Lot sizing with inventory gains

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Abstract

This paper introduces the single item lot sizing problem with inventory gains. This problem is a generalization of the classical single item capacitated lot sizing problem to one in which stock is not conserved. That is, the stock in inventory undergoes a transformation in each period that is independent of the period in which the item was produced. A 0–1 mixed integer programming formulation of the problem is given. It is observed, that by projecting the demand in each period to a distinguished period, that an instance of this problem can be polynomially transformed into an instance of the classical problem. As a result, existing results in the literature can be applied to the problem with inventory gains. The implications of this transformation for problems involving different production capacity limitations as well as backlogging and multilevel production are discussed. In particular, it is shown that the polynomially solvable classical constant capacity problems become NP-hard when stock is not conserved.

Key words: Lot sizing, production planning, inventory gains, perishable inventory, deteriorating inventory, fixed charge networks, generalized networks, networks with gains.

1 Introduction

The single item lot sizing problem is a classical production planning and scheduling problem that has been extensively studied in the literature. An instance of the capacitated version of this problem consists of n time periods, and a production capacity C_t and a demand d_t for a single item in each time period $t \in \{1, \ldots, n\}$. Once produced, the item can be held in inventory to satisfy demand in a later period. A schedule is a set of setup time periods and production levels. A feasible schedule is a schedule such that production does not exceed capacity, production only occurs when the item is set up, and the demand for the item is satisfied in each time period. The problem is to find a feasible schedule that minimizes the fixed setup costs and per unit production and inventory costs. This classical single item capacitated lot sizing problem is denoted LS-C using the three field PROB-CAP-VAR classification of single item lot sizing problems due to Wolsey (2002).

This paper introduces the single item lot sizing problem with inventory gains (LS-C-G). This problem is a generalization of the classical problem LS-C to one in which stock is not conserved. Associated with the stock in inventory at the end of period t is a strictly positive rational multiplier or gain, $\mu_t > 0$. If one unit of stock remains at the end of period $t-1$, then μ_{t-1} units of stock are available in period t. If $\mu_t < 1$, then there is a loss of inventory. If $\mu_t > 1$, then there is an increase in inventory. LS-C-G is a generalization of LS-C in the sense that if every multiplier has value one, then LS-C-G reduces to LS-C.

Lot sizing problems are special cases of integer networks and fixed charge network flow problems. This paper considers lot sizing problems in which flow in the underlying network is not conserved. Such networks are called networks with gains, or generalized networks. They are the generalization of the classical pure network flow problem to one in which a strictly positive multiplier, or gain, is associated with each arc. The flow entering the arc is multiplied by the value of the multiplier to produce the quantity of flow leaving the arc. If a multiplier is strictly less than one, then there is a loss of flow along the arc. If a multiplier is strictly greater than one, then there is an increase in flow along the arc.

Common interpretations of these multipliers are that they modify the amount of flow of some particular item, or they transform one type of item into another. The former interpretation corresponds to physical or administrative transformations such as evaporation, seepage, deterioration, breeding, amplification, purification and efficiency. The latter interpretation has applications in manufacturing, production, energy conversion, blending, manpower planning and currency exchange problems (Glover, Hultz, Klingman and Stultz, 1978; Glover, Klingman and Phillips, 1990; Ahuja, Magnanti and Orlin, 1993). A diverse range of significant practical appplications have been modelled as generalized networks, or as one of their discrete variants which include integer generalized networks and fixed charge generalized networks. These models and related specialized solution techniques are well documented in the literature. See, for example, Glover et al. (1978), Aronson (1989), Glover et al. (1990), and Ahuja et al. (1993), and the references cited therein.

This study of lot sizing problems with inventory gains was motivated by the problem of scheduling the shutdown of nuclear reactors for refuelling and maintenance (Waterer, 2001). Glover, Klingman and Phillips (1989) describe a fixed charge network model that determines the batch size and timing for nuclear plant refuelling, as well as how much energy should be produced by nuclear and non-nuclear units in each time period so as to satisfy the forecasted demand over the planning horizon with minimum total operating cost. Due to the lack of data, and the need to benchmark against existing models, the authors considered a pure network model. However, they make the comment that the realism of the model would be enhanced by considering a generalized network model, in which multipliers on the arcs represented the half-life of the uranium being used and transmission line losses. In this case, the problem becomes a fixed charge generalized network model and the subproblem associated with a single nuclear unit is a lot sizing problem with inventory gains. These problems are not tractable using direct methods due to their large size, and so decomposition and cutting plane approaches are used which require algorithms to optimize or separate over such subproblems (see, e.g., Fourcade, Eve and Socroun, 1997; Fourcade, Johnson, Bara and Cortey-Dumont, 1997).

Multilevel lot sizing problems associated with requirements planning systems have been modelled as fixed charge generalized network flow problems (see, e.g., Steinberg and Napier, 1980; Rao and McGinnis, 1983). The bill of material structure combines different amounts of raw materials in the assembly or production of other components, which in turn may be combined in differing amounts, in order to satisfy the demands for one or more final, or finished, products. These production gains within a bill of material are inversely related to the multiplier on the production arc of the corresponding underlying network. For single level single item lot sizing problems it is sufficient to consider multipliers on just the inventory arcs as, without loss of generality, it may be assumed that the multipliers on production arcs are equal to one.

Lot sizing problems with inventory losses, that is, where inventory gains are at most one, are subtly different from those problems involving perishable or deteriorating inventory. This is due to the fact that the gains are strictly positive, and the implicit assumption that all stock in a given period undergoes the same transformation, independent of the age of the item or the period in which it was produced. This assumption is reasonable as an approximation to age dependent problems, or for items that are cheap to produce.

Perishable and deteriorating inventory has been extensively studied in the context of continuous time, infinite time horizon, economic order quantity models. See Nahmias (1982) for a review. A lot sizing model in which the inventory deterioration rate depends upon the age of the item, but is independent of the period in which the item was produced, has been considered by Friedman and Hoch (1978). Hsu (2000) gives a polynomially solvable uncapacitated model in which the stock deterioration rates depend on both the age of the item and the period in which the item was produced. Furthermore, the inventory costs are age-dependent, and the production and inventory cost functions are general concave functions. This model is generalized to include backlogging in Hsu (2002).

The remainder of this paper is arranged as follows. A 0–1 mixed integer programming (MIP) formulation of LS-C-G is given in Section 2. A polynomial transformation of an instance of LS-C-G to an instance of LS-C by way of projecting the demand in each period to a distinguished period is given in Section 3. As a result of this transformation, existing results in the literature can be applied to problems with inventory gains. Section 4 discusses some implications of this transformation for problems subject to different production capacity limitations, as well as for problems that include backlogging and multilevel production. Some concluding remarks are given in Section 5.

2 Formulation

Let the index set N denote the set of periods $\{1, \ldots, n\}$. Let p_t and h_t denote the per unit production and holding costs, and f_t denote the fixed setup cost, for period $t \in N$. To avoid uninteresting special cases we assume that $d_t > 0$ for all periods $t \in N$, and that there is no beginning or ending inventory. Without loss of generality, let $\mu_0 = 1$.

LS-C-G can be formulated as a 0–1 MIP with variables $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ indexed by the periods $t \in N$. The variable x_t denotes the amount produced in period t. The variable $y_t = 1$ indicates that the item is set up for production in period t and $y_t = 0$ otherwise. The variable s_t denotes the amount of stock in inventory remaining at the end of period t. The 0–1 MIP formulation of LS-C-G that is considered is the following.

$$\text{(1)} \quad \text{minimize} \sum_{t \in N} p_t x_t + \sum_{t \in N} f_t y_t + \sum_{t \in N} h_t s_t$$

$$
\begin{array}{lll}
\text{(2)} \quad \text{subject to} & \mu_{t-1} s_{t-1} + x_t = d_t + s_t, & t \in N \\
\text{(3)} & x_t \leq C_t y_t, & t \in N \\
\text{(4)} & s_0 = s_n = 0, & \\
\text{(5)} & y_t \leq 1, & t \in N \\
\text{(6)} & x_t,\ y_t,\ s_t \geq 0, & t \in N \\
\text{(7)} & y_t \text{ integer}, & t \in N
\end{array}
$$

The objective function (1) contains per unit production and inventory costs, and a fixed setup cost. The balance constraints (2) ensure flow balance subject to the gains on the stock in inventory. The setup constraints (3) ensure that the item is only produced when set up and that production does not exceed capacity. The stock constraints (4) ensure that there is no beginning or ending inventory. Note that if $\mu_t = 1$ for all periods $t \in N$, then (1)–(7) is a 0–1 MIP formulation of LS-C.

3 A polynomial transformation

An instance of LS-C-G can be transformed into an instance of LS-C by projecting the demand in each period to a distinguished period $q \in N$. This transformation can be though of as a scaling of the 0–1 MIP formulation, followed by a change of variables. Let

$$\mu_{kl} = \prod_{t=k}^{l-1} \mu_t$$

where $\mu_{kl} = 1$ if $l \leq k$. Given a nonzero rational n-vector μ of multipliers, a distinguished period $q \in N$, and an arbitrary n-vector $\mathbf{u}$, let the function $\phi_{\mu,q}(\mathbf{u}) = \hat{\mathbf{u}}$ where $\hat{u}_t = \mu_{tq} u_t$ if $t \leq q$ and $\hat{u}_t = u_t / \mu_{qt}$ otherwise. Let $\phi^{-1}_{\mu,q}(\hat{\mathbf{u}}) = \mathbf{u}$ denote the inverse of this function. For convenience, let $\phi(\mathbf{u}) = \phi_{\mu,q}(\mathbf{u})$ and $\phi^{-1}(\mathbf{u}) = \phi^{-1}_{\mu,q}(\mathbf{u})$.

Given a distinguished period $q \in N$, the instance is scaled by multiplying the constraint corresponding to period t from the constraints (2) and (3) by μ_{tq} if $t \leq q$ and by $1/\mu_{qt}$ otherwise. Following the change of variables $(\hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{s}}) = (\phi(\mathbf{x}), \mathbf{y}, \phi(\mathbf{s}))$, a 0–1 MIP formulation of LS-C is obtained with projected demands $\hat{\mathbf{d}} = \phi(\mathbf{d})$, transformed production capacities $\hat{\mathbf{C}} = \phi(\mathbf{C})$, and transformed per unit production and holding costs $(\hat{\mathbf{p}}, \hat{\mathbf{h}}) = (\phi^{-1}(\mathbf{p}), \phi^{-1}(\mathbf{h}))$. The function $\phi(\cdot)$ provides a one-to-one correspondence between the data and feasible solutions of an instance of LS-C-G and the data and the feasible solutions of an instance of LS-C respectively. Furthermore, the objective function values of any two corresponding solutions $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ and $(\hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{s}})$ will be the same.

4 Implications of the transformation

Variants on the general lot sizing problem (LS) include those whose costs satisfy the Wagner-Whitin condition (WW) or are discrete lot sizing problems (DLS,DLSI). Each of these problems may have varying production capacities (C), constant capacities (CC) or be altogether uncapacitated (U). Extensions to these basic variants that are relevant to this discussion on inventory gains include problems with backlogging (B) and multilevel production (NL). Consequently, the problems PROB-CAP-VAR discussed in this paper will be restricted to those in which $\mathrm{PROB} \in \{\mathrm{LS}, \mathrm{WW}, \mathrm{DLS}, \mathrm{DLSI}\}$, $\mathrm{CAP} \in \{\mathrm{C}, \mathrm{CC}, \mathrm{U}\}$, and $\mathrm{VAR} \subseteq \{\mathrm{B}, \mathrm{G}, \mathrm{NL}\}$. For more details on this classification and a comprehensive list of lot sizing references the reader is referred to Wolsey (2002).

Important simplifications to lot sizing problems arise when production and inventory costs satisfy the Wagner-Whitin condition. Specifically, it is always optimal to produce as late as possible among those periods in which a setup is performed. An instance of LS-C with per unit production and holding costs $(\hat{\mathbf{p}}, \hat{\mathbf{h}})$ is said to have Wagner-Whitin costs if $\hat{p}_t + \hat{h}_t \geq \hat{p}_{t+1}$ for all periods $t \in N \setminus \{n\}$. For LS-C-G with per unit costs $(\mathbf{p}, \mathbf{h})$, this condition becomes $p_t + h_t \geq \mu_t p_{t+1}$ for all periods $t \in N \setminus \{n\}$. The cost of producing and holding one item in period t is at least the cost of producing μ_t units in period $t+1$. This condition is consistent with the transformation given in Section 3 and holds regardless of which distinguished period the demand is projected to.

4.1 Capacitated lot sizing

The four varying capacity problems PROB-C are known to be NP-hard (see, e.g., Bitran and Yanasse, 1982). Consequently, the problems PROB-C-G are NP-hard as they each have PROB-C as a special case. The problems PROB-C are said to have constant capacity if $C_t = C$ for all periods $t \in N$. The four constant capacity problems PROB-CC are solvable in polynomial time (Florian and Klein, 1971; van Hoesel and Wagelmans, 1996; Van Vyve, 2003). However, it will be shown that the constant capacity problems with inventory gains (PROB-CC-G) are NP-hard.

Consider an instance of a problem PROB-C-G with varying capacities $\hat{\mathbf{C}}$. Arbitrarily choose a distinguished period $q \in N$ and suppose that $\mathbf{C} = \phi^{-1}(\hat{\mathbf{C}})$ where $C_t = C = \hat{C}_q$ for all periods $t \in N$. Let $m_t = \hat{C}_t / C > 0$ for each period $t \in N$. It follows from the definition of the function $\phi(\cdot)$ in Section 3, that the multipliers for which this statement will be true are unique. These multipliers can be determined using back and forward substitution to solve the two triangular systems $\mu_{tq} = m_t$ for periods $t = 1, \ldots, q-1$, and $\mu_{qt} m_t = 1$ for periods $t = q+1, \ldots, n$, respectively. Given the multipliers μ, the function $\phi(\cdot)$ is defined. Let $(\mathbf{d}, \mathbf{p}, \mathbf{h}) = (\phi^{-1}(\hat{\mathbf{d}}), \phi(\hat{\mathbf{p}}), \phi(\hat{\mathbf{h}}))$. Similarly, the function $\phi(\cdot)$ provides a one-to-one mapping between a feasible solution $(\hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{s}})$ to the problem PROB-C and a feasible solution $(\mathbf{x}, \mathbf{y}, \mathbf{s}) = (\phi^{-1}(\hat{\mathbf{x}}), \mathbf{y}, \phi^{-1}(\hat{\mathbf{s}}))$ to the problem PROB-CC-G. Furthermore, the objective function values of these two solutions will be the same. Therefore, a problem PROB-C-G has an optimal solution with objective function value at most K, if and only if the corresponding problem PROB-CC-G has an optimal solution with objective function value at most K.

The above provides reductions of the problems PROB-CC-G from the problems PROB-C and proves that the four constant capacity problems with inventory gains PROB-CC-G are NP-hard.

Theorem 1. *The problems LS-CC-G, WW-CC-G, DLS-CC-G and DLSI-CC-G are NP-hard.* □

Now consider, in a sense, the reverse situation in which after applying the transformation in Section 3 to the problem PROB-C-G, the resulting problem PROB-C has constant capacities $\hat{C}_t = \hat{C}$ for all periods $t \in N$. Given the distinguished period $q \in N$ into which the demand is to be projected, the multipliers μ must be such that $\hat{C}\mathbf{e} = \phi(\mathbf{C})$ where $\hat{C} = C_q$ and $\mathbf{e}$ is the unit n-vector.

4.2 Uncapacitated lot sizing

The problem PROB-C-G is said to be uncapacitated if

$$C_t \geq \sum_{\tau=t}^{n} \frac{d_\tau}{\mu_{t\tau}}$$

for all periods $t \in N$. The uncapacitated problems PROB-U have been extensively studied in the literature.

The implications of the transformation given in Section 3 on the uncapacitated lot sizing problem with inventory gains (LS-U-G) are discussed. These results can be obtained by applying the transformation in reverse to existing results in the literature for the problem LS-U. Direct proofs of the structure of the optimal solutions to LS-U-G and the linear inequality description of the convex hull of the set of feasible solutions are given in Waterer (2001).

The structure of the optimal solutions to LS-U-G generalizes the result of Zangwill (1968) for LS-U.

Proposition 1. *There exists an optimal solution to LS-U-G such that*

1. $s_{t-1}x_t = 0$ *for all* $t \in N$*, and*
2. *if* $x_t > 0$*, then*

$$x_t = \sum_{\tau=t}^{t+k} \frac{d_\tau}{\mu_{t\tau}}$$

for some $t \in N$ *and* $k \geq 0$. □

The $O(n^2)$ and $O(n \log n)$ algorithms for solving LS-U of Wagner and Whitin (1958) and Wagelmans, van Hoesel and Kolen (1992), respectively, can be used to solve LS-U-G. Furthermore, there exist $O(n^2)$ constraint and $O(n^2)$ variable facility location (Krarup and Bilde, 1977), and $O(n)$ constraint and $O(n^2)$ variable shortest path (Eppen and Martin, 1987) reformulations of LS-U that can be used to reformulate LS-U-G. In the case of the facility location reformulation of LS-U-G, the transportation costs in the objective function exhibit the structure that Krarup and Bilde (1977) showed to be sufficient for the linear programming relaxation to have an optimal solution with $\mathbf{y}$ integer.

The linear inequality description of the convex hull of the set of feasible solutions to LS-U-G generalizes the result of Barany, Van Roy and Wolsey (1984) for LS-U.

Theorem 2. *The inequalities*

$$
\begin{aligned}
&\mu_{t-1}s_t + x_t = d_t + s_t, && t \in N \\
(8)\quad &\sum_{t\in S}\mu_{tl}x_t \le \sum_{t\in S}\left(\sum_{\tau=t}^{l}\mu_{\tau l}d_\tau\right)y_t + s_l, && l \in N,\ S \subseteq L = \{1,\ldots,l\} \\
&y_t \le 1, && t \in N \\
&s_0 = s_n = 0 && \\
&y_1 = 1 && \\
&x_t, y_t, s_t \ge 0, && t \in N
\end{aligned}
$$

describe the convex hull of the set of feasible solutions to LS-U-G. □

The LS-U-G (l, S) inequalities (8) can be separated in $O(n \log n)$ time using a similar approach to that of Barany et al. (1984) for the LS-U (l, S) inequalities.

LS-U-G is a fixed charge uncapacitated generalized network flow problem in which the feasible solutions correspond to flows in the underlying generalized network. Thus, LS-U-G is a network design problem in which one must choose which production arcs to open and then find a minimum cost flow through the resulting network. An alternate proof of the structure of the optimal solutions to this problem follows from the observation that this generalized network flow problem is a gain-free pre-Leontief substitution flow problem and therefore has acyclic extreme solutions (see, e.g., Jeroslow, Martin, Rardin and Wang, 1992).

Rardin and Wolsey (1993) introduced a family of dicut collection inequalities that completely describe the projection onto the original variables of the linear programming relaxation of the multicommodity formulation of single source fixed charge uncapacitated network flow problems. Hassin (1981) gives a necessary and sufficient condition for a feasible flow in a generalized network which could be used to determine a family of generalized dicut collection inequalities that are valid for single source fixed charge uncapacitated generalized network flow problems.

4.3 Lot sizing with backlogging

LS-C with backlogging (LS-C-B) is a variant of LS-C in which stock may be backlogged. LS-C-G with backlogging (LS-C-{B,G}) is a generalization of LS-C-B to one in which stock, either in inventory or in backlog, is not conserved. In the same way that there is a multiplier $\mu_t > 0$ associated with the stock in inventory at the end of period t, there is a nonzero rational multiplier $\eta_t > 0$ associated with the stock in backlog at the end of period t. If one unit of stock in backlog is available at the beginning of period $t+1$, then η_t units of stock in backlog are available in period t. Again, it is assumed that the transformation that the stock undergoes in each period is independent of the period in which the item was produced. This problem is a generalization of LS-C-B in the sense that if every multiplier has value one, then LS-C-{B,G} reduces to LS-C-B.

Let b_t denote the per unit backlogging cost for period $t \in N$. LS-C-{B,G} can be formulated as a 0–1 MIP with variables $(\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{r})$ indexed by the periods $t \in N$. The variable r_t denotes the amount of stock in backlog at the end of period t. The 0–1 MIP formulation of LS-C-{B,G} that is considered is the following.

$$(9) \quad \text{minimize} \sum_{t\in N} p_t x_t + \sum_{t\in N} f_t y_t + \sum_{t\in N} h_t s_t + \sum_{t\in N} b_t r_t$$

$$
\begin{array}{lll}
(10) & \text{subject to } \mu_{t-1}s_{t-1} + \eta_t r_t + x_t = d_t + s_t + r_{t-1}, & t \in N \\
(11) & x_t \le C_t y_t, & t \in N \\
(12) & s_0 = s_n = 0,\ r_0 = r_n = 0, & \\
(13) & y_t \le 1, & t \in N \\
(14) & x_t,\ y_t,\ s_t,\ r_t \ge 0, & t \in N \\
(15) & y_t \text{ integer}, & t \in N
\end{array}
$$

The objective function (9) contains per unit production, inventory and backlogging costs, and a fixed setup cost. The balance constraints (10) ensure flow balance subject to the gains on the stock in inventory and in backlog. The setup constraints (11) ensure that the item is produced only when set up and that production does not exceed capacity. The stock constraints (12) ensure that there is no beginning or ending inventory or backlog. Note that if $\mu_t = \eta_t = 1$ for all periods $t \in N$, then (9)–(15) is a 0–1 MIP formulation of LS-C-B.

An instance of LS-C-B with per unit production, holding and backlogging costs $(\hat{\mathbf{p}}, \hat{\mathbf{h}}, \hat{\mathbf{b}})$ is said to have Wagner-Whitin costs if both $\hat{p}_t + \hat{h}_t \ge \hat{p}_{t+1}$ and $\hat{p}_{t+1} + \hat{b}_t \ge \hat{p}_t$ for all periods $t \in N \setminus \{n\}$. For LS-C-{B,G} with per unit costs $(\mathbf{p}, \mathbf{h}, \mathbf{b})$, these conditions become $p_t + h_t \ge \mu_t p_{t+1}$, and $p_{t+1} + b_t \ge \eta_t p_t$, for all periods $t \in N \setminus \{n\}$. These conditions hold regardless of which distinguished period the demand is projected to.

The introduction of backlogging gains to the problem introduces directed cycles to the underlying generalized network. These cycles are not necessarily gain-free. That is, there may exist an interval in time $\{k, \dots, l-1\} \subseteq N$ for which the net gain around the corresponding cycle in the underlying network is $\mu_{kl}\eta_{kl} > 1$. Such a cycle is said to be flow generating. Fortunately, the problem remains bounded for all objective functions due to the finite demand and constraint (12). If the underlying network contains no flow-generating cycles, then it can be transformed into an equivalent "lossy" network in which the modified gains are at most one (Truemper, 1977).

It is not clear that the transformation in Section 3 can be extended to accommodate backlogging in the general case. If $\eta_t = 1/\mu_t$ for all periods $t \in N$, then the transformation in Section 3 can be applied directly. Similarly, consider the case in which either, the multipliers $\mu_t\eta_t \le 1$ for all periods $t \in N$, or the multipliers $\mu_t\eta_t \ge 1$ for all periods $t \in N$, but not both. Suppose that the problem has been scaled by multiplying the constraints (10) and (11) by $1/\eta_{tq}$ if $t \le q$ and by η_{qt} otherwise, in the former case, and by μ_{tq} if $t \le q$ and by $1/\mu_{qt}$ otherwise, in the latter case. With an appropriate change of variables, constraints (10) can be written as

$$\hat{s}_{t-1} + \mu_t\eta_t\hat{r}_t + \hat{x}_t = \hat{d}_t + \hat{s}_t + \hat{r}_{t-1}, \quad t \in N$$

in the former case, and as

$$\mu_{t-1}\eta_{t-1}\hat{s}_{t-1} + \hat{r}_t + \hat{x}_t = \hat{d}_t + \hat{s}_t + \hat{r}_{t-1}, \quad t \in N$$

in the latter. By treating $\hat{r}_t \geq 0$, respectively $\hat{s}_t \geq 0$, as a slack variable, and making the substitution,

$$\sigma_l = \sum_{t=1}^{l-1}(1-\mu_t\eta_t)\hat{s}_t + \hat{s}_l \geq 0, \quad \text{or} \quad \gamma_l = \sum_{t=1}^{l-1}(\mu_t\eta_t - 1)\hat{r}_t + \mu_l\eta_l\hat{r}_l \geq 0$$

respectively, for all periods $l \in N$, nested single node flow model relaxations are obtained (see, e.g., Padberg, Van Roy and Wolsey, 1985).

$$\begin{array}{llcll}
\sum_{t=1}^{l}\hat{x}_t \leq \sum_{t=1}^{l}\hat{d}_t + \sigma_l, & l \in N & \qquad & \gamma_l + \sum_{t=1}^{l}\hat{x}_t \geq \sum_{t=1}^{l}\hat{d}_t, & l \in N \\
\hat{x}_t \leq \hat{C}_t y_t, & t \in N & & \hat{x}_t \leq \hat{C}_t y_t, & t \in N \\
y_t \leq 1, & t \in N & & y_t \leq 1, & t \in N \\
\hat{x}_t, y_t, \sigma_t \geq 0, & t \in N & & \hat{x}_t, y_t, \gamma_t \geq 0, & t \in N \\
y_t \text{ integer}, & t \in N & & y_t \text{ integer}, & t \in N
\end{array}$$

Note that in these relaxations, no assumptions are made regarding the relationship between successive σ_l, respectively γ_l, variables. Valid inequalities derived for these relaxations will lead to valid inequalities for LS-C-{B,G} (see, e.g., Barany et al., 1984; Pochet and Wolsey, 1994; Pochet, 1988; Louveaux and Wolsey, 2003).

4.4 Lot sizing with multilevel production

LS-C with multilevel production (LS-C-NL) is a variant of LS-C in which intermediate items, or components, are required to produce a single end item according to a prescribed product structure. LS-C-G with multilevel production (LS-C-{G,NL}) is a generalization of the problem LS-C-NL to one in which the stock in inventory of each component is not conserved. Hence, there is a nonzero rational multiplier $\mu_t^j > 0$ associated with the stock in inventory of component j at the end of period t. Again, it is assumed that the transformation that the stock undergoes in each period is independent of the period in which the component was produced. This problem is a generalization of LS-C-NL in the sense that if every multiplier has value one, then LS-C-{G,NL} reduces to LS-C-NL.

Let the index set J denote the set of components. Let component $1 \in J$ denote the end item. Let p_t^j and h_t^j denote the per unit production and holding costs, and f_t^j denote the fixed setup cost, for component $j \in J$ in period $t \in N$. Let S_j denote the set of immediate successors of component j and ρ^{jk} denote the number of units of component j that are required to produce one unit of component $k \in S_j$ for all components $j \in J \setminus \{1\}$. The multipliers ρ can be interpreted as production gains. The multiplier ρ^{jk} and the gain on the corresponding production arc in the underlying network are inversely related. Without loss of generality, let $\mu_0^j = 1$ for all components $j \in J$.

LS-C-{G,NL} can be formulated as a 0–1 MIP with variables $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ indexed by the components $j \in J$ and the periods $t \in N$. The variable x_t^j denotes the amount of component j produced in period t. The variable $y_t^j = 1$ indicates that component j is set up for production in period t and $y_t^j = 0$ otherwise.

The variable s_t^j denotes the amount of stock in inventory of component j remaining at the end of period t. The 0–1 MIP formulation of LS-C-{G,NL} that is considered is the following.

$$\text{(16)}\quad \text{minimize} \sum_{j\in J}\sum_{t\in N} p_t^j x_t^j + \sum_{j\in J}\sum_{t\in N} f_t^j y_t^j + \sum_{j\in J}\sum_{t\in N} h_t^j s_t^j$$

$$\begin{array}{lll}
\text{(17)} & \text{s.t. } \mu_{t-1}^j s_{t-1}^j + x_t^j = \sum_{k\in S_j} \rho^{jk} x_t^k + s_t^j, & j \in J\setminus\{1\},\ t\in N \\
\text{(18)} & \mu_{t-1}^1 s_{t-1}^1 + x_t^1 = d_t + s_t^1, & t \in N \\
\text{(19)} & x_t^j \le C_t^j y_t^j, & j\in J,\ t\in N \\
\text{(20)} & s_0^j = s_n^j = 0 & j \in J \\
\text{(21)} & y_t^j \le 1, & j\in J,\ t\in N \\
\text{(22)} & x_t^j,\ y_t^j,\ s_t^j \ge 0, & j\in J,\ t\in N \\
\text{(23)} & y_t^j \text{ integer}, & j\in J,\ t\in N
\end{array}$$

The objective function (16) contains per unit production and inventory costs, and a fixed setup cost. The balance constraints (17) and (18) ensure flow balance subject to the product structure and the gains on the stock in inventory. The setup constraints (11) ensure that a component is produced only when set up and that production does not exceed capacity. The stock constraints (12) ensure that there is no beginning or ending inventory. Note that if $\mu_t^j = 1$ for all periods $t \in N$, then (16)–(23) is a 0–1 MIP formulation of LS-C-NL.

A transformation similar to that in Section 3 can be applied. The problem is rescaled by multiplying the constraints corresponding to component j in period t from the constraints (17)–(19) by μ_{tq}^j if $t \le q$ and by $1/\mu_{qt}^j$ otherwise. With an appropriate change of variables an instance of LS-C-NL is obtained with time dependent production multipliers $\hat{\rho}_t^{jk} = \mu_{tq}^j \rho^{jk}/\mu_{tq}^k$ if $t \le q$ and $\hat{\rho}_t^{jk} = \mu_{tq}^k \rho^{jk}/\mu_{tq}^j$ otherwise. By introducing echelon stock variables (Clark and Scarf, 1960)

$$e_t^j = \hat{s}_t^j + \sum_{k\in S_j} \hat{\rho}_t^{jk} e_t^k \ge 0,$$

the problem can be reformulated so that there is a single item lot sizing problem for each component. The echelon stock of e_t^j of component j in period t is the total number of component j in stock anywhere in the system, in particular, in components appearing further on in the production process.

5 Concluding remarks

The single item lot sizing problem with inventory gains LS-C-G was introduced. This problem generalizes the classical single item capacitated lot sizing problem LS-C to one in which stock is not conserved. Stock in inventory undergoes a transformation in each period that is independent of the period in which the item was produced. A 0–1 MIP formulation of the problem was given in Section 2. The observation that if the demand in each period is projected to a distinguished period, then an instance of PROB-C-G can be polynomially transformed into an instance of the PROB-C was discussed in Section 3. As a result, existing results in the literature can be applied to PROB-C-G. In Section 4.1 it was shown that the polynomially solvable problems PROB-CC become NP-hard when stock is not conserved. The problems PROB-U-G were discussed in Section 4.2. It is not clear how to extend the transformation given in Section 3 in the general case of the problem PROB-C-{B,G}. However, the special case in which $\eta_t = 1/\mu_t$ for all periods $t \in N$, and the special case in which either the multipliers $\mu_t\eta_t \leq 1$ for all periods $t \in N$, or the multipliers $\mu_t\eta_t \geq 1$ for all periods $t \in N$, but not both, were discussed in Section 4.3. In Section 4.4, an echelon stock reformulation of the problem PROB-C-{G,NL} was described that involved time dependent production multipliers.

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