Some remarks about the transformation of Charnes and Cooper

by

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Abstract

In this paper we extend in a simple way the transformation of Charnes and Cooper to the case where the functional ratio to be considered are of similar polynomial.

Keywords: programming, transformation, Charnes and Cooper, fractional programming.

Introduction

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Charnes and Cooper in [3] have introduced an important transformation. This regards the equivalence about programming with linear fractional functional and linear ones. Members of this class have been encountered in a variety of contexts. One such occurrence [3] involved situations in which the more usual sensitivity analyses were extended to problems involving plans for optimal data changes. In these instances, linear programming inequalities were to be considered relative to a functional formulated as a ratio of two variables wherein one variable, in the numerator, represented the volume changes that might attend the possible variations of a particular cost coefficient. Another example was dealt with by M. Klein in [8]. To handle the problem of the fractional function, Klein applied a square-root transformation (which be attributed to C. Derman [5]) in order to effect a reduction to an equivalent linear programming problem. Finally, a special instance of our general case was treated by J. R. Isbell and W. H. Marlow in their article on "Attrition Games, "[7]. In considering a ratio, Isbell and Marlow were able to establish a convergent iterative process which involved replacing the ratio by the problem of optimizing a sequence of different linear functionals. The linear functional at any stage in the iterations was determined by optimization of the linear functional at the preceding stage.

The objective of the present short paper is to extend the Charnes and Cooper transformation to some other problems where the functional involves are quotient of linear ones. As a particular case of it we get the result that a general non-linear problem is equivalent to a linear one.

General Fractional Models

Consider a general class of fractional models arising in programming, which are rendered in the following form:

$$(1.1) \qquad \begin{aligned} Ax &\le b \\ x &\ge 0 \end{aligned}$$

where A is an mxn matrix and b is an $mx1$ vector so that the two sets of constants for the constrains are related by the $nx1$ vector of variables, x.

It is assumed, unless otherwise noted, that the constraints of *(1.1)* are regular, so that the solution set

$$F_1 = \{ x \in R^n : Ax \leq b,\ x \geq 0 \}$$

in nonempty and bounded.

Here R indicates the reals.

The general class of fractional models to be considered in this paper is of the form

$$(P_1) \qquad \min f(x) = \frac{\prod_{i=1}^{k} \left(c_i^T x + a_i\right) \left(c^T x + a\right)}{\prod_{i=1}^{k} \left(d_i^T x + b_i\right) \left(d^T x + b\right)}$$

subject to (1.1):

$$(F_1) \qquad \begin{aligned} Ax &\leq b \\ x &\geq 0 . \end{aligned}$$

Where c^T, c_i^T, d^T, d_i^T, are transpose of the $nx1$ vectors of coefficients, where $i : 0, \cdots, k$. If $k = 0$ we have the case studied by Charnes and Cooper [4].

By reason of simplicity we assume that for each $x \in F_1$, we have the last part of the denominator

$$\prod_{i=1}^{k} \left(d_i^T\ x + b_i\right) \left(d^T\ x + b\right) > 0 .$$

Thus since *f(x)* is a continuous function defined on a non-empty compact set (F_1), it reaches its minimum. That is to say the problem (P_1) is solvable.

Following the ideas presented by Charnes and Cooper in [4], we will prove that the guarantee the equivalence between (P_1) and the programming program with $n+1$ variables $y_1, \ldots, y_n, z$ and the feasible set (F_2):

$$(P_2) \qquad \min \bar{f}(y,z) = \frac{\prod_{i=1}^{k} \left(c_i^T y + a_i\ z\right) \left(c^T y + a\ z\right)}{\prod_{i=1}^{k} \left(d_i^T y + b_i\ z\right)}$$

subject to

$$(F_2) \qquad \begin{aligned} &A\,y - z\,b \le 0 \\ &d\;\; y - b\;\; z = 1 \\ &y \ge 0, z \ge 0. \end{aligned}$$

We now proceed to prove:

Lemma 1: If $(y, z) \in F_2$, then $z > 0$.

Proof: Assume that $z = 0$, then $A\,y \le 0$. Take $\hat{x} \in F_1$ and $m > 0$ and arbitrary, we obtain

$$A(\hat{x} + m\,y) \le A\,\hat{x} \le b.$$

But $\hat{x} \ge 0$ and $y \ge 0$, F_1 contains the half straight line $\{\hat{x} + l\;y \,/\, l\; \ge 0\}$ if $y \ne 0$. This is contradictory with the fact that F_1 is bounded. Then $y = 0$. But this last fact provides a contradiction with the second restriction of (P_2). Since $z \ge 0$ and $z \ne 0$, we have $z > 0$. (Q. E. D.).

As an immediate consequence of the previous fact, we have that, if $(y, z) \in F_2$ then $x : {}^{y}\!/_{z} \in F_1$.

Lemma 2: If $x \in F_1$, let us define

$$y := \frac{x}{\left(d^T x + b\right)} \text{ and } z := \frac{1}{\left(d^T x + b\right)}$$

then $(y, z) \in F_2$.

Proof: Since $\left(d^T x + b\right) > 0$, we get $y \ge 0$ and $z \ge 0$ for each $x \in F_1$.

On the other hand, if we divide the inequality $A\,x \le b$ by $\left(d^T x + b\right)$ we obtain

$$A \frac{x}{\left(d^T x + b\right)} \le b \frac{1}{\left(d^T x + b\right)}$$

which is equivalent to

$$A\,y - z\,b \le 0\,.$$

Finally

$$d^T y + b\;\; z = d^T \frac{x}{d^T x + b} + b\;\; \frac{1}{q^T x + b} = 1$$

hence $(y, z) \in F_2$. (Q. E. D.).

Theorem 3: P_1 and P_2 are solvable. Moreover, if $\bar{x}$ is an optimal solution of P_1,

$$\left(\frac{\bar{x}}{d^T x + b}, \frac{1}{d^T \bar{x} + b} \right)$$

is an optimal solution of P_2, and reciprocally, if $(\bar{y}, \bar{z})$ is an optimal solution of P_2, then $\bar{x} = \bar{y} / \bar{z}$ is an optimal solution of P_1.

Proof: We know that P_1 is solvable. Let $\bar{x}$ an optimal solution of P_1. Giving a pair $(y, z) \in F_2$ then $z > 0$ and $x = y / z \in F_1$ by Lemma 1 and the comment after it. Since we assume that $f(x) \geq f(\bar{x})$ for each $x \in F_1$, if we now associate to $\bar{x}$ the vector

$$(\bar{y}, \bar{z}) = \left(\frac{\bar{x}}{d^T \bar{x} + b}, \frac{1}{d^T \bar{x} + b} \right) \in F_2,$$

we have

$$\bar{f}(y,z) = \frac{\prod_{i=1}^{k} \left(c_i^T y + a_i\, z\right)}{\prod_{i=1}^{k} \left(d_i^T y + b_i\, z\right)} \left(c^T y + a\, z\right) = \frac{\prod_{i=1}^{k} \left(c_i^T y + a_i\, z\right) \left(c^T y + a\, z\right)}{\prod_{i=1}^{k} \left(d_i^T y + b_i\, z\right) \left(d^T y + b\, z\right)}$$

$$= \frac{\prod_{i=1}^{k} \left(c_i^T x + a_i\right) \left(c^T x + a\right)}{\prod_{i=1}^{k} \left(d_i^T x + b_i\right) \left(d^T x + b\right)} \geq \frac{\prod_{i=1}^{k} \left(c_i^T \bar{x} + a_i\right) \left(c^T \bar{x} + a\right)}{\prod_{i=1}^{k} \left(d_i^T \bar{x} + b_i\right) \left(d^T \bar{x} + b\right)}$$

$$= \frac{\prod_{i=1}^{k} \left(c_i^T \bar{y} + a_i\, \bar{z}\right) \left(c^T \bar{y} + a\, \bar{z}\right)}{\prod_{i=1}^{k} \left(d_i^T \bar{y} + b_i\, \bar{z}\right) \left(d^T \bar{y} + b\, \bar{z}\right)} =$$

$$= \frac{\prod_{i=1}^{k} \left(c_i^T \bar{y} + a_i\, \bar{z}\right) \left(c^T \bar{y} + a\, \bar{z}\right)}{\prod_{i=1}^{k} \left(d_i^T \bar{y} + b_i\, \bar{z}\right)} = \bar{f}(\bar{y}, \bar{z})$$

Then we have proved that $(\bar{y}, \bar{z})$ is an optimal solution of P_2. Therefore P_2 is also solvable.

Assume now that $(\bar{y},\bar{z})$ is an optimal solution of P_2. Consider a vector $x \in F_1$ and let the transformation

$$y = \frac{x}{d^T x + b} \text{ and } z = \frac{1}{d^T x + b}.$$

Applying the result after the Lemma 1, we have

$$f(x) = \bar{f}(y,z) \geq \bar{f}(\bar{y},\bar{z}) = f(\bar{x})$$

which it says that $\bar{x}$ is a solution of P_1. (Q. E. D.).

Our result it turns out to be of important interest in the case of $k = 0$. In this instance it reduces to the important transformation of Charnes and Cooper [4].

In more general cases, it reduces the degree of complexity of the polynomial.

Conclusions

Here we have obtained an extension of the Charnes and Cooper transformation to the case when we have a polynomial form in the numerator and the denominator. The case when some d_i and b_i are equal it reduces the complexity in more degrees in the denominator. Furthermore, there exist some varieties of situations that the simplicity of the new programs appears naturally.

For example the case where $d_i = d$ for $i \in S \subset N$. Then in such a case we obtain good reduction. Moreover if all $c_i = 0$ and the $d_i = d$, $i = 1,\ldots,k$ we have obtained an interesting result, since the transformed program is linear.

For further comments we subject the reader to refer to the original paper [4]. There are also further extensions which we shall treat elsewhere.

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