, \overline{\gamma}]$, where $0 < \underline{\gamma} < \overline{\gamma} < 1$ are given parameters, while in a predictor step we take

$$(3.3) \qquad \gamma = 0, \quad \text{and } \epsilon = \begin{cases} 0, & \text{if HLCP is nondegenerate} \\ 1, & \text{if HLCP is degenerate} \end{cases}.$$

We note that in the corrector step, where we have $\epsilon = 0$, w^1 is the affine scaling direction if $\gamma = 0$, and the classical centering direction if $\gamma = 1$. In system (3.2), w^1 is a convex combination of the affine scaling and the centering directions. The directions w^i are related to the higher derivatives of the central path [19]. Due to the fact that the m linear systems in (3.2) have the same left hand matrix, only one matrix factorization and m backsolves are needed. Therefore it involves $O(n^3) + O(m\,n^2)$ arithmetic operations. We take $m = m_c$ in the corrector step, and $m = m_p$ in the predictor step. From (3.1) and (3.2) it follows that

$$x(\theta)s(\theta) = (1 - \theta)^{1+\epsilon}xs + \gamma\theta\mu e + \sum_{i=m+1}^{2m} \theta^i h^i,$$

$$\mu(\theta) = (1 - \theta)^{1+\epsilon}\mu + \gamma\theta\mu + \sum_{i=m+1}^{2m} \theta^i(e^T h^i/n),$$

(3.4) $$\text{where } h^i = \sum_{j=i-m}^{m} u^j v^{i-j}.$$

In the development of our algorithm, we want to preserve positivity of each iterated point. We thus give an upper bound θ_0 for the step length taken both in the predictor and the corrector step:

(3.5) $$\theta_0 = \sup\{\hat{\theta}_0 \ : \ x(\theta) > 0, \ s(\theta) > 0, \ \forall\theta \in [0, \hat{\theta}_0]\}.$$

We introduce the following notation, which will be used in describing both the corrector and predictor steps:

(3.6) $$p(\theta) = \frac{x(\theta)s(\theta)}{\mu(\theta)}, \quad f(\theta) = \min_{i=1,\dots,n} p_i(\theta).$$

The corrector. The corrector step is obtained by taking $\epsilon = 0$, and $0 < \gamma < 1$ in (3.1)-(3.2). The main purpose of the corrector step is to increase proximity to the central path. However, we also improve the normalized complementarity gap $\mu(\theta)$ at the same time. We choose $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, where $0 < \underline{\sigma} < \overline{\sigma} < 1$ are given parameters, and define

(3.7) $$\theta_1 = \sup\{\hat{\theta}_1 \ : \ 0 \le \hat{\theta}_1 \le \theta_0, \mu(\theta) \le (1 - \sigma(1-\gamma)\theta)\mu, \ \forall\theta \in [0, \hat{\theta}_1]\}.$$

The step-length of the corrector is obtained as

(3.8) $$\theta_c = \operatorname{argmax}\{f(\theta) \ : \ \theta \in [0, \theta_1]\}.$$

As a result of the corrector step we obtain the point

(3.9) $$\overline{z} = \lceil\overline{x}, \overline{s}\rfloor := z(\theta_c).$$

We have clearly $\overline{z} \in \mathcal{D}(\beta_c)$ with $\beta_c > \beta$. While the parameter β is fixed during the algorithm, the positive quantity β_c varies from iteration to iteration. However, we will prove that there is a constant $\beta_c^* > \beta$, such that $\beta_c > \beta_c^*$ in all iterations.

The predictor. The predictor is obtained by taking $z = \overline{z}$, where $\overline{z}$ is the result of the corrector step, and $\gamma = 0$ in (3.1)-(3.2). The aim of the predictor step is to decrease the normalized complementarity gap as much as possible while keeping the iterate in $\mathcal{D}(\beta)$. We define the predictor step length as

(3.10) $$\theta_p = \operatorname{argmin}\{\mu(\theta) \ : \ \theta \in [0, \theta_2]\},$$

where

(3.11) $$\theta_2 = \max\{\hat{\theta}_2 \ : \ z(\theta) \in \mathcal{D}(\beta), \ \forall\theta \in [0, \hat{\theta}_2]\}.$$

A standard continuity argument can be used to show that $z(\theta) > 0, \ \forall\theta \in [0, \theta_2]$. As a result of the predictor step, we obtain a point

(3.12) $$z^+ = \lceil x^+, s^+\rfloor := z(\theta_p).$$

By construction we have $z^+ \in \mathcal{D}(\beta)$, so that a new corrector step can be applied. Summing up we can formulate the following iterative procedure:

Algorithm 1
Given real parameters $0 < \beta < 1$, $0 < \underline{\gamma} < \overline{\gamma} < 1$, $0 < \underline{\sigma} < \overline{\sigma} < 1$, integers $m_c, m_p \geq 1$, and a vector $z^0 \in \mathcal{D}(\beta)$:

    Set $k \leftarrow 0$;
    repeat
        (corrector step)
        Set $z \leftarrow z^k$;
        Choose $\gamma \in [\underline{\gamma}, \overline{\gamma}]$ and set $m = m_c$;
        Compute directions $w^i = \lceil u^i, v^i \rfloor$, $i = 1, \ldots, m$, by solving (3.2);
        Compute θ_0 from (3.5)
        If HLCP is skew-symmetric, set $\sigma = 1$ and $\theta_1 = \theta_0$;
            Else, choose $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, and compute θ_1 from (3.7);
        Compute corrector step-length θ_c from (3.8);
        Compute $\overline{z}$ from (3.9);
        Set $\overline{z}^k \leftarrow \overline{z}$, $\overline{\mu}_k \leftarrow \overline{\mu} = \mu(\overline{z})$.
        (predictor step)
        Set $z \leftarrow \overline{z}^k$, $\gamma = 0$, and $m = m_p$;
        Compute directions $w^i = \lceil u^i, v^i \rfloor$, $i = 1, \ldots, m$, by solving (3.2);
        Compute θ_p from (3.10);
        Compute z^+ from (3.12);
        Set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu^+ = \mu(z^+)$, $k \leftarrow k+1$.
    continue

The computation of the exact values of θ_c and θ_p is quite involved, so that in practice good estimates of θ_c and θ_p are obtained by appropriate line search procedures. In particularly, by adopting the line search procedure from [14] we can preserve both the computational complexity and superlinear convergence of the theoretical algorithm. In fact the convergence properties can be proved by using the explicit lower bounds in the next section.

4. Polynomial complexity. We analyze in this section the computational complexity of Algorithm 1. In the proof of the complexity results, we will use the following lemmas, which were proved in [16].

LEMMA 4.1. *Assume that HLCP (2.1) is* $P_*(\kappa)$, *and let* $w = \lceil u, v \rfloor$ *be the solution of the following linear system*

$$
\begin{aligned}
su + xv &= a \\
Qu + Rv &= 0
\end{aligned}
$$

where $z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_{++}$ *and* $a \in \mathbb{R}^n$ *are given vectors, and consider the index sets:*

$$
\mathcal{I}^+ = \{i : u_i v_i > 0\}, \quad \mathcal{I}^- = \{i : u_i v_i < 0\}.
$$

Then the following inequalities are satisfied:

$$
\frac{1}{1 + 4\kappa} \, \| uv \|_\infty \leq \sum_{i \in \mathcal{I}_+} u_i v_i \leq \frac{1}{4} \left\| (xs)^{-1/2} a \right\|_2^2 .
$$

LEMMA 4.2. *Assume that HLCP (2.1) is* $P_*(\kappa)$, *and let* $w = \lceil u, v \rfloor$ *be the solution of the following linear system*

$$
\begin{aligned}
su + xv &= a \\
Qu + Rv &= 0 ,
\end{aligned}
$$

where $z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_{++}$ and $a \in \mathbb{R}^n$ are given vectors. Then the following inequality holds:

$$(4.1) \qquad u^T v \geq -\kappa \left\| (xs)^{-1/2} a \right\|_2^2 .$$

Let us denote

$$(4.2) \qquad \eta_i = \left\| Du^i + D^{-1}v^i \right\|_2 , \quad \text{where} \quad D = X^{-1/2} S^{1/2} .$$

The following lemma is a slight improvement over the corresponding results in [16] and a generalization to sufficient HLCP of the corresponding results in [14].

LEMMA 4.3. *If HLCP (2.1) is sufficient and $z = \lceil x, s \rfloor \in \mathcal{D}(\beta)$, then for $n \geq 8$, the solution of (3.2) satisfy*

$$(4.3) \quad \frac{1}{\sqrt{1+2\kappa}} \sqrt{ \left\| Du^i \right\|_2^2 + \left\| D^{-1}v^i \right\|_2^2 } \leq \eta_i \leq \frac{2}{1+2\kappa} \alpha_i \sqrt{\beta\mu} \left(\frac{(1+2\kappa)\tau}{4} \sqrt{n} \right)^i ,$$

where

$$(4.4) \qquad \tau = \frac{2\sqrt{\beta(1+\epsilon-\gamma)^2 + (1-\beta)\gamma^2}}{\beta} ,$$

and the sequence

$$\alpha_i = \frac{1}{i} \begin{pmatrix} 2i-2 \\ i-1 \end{pmatrix} \leq \frac{1}{i} 4^i$$

is the solution of the following recurrence scheme

$$\alpha_1 = 1, \quad \alpha_i = \sum_{j=1}^{i-1} \alpha_j \, \alpha_{i-j} .$$

Proof. The first part of the inequality follows immediately, since by using (3.2) and Lemma 4.2 we have

$$\left\| Du^i + D^{-1}v^i \right\|_2^2 = \left\| Du^i \right\|_2^2 + 2u^{i\,T}v^i + \left\| D^{-1}v^i \right\|_2^2$$

$$\geq \left\| Du^i \right\|_2^2 + \left\| D^{-1}v^i \right\|_2^2 - 2\kappa \left\| Du^i + D^{-1}v^i \right\|_2^2 .$$

By multiplying the first equations of (3.2) with $(xs)^{-1/2}$ we obtain

$$Du^1 + D^{-1}v^1 = - \left((1+\epsilon)(xs)^{1/2} - \gamma\mu(xs)^{-1/2} \right)$$

$$Du^2 + D^{-1}v^2 = - \left(\epsilon(xs)^{1/2} - (xs)^{-1/2}u^1v^1 \right)$$

$$Du^i + D^{-1}v^i = -(xs)^{-1/2} \sum_{j=1}^{i-1} Du^j D^{-1}v^{i-j} , \quad 3 \leq i \leq m .$$

Because $z \in \mathcal{D}(\beta)$ we have $(xs)^{-1/2} \leq (1/\sqrt{\beta\mu})e$, and we deduce that

$$\eta_1 = \| (1+\epsilon)(xs)^{1/2} - \gamma\mu(xs)^{-1/2} \|, \quad \eta_2 = \| \epsilon(xs)^{1/2} - (xs)^{-1/2}u^1v^1 \|,$$

and

$$(4.5) \qquad \eta_i \le \frac{1}{\sqrt{\beta\mu}} \sum_{j=1}^{i-1} \left\| Du^j \right\|_2 \left\| D^{-1}v^{i-j} \right\|_2, \ 3 \le i \le m.$$

We have

$$\eta_1^2 = \|(1+\epsilon)(xs)^{1/2} - \gamma\mu(xs)^{-1/2}\|^2 = \sum_{j=1}^n \left((1+\epsilon)^2 x_j s_j - 2(1+\epsilon)\gamma\mu + \frac{\gamma^2\mu^2}{x_j s_j} \right)$$

$$= ((1+\epsilon)^2 - 2(1+\epsilon)\gamma)\mu n + \gamma^2\mu^2 \sum_{j=1}^n \frac{1}{x_j s_j}$$

$$\le \mu n \left((1+\epsilon)^2 - 2(1+\epsilon)\gamma + \frac{\gamma^2}{\beta} \right) = \frac{\beta\mu n\tau^2}{4},$$

which shows that the second inequality in (4.3) is satisfied for $i = 1$. We next show that the inequality also holds for $i = 2$, i.e., we want to prove that

$$\eta_2^2 \le (\frac{1+2\kappa}{8})^2 \beta\mu n^2\tau^4 = \frac{1}{128}\beta\mu n^2\tau^4(2 + 8\kappa + 8\kappa^2).$$

Using Lemma4.2, Corollary 2.3 of [13], and the fact $z \in \mathcal{D}(\beta)$, we have

$$\eta_2^2 = \|\epsilon(xs)^{1/2} - (xs)^{-1/2}u^1v^1\|^2 = \sum_{j=1}^n \left(\epsilon^2 x_j s_j - 2\epsilon u_i^1 v_i^1 + \frac{(u_i^1 v_i^1)^2}{x_j s_j} \right)$$

$$\le \epsilon^2 n\mu + 2\epsilon\kappa\eta_1^2 + \frac{\eta_1^4}{8\beta\mu}(1 + 4\kappa + 8\kappa^2) \le \epsilon^2 n\mu + \frac{\epsilon\kappa\beta\mu n\tau^2}{2} + \frac{\beta\mu n^2\tau^4}{128}(1 + 4\kappa + 8\kappa^2).$$

Therefore, it remains to show that

$$\epsilon^2 n\mu + \frac{\epsilon\kappa\beta\mu n\tau^2}{2} \le \frac{\beta\mu n^2\tau^4}{128}(1 + 4\kappa),$$

which holds trivially for $\epsilon = 0$. The inequality holds for $\epsilon = 1$ provided

$$\beta n\tau^4 \ge 128, \quad n\tau^2 \ge 16.$$

Using the definition of τ (4.4), this reduces to

$$\frac{n(\beta(2-\gamma)^2 + (1-\beta)\gamma^2)^2}{\beta^3} \ge 8, \quad \frac{n(\beta(2-\gamma)^2 + (1-\beta)\gamma^2)}{\beta^2} \ge 4.$$

Since the minimum over $0 \le \beta, \gamma \le 1$ of both left hand side functions are obtained at $\beta = \gamma = 1$, we conclude that that the second inequality in (4.3) is satisfied for $i = 1$ whenever $n \ge 8$. For $i \ge 3$ and $1 \le j < i$ we obtain the following inequality by using the first inequality in (4.3)

$$\left\| Du^j \right\|_2 \left\| D^{-1}v^{i-j} \right\|_2 + \left\| Du^{i-j} \right\|_2 \left\| D^{-1}v^j \right\|_2$$

$$\le \left(\left\| Du^j \right\|_2^2 + \left\| D^{-1}v^j \right\|_2^2 \right)^{1/2} \left(\left\| Du^{i-j} \right\|_2^2 + \left\| D^{-1}v^{i-j} \right\|_2^2 \right)^{1/2}$$

$$\le (1+2\kappa)\eta_j \eta_{i-j},$$

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we thus have from (4.5)

$$\eta_i \le \frac{1+2\kappa}{2\sqrt{\beta\mu}} \sum_{j=1}^{i-1} \eta_j\,\eta_{i-j}\,, \quad i = 2,\dots, m\,.$$

The required inequalities are then easily proved by mathematical induction. ∎

By virtue of Lemma 4.3 we obtain the following bound for $\|h^i\|$.

LEMMA 4.4. *If HLCP (2.1) is sufficient and $z = \lceil x,\,s \rceil \in \mathcal{D}(\beta)$ then for $n \ge 8$, the directions computed in (3.4) satisfy*

$$(4.6) \qquad \zeta_i := \|h^i\| \le \frac{2\beta\mu}{(1+2\kappa)i}\big((1+2\kappa)\tau\sqrt{n}\,\big)^i\,, \; i = m+1,\dots, 2m\,.$$

Proof. For any $m+1 \le i \le 2m$, we have

$$\|\,h^i\,\|_2 \le \sum_{j=i-m}^{m} \|\,Du^j\,\|_2\,\|\,D^{-1}v^{i-j}\,\|_2 \le \sum_{j=1}^{i-1} \|\,Du^j\,\|_2\,\|\,D^{-1}v^{i-j}\,\|_2$$

$$= \frac{1}{2}\sum_{j=1}^{i-1}\big(\|\,Du^j\,\|_2\,\|\,D^{-1}v^{i-j}\,\|_2 + \|\,Du^{i-j}\,\|_2\,\|\,D^{-1}v^j\,\|_2\big)$$

$$\le \frac{1}{2}\sum_{j=1}^{i-1} \sqrt{\|\,Du^j\,\|_2^2 + \|\,D^{-1}v^j\,\|_2^2}\,\sqrt{\|\,Du^{i-j}\,\|_2^2 + \|\,D^{-1}v^{i-j}\,\|_2^2}$$

$$\le \frac{1+2\kappa}{2}\sum_{j=1}^{i-1}\eta_j\,\eta_{i-j} \le \frac{2\beta\mu}{1+2\kappa}\left(\frac{(1+2\kappa)\tau\sqrt{n}}{4}\right)^i \sum_{j=1}^{i-1}\alpha_j\,\alpha_{i-j}$$

$$= \frac{2\beta\mu}{1+2\kappa}\left(\frac{(1+2\kappa)\tau\sqrt{n}}{4}\right)^i \alpha_i \le \frac{2\beta\mu}{(1+2\kappa)i}\big((1+2\kappa)\tau\sqrt{n}\,\big)^i\,,$$

where the last inequality follows from the fact that $\alpha_i \le \frac{1}{i}4^i$. ∎

From the above lemmas we obtain the following result.

COROLLARY 4.5. *If HLCP (2.1) is sufficient and $z = \lceil x,\,s \rceil \in \mathcal{D}(\beta)$ then the following relations hold for any $\alpha > 0$, $\kappa \ge 0$, and $n \ge 8$:*

$$(4.7)\frac{\alpha}{\mu}\sum_{i=m+1}^{2m}\theta^i\,\|\,h^i\,\|_2 < 1,\; \forall\, 0 \le \theta \le \frac{1}{(1+2\kappa)\tau\sqrt{n}}\min\left\{1,\,\left(\frac{1.4\,\alpha\beta}{1+2\kappa}\right)^{\frac{-1}{m+1}}\right\},$$

$$(4.8)\frac{\alpha}{\mu\sqrt{n}}\sum_{i=m+1}^{2m}\theta^i\,\|\,h^i\,\|_2 < \theta,\; \forall\, 0 \le \theta \le \frac{1}{(1+2\kappa)\tau\sqrt{n}}\min\left\{1,\,(1.4\,\alpha\beta\tau\,)^{\frac{-1}{m}}\right\},$$

Proof. For any $t \in (0,1]$ we have

$$\sum_{i=m+1}^{2m}\frac{t^i}{i} \le t^{m+1}\sum_{i=m+1}^{2m}\frac{1}{i} < t^{m+1}\int_m^{2m}\frac{du}{u} = t^{m+1}\log 2 < .7\,t^{m+1}\,.$$

Using Lemma 4.4 and the above inequality we obtain

$$(4.9)\sum_{i=m+1}^{2m}\theta^i\,\|\,h^i\,\|_2 < \frac{1.4\beta\mu}{1+2\kappa}((1+2\kappa)\tau\sqrt{n}\theta)^{m+1}\,, \quad \forall\,\theta \in \left(0,\,\frac{1}{(1+2\kappa)\tau\sqrt{n}}\right]\,.$$

Therefore,

$$\frac{\alpha}{\mu} \sum_{i=m+1}^{2m} \theta^i \left\| h^i \right\|_2 < \frac{1.4\alpha\beta}{1+2\kappa}((1+2\kappa)\tau\sqrt{n}\theta)^{m+1} \leq 1,$$

$$\forall\,\theta \in \left(0,\ \frac{1}{(1+2\kappa)\tau\sqrt{n}}\ \min\left\{1,\ \left(\frac{1.4\,\alpha\beta}{1+2\kappa}\right)^{\frac{-1}{m+1}}\right\}\right].$$

(4.9) also implies that

$$\frac{\alpha}{\mu\sqrt{n}} \sum_{i=m+1}^{2m} \theta^i \left\| h^i \right\|_2 < \frac{1.4\alpha\beta}{(1+2\kappa)\sqrt{n}}((1+2\kappa)\tau\sqrt{n}\theta)^{m+1} = 1.4\alpha\beta\tau\theta\,((1+2\kappa)\tau\sqrt{n}\theta)^m \leq \theta,$$

$$\forall\,\theta \in \left(0,\ \frac{1}{(1+2\kappa)\tau\sqrt{n}}\ \min\left\{1,\ (1.4\,\alpha\beta\tau)^{\frac{-1}{m}}\right\}\right].$$

∎

From the definition of τ (4.4) it follows that

(4.10) $\quad \dfrac{2(1+\epsilon)\sqrt{1-\beta}}{\sqrt{\beta}} \leq \tau \leq 2\max\left\{\dfrac{1+\epsilon}{\sqrt{\beta}},\ \dfrac{\sqrt{1-\beta+\beta\epsilon^2}}{\beta}\right\} < \dfrac{2(1+\epsilon)}{\beta}.$

In the corrector step we take $\epsilon = 0$, therefore we will use the bound $\tau < 2/\beta$ in the analysis below.

THEOREM 4.6. *If HCLP (2.1) is sufficient, then Algorithm 1 is well defined and the following relations hold for any $\kappa \geq 0$ and $n \geq 8$:*

$$\overline{z}^k, z^k \in \mathcal{D}(\beta),$$

$$\mu_{k+1} \leq \left(1 - \frac{\chi}{(1+2\kappa)n^{\frac{1}{2}+\frac{m_c+1}{2m_c(m_p+1)}}}\right)\overline{\mu}_k,\quad k = 0, 1, \dots,$$

$$\overline{\mu}_{k+1} \leq \left(1 - \frac{\overline{\chi}}{(1+2\kappa)n^{\frac{1}{2}+\upsilon}}\right)\overline{\mu}_k,\quad k = 0, 1, \dots,$$

where χ, $\overline{\chi}$ are constants depending only on $\beta, \underline{\gamma}, \overline{\gamma}, \underline{\sigma}, \overline{\sigma}$, and

(4.11) $$\upsilon := \min\left\{\frac{1}{2m_c},\ \frac{m_c+1}{2m_c(m_p+1)}\right\}.$$

Proof. **Analysis of the corrector.** On the corrector we have $m = m_c$, $\epsilon = 0$, $0 < \underline{\gamma} < \gamma < \overline{\gamma} < 1$, $0 < \underline{\sigma} < \gamma < \overline{\sigma} < 1$, and $\tau < 2/\beta$.

First, we prove that if $z \in \mathcal{D}(\beta)$, then the quantities θ_0 defined in (3.5) satisfy

(4.12) $$\theta_0 \geq \theta_3 := \frac{\beta}{2(1+2\kappa)\sqrt{n}}\left(\frac{2.8}{1+2\kappa}\right)^{-\frac{1}{m_c+1}},$$

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This can be shown by using (4.7) with $\alpha = 2/\beta$, and the fact that $\theta_3 < 1/2$,

$$\frac{x(\theta)s(\theta)}{\mu} > (1-\theta)\frac{xs}{\mu} + \frac{1}{\mu}\sum_{i=m+1}^{2m} \theta^i h^i \geq \frac{\beta}{2}e - \frac{1}{\mu}\sum_{i=m+1}^{2m} \theta^i \left\| h^i \right\|_2 e > 0, \forall \theta \in [0, \theta_3].$$

Since $x(0) > 0, s(0) > 0$, we can use a standard continuity argument to show that $x(\theta) > 0, s(\theta) > 0, \forall \theta \in [0, \theta_3]$, which proves that $\theta_0 \geq \theta_3$.

Next we show that the quantities θ_1 defined in (3.7), satisfy

$$(4.13) \qquad \theta_1 \geq \theta_4 := \frac{\beta}{2(1+2\kappa)\sqrt{n}}\left(\frac{(1-\overline{\sigma})(1-\overline{\gamma})}{2.8}\right)^{\frac{1}{m_c}}.$$

By using (4.8) with $\alpha = 1/((1-\overline{\sigma})(1-\overline{\gamma}))$, we deduce that the following inequalities hold for any $\theta \in [0, \theta_4]$:

$$\frac{\mu(\theta) - (1-\sigma(1-\gamma)\theta)\mu}{\mu} = -(1-\sigma)(1-\gamma)\theta + \frac{1}{\mu n}\sum_{i=m+1}^{2m} \theta^i e^T h^i$$

$$\leq -(1-\overline{\sigma})(1-\overline{\gamma})\theta + \frac{1}{\mu\sqrt{n}}\sum_{i=m+1}^{2m} \theta^i \left\| h^i \right\|_2 \leq 0,$$

which shows that $\theta_1 \geq \theta_4$.

At last, we show that if $z \in \mathcal{D}(\beta)$, then

$$(4.14) \qquad f(\theta) \geq \beta + \frac{1}{2}(1-\beta)\gamma\,\theta \geq \beta + \frac{1}{2}(1-\beta)\underline{\gamma}\,\theta, \quad \forall \theta \in [0, \theta_5],$$

where

$$(4.15)\,\theta_5 := \min\left\{\theta_4\,,\; \frac{\beta}{2(1+2\kappa)\,n^{\frac{1}{2}+\frac{1}{2m_c}}}\left(\frac{(1-\beta)\underline{\gamma}}{5.6}\right)^{\frac{1}{m_c}}\right\} \geq \frac{\chi_5}{(1+2\kappa)n^{\frac{1}{2}+\frac{1}{2m_c}}},$$

$$(4.16)\chi_5 := \frac{\beta}{2}\min\left\{\frac{(1-\overline{\sigma})(1-\overline{\gamma})}{2.8}\,,\; \frac{(1-\beta)\underline{\gamma}}{5.6}\right\}.$$

It is easily seen that

$$p(\theta) = \frac{x(\theta)s(\theta)}{\mu(\theta)} = \frac{(1-\theta)xs + \gamma\theta\mu e + \sum_{i=m+1}^{2m}\theta^i h^i}{(1-\theta)\mu + \gamma\theta\mu + \sum_{i=m+1}^{2m}\theta^i e^T h^i/n}$$

$$\geq \frac{(1-\theta)\beta\mu e + \gamma\theta\mu e + \sum_{i=m+1}^{2m}\theta^i h^i}{(1-\theta)\mu + \gamma\theta\mu + \sum_{i=m+1}^{2m}\theta^i e^T h^i/n}$$

$$= \beta e \frac{(1-\beta)\gamma\theta\mu e + \beta\sum_{i=m+1}^{2m}\theta^i(h^i - (e^T h^i/n)e) + (1-\beta)\sum_{i=m+1}^{2m}\theta^i h^i}{(1-\theta)\mu + \gamma\theta\mu + \sum_{i=m+1}^{2m}\theta^i e^T h^i/n}$$

$$\geq \beta e + \frac{(1-\beta)\gamma\theta\mu - \sum_{i=m+1}^{2m}\theta^i\left\| h^i \right\|_2}{\mu(\theta)}e$$

$$(4.17) \qquad \geq \beta e + (1-\beta)\gamma\theta - \frac{1}{\mu}\sum_{i=m+1}^{2m}\theta^i\left\| h^i \right\|_2 e, \quad \forall \theta \in [0, 1].$$

The last inequality follows from the fact that

$$\mu(\theta) \leq (1 - \sigma(1 - \gamma)\theta)\mu \leq \mu, \ \forall \theta \in [0, \theta_4].$$

According to (4.8), with α replace by $2\sqrt{n}/((1-\beta)\gamma)$, and τ replaced by $2/\beta$, we have

$$\frac{1}{\mu} \sum_{i=m+1}^{2m} \theta^i \left\| h^i \right\|_2 < \frac{1}{2}(1 - \beta)\gamma\,\theta, \quad \forall \theta \leq \frac{\beta}{2(1 + 2\kappa)\sqrt{n}} \left(\frac{(1-\beta)\gamma}{5.6\sqrt{n}} \right)^{\frac{1}{m}}.$$

and (4.14) follows from the above inequality and (4.17).
Relation (4.14) shows that if $z \in \mathcal{D}(\beta)$, then the point $\bar{z}$ obtained in the corrector step of Algorithm 1 belongs to $\mathcal{D}(\beta + \delta)$, where

$$(4.18) \qquad\qquad \delta = \frac{1}{2}(1 - \beta)\gamma\,\theta_5.$$

As we mentioned before, the main purpose of the corrector is to increase proximity to the central path. However, it turns out that if the corrector step-length θ_c is large enough then we also obtain a significant reduction of the duality gap during the corrector step. In what follows we find a lower bound for θ_c in case the point $z \in \mathcal{D}(\beta)$ is not very well centered. More precisely we show that

$$(4.19) \qquad\qquad \exists j \quad \text{such that } p_j := \frac{x_j s_j}{\mu} \leq \beta + .44\,\delta \ \Rightarrow\ \theta_c > .2\,\theta_5.$$

Let us denote

$$\lambda = .44\,\delta = .22(1 - \beta)\gamma\,\theta_5,\ q^i = \frac{h^i}{\mu},\ i = m+1, \ldots, 2m.$$

For any $\theta \in [0, 1]$ we have

$$
\begin{aligned}
p_j(\theta) = \frac{x_j(\theta)s_j(\theta)}{\mu(\theta)} &= \frac{(1-\theta)p_j + \gamma\theta + \sum_{i=m+1}^{2m} \theta^i q_j^i}{(1-\theta) + \gamma\theta + \sum_{i=m+1}^{2m} \theta^i e^T q^i/n} \\
&< \frac{(1-\theta)(\beta + \lambda) + \gamma\theta + \sum_{i=m+1}^{2m} \theta^i q_j^i}{(1-\theta) + \gamma\theta + \sum_{i=m+1}^{2m} \theta^i e^T q^i/n} \\
&= \beta + \lambda + \frac{\gamma(1 - \beta - \lambda)\theta - (\beta + \lambda)\sum_{i=m+1}^{2m} \theta^i e^T q^i/n + \sum_{i=m+1}^{2m} \theta^i q_j^i}{1 - (1 - \gamma)\theta + \sum_{i=m+1}^{2m} \theta^i e^T q^i/n} \\
&\leq \beta + \lambda + \frac{\gamma(1 - \beta - \lambda)\theta + (1 + \frac{\beta+\lambda}{\sqrt{n}})\sum_{i=m+1}^{2m} \theta^i \left\| q^i \right\|_2}{1 - (1 - \gamma)\theta - \frac{1}{\sqrt{n}}\sum_{i=m+1}^{2m} \theta^i \left\| q^i \right\|_2} \\
&\leq \beta + \lambda + \frac{\gamma(1 - \beta)\theta + 2\sum_{i=m+1}^{2m} \theta^i \left\| q^i \right\|_2}{1 - (1 - \gamma)\theta - \sum_{i=m+1}^{2m} \theta^i \left\| q^i \right\|_2}.
\end{aligned}
$$

Assume now that $\theta \in [0, .2\,\theta_5]$ and set $\theta = .2\phi$. Since $\phi \in [0, \theta_5]$, by virtue of (4.8), we can write

$$
\begin{aligned}
\sum_{i=m+1}^{2m} \theta^i \left\| q^i \right\|_2 = \sum_{i=m+1}^{2m} .2^i \phi^i \left\| q^i \right\|_2 &\leq .2^{m+1} \sum_{i=m+1}^{2m} \phi^i \left\| q^i \right\|_2 \\
&< \frac{.2^{m+1}}{2}\gamma(1 - \beta)\phi = \frac{.2^m}{2}\gamma(1 - \beta)\theta \leq .1\gamma(1 - \beta)\theta.
\end{aligned}
$$

Using the fact that $\theta_5 < .5$, $\forall\, n \geq 1$, we obtain

$$p_j(\theta) < \beta + \lambda + \frac{1.2\gamma(1-\beta)\theta}{1 - (1 - \gamma + .1\,\gamma(1-\beta))\,\theta} \leq \beta + \lambda + \frac{1.2\gamma(1-\beta)\theta}{1-\theta}$$
$$< \beta + \lambda + 1.4\gamma(1-\beta)\theta \leq \beta + \lambda + .28\gamma(1-\beta)\theta_5 = \beta + \delta, \quad \forall\, \theta \in [0, .2\,\theta_5].$$

It follows that $f(\theta_c) \geq \beta + \delta > \max_{0 \leq \theta \leq .2\,\theta_5} f(\theta)$, wherefrom we deduce that $\theta_c > .2\,\theta_5$.

Analysis of the predictor. In the predictor step we have $\gamma = 0$ and $m = m_p$. From (4.4) it follows $\tau = 2(1 + \epsilon)/\sqrt{\beta} \leq 4/\sqrt{\beta}$. Since the predictor step follows a corrector step, we have $z \in \mathcal{D}(\beta + \delta) \subset \mathcal{D}(\beta)$.

First, we study the behavior of the normalized duality gap in the predictor step. We start by proving that

$$(1 - 2.5\,\theta)\mu \leq \mu(\theta) \leq (1 - .5\,\theta)\mu, \,\forall\, 0 \leq \theta \leq \theta_6 := \frac{\sqrt{\beta}}{4(1+2\kappa)\sqrt{n}} \min\left\{1, \left(11.2\sqrt{\beta}\right)^{\frac{-1}{m_p}}\right\}.$$
(4.20)

Due to the obvious fact that

$$(1 - 2\theta) \leq (1 - \theta)^2 \leq (1 - \theta), \quad \forall\, \theta \in [0, 1],$$

we have

$$(1 - 2\theta)\mu + \sum_{i=m+1}^{2m} \theta^i(e^T h^i/n) \leq \mu(\theta) \leq (1 - \theta)\mu + \sum_{i=m+1}^{2m} \theta^i(e^T h^i/n).$$

Using (4.8), with $\alpha = 2$ and $\tau = 4/\sqrt{\beta}$, we obtain

$$\left| \sum_{i=m+1}^{2m} \theta^i(e^T h^i/n) \right| \leq \frac{1}{\sqrt{n}} \sum_{i=m+1}^{2m} \theta^i \left\| h^i \right\|_2 < .5\,\theta\mu,$$

for all $\theta \in [0, \theta_6]$. Using Lemma 4.4 and the sum of a geometric series with ratio .1, we deduce that for any $\theta \in [0, \frac{\sqrt{\beta}}{40(1+2\kappa)\sqrt{n}}]$ holds

$$\mu'(\theta) = -(1 + \epsilon - 2\epsilon\theta)\mu + \sum_{i=m+1}^{2m} i\theta^{i-1}(e^T h^i/n) \leq -\mu + \frac{1}{\sqrt{n}} \sum_{i=m+1}^{2m} i\theta^{i-1} \left\| h^i \right\|_2$$
$$\leq -\mu + 8\mu\sqrt{\beta} \sum_{i=m}^{2m-1} \left(\frac{4(1+2\kappa)\theta\sqrt{n}}{\sqrt{\beta}} \right)^i < -\mu + 8\mu\sqrt{\beta}\frac{.1^m}{1 - .1} < -\mu + 8\mu\frac{.1^m}{.9} < 0.$$

Since $\theta_6 > \frac{\sqrt{\beta}}{44.8(1+2\kappa)\sqrt{n}} > \frac{\sqrt{\beta}}{50(1+2\kappa)\sqrt{n}}$, we conclude that

(4.21) $$(1 - 2.5\,\theta)\mu \leq \mu(\theta) \leq (1 - .5\,\theta)\mu \quad \text{and} \quad \mu'(\theta) < 0,$$

(4.22) $$\forall\, \theta \in \left[0, \frac{\sqrt{\beta}}{50(1+2\kappa)\sqrt{n}}\right].$$

Next, we claim that the quantity θ_2 from (3.11) used in the computation of the predictor step-length satisfies

$$\theta_2 \geq \theta_7 := \frac{\sqrt{\beta}}{4(1+2\kappa)\sqrt{n}} \min\left\{1, \left(11.2\sqrt{\beta}\right)^{\frac{-1}{m_p}}, \left(\frac{(1+2\kappa)\delta}{2\beta}\right)^{\frac{1}{m_p+1}}\right\}$$

$$(4.23) \qquad \geq \frac{\chi_7}{(1+2\kappa)n^{\frac{1}{2}+\frac{m_c+1}{2m_c(m_p+1)}}},$$

$$(4.24) \qquad \chi_7 := \frac{1}{4}\min\left\{\frac{1}{11.2}, \left(\frac{(1-\beta)\underline{\gamma}\chi_5}{4}\right)^{\frac{1}{2}}\right\}.$$

Using (4.20) with $n \geq 8$ we obtain

$$\mu(\theta) \geq (1 - 2.5\theta_6)\mu \geq (1 - \frac{2.5}{8\sqrt{2}})\mu \geq .7\mu, \ \forall\, \theta \in [0, \theta_6].$$

By taking $\gamma = 0$, and $\beta + \delta$ instead of β, in (4.17), using (4.7) with $\alpha = 1/(.7\delta)$, we deduce that

$$f(\theta) \geq \beta + \delta - \frac{\sum_{i=m+1}^{2m}\theta^i \|h^i\|_2}{\mu(\theta)} \geq \beta + \delta - \frac{1}{.7\mu}\sum_{i=m+1}^{2m}\theta^i\|h^i\|_2 \geq \beta, \ \forall\, \theta \in [0, \theta_7],$$

which proves that $\theta_2 \geq \theta_7$. From the definition of θ_7 it follows

$$\theta_7 \geq \frac{\sqrt{\beta}}{4(1+2\kappa)\sqrt{n}}\min\left\{\frac{1}{11.2\sqrt{\beta}}, \frac{1}{\sqrt{\beta}}\left(\frac{(1+2\kappa)\delta}{2}\right)^{\frac{1}{m_p+1}}\right\}$$

$$\geq \frac{1}{4(1+2\kappa)\sqrt{n}}\min\left\{\frac{1}{11.2}, \left(\frac{(1-\beta)\underline{\gamma}\theta_5}{4}\right)^{\frac{1}{m_p+1}}\right\}$$

$$\geq \frac{\sqrt{\beta}}{4(1+2\kappa)\sqrt{n}}\min\left\{\frac{1}{11.2}, \left(\frac{(1-\beta)\underline{\gamma}\chi_5}{4n^{\frac{m_c+1}{2m_c}}}\right)^{\frac{1}{m_p+1}}\right\}$$

$$\geq \frac{\chi_7}{(1+2\kappa)n^{\frac{1}{2}+\frac{m_c+1}{2m_c(m_p+1)}}}.$$

Bounding the decrease of the duality gap. Due to the fact that the duality gap decreases in both predictor step and corrector step, a complete analysis of the decreases of the duality gap has to be done by studying a succession of corrector-predictor-corrector steps. Assume that we are at iteration k and we have a point $z^k \in \mathcal{D}(\beta)$, with normalized duality gap μ_k. We follow the notations of Algorithm 1. The corrector step produces a point $\bar{z}^k \in \mathcal{D}(\beta + \delta)$, with δ given by (4.18). The corresponding normalized duality gap clearly satisfies $\bar{\mu}_k \leq \mu_k$, but a bound on the decrease of the duality gap cannot be given at this stage. The corrector is followed by a predictor that produces a point $z^{k+1} = z(\theta_p) \in \mathcal{D}(\beta)$ with duality gap $\mu_{k+1} = \mu(\theta_p) = \min_{0\leq\theta\leq\theta_2}\mu(\theta)$. We have $\theta_7 \leq \theta_2$ and $\theta_7 \leq \theta_6$, so that according to (4.20)

$$(4.25) \ \mu_{k+1} \leq \mu(\theta_7) \leq (1 - .5\,\theta_7)\bar{\mu}_k \leq \left(1 - \frac{\chi_7}{2\,n^{\frac{1}{2}+\frac{m_c+1}{2m_c(m_p+1)}}}\right)\bar{\mu}_k, \quad \bar{\mu}_k \leq \mu_k.$$

The above relation is sufficient for proving polynomial complexity, but it does not take into account the contribution of the corrector step. A finer analysis is needed in order to account for that. We distinguish two cases:

(a) $\theta_2 \geq \frac{\sqrt{\beta}}{50(1+2\kappa)\sqrt{n}}$. According to (4.21), in this case we have

$$\mu_{k+1} = \min_{0 \leq \theta \leq \theta_2} \mu(\theta) \leq \left(1 - \frac{\sqrt{\beta}}{100(1+2\kappa)\sqrt{n}}\right) \overline{\mu}_k ;$$

(b) $\theta_2 < \frac{\sqrt{\beta}}{50(1+2\kappa)\sqrt{n}}$. In this case $\mu(\theta)$ is decreasing on the interval $[0, \theta_2]$, by virtue of (4.21), and by using (4.23) we deduce that $\theta_p = \theta_2, f(\theta_p) = \beta$. The latter equality must be true, since if $f(\theta_p) > \beta$, then, by a continuity argument, it follows that $\theta_2 > \theta_p$ which is a contradiction (see the definition of θ_2 (3.11)). But if $f(\theta_p) = \beta$, then, according to (3.8), in the next corrector step we have $\theta_c > .2\,\theta_5$, so that

$$\overline{\mu}_{k+1} < (1 - .2\,\underline{\sigma}(1-\overline{\gamma})\theta_5)\,\mu_{k+1} \leq \left(1 - \frac{\underline{\sigma}(1-\overline{\gamma})\chi_5}{5(1+2\kappa)\,n^{\frac{1}{2}+\frac{1}{2m_c}}}\right) \mu_{k+1} .$$

In conclusion, for any $k \geq 0$ we have

$$\overline{\mu}_{k+1} \leq \mu_{k+1} \leq \left(1 - \frac{\sqrt{\beta}}{100(1+2\kappa)\sqrt{n}}\right) \overline{\mu}_k$$

or

$$\overline{\mu}_{k+1} < \left(1 - \frac{\underline{\sigma}(1-\overline{\gamma})\chi_5}{5(1+2\kappa)\,n^{\frac{1}{2}+\frac{1}{2m_c}}}\right)\left(1 - \frac{\chi_7}{2(1+2\kappa)\,n^{\frac{1}{2}+\frac{m_c+1}{2m_c(m_p+1)}}}\right) \overline{\mu}_k .$$

By taking

$$\overline{\chi} := \min\left\{\frac{\sqrt{\beta}}{100}, \frac{\underline{\sigma}(1-\overline{\gamma})\chi_5}{5}, \frac{\chi_7}{2}\right\},$$

we deduce that

$$\overline{\mu}_{k+1} \leq \left(1 - \frac{\overline{\chi}}{(1+2\kappa)n^{\frac{1}{2}+\upsilon}}\right) \overline{\mu}_k , \quad k = 0, 1, \dots,$$

where υ is given by (4.11). The proof is complete ∎

As an immediate consequence of the above theorem we obtain the following complexity result:

COROLLARY 4.7. *Algorithm 1 produces a point $z = \lceil x, s \rfloor \in \mathcal{D}(\beta)$ with $x^T s \leq \varepsilon$, in at most $O\left((1+\kappa)n^{1/2+\upsilon}\log\left(x^{0\,T}s^0/\varepsilon\right)\right)$ iterations, where υ is given by (4.11).* It follows that if the order of either the corrector or the predictor is larger than a multiple of $\log n$, then Algorithm 1 has $O\left((1+\kappa)\sqrt{n}L\right)$-iteration complexity.

COROLLARY 4.8. *If $\max\{m_c, m_p\} = \Omega(\log n)$, then Algorithm 1 produces a point $z = \lceil x, s \rfloor \in \mathcal{D}(\beta)$ with $x^T s \leq \varepsilon$, in at most $O\left((1+\kappa)\sqrt{n}\log\left(x^{0\,T}s^0/\varepsilon\right)\right)$.*

Proof. Under the hypothesis of the corollary there is a constant ϑ, such that $\upsilon \leq \vartheta/\log n$. Hence $n^{1/2+\upsilon} \leq n^{\frac{\vartheta}{\log n}}\sqrt{n} = e^{\vartheta}\sqrt{n}$. ∎

Due to the fact that $\lim_{n\to\infty} n^{1/n^{\omega}} = 1$ for any $\omega \in (0,1)$, in applications we can choose $m_p = \lceil n^{\omega}\rceil$ for some value of $\omega \in (0,1)$. This choice was initially suggested by Roos (private communication) and subsequently used in [24] and [16]. A correspondence between n and $\lceil n^{\omega}\rceil$ with $\omega = 0.1$ is shown in table 4.

TABLE 4.1

n	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}
$\lceil n^{.1} \rceil$	3	4	4	6	7	8	11

5. Superlinear convergence. In this section we show that the duality gap of the sequence produced by Algorithm 1 is superlinearly convergent. The result is based on the following lemma which is a consequence of the results about the analyticity of the central path from [19].

LEMMA 5.1. *If HLCP (2.1) is sufficient then the solution of (3.2) with $\gamma = 0$ satisfies*

$$u^i = O(\mu^i), \quad v^i = O(\mu^i), \quad i = 1, \ldots, m, \ \ if \ \ HLCP \ (2.1) \ is \ nondegenerate,$$

and

$$u^i = O(\mu^{i/2}), \quad v^i = O(\mu^{i/2}), \quad i = 1, \ldots, m, \ \ if \ HLCP \ (2.1) \ is \ degenerate.$$

By using the above lemma we obtain the following superlinear convergence result, which is a trivial extension of the corresponding result in [14] to sufficient linear complementarity problems.

THEOREM 5.2. *The sequence μ_k produced by Algorithm 1 satisfies*

$$\mu_{k+1} = O(\mu_k^{m_p+1}), \ \ if \ \ HLCP \ (2.1) \ is \ nondegenerate,$$

and

$$\mu_{k+1} = O(\mu_k^{(m_p+1)/2}), \ \ if \ \ HLCP \ (2.1) \ is \ degenerate.$$

6. Conclusions. We have presented a corrector-predictor interior point algorithm for sufficient horizontal linear complementarity problems acting in a wide neighborhood of the central path.

The corrector of order m_c is used to improve both the centrality and the complementarity gap. The predictor of order m_p follows each corrector step to further decrease the complementarity gap. If $\max\{m_c, m_p\} = \Omega(\log n)$, then the iteration complexity of the algorithms is $O((1 + \kappa)\sqrt{n}L)$. Although the complexity of our algorithm depends on κ, the algorithm itself does not, so that the algorithm works for the class of sufficient HLCP. Our algorithm has the best known iteration complexity for sufficient linear complementarity problems and is superlinearly convergent even for degenerate problems. The cost of implementing one iteration of our algorithm is $O(n^3)$ arithmetic operations.

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