A note on polyhedral aspects of a robust knapsack problem

Olivier Klopfenstein[1,2], Dritan Nace[2]
[1]France Télécom R&D, 38-40 rue du gl Leclerc, 92794 Issy-les-Moux cedex 9, France
[2]Université de Technologie de Compiègne, Laboratoire Heudiasyc UMR CNRS 6599, 60205 Compiègne Cedex, France.
olivier.klopfenstein@francetelecom.com, nace@utc.fr

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Abstract

The robust optimization framework proposed by Bertsimas and Sim can take account of data uncertainty in linear programs. The current paper investigates the polyhedral impacts of this robust model for the 0-1 knapsack problem. In particular, classical cover cuts are adapted to provide valid inequalities for the robust knapsack problem. The strength of the proposed inequalities is highlighted from both theoretical and practical viewpoints.

Keywords: Robust optimization; Knapsack problem; Polyhedral analysis; Knapsack cover inequalities; 90C57

1 Introduction

Many real-life problems require uncertainty on input data to be taken into account. Earlier work on robust optimization has provided interesting ways to handle uncertainty on input data for linear problems (see Ben Tal and Nemirovski (2000), Bertsimas and Sim (2004)). That is, under some weak probabilistic assumptions, a tractable robust model is proposed, whose solutions are feasible with probability at least $1 - \varepsilon$ (where $\varepsilon > 0$ is a given unfeasibility tolerance). The robust framework of Bertsimas and Sim (2004) has the advantage of preserving the linearity of the initial problem. Hence, the robust version of an integer linear problem is also an integer linear problem. This is important in practice, since integer linear solvers are nowadays very efficient. Despite its practical interest, the application of this general model of robustness to integer linear problems has not been extensively investigated so far. In the literature, the theoretical studies on applications of this framework for robust integer programming have mainly focused on cases where only objective coefficients are uncertain. Within this restriction, Bertsimas and Sim (2003) have provided complexity results, Pinar (2004) proves a probability bound easier to compute than that proposed in Bertsimas and Sim (2004). Atamtürk (2004) investigates some reformulations for this particular case of uncertain objective coefficients.

However, many uncertain problems also require uncertain constraint coefficients to be considered. With respect to the robust framework of Bertsimas and Sim, this uncertainty impacts the polyhedron of feasible solutions. Because of its fundamental and practical importance for integer programming, this paper addresses the 0-1 knapsack problem with uncertain weights and capacity. The stress is put on adaptating classical results to obtain strong polyhedral descriptions for robust problems. In particular, this paper shows how the widely used knapsack cover cuts can be adapted to provide strong inequalities for the robust knapsack problem.

2 Problems and formulations

2.1 The knapsack problem

Let $I = \{1, \dots, n\}$. The classical knapsack polyhedron is:

$$\mathcal{K} = conv\left\{x \in \{0,1\}^n | \sum_{i\in I} w_i x_i \le c\right\} \tag{1}$$

where coefficients w_i (weights) and c (knapsack capacity) are assumed to be non-negative integers. Given a profit vector p, the knapsack problem is: $\max\{px | x \in \mathcal{K}\}$.

For an extensive study of the knapsack problem and its variants, we refer to Martello and Toth (1990), and Kellerer, Pferschy and Pisinger (2004). The problem is known to be NP-hard, even though there exists a pseudo-polynomial dynamic programming algorithm to solve it. While branch-and-cut algorithms are often not the best suited for solving this problem, its polyhedral comprehension is of interest. Indeed, knapsack appears as a sub-problem of a lot of other larger problems which are classically processed with branch-and-cut (such as capacitated location problems, flow routing in capacitated networks).

2.2 A robust knapsack problem

From now on, we assume that the coefficients w_i lie in intervals $[\overline{w}_i - \hat{w}_i, \overline{w}_i + \hat{w}_i]$: $\overline{w}_i > 0$ is the nominal value of the weight i, $\hat{w}_i \ge 0$ being the possible variation of w_i from this expected value. Both $\overline{w}$ and $\hat{w}$ are assumed to be integral. To each weight is associated a random variable; for the sake of simplicity, the random variables and their realizations are denoted by the same symbol w_i.

Let $\Gamma \in \{0, \dots, n\}$. The idea of the robust formulation is to find a solution which is feasible even though up to Γ coefficients of w have their largest possible value. Thus, for instance, if $\Gamma = 0$, we consider only the nominal scenario; if $\Gamma = n$, the worst case is taken into account. Within this framework, let us introduce the robust knapsack polyhedron, parameterized by Γ:

$$\mathcal{K}(\Gamma) = conv\left\{x \in \{0,1\}^n | \forall S \subseteq I \text{ s.t. } |S| = \Gamma : \sum_{i\in I\setminus S} \overline{w}_i x_i + \sum_{i\in S} (\overline{w}_i + \hat{w}_i) x_i \le c\right\}. \tag{2}$$

Observe that $\mathcal{K}(\Gamma)$ is a multidimensional knapsack polyhedron. For any subset $S \subseteq I$, let us denote:

$$\mathcal{K}_S = \left\{x \in \{0,1\}^n | \sum_{i\in S} (\overline{w}_i + \hat{w}_i) x_i + \sum_{i\in I\setminus S} \overline{w}_i x_i \le c\right\}. \tag{3}$$

We have:

$$\mathcal{K}(\Gamma) = conv\left(\bigcap_{S\subseteq I, |S|=\Gamma} \mathcal{K}_S\right). \tag{4}$$

The proposed robust model is relevant in many ways. First, the probability results of Bertsimas and Sim (2004) lead to:

Proposition 1 *Assume that the random variables w_i are independent and symmetrically distributed. Let $\varepsilon \in (0,1)$. If $\Gamma \ge \sqrt{-2n\ln(\varepsilon)}$, then any $x \in \mathcal{K}(\Gamma)$ is feasible with probability at least $1 - \varepsilon$.*

Note that in Bertsimas and Sim (2004), the result is stated only for optimal solutions of an integer linear program. The extension to all feasible points of $\mathcal{K}(\Gamma)$ is straightforward (see e.g. Klopfenstein and Nace (2005)).

Secondly, the set of feasible events w associated with the set of solutions (4) would seem to be relevant, since in particular the worst-case, where all weights take their maximal values $\overline{w}_i + \hat{w}_i$, is discarded. A deeper motivation for studying this robust knapsack problem comes from Klopfenstein and Nace (2005), where it is shown that it can be effectively used to provide good solutions to the knapsack problem with probability constraint $P\left(\sum_{i\in I} w_i x_i \le c\right) \ge 1 - \varepsilon$.

Finally, a compact 0-1 linear formulation is provided in Bertsimas and Sim (2004), and makes the computational resolution tractable. The formulation (2) of $\mathcal{K}(\Gamma)$ has an exponential number of constraints. However, it can alternatively be written:

$$\mathcal{K}(\Gamma) = conv\Big\{x \in \{0,1\}^n \mid \sum_{i\in I} \overline{w}_i x_i + \max\{\sum_{i\in S} \hat{w}_i x_i | S \subseteq I \text{ s.t. } |S| = \Gamma\} \leq c\Big\}.$$

Then, by linear duality, it is shown in Bertsimas and Sim (2004) that for all $x \in \{0,1\}^n$:

$$\max\Big\{ \textstyle\sum_{i\in S} \hat{w}_i x_i \;\mid\; S \subseteq I, |S| = \Gamma\Big\} = \\ \min\Big\{ \textstyle\sum_{i\in I} u_i + \Gamma v \;\mid\; u \geq 0, v \geq 0 \text{ and } \forall i \in I : u_i + v \geq \hat{w}_i x_i\Big\}.$$

Hence, as a direct consequence:

$$\mathcal{K}(\Gamma) = conv\left\{ x \in \{0,1\}^n \;\mid\; \exists u \geq 0, v \geq 0 \text{ s.t. } \begin{array}{l} \sum_{i\in I} \overline{w}_i x_i + \sum_{i\in I} u_i + \Gamma v \leq c \\ \forall i \in I, u_i + v \geq \hat{w}_i x_i \end{array} \right\} \quad (5)$$

3 Polyhedral aspects of $\mathcal{K}(\Gamma)$

In this section, we show how classical results related to the knapsack polyhedron can be adapted for its robust version.

3.1 General facts

From (2), we easily deduce the following result:

Lemma 1 *Let $\Gamma \geq 1$. $\mathcal{K}(\Gamma)$ is full dimensional if, and only if, for all $i \in I$, $\overline{w}_i + \hat{w}_i \leq c$.*

From now on, we assume that none of the robust weights $\overline{w}_i + \hat{w}_i$ exceeds the knapsack capacity.

Lemma 2 *Let $S \subseteq I$ such that $|S| \leq \Gamma$, any inequality valid for $\mathcal{K}_S$ is valid for $\mathcal{K}(\Gamma)$.*

This is clear from (4). However, most of the time: $\mathcal{K}(\Gamma) = conv\left(\bigcap_{S\subseteq I,|S|=\Gamma} \mathcal{K}_S\right) \subset \bigcap_{S\subseteq I,|S|=\Gamma} conv(\mathcal{K}_S)$. This simple fact is illustrated with the following example. Consider $P_1 = \{x \in \{0,1\}^2 | x_1 + x_2 \leq 1\}$ and $P_2 = \{x \in \{0,1\}^2 | x_2 - x_1 \leq 0\}$, we have: $conv(P_1 \cap P_2) = \{x \in [0,1]^2 | x_2 = 0\}$, and: $conv(P_1) \cap conv(P_2) = \{x \in [0,1]^2 | x_1 + x_2 \leq 1 \text{ and } x_2 - x_1 \leq 0\}$. There are therefore inequalities valid for $\mathcal{K}(\Gamma)$ which are valid for none of the classical knapsack sets $\{\mathcal{K}_S\}_{S\subseteq I,|S|=\Gamma}$.

3.2 Robust cover cuts

When dealing with the classical knapsack problem, the most frequently used cutting inequalities are the so-called *knapsack cover inequalities* (or cover inequalities for short). A set $\mathcal{C} \subseteq I$ is said to be a *cover* if: $\sum_{i\in\mathcal{C}} w_i > c$. Then, the following inequality is valid for $\mathcal{K}$:

$$\sum_{i\in\mathcal{C}} x_i \leq |\mathcal{C}| - 1.$$

The idea is that the elements of a cover cannot all be present in the knapsack together. A cover $\mathcal{C}$ is said to be *minimal* if for any $i \in \mathcal{C}$, $\mathcal{C} \setminus \{i\}$ is not a cover. Minimal cover inequalities clearly dominate other inequalities.

A cover inequality can be sequentially lifted, but this process requires the solution of new knapsack problems (see Gu, Nemhauser and Savelsbergh (1998), Nemhauser and Wolsey (1999)). Nevertheless, there is a very easy means to strengthen cover inequalities. Given a cover $\mathcal{C}$, we call the set $E(\mathcal{C}) = \mathcal{C} \cup \{k \in I | w_k \geq \max_{i\in\mathcal{C}} w_i\}$ the *extended cover*. Then, the inequality:

$$\sum_{i\in E(\mathcal{C})} x_i \leq |\mathcal{C}| - 1$$

is also valid for $\mathcal{K}$, since any subset of $|\mathcal{C}|$ (or more) elements in $E(\mathcal{C})$ will give an amount exceeding the knapsack capacity. Extended cover inequalities are often proved to define facets of $\mathcal{K}$ (see Nemhauser and Wolsey (1999)), and are effectively shown in practice to strengthen the knapsack linear relaxation (see for instance Gabrel and Minoux (2002)).

The classical concepts related to cover inequalities can be adapted to our robust framework:

Definition 1 *The set $\mathcal{C} \subseteq I$ is a robust cover if there exists $S \subseteq \mathcal{C}$ such that $|S| = \min\{\Gamma, |\mathcal{C}|\}$ and:*

$$\sum_{i \in \mathcal{C} \setminus S} \overline{w}_i + \sum_{i \in S} (\overline{w}_i + \hat{w}_i) > c.$$

Definition 2 *A robust cover $\mathcal{C} \subseteq I$ is said minimal if for any $i \in \mathcal{C}$, $\mathcal{C} \setminus \{i\}$ is not a robust cover.*

For any robust cover $\mathcal{C}$, the following inequality is valid for $\mathcal{K}(\Gamma)$:

$$\sum_{i \in \mathcal{C}} x_i \leq |\mathcal{C}| - 1.$$

This is clear, since the above equation is valid for at least one of the sets $\mathcal{K}_S$ (cf Lemma 2). On the other hand, the classical concept of *extended cover* can be adapted to our robust framework. A robust cover $\mathcal{C}$ can be extended into $E(\mathcal{C})$:

$$E(\mathcal{C}) = \begin{cases} \mathcal{C} \cup \{i \in I \,|\, \overline{w}_i + \hat{w}_i \geq \max_{k \in \mathcal{C}} (\overline{w}_k + \hat{w}_k)\}\,, & \text{if } |\mathcal{C}| \leq \Gamma \\ \mathcal{C} \cup \{i \in I \,|\, \overline{w}_i \geq \max_{k \in \mathcal{C}} \overline{w}_k, \text{ and: } \overline{w}_i + \hat{w}_i \geq \max_{k \in \mathcal{C}} (\overline{w}_k + \hat{w}_k)\}\,, & \text{if } |\mathcal{C}| \geq \Gamma + 1 \end{cases}. \tag{6}$$

Proposition 2 *Let $\mathcal{C}$ be a robust cover. Then for any extended robust cover $E(\mathcal{C})$ the following inequality is valid for $\mathcal{K}(\Gamma)$:*

$$\sum_{i \in E(\mathcal{C})} x_i \leq |\mathcal{C}| - 1. \tag{7}$$

Proof: Let us first prove that any subset $\mathcal{C}' \subseteq E(\mathcal{C})$ such that $|\mathcal{C}'| = |\mathcal{C}|$ is a robust cover. Let $k \in E(\mathcal{C}) \backslash \mathcal{C}$. Observe that for any $i \in \mathcal{C}$, the set $(\mathcal{C} \setminus \{i\}) \cup \{k\}$ is a robust cover (this follows directly from (6) and Definition 1). Hence, $\mathcal{C}'$ can be obtained from $\mathcal{C}$ through a finite sequence of replacements of elements of $\mathcal{C}$ by elements of $E(\mathcal{C}) \setminus \mathcal{C}$. This shows that $\mathcal{C}'$ is a robust cover.

Suppose now that there exists a 0-1 point $x \in \mathcal{K}(\Gamma)$ such that $\sum_{i \in E(\mathcal{C})} x_i \geq |\mathcal{C}|$. Let us consider a 0-1 vector $\bar{x} \leq x$ such that $\sum_{i \in E(\mathcal{C})} \bar{x}_i = |\mathcal{C}|$: $\bar{x} \in \mathcal{K}(\Gamma)$. Then, let $\mathcal{C}'$ be the set whose characteristic vector is $\bar{x}$, i.e.: $i \in \mathcal{C}' \Leftrightarrow \bar{x}_i = 1$. As $|\mathcal{C}'| = |\mathcal{C}|$, $\mathcal{C}'$ is a robust cover and as a consequence $\sum_{i \in \mathcal{C}'} x_i \leq |\mathcal{C}| - 1$ (contradiction). □ □

While robust cover cuts are valid for some knapsack sets $\mathcal{K}_S$, this is not the case, in general, for extended robust cover inequalities. The following examples show this for both types of extensions given in (6).

Example 1: Consider a robust knapsack problem with three elements (i.e. $n = 3$) of respective sizes $w_1 \in [8, 12]$, $w_2 \in [9, 13]$ and $w_3 \in [10, 14]$, with respect to our data uncertainty model, and a knapsack of capacity $c = 24$. With $\Gamma = 2$, the robust knapsack polyhedron is the convex hull of the following set:

$$\begin{cases} 10x_1 + 13x_2 + 14x_3 \leq 24 & (i) \\ 12x_1 + 11x_2 + 14x_3 \leq 24 & (ii) \\ 12x_1 + 13x_2 + 12x_3 \leq 24 & (iii) \\ x_1, x_2, x_3 \in \{0, 1\} \end{cases}$$

$\mathcal{C} = \{1, 2\}$ is a robust cover. Observing that $|\mathcal{C}| = \Gamma$, this robust cover can be extended into $E(\mathcal{C}) = \{1, 2, 3\}$. This leads to the inequality: $x_1 + x_2 + x_3 \leq 1$, which is not valid for any of the knapsack problems associated with single inequalities (i), (ii) or (iii), but valid for the global problem.

Example 2: Consider a robust knapsack problem with three elements (i.e. $n = 3$) of respective sizes $w_1 \in [7, 13]$, $w_2 \in [8, 14]$ and $w_3 \in [9, 15]$, with respect to our data uncertainty model, and a knapsack of capacity $c = 23$. With $\Gamma = 1$, the robust knapsack polyhedron is the convex hull of the following set:

$$\begin{cases} 13x_1 + 11x_2 + 12x_3 \leq 23 & (i) \\ 10x_1 + 14x_2 + 12x_3 \leq 23 & (ii) \\ 10x_1 + 11x_2 + 15x_3 \leq 23 & (iii) \\ x_1, x_2, x_3 \in \{0, 1\} \end{cases}$$

$\mathcal{C} = \{1, 2\}$ is a robust cover. Observing that $|\mathcal{C}| = \Gamma + 1$, it can be extended into $E(\mathcal{C}) = \{1, 2, 3\}$. This leads to the inequality: $x_1 + x_2 + x_3 \leq 1$, which is not valid for any of the knapsack problems associated to (i), (ii) or (iii), but valid for the global problem.

Thus, it is shown that the extension procedure is likely to provide cuts which are not valid for any of the sets $\mathcal{K}_S$ ($|S| = \Gamma$) of (2). This is an indication of their strength. However, the above observation does not hold for extended robust covers $E(\mathcal{C})$ with cardinality not greater than Γ:

Lemma 3 *Let $\mathcal{C}$ be a robust cover. If $|E(\mathcal{C})| \leq \Gamma$, then there exists $S \subseteq I$ with $|S| = \Gamma$ such that the inequality (7) is valid for $\mathcal{K}_S$.*

Proof: Consider $S \supseteq E(\mathcal{C})$ of cardinality Γ. $\mathcal{C}$ is a classical cover for $\mathcal{K}_S$ such that $|\mathcal{C}| \leq \Gamma$. Thus, from (6), and since $E(\mathcal{C}) \subseteq S$, $E(\mathcal{C})$ is a classical extended cover cut for the knapsack polyhedron $\mathcal{K}_S$. □ □

Let us observe that apart from providing stronger cuts for the robust problem the theoretical setting proposed enables to characterize them in a global way. That is, considering all the classical knapsack sets $\mathcal{K}_S$ is avoided and effectively replaced by a more general analysis. This is of primary importance from both the theoretical and practical points of view.

3.3 Strength of robust cover cuts

Most of the classical polyhedral results available for cover cuts (see for instance Nemhauser and Wolsey (1999)) can be transposed directly in our robust framework.

Proposition 3 *Suppose that the set I is ordered so that: $i < j \Rightarrow \begin{cases} \overline{w}_i \geq \overline{w}_j \\ \overline{w}_i + \hat{w}_i \geq \overline{w}_j + \hat{w}_j \end{cases}$. Let $\mathcal{C} = \{i_1, ..., i_r\}$ be a minimal robust cover, with $i_1 < i_2 < ... < i_r$. If any of the following conditions holds, then (7) is a facet of $\mathcal{K}(\Gamma)$:*

(i) $\mathcal{C} = I$,

(ii) $E(\mathcal{C}) = I$, and $(\mathcal{C} \setminus \{i_1, i_2\}) \cup \{1\}$ is not a robust cover,

(iii) $E(\mathcal{C}) = \mathcal{C}$, and $(\mathcal{C} \setminus \{i_1\}) \cup \{p\}$ is not a robust cover, where $p = \min\{i \in I \setminus \mathcal{C}\}$,

(iv) $\mathcal{C} \subset E(\mathcal{C}) \subset I$, and neither $(\mathcal{C} \setminus \{i_1, i_2\}) \cup \{1\}$ nor $(\mathcal{C} \setminus \{i_1\}) \cup \{p\}$, with $p = \min\{i \in I \setminus \mathcal{C}\}$, are robust covers.

Proof: Given a subset $U \subseteq I$, we denote $x^U \in \{0, 1\}^n$ the characteristic vector of U: $x^U_i = 1 \Leftrightarrow i \in U$. For any minimal robust cover $\mathcal{C} = \{i_1, ..., i_r\}$, consider the following sets U_j:

a. for all $j \in \mathcal{C}$, $U_j = \mathcal{C} \setminus \{j\}$,
b. for all $j \in E(\mathcal{C}) \setminus \mathcal{C}$, $U_j = (\mathcal{C} \setminus \{i_1, i_2\}) \cup \{j\}$,
c. and for all $j \in I \setminus E(\mathcal{C})$, $U_j = (\mathcal{C} \setminus \{i_1\}) \cup \{j\}$.

For any $j \in I$, observe that: $\sum_{i \in E(\mathcal{C})} x_i^{U_j} = |E(\mathcal{C}) \cap U_j| = |\mathcal{C}| - 1$. Moreover, the points $\{x^{U_j}\}_{j \in I}$ are linearly independent. Finally, we need to check that these points belong to $\mathcal{K}(\Gamma)$.

For (i), the sets of a. provide $|\mathcal{C}| = n$ independent points in $\mathcal{K}(\Gamma)$, since $|\mathcal{C}|$ is a minimal robust cover. Under conditions (ii), the sets of b. correspond also to feasible points, since for all $j \in I$, U_j is not a robust cover; thus, the sets a. and b. provide n independent points in $\mathcal{K}(\Gamma)$. Similarly, with conditions (iii), the sets U_j of c. are not robust covers, and thus the points associated with a. and c. belong to $\mathcal{K}(\Gamma)$. The result for (iv) is obtained by considering the points associated with a., b. and c.. □ □

3.4 Separation of robust cover cuts

Considering the classical knapsack polyhedron $\mathcal{K}$, the separation of cover inequalities is known also to be a knapsack problem. Let us denote $\tilde{\mathcal{K}} = \{x \in [0,1]^n | \sum_{i\in I} w_i x_i \leq c\}$ the linear relaxation of $\mathcal{K}$. For a non-integral point $\tilde{x} \in \tilde{\mathcal{K}}$, a most violated cover cut is obtained by solving the following problem:

$$\begin{array}{ll} \min & \sum_{i\in I}(1-\tilde{x}_i)r_i \\ \text{s.t.} & \sum_{i\in I} w_i r_i \geq c+1 \\ & r \in \{0,1\}^n \end{array}$$

If the optimal value of this problem is (strictly) less than 1, a violated cover inequality is found. Otherwise, there is no violated cover cut. Moreover, if for all $i \in I$, $\tilde{x}_i < 1$, then we obtain a minimal cover.

This separation problem can be adapted for the robust knapsack polyhedron $\mathcal{K}(\Gamma)$. Let $\tilde{\mathcal{K}}(\Gamma)$ denote the continuous relaxation of $\mathcal{K}(\Gamma)$. Given a non-integral point $\tilde{x} \in \tilde{\mathcal{K}}(\Gamma)$, a most violated robust cover inequality can be generated by solving:

$$\begin{array}{ll} \min & \sum_{i\in I}(1-\tilde{x}_i)r_i \\ \text{s.t.} & \sum_{i\in I}(\overline{w}_i + \hat{w}_i s_i) r_i \geq c+1 \\ & \sum_{i\in I} s_i \leq \Gamma \\ & r \in \{0,1\}^n, s \in \{0,1\}^n \end{array} \tag{8}$$

This can be linearized:

$$\begin{array}{lll} \min & \sum_{i\in I}(1-\tilde{x}_i)r_i & \\ \text{s.t.} & \sum_{i\in I}(\overline{w}_i r_i + \hat{w}_i t_i) \geq c+1 & \\ & \sum_{i\in I} s_i \leq \Gamma & \\ & t_i \leq r_i, & \forall i \in I, \\ & t_i \leq s_i, & \forall i \in I, \\ & r \in \{0,1\}^n, s \in \{0,1\}^n, t \geq 0 & \end{array} \tag{9}$$

Lemma 4 *(r,s) is a feasible solution of (8) if and only if there exists $t \geq 0$ such that (r,s,t) is a feasible solution of (9).*

Proof: Let (r,s) be a feasible solution of (8), then t with $t_i = r_i s_i$ is such that (r,s,t) is a feasible solution of (9). Reciprocally, let (r,s,t) be a feasible solution of (9). Let us consider t' such that $t'_i = \min\{r_i, s_i\}$. As $t' \geq t$, (r,s,t') is also a feasible solution of (9). Observing that $t'_i = r_i s_i$, we deduce that (r,s) is a feasible solution of (8). □ □

The optimal value of (9) is (strictly) less than 1 if and only if a violated robust cover inequality exists. In this case, the solution vector r characterizes a violated cover. This separation problem is NP-hard, since $\Gamma = 0$ leads to a knapsack problem. As for classical separation of cover cuts:

Lemma 5 *If for all $i \in I$, $\tilde{x}_i < 1$, then the robust cover produced by (9) is minimal.*

3.5 Uncertainty on the knapsack capacity

Until now, it has been assumed that the knapsack capacity c was known with accuracy. In this paragraph, our model of data uncertainty is also applied to c in the same way as for coefficients w_i. Thus c is assumed to belong to an interval $[\overline{c} - \hat{c}, \overline{c} + \hat{c}]$ such that $\overline{c} - \hat{c} \geq 0$. Then, the associated robust polyhedron can be modeled as previously by considering a fixed knapsack capacity $\overline{c} + \hat{c}$, and by including in the model a supplementary element of uncertain weight characterized by $\overline{w}_{n+1} = \hat{w}_{n+1} = \hat{c}$. This new element is introduced to capture the uncertainty of the knapsack capacity.

More precisely, let us denote $I' = I \cup \{n+1\}$. For a given robust parameter $\Gamma \in \{0, \ldots, n+1\}$, a vector $x \in \{0,1\}^n$ will be said feasible for the robust knapsack problem with uncertain capacity if and only if there exists $x_{n+1} \in \{0,1\}$ such that:

$$\forall S \subseteq I' \text{ s.t. } |S| = \Gamma : \sum_{i \in I' \setminus S} \overline{w}_i x_i + \sum_{i\in S}(\overline{w}_i + \hat{w}_i) x_i \leq \overline{c} + \hat{c}.$$

Note in particular that Proposition 1 remains valid for a solution $x \in \{0,1\}^n$ by considering $n+1$ instead of n in the probability bound, provided that the r.v. c is symmetrically distributed and independent of weights $\{w_i\}_{i\in I}$. Finally, the polyhedral analysis performed for certain capacity can be directly re-used here.

4 Numerical tests

The robust cover cuts proposed have been tested on several robust knapsack instances. Following the analysis of Pisinger (2005), hard instances were built with the following rules:

- each nominal integer weight $\overline{w}_i$ is randomly chosen in $[1, 100]$;
- the integer knapsack capacity c is randomly chosen between $1/3. \sum_{i\in I} \overline{w}_i$ and $2/3. \sum_{i\in I} \overline{w}_i$;
- an objective function $\sum_{i\in I} p_i x_i$ has to be maximized for $x \in \mathcal{K}(\Gamma)$, with each profit coefficient satisfying: $p_i = \overline{w}_i + 10$ (cf the *strongly correlated* instances of Pisinger (2005).

The weight uncertainties are 10% of the nominal values: $\hat{w}_i = \overline{w}_i/10$. No uncertainty is considered on c.

We used the integer linear formulation 5 for our computational tests. Although greedy heuristics can effectively be used to generate violated robust cover cuts, we preferred to work with an "exact" separation process. This means that a violated cut will be found if one exists (there is no such certainty when considering heuristic separation procedures). This choice was motivated by our aim of achieving better assessment of the cut strengths. Hence, the problem (9) is solved to optimality at each branching node to find violated robust cover cuts. Then, depending on the optimization strategy tested, the obtained cuts are possibly extended.

Ten knapsack instances of 20 objects ($n = 20$) were generated, and three values of Γ were considered: $\Gamma = 5$, $\Gamma = 10$ and $\Gamma = 15$. For each problem, three different branch-and-bound strategies were tested. The first corresponds to the classical strategy, where the upper bound at a branching node is obtained by solving the linear relaxation without any additional inequality. In the second strategy, the bound at each node is the value of the linear relaxation when adding successively as many robust cover cuts as needed. The third method is similar, using extended robust cover cuts. Note that these resolution schemes are not standard branch-and-cut algorithms. Only the bound calculation differs from one strategy to another. Our particular computer implementation implies that the branching tree is preserved: the branching tree of the second optimization strategy is a subtree of the branching tree of the first strategy; similarly, the third branching tree is a subtree of the second. As a result, the number of computation nodes processed is a direct measure of the strength of the proposed cuts. The computer program was written in C++ with CPLEX 9.0, used as a linear problem solver (simplex algorithm) at each branching node, and as an integer linear problem solver for solving the separation problems.

Note that exact resolution of the separation problem (9) is time consuming. Thus, the total resolution time is not indicated, since it is not relevant here: the only comparison criterion between the different methods is the total number of branching nodes. Tables 1, 2 and 3 present the results for the three values of the robust parameter; the last column gives the number of inequalities which were actually strengthened through extension. Table 4 provides the average improvement when using cuts for each value of Γ. The introduction of robust cover inequalities reduces the number of computation nodes by about 45% with respect to the optimization with no cuts. When considering extended inequalities, this decrease is about 65%. This shows the strong impact of the proposed inequalities in strengthening the linear relaxation of a robust knapsack polyhedron.

Our goal was not to describe a time-effective resolution algorithm of the robust knapsack problem. The tests were run to illustrate the impact of robust cover inequalities for cutting the relaxed polyhedron. However, these cuts will probably in practice be of considerable importance for solving in a shorter time many real-life problems, where knapsack inequalities are often involved.

5 Conclusion

A robust knapsack problem has been studied, with respect to the robust framework of Bertsimas and Sim. The emphasis has been put on describing the associated polyhedron by adapting classical results available

instance	no cuts	robust cover cuts	extended robust cover cuts	
	nb of nodes	nb of nodes	nb of nodes	nb of extensions
1	2403	1637	827	56%
2	3273	1463	689	70%
3	743	337	257	58%
4	1353	645	381	71%
5	4279	2047	1277	59%
6	1235	589	427	37%
7	1843	845	335	100%
8	933	381	237	58%
9	1361	709	409	55%
10	1247	821	627	34%

Table 1: Knapsack instances solved with $n = 20$ and $\Gamma = 5$

instance	no cuts	robust cover cuts	extended robust cover cuts	
	nb of nodes	nb of nodes	nb of nodes	nb of extensions
1	1427	1203	883	74%
2	1967	1333	947	73%
3	899	627	519	23%
4	687	397	243	88%
5	3623	2397	1961	38%
6	1005	669	539	34%
7	3239	1783	1351	93%
8	1253	633	431	75 %
9	1351	579	427	42%
10	899	557	441	36%

Table 2: Knapsack instances solved with $n = 20$ and $\Gamma = 10$

instance	no cuts	robust cover cuts	extended robust cover cuts	
	nb of nodes	nb of nodes	nb of nodes	nb of extensions
1	1377	965	659	89%
2	1357	769	281	94%
3	441	245	131	73%
4	2627	1151	569	60%
5	3109	1319	871	67%
6	787	465	327	50%
7	3831	1873	947	96%
8	1235	595	253	76%
9	1445	517	339	61%
10	445	309	163	99%

Table 3: Knapsack instances solved with $n = 20$ and $\Gamma = 15$

	$\Gamma = 5$	$\Gamma = 10$	$\Gamma = 15$
robust cover cuts	49%	38%	47%
extended robust cover cuts	69%	63%	71%

Table 4: Average reduction in the number of computation nodes

for the knapsack polyhedron. In particular, the classical extended cover inequalities have been adapted to this robust knapsack problem. The analysis performed leads to new specific robust inequalities. Furthermore, suitable models are given to separate effectively these robust cover inequalities. From both theoretical and practical viewpoints, these inequalities are shown to strengthen effectively the linear relaxation of a robust knapsack polyhedron. Finally, the current paper has underlined the need for specific polyhedral studies when dealing with robust integer linear problems. Further works should be devoted to other studies on specific polyhedra of practical importance. It would also be interesting to provide more general results for non-specific robust polyhedra.

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