New Inequalities for Finite and Infinite Group Problems from Approximate Lifting

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May 17, 2006

Abstract

In this paper, we derive new families of piecewise linear facet-defining inequalities for the finite group problem and extreme inequalities for the infinite group problem using approximate lifting. The new valid inequalities for the finite group problem are two- and three-slope facet-defining inequalities as well as the first family of four-slope facet-defining inequalities. The new valid inequalities for the infinite group problem are families of two- and three-slope extreme inequalities, including nontrivial inequalities that are not continuous. These new inequalities not only illustrate the diversity of strong inequalities for the finite and infinite group problems, but also provide a large variety of new cutting planes for solving integer and mixed-integer programming problems.

1 Introduction

The use of cutting planes is extremely valuable in solving integer and mixed integer programs. By adding cuts to an integer programming formulation, the resulting linear programming relaxation becomes tighter and the computational time needed to solve the problem to optimality is usually decreased as compared to pure branch-and-bound algorithms. The quality of the cuts generated and the effort required to generate them are key factors to overall computational savings. For this reason, there have been many polyhedral studies of practical problems whose mixed-integer programming formulations have specific structures. For unstructured problems or problems with unknown structure, generating cuts is typically more difficult. The well-known general approaches proposed to generate cuts for such problems include lifting techniques, disjunctive methods and group-theoretic approaches. We refer to Balas [2] for a description of the disjunctive approach and to Louveaux and Wolsey [16] for a description of the lifting approach.

Group-theoretic approaches for the generation of cutting planes in integer and mixed-integer programming were introduced by Gomory [8]. These approaches require the study of the convex hull of solutions to modular relaxations

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of integer programs. Strong valid inequalities and/or facet-defining inequalities can be characterized for the so-called group relaxations. These inequalities can then be used as cutting planes for general integer and mixed-integer programs. We call these cutting planes *group cuts*.

Even though the group approach can be applied with multiple constraints and various types of groups, it typically utilizes just a single constraint (a knapsack relaxation) of the original formulation and either a finite cyclic group or the group of real numbers modulo 1. We refer to the former as the *finite group problem* and the latter as the *infinite group problem*. A thorough study of these problems can be found in Gomory and Johnson [9, 10] and Johnson [14].

For the finite group problem, the convex hull of solutions to the modular relaxation is polyhedral and its facets are in one-to-one correspondance with the extreme rays of a particular polyhedral cone. Thus, generating facet-defining inequalities for the finite group problem requires the solution of a linear program over the given polyhedral cone. Because the number of inequalities in the description of the cone is an increasing function of the size of the group, this approach is practical only for small groups. Unfortunately, the size of the group used in practice is typically large as it is determined by the determinant of the basis of the current LP relaxation, and therefore make the solution of an LP prohibitive, especially if multiple rounds of cut generation are used. This is the reason that many different families of inequalities were explicitely obtained from the description of the polyhedral cone; see Gomory [8] and Gomory and Johnson [9, 10]. Also see Araóz et al. [1] for a more recent study.

The computational difficulties associated with using finite groups motivated Gomory and Johnson to introduce a variant of the approach [9, 10]. In this variant, they use a group with an infinite number of elements consisting of all the real numbers taken modulo 1. The main advantage is that the inequalities derived from the infinite group problem can be used easily for all integer programs. One of the major difficulties is that deriving extreme inequalities for the infinite group problem is much harder than for the finite group problem because there is no complete and nice characterization of their extreme inequalities. To date, the main method used to obtain extreme inequalities for the infinite group problem is through the identification of families of inequalities that remain facet-defining for the finite group problem when the group size becomes large. The fact that two-slope inequalities for the finite group problem are facet-defining led Gomory and Johnson to prove that all continuous two-slopes inequalities are extreme for the infinite group problem [10, 11]. Similarly, Gomory and Johnson used certain three-slope facet-defining inequalities for the finite group problem to derive the first family of 3-slope extreme inequalities for the infinite group problem in [11]. More recent studies of inequalities for the infinite group problem include Dash and Günlük [4], where it is shown that some two-slope inequalities can be obtained using two steps of the mixed integer rounding (MIR) procedure, and Dey et al. [5], where it is proven that certain families of discontinuous inequalities can be extreme for the infinite group problem.

Richard et al. [18] showed that many group cuts can be derived through an approximate lifting procedure starting from a rounding inequality on a single variable. This result is interesting as it yields a procedure to identify potentially strong inequalities for group problems without relying on the cone description. It was shown in [18] that representatives of all the well-known families of continuous facets for the finite group problem can be derived in this way. The approach was also used to obtain strong discontinuous inequalities, which were later proven to be extreme for the infinite group problem; see Dey et al. [5].

In this paper, we use the approximate lifting scheme developed in [18] to derive new families of strong inequalities for the group problem and integer programs. In Section 2, we present inequalities for integer programs that can obtained through the lifting procedure of [18]. We then focus on a family of continuous piecewise linear lifting functions that have n independent parameters (CPL_n functions) and characterize the strongest inequalities in this family. In Section 3, we study a specific set of CPL_3 functions. We derive simple analytical forms for all eighteen classes of undominated inequalities in this family. In Section 4, we show how these inequalities can be converted into valid inequalities for the finite and infinite group problems. For each class, we derive a lower bound on the dimension of the face it induces on the finite group polyhedron and give conditions when these faces are facets. Then, we present conditions under which these inequalities are extreme for the infinite group problem. We conclude in Section 5 with some remarks and future research directions.

The main contributions of this paper can be summarized as follows. First, we theoretically validate the usefulness of the approximate lifting scheme presented in [18] by showing that the inequalities generated from well-chosen CPL_3 functions correspond to strong inequalities for group problems. Second, we introduce a large variety of new inequalities for group problems. They include the first family of four-slope inequalities for the finite group problem, new families of three-slope extreme inequalities for the infinite group problem, and several discontinuous functions for the infinite group problem. This is important because very few inequalities are known to be extreme for the infinite group problem. Third, the new extreme inequalities for the infinite group problem give a new set of strong and easy-to-apply inequalities for solving unstructured integer and mixed-integer programs.

2 Strong inequalities from approximate lifting

In this section, we present inequalities that are derived using the approximate lifting procedure we proposed in [18]. Proofs for all results in this section can be found in [18].

The objective of the lifting procedure is to generate strong valid inequalities for mixed integer sets of the form

$$S = \{(x, y) \in \mathbb{Z}^m_+ \times [0, 1]^n \mid \sum_{i\in M} a_i x_i + \sum_{j\in N} b_j y_j \leq a_0\}, \tag{1}$$

where $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$, $a_i \in \mathbb{Z}$ for $i \in M \cup \{0\}$ and $b_j \in \mathbb{R}$ for $j \in N$. It is assumed that $a_i \neq 0$ for some $i \in M$ since otherwise, the structure of S is trivial. Without loss of generality, we assume that $a_1 \neq 0$. Finally, note that the continuous variables are assumed to be bounded. However, this assumption is usually not restrictive; see Richard et al. [17]. We also let $N^- = \{j \in N | b_j < 0\}$ be the indices of continuous variables with negative coefficients. Let PS be the convex hull of S.

For any fixed positive real number K, we define, for $i \in M \cup \{0\}$, $(q_i, r_i) \in \mathbb{Z} \times \mathbb{R}$ to be the unique pair of numbers such that $a_i = Kq_i + r_i$ with $0 \leq r_i < K$.

In the remainder of this paper, K is chosen such that $q_1 \neq 0$. This can be achieved by choosing sufficiently small K.

Our approximate lifting procedure utilizes lifting functions that satisfy certain properties, including superadditivity. Superadditivity is sufficient to guarantee that the resulting inequalities are valid. Here, we will consider only the following family of potential lifting functions.

Definition 1 *Let $K \in \mathbb{R}_+$ and $r_0 \in (0, K)$. Let $n \in \mathbb{Z}_+$, $z = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n_+$, and $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n_+$ be such that $\sum_{j=1}^n z_j = \frac{K - r_0}{2}$ and $\sum_{j=1}^n \theta_j = \frac{1}{2}$. A function $\phi(u)$ is said to be a $CPL_n(K; r_0; z; \theta)$ function if, when u is restricted in $[0, K)$,*

$$\phi(u) = \begin{cases} 0, & \text{if} \quad u \in [0, r_0], \\ \Theta_{i-1} + \frac{\theta_i}{z_i}(u - r_0 - Z_{i-1}), & \text{if} \quad u \in (r_0 + Z_{i-1}, r_0 + Z_i], \\ 1 - \Theta_i + \frac{\theta_i}{z_i}(u - K + Z_i), & \text{if} \quad u \in (K - Z_i, K - Z_{i-1}], \end{cases}$$

where $Z_0 = 0$, $Z_i = \sum_{j=1}^i z_j$, $\Theta_0 = 0$ and $\Theta_i = \sum_{j=1}^i \theta_j$ for $i = 1, \ldots, n$. For the sake of brevity, we call a $CPL_n(K; r_0; z; \theta)$ function a CPL_n function. □

An example of a CPL_3 function is illustrated in Figure 1. CPL_n functions are continuous, nondecreasing, and have $2n$ distinct intervals on $[0, K]$ over which they are linear. We denote these intervals as J_1, …, J_{2n}.

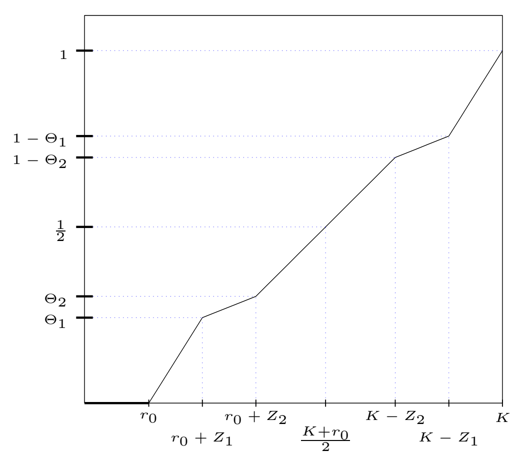

Figure 1: A CPL_3 function

Although CPL_n functions satisfy many desirable properties described in [18] that guarantee that the resulting inequalities are strong, they are not necessarily superadditive. The following result gives necessary and sufficient conditions for CPL_n functions to be superadditive.

Theorem 2 *A CPL_n function $\phi(u)$ is superadditive if and only if*

$$\begin{aligned} \phi(r_0 + Z_i) + \phi(r_0 + Z_j) &\leq \phi(2r_0 + Z_i + Z_j), && 0 \leq i \leq j \leq n-1, \\ \phi(r_0 + Z_i) + 1 &\leq \phi(r_0 + K + Z_i - Z_j) + \phi(r_0 + Z_j), && 0 \leq i, j \leq n-1, \\ \phi(r_0 + Z_i + Z_j) &\leq \phi(r_0 + Z_i) + \phi(r_0 + Z_j), && 0 \leq i \leq j \leq n-1. \end{aligned}$$

□

Because the expression of the function ϕ is linear in θ over all intervals J_i of the function when z is fixed, the set of function parameters θ that yield superadditive functions is a polyhedron, as described below.

Corollary 3 *Let $z \in \mathbb{R}^n_+$ be such that $Z_n = \frac{K-r_0}{2}$. Parameter θ defines a superadditive CPL$_n$ function if and only if θ belongs to the polyhedron*

$$\begin{aligned} P\Theta_n(z) := \{\theta \in \mathbb{R}^{n-1}_+ \mid & \Theta_i + \Theta_j \le \phi(2r_0 + Z_i + Z_j), && 0 \le i, j \le n-1, \\ & \Theta_i - \Theta_j \le \phi(r_0 + K + Z_i - Z_j) - 1, && 0 \le i, j \le n-1, \\ & \Theta_i + \Theta_j \ge \phi(r_0 + Z_i + Z_j), && 0 \le i, j \le n-1, \\ & \Theta_{n-1} \le \frac{1}{2} \ \}. \end{aligned}$$

□

Therefore, if θ is chosen in $P\Theta_n(z)$, the corresponding CPL$_n$ function can be used to generate strong valid inequalities for *PS* as shown in the following theorem.

Theorem 4 *If ϕ is a CPL$_n$ function with $z \in \mathbb{R}^n_+$, $Z_n = \frac{K-r_0}{2}$, and $\theta \in P\Theta_n(z)$, then*

$$q_1 x_1 + \sum_{i \in M \setminus \{1\}} (q_i + \phi(r_i)) x_i + \sum_{j \in N^-} \phi'_-(K) b_j y_j \le q_0. \tag{2}$$

is valid for PS, where $\phi'_-(K) = \lim_{\epsilon \to 0^+} \frac{1-\phi(K-\epsilon)}{\epsilon}$. □

Theorem 4 motivates the following definition.

Definition 5 *A CPL$_n$ inequality is a valid inequality (2) that is generated using a CPL$_n(K; r_0; z; \theta)$ function with $\theta \in P\Theta_n(z)$.* □

Observe, however, that not all the inequalities corresponding to points in $P\Theta_n(z)$ are equally strong. In particular, only the extreme points of the polyhedron $P\Theta_n(z)$ should be considered when developing CPL$_n$ inequalities, as the other inequalities can be generated from convex combinations of inequalities derived from the extreme points of $P\Theta_n(z)$.

Proposition 6 *$\theta = \alpha\hat{\theta} + (1-\alpha)\tilde{\theta}$ for some $\hat{\theta}, \tilde{\theta} \in P\Theta_n(z)$ if and only if $\phi_\theta = \alpha\phi_{\hat{\theta}} + (1-\alpha)\phi_{\tilde{\theta}}$ for some superadditive CPL$_n$ functions $\phi_{\hat{\theta}}, \phi_{\tilde{\theta}}$, where $0 \le \alpha \le 1$.* □

The CPL$_n$ functions and inequalities that correspond to the extreme points of $P\Theta_n(z)$ are extreme within their family. We call them CPL$_n$-extreme functions and inequalities. Note that these inequalities may not be extreme among all the valid inequalities derived from valid superadditive functions. Fortunately, we show in Section 4 that many of the CPL$_n$-extreme inequalities correspond to facets or high-dimensional faces of the master cyclic group problem and/or extreme inequalities of the infinite group problem. This result strengthens preliminary results obtained for $n = 2$ in [18] where it was shown that the four extreme points of $P\Theta_2(z)$ correspond to Gomory's mixed integer cut [7], Gomory and Johnson's two-slope inequality [10], Gomory and Johnson's three-slope inequality [11], and a new class of three-slope inequalities, which are all strong for group problems.

3 $CPL_3^=$-extreme functions and inequalities

In this section, we apply the results presented in Section 2 to study the polytope $P\Theta_3(z)$ with $z_1 = z_2$. We denote this polytope $P\Theta_3^=(z_1)$. The CPL_3 functions and inequalities that correspond to the extreme points of $P\Theta_3^=(z_1)$ are called $\text{CPL}_3^=$-extreme functions and inequalities. An automated code is under development to study the more complex $P\Theta_3(z_1, z_2)$ and is the basis for future work. However, assuming that $z_1 = z_2$ significantly decreases the number of cases, so they can be analyzed manually. We show next that there are only eighteen functions that correspond to extreme points of the polyhedron $P\Theta_3^=(z_1)$. We analyze these functions in more details in Section 4.

3.1 Inequality Description of $P\Theta_3^=(z_1)$

It follows from Corollary 3 that, for all values of z_1 such that $r_0 + 4z_1 \leq K$,

$$
\begin{aligned}
P\Theta_3^=(z_1) = \{(\theta_1, \theta_2) \in \mathbb{R}^2 \quad | \quad & \theta_2 \geq -\phi(r_0 - z_1), \\
& 2\theta_1 \leq \phi(2r_0 + 2z_1), \\
& 2\theta_1 + \theta_2 \geq \phi(r_0 + 3z_1), \\
& 2\theta_1 + \theta_2 \leq \phi(2r_0 + 3z_1), \\
& 2\theta_1 + 2\theta_2 \geq \phi(r_0 + 4z_1), \\
& 2\theta_1 + 2\theta_2 \leq \phi(2r_0 + 4z_1), \\
& \theta_1 - \theta_2 \geq 0, \\
& \theta_1 \geq 0, \theta_2 \geq 0\}.
\end{aligned}
$$

Observe that the inequality $\theta_1 + \theta_2 \leq \frac{1}{2}$ does not need to be included in the system as it is dominated by other inequalities in the description of $P\Theta_3^=(z_1)$. In order to obtain an explicit description of this polytope, we need to determine the intervals J_i that the points $r_0 - z_1$, $2r_0 + 2z_1$, $r_0 + 3z_1$, $2r_0 + 3z_1$, $r_0 + 4z_1$ and $2r_0 + 4z_1$ belong to so that we can determine the values of the function ϕ at these points. In fact, the function values at these points depend on the relative magnitude of the parameters r_0, z_1 and K. As a result, there are 53 different cases that are summarized in Table 1. The inequalities defining $P\Theta_3^=(z_1)$ typically take a different form for each one of these cases. Therefore, the extreme points of the polytope $P\Theta_3^=(z_1)$ are typically different from case to case. For the sake of brevity, we will only present the analysis of Subcase 1 of Case 1. The analysis for other cases is given in an online appendix and is also available by contacting the authors.

3.2 Analysis of Subcase 1 of Case 1

In this case, we have $0 < r_0 < z_1$ and $2r_0 + 6z_1 \leq K$. Now we can establish that

$$\begin{aligned} r_0 - z_1 &\in [-z_1, 0], \\ 2r_0 + 2z_1 &\in [r_0 + 2z_1, K - 2z_1], \\ r_0 + 3z_1 &\in [r_0 + 2z_1, K - 2z_1], \\ 2r_0 + 3z_1 &\in [r_0 + 2z_1, K - 2z_1], \\ r_0 + 4z_1 &\in [r_0 + 2z_1, K - 2z_1], \\ 2r_0 + 4z_1 &\in [r_0 + 2z_1, K - 2z_1]. \end{aligned}$$

Therefore, the constraints of $P\Theta_3^=(z_1)$ are

$$\begin{array}{rrlr} (-r_0 + z_1)\theta_1 + & (-z_1)\theta_2 \leq & 0 & (3) \\ (K + r_0 - 4z_1)\theta_1 + & (-K + 3r_0 + 4z_1)\theta_2 \leq & r_0 & (4) \\ (-K + r_0 + 2z_1)\theta_1 + & (-2z_1)\theta_2 \leq & -z_1 & (5) \\ (K + r_0 - 2z_1)\theta_1 + & (2r_0 + 2z_1)\theta_2 \leq & r_0 + z_1 & (6) \\ (-K + r_0)\theta_1 + & (-K + r_0)\theta_2 \leq & -2z_1 & (7) \\ (K + r_0)\theta_1 + & (K + r_0)\theta_2 \leq & r_0 + 2z_1 & (8) \\ (-1)\theta_1 + & (1)\theta_2 \leq & 0 & (9) \\ \theta_1 \geq 0, & \theta_2 \geq 0. & & (10) \end{array}$$

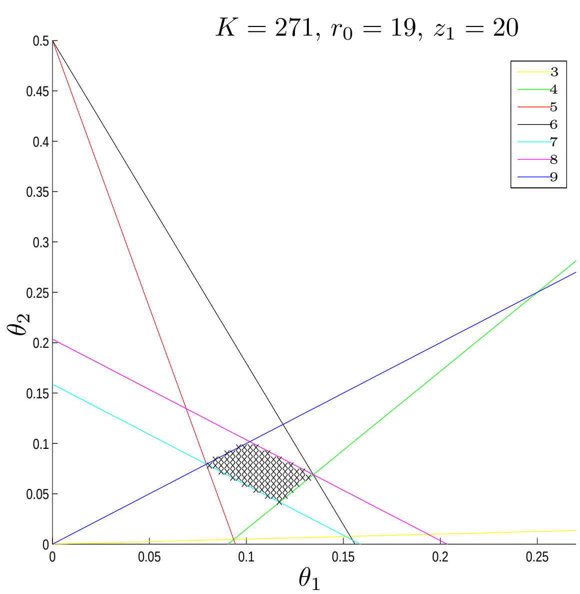

Figure 2: Polyhedron $P\Theta_3^=(z_1)$

Polytope $P\Theta_3^=(z_1)$ is illustrated in Figure 2 for a particular choice of values of the parameters r_0, z_1 and K. We are interested in determining the extreme points of $P\Theta_3^=(z_1)$ since they correspond to $\mathrm{CPL}_3^=$-extreme functions. A simple procedure to compute all the extreme points of $P\Theta_3^=(z_1)$ is to first derive the pairwise intersection points of the inequalities (3)-(10) and then construct the subset of the intersection points that are feasible for the system (3)-(10). It is easy to verify that this subset is the set of extreme points of $P\Theta_3^=(z_1)$. We observe that this procedure becomes simpler if the redundant inequalities are removed from the description of $P\Theta_3^=(z_1)$. In Proposition 7 we identify those inequalities that are redundant in the description of $P\Theta_3^=(z_1)$, and in Proposition 8 we give all the extreme points of $P\Theta_3^=(z_1)$.

Proposition 7 *In Subcase 1 of Case 1, inequalities (3), (5), (6), (10a) and (10b) are redundant in the description of $P\Theta_3^=(z_1)$.*

Proof

1. The sum of the inequalities obtained by multiplying (4) by $\frac{2z_1-r_0}{2(K-r_0-4z_1)} > 0$ and (7) by $\frac{r_0(K-3r_0)}{2(K-r_0)(K-r_0-4z_1)} > 0$ is

$$(-r_0 + z_1)\theta_1 + (-z_1)\theta_2 \leq \frac{-r_0^2}{2(K-r_0)},$$

   which dominates (3) since $r_0 > 0$ and $(K - r_0) > 0$.

2. The sum of the inequalities obtained by multiplying (7) by $\frac{1}{2}$ and (9) by $\frac{K-r_0-4z_1}{2} > 0$ is exactly (5).

3. The sum of the inequalities obtained by multiplying (4) by $\frac{1}{2}$ and (8) by $\frac{1}{2}$ is exactly (6).

4. The sum of the inequalities obtained by multiplying (7) by 1 and (9) by $(K - r_0) > 0$ is

$$\theta_1 \geq \frac{z_1}{K-r_0},$$

   which dominates (10a) since $z_1 > 0$ and $(K - r_0) > 0$.

5. The sum of the inequalities obtained by multiplying (3) by 1 and (9) by $(z_1 - r_0) > 0$ is exactly (10b) since $r_0 > 0$. □

We conclude from Proposition 7 that

$$P\Theta_3^=(z_1) = \{(\theta_1, \theta_2) \in \mathbb{R}^2 \mid (4), (7), (8), (9)\}.$$

Since a 2-dimensional polytope defined by four constraints has at most four extreme points, the following proposition can easily be proven by verifying that the given points are feasible for $P\Theta_3^=(z_1)$.

Proposition 8 *In Subcase 1 of Case 1, the polyhedron $P\Theta_3^=(z_1)$ has the following four extreme points*

1. *Point (a):* $(\frac{z_1}{K-r_0}, \frac{z_1}{K-r_0})$ *(intersection of (7) and (9))*
2. *Point (b):* $(\frac{r_0+2z_1}{2K+2r_0}, \frac{r_0+2z_1}{2K+2r_0})$ *(intersection of (8) and (9))*
3. *Point (c):* $(\frac{r_0+z_1}{K+r_0}, \frac{z_1}{K+r_0})$ *(intersection of (4) and (8))*
4. *Point (d):* $(\frac{r_0+2z_1}{2K-2r_0}, \frac{2z_1-r_0}{2K-2r_0})$ *(intersection of (4) and (7))* □

3.3 Extreme points of $P\Theta_3^=(z_1)$ in all cases

Following a discussion similar to that in Section 3.2, we can determine the extreme points of $P\Theta_3^=(z_1)$ for all admissible values of z_1. We obtain 18 distinct extreme points that are summarized in Table 2. In this table, we give each of these extreme points an identifier label 'Extreme Point' and we present its coordinates 'θ_1' and 'θ_2'. The valid ranges of the parameters K, r_0 and z_1 under which each point is extreme are reported under 'Range of r_0' and 'Range of K'. Remember also that CPL$_3$ functions with $z_1 = z_2$ always have parameters that satisfy $r_0 > 0$, $z_1 > 0$ and $r_0 + 4z_1 \le K$.

The extreme points presented in Table 2 correspond to all CPL$_3^=$-extreme inequalities. Note that each of these extreme points can be used to derive a strong inequality for an integer program, as demonstrated in the following example.

Example 9 *Consider the following single-constraint integer program:*

$$\begin{aligned} \textit{min} \quad & 2x_1 + 17x_2 + 11x_3 + 12x_4 + 12x_5 \\ \textit{s.t.} \quad & x_1 + 7.3x_2 + 2.4x_3 + 3.7x_4 + 2.9x_5 = 32.2 \\ & x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_+. \end{aligned}$$

This integer program has the following five linearly independent feasible solutions: $(x_1, x_2, x_3, x_4, x_5) = (22, 1, 0, 0, 1)$, $(9, 0, 0, 0, 8)$, $(14, 0, 0, 1, 5)$, $(24, 0, 1, 0, 2)$, $(3, 4, 0, 0, 0)$. *Therefore, the dimension of the convex hull of feasible solutions is* 4.

We will show next how to generate strong cuts for this set using CPL$_3^=$-extreme inequalities corresponding to extreme points (a), (b), (c), and (d). Note that since $r_0 = 0.2$, choosing $z_1 = 0.1$ allows us to use the points (a), (b), (c) and (d) to generate extreme inequalities. Through simple calculations, we obtain the following cutting-planes:

$$\begin{aligned} &\textit{Inequality (a):} \quad && \frac{7}{8}x_2 + \frac{3}{4}x_3 + \frac{3}{8}x_4 + \frac{1}{8}x_5 \ge 1; \\ &\textit{Inequality (b):} \quad && \frac{2}{3}x_2 + \frac{1}{3}x_3 + \frac{7}{12}x_4 + \frac{1}{3}x_5 \ge 1; \\ &\textit{Inequality (c):} \quad && \frac{1}{4}x_2 + \frac{1}{3}x_3 + \frac{7}{12}x_4 + \frac{3}{4}x_5 \ge 1; \\ &\textit{Inequality (d):} \quad && \frac{1}{4}x_2 + \frac{3}{4}x_3 + \frac{3}{8}x_4 + \frac{3}{4}x_5 \ge 1. \end{aligned}$$

Observe that the inequality (a) is the Gomory mixed integer cut. In fact, it is facet-defining since it is satisfied at equality by four affinely independent solutions to the integer program. These solutions are $(22, 1, 0, 0, 1)$, $(9, 0, 0, 0, 8)$, $(14, 0, 0, 1, 5)$ *and* $(24, 0, 1, 0, 2)$. *Inequality (b) is satisfied at equality by three affinely independent solutions:* $(25, 0, 3, 0, 0)$, $(22, 1, 0, 0, 1)$ *and* $(24, 0, 1, 0, 2)$. *Inequality (c) is satisfied at equality by three affinely independent solutions:* $(25, 0, 3, 0, 0)$, $(22, 1, 0, 0, 1)$ *and* $(3, 4, 0, 0, 0)$. *Finally, inequality (d) is satisfied at equality by two affinely independent solutions:* $(22, 1, 0, 0, 1)$ *and* $(3, 4, 0, 0, 0)$.

The optimal solution to the integer program is $(22, 1, 0, 0, 1)$, *and the optimal objective value is* 73. *The optimal objective value of the linear programming* *relaxation is* 64.4. *When inequality (a) is added as a cut to the linear programming relaxation, the optimal objective value increases to* 67.1429. *When inequalities (a) and (b) are both added, the optimal objective value increases to* 68. *When inequalities (a), (b) and (c) are added, the optimal objective value increases to* 72.5. *When all four inequalities are added as cuts, we obtain the optimal solution to the integer program.*

It can be verified that the addition of just one, or even any two, of these inequalities is not sufficient to expose the optimal solution. There are only two ways to obtain the optimal solution by adding three cuts. The first is with the addition of (a),(c),and (d), and the second is with the addition of (b),(c), and (d). It is interesting to observe that the GMIC is not particularly helpful in this example. □

In general, it is difficult to guarantee that the inequalities generated under any approximate lifting scheme are strong. Typical notions of strength include nondomination and maximality; see [13]. Because of the particular nature of our approximate lifting scheme, we can study the strength of the inequalities within the framework of the group problem as we discuss in the following section.

4 New inequalities for group problems from $CPL_3^=$-extreme functions

In this section, we consider the eighteen extreme points that are derived in Section 3 and determine whether the corresponding extreme functions yield strong valid inequalities for finite and infinite group problems.

In Section 4.1, we first give an overview of some basic concepts and results on finite and infinite group problems, and then show how a valid inequality (2) obtained by the approximate lifting procedure is converted into a minimal valid inequality for group problems. In Section 4.2, we study the strength of the $\mathrm{CPL}_3^=$-extreme inequalities for the finite group problem. Because these inequalities are valid for the master cyclic group problem, they induce faces of the convex hull of its solutions. For each of the eighteen $\mathrm{CPL}_3^=$-extreme inequalities, we derive a lower bound on the dimension of the supporting face and give conditions under which it is facet-defining. As a consequence, we obtain several families of facets for the finite group problem, including some new families of two-, three- and four-slope inequalities. In Section 4.3, we reconsider these inequalities in the framework of the infinite group problem and give conditions under which they are extreme. These new families of two- and three-slope extreme inequalities add to the few known families of extreme inequalities given in the literature.

We note that, in this section, $J_i = (a, b]$ (as defined in Section 2) is used to denote the set of integers $\{a+1, a+2, \ldots, b-1, b\}$ when finite groups are discussed (Section 4.2) and is used to denote the continuous interval $(a, b]$ when infinite groups are discussed (Section 4.3). Furthermore, in the case of the finite group problem, K, r_0 and z_1 are always considered to be integer, while in the case of the infinite group problem, $K = 1$ and $0 < r_0, z_1 < 1$.

4.1 Group problems and valid inequalities

Consider the simple integer set

$$\hat{S} = \{x \in \mathbb{Z}_+^m \mid \sum_{i \in M} a_i x_i = a_0\}, \tag{11}$$

defined with the same assumptions used for S in (1).

Let ϕ be a CPL$_n$-function for some $K \in \mathbb{Z}_+$ such that $q_1 \neq 0$. An equivalent form of (2) for $\hat{S}$ is

$$\sum_{i \in M} \frac{r_i - K\phi(r_i)}{r_0} x_i \geq 1, \tag{12}$$

which is obtained by subtracting K times (2) from the defining equation of $P\hat{S}$. This inequality is therefore valid for $P\hat{S}$.

Consider now the function

$$f(u) = \frac{r(u) - K\phi(r(u))}{r_0},$$

where $r(u)$ is the remainder of division of u by K. We call $f(u)$ the *group representation* of ϕ. An illustration of the group representation of the CPL$_n$ function of Figure 1 is given in Figure 3. The next proposition summarizes some properties of the group representation of CPL$_n$ functions.

Proposition 10 *[18] The group representation $f(u) = \frac{r(u)-K\phi(r(u))}{r_0}$ of a CPL$_n$ function ϕ satisfies the following properties:*

(i) *$f(u)$ is continuous.*

(ii) *$f(u)$ is linear over $2n$ intervals.*

(iii) *The slope of $f(u)$ in interval J_{i+1} is identical to its slope in interval J_{2n+1-i} for $i = 1, \ldots, n-1$.*

(iv) *$f(u) \geq 0$, $\forall u \in [0, K)$.*

(v) *$f(u) = \frac{u}{r_0}$, $\forall u \in [0, r_0]$.*

(vi) *$f(u) + f(v) \geq f((u+v) \pmod K)$, $\forall u, v \in [0, K)$.*

(vii) *$f(u) + f((r_0 - u) \pmod K) = f(r_0)$, $\forall u \in [0, K)$.* □

The number of different slopes of a group representation $f(u)$ is a typical method of characterization. Therefore, we computed the slopes of the group representation of each CPL$_3^=$-extreme function. The slopes are denoted by s_1, s_2, s_3, s_4, s_5 and s_6 and are presented in Table 3. We observe from Table 3 that the group representations induced by extreme points a, h, q and r are always two-slope functions, and the group representations induced by extreme points b, c, i, j, m and n are always three-slope functions. Depending on the relations among z_1, r_0 and K, the group representations induced by extreme points e, f, k, l, o and p can be two- or three-slope functions, while the group

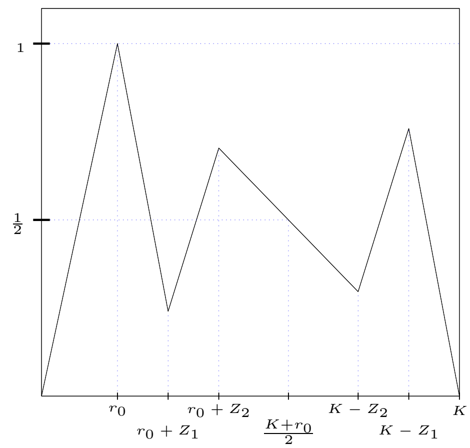

Figure 3: Group representation of a CPL_3 function

representations induced by extreme points d and g can be three- or four-slope functions. We will show in Section 4.2 that the subadditive functions corresponding to Gomory's mixed integer cut [8] and Gomory and Johnson's 2-slope and 3-slope cuts [9, 11] are CPL_n functions.

We now study the function $f(u)$ in relation to the *group problem.* Group problems are modular relaxations of integer programs, defined as follows.

Definition 11 *Given a group $\mathbb{G}$ and an element $r_0 \in \mathbb{G}$, the group problem $PI(\mathbb{G}, r_0)$ on $\mathbb{G}$ with right-hand side r_0 is defined as the set of functions $t : \mathbb{G} \rightarrow \mathbb{Z}_+$ that have finite support and for which $\sum_{g \in \mathbb{G}} gt(g) = r_0$.* □

Group problems can be defined for various groups. The group considered in this paper are the finite cyclic group C_n, where n is the order of the group, and the group I of real numbers with the addition performed modulo one. Group problems are important because valid inequalities for their feasible sets can be used as cutting planes for general integer programs; see [8]. We next give several relevant definitions and results from [8, 9] regarding valid inequalities for group problems.

Proposition 12

1. *A function $f : \mathbb{G} \rightarrow \mathbb{R}_+$ is a valid inequality for $PI(\mathbb{G}, r_0)$ if $f(0) = 0$, $f(r_0) = 1$ and $\sum_{g \in \mathbb{G}} f(g)t(g) \geq 1$ for all $t \in PI(\mathbb{G}, r_0)$.*

2. *A valid inequality f for $PI(\mathbb{G}, r_0)$ is minimal for $PI(\mathbb{G}, r_0)$ if and only if f is subadditive (i.e., it satisfies $f(g_1) + f(g_2) \geq f(g_1 + g_2)$ for $g_1, g_2 \in \mathbb{G}$) and $f(u) + f(r_0 - u) = 1 \ \forall u \in \mathbb{G}$.*

3. *A valid inequality f for $PI(\mathbb{G}, r_0)$ is extreme if it cannot be expressed as a convex combination of other valid inequalities for $PI(\mathbb{G}, r_0)$.* □

Proposition 10 implies that the inequality $\sum_{g \in \mathbb{G}} f(g)t(g) \geq 1$ given by a group representation f of a CPL_n function is a minimal valid inequality for the group problem $PI(\mathbb{G}, r_0)$, where $\mathbb{G} = C_K$ or I. However, characterizing extreme inequalities for group problems is not always simple. Although there exists a nice characterization of extreme inequalities for the finite cyclic group problem $PI(C_n, r_0)$ from Gomory [8], such an elegant result does not exist for I. In the following subsections, we will give conditions under which $\text{CPL}_3^=$-extreme inequalities are also extreme inequalities for the corresponding group problems.

4.2 New facet-defining inequalities for the finite group problem

When the group problem is defined on a finite cyclic group C_n, it reduces to the *master cyclic group problem* introduced by Gomory [8]:

$$C_{K,r_0} = \{x \in \mathbb{Z}_+^{K-1} | \sum_{i=1}^{K-1} ix_i \equiv r_0 \pmod{K}\}.$$

The convex hull of C_{K,r_0}, $P(C_{K,r_0})$, is a polyhedron and is known as the *master cyclic group polyhedron*. Therefore, the notion of an extreme inequality for this set reduces to that of a facet-defining inequality. Furthermore, Gomory obtained the following implicit characterization of all the facet-defining inequalities of $P(C_{K,r_0})$ for $1 \leq r_0 \leq K-1$. Note that he also gave a similar (but different) description for the case when $r_0 = 0$. We do not consider this case here.

Theorem 13 ([6, 8]) *For integers r_0 and K, where $1 \leq r_0 \leq K-1$, the facet-defining inequalities $\sum_{i=1}^{K-1} \pi_i x_i \geq \gamma$ of the master cyclic group polyhedron $P(C_{K,r_0})$ are $x_i \geq 0$ for $i = 1, \ldots, K-1$ and the extreme rays of the cone S_{K,r_0} defined by the vectors $\pi \in \mathbb{R}^{K-1}$ that satisfy the following sets of relations:*

Nonnegativity:	$\pi_i \geq 0,$	$1 \leq i \leq K-1,$
Subadditivity:	$\pi_i + \pi_j \geq \pi_k,$	$1 \leq i, j, k \leq K-1,\ (i+j) \equiv k \ (mod\ K),$
Complementarity:	$\pi_i + \pi_j = \gamma,$	$1 \leq i, j \leq K-1,\ (i+j) \equiv r_0 \ (mod\ K),$
Scalability:	$\pi_{r_0} = \gamma.$	

□

Note that in this case, we can perform an analysis that is deeper than simply determining if an inequality is facet-defining for $P(C_{K,r_0})$. In fact, even if a valid inequality is not facet-defining for the master cyclic group polyhedron, the dimension of the face it induces may be large. In the following, we build on Gomory's result to characterize the d-dimensional faces of $P(C_{K,r_0})$. We first give a general result on the dimension of the supporting face of $P(C_{K,r_0})$ induced by a valid inequality (*supporting-face dimension*). We then use this result to derive lower bounds on the supporting-face dimension of each $\text{CPL}_3^=$-extreme inequality for C_{K,r_0}. Finally, we give conditions under which each of these inequalities is facet-defining. In particular, we obtain several new families of facet-defining inequalities for the master cyclic group problem, as well as alternate derivations of some known families.

Definition 14 *An equality $\alpha x = \gamma$ is said to be a supporting equality of a μ-dimensional face of polyhedron P if $P \subseteq \{x \in \mathbb{R}^n | \alpha x \geq \gamma\}$ and $dim(P \cap \{x \in \mathbb{R}^n | \alpha x = \gamma\}) \geq \mu$. The polyhedron $P \cap \{x \in \mathbb{R}^n | \alpha x = \gamma\}$ is called the supporting face of P induced by the inequality $\alpha x \geq \gamma$.* □

The following theorem, which generalizes Theorem 13, gives a way to calculate the supporting-face dimension of a valid inequality of C_{K,r_0}.

Theorem 15 *Let $r_0, K \in \mathbb{Z}_+$, where $1 \le r_0 \le K-1$, $\gamma > 0$ and $\mu \le K-1$. If there are μ constraints of S_{K,r_0} that are satisfied as linearly independent equations by $(\pi,\gamma) \in S_{K,r_0}$, then $\pi x = \gamma$ is a supporting equality of a $(\mu-1)$-dimensional face of $P(C_{K,r_0})$.*

Proof There must be μ linearly independent scalability, complementarity, non-negativity, and subadditivity constraints satisfied at equality. Of these, the subadditivity and non-negativity equations have the form

$$\pi_i + \pi_j - \pi_k = 0, \text{ for } 1 \le i,j,k \le K-1,\ (i+j) \equiv k \pmod K \qquad (13)$$

and

$$\pi_i = 0, \text{ for } 1 \le i \le K-1, \qquad (14)$$

respectively. We can add $\pi_k + \pi_q = \gamma$, where $k + q \equiv r_0 \pmod K$, to each equation of the form (13), to construct a new and independent equation

$$\pi_i + \pi_j + \pi_q = \gamma.$$

For each equation of the form (14), we can first multiply it by the smallest positive integer $g(i)$ satisfying $g(i)i \equiv 0 \pmod K$ and then add it to $\pi_{r_0} = \gamma$ to obtain a new and independent equation

$$g(i)\pi_i + \pi_{r_0} = \gamma.$$

Now the μ independent equations obtained above are all in the form $\sum_{i=1}^{K-1} s^j(i)\pi_i = \gamma$, $1 \le j \le \mu$, where all the $s^j(i)$ are nonnegative integers and satisfy $\sum_{i=1}^{K-1} s^j(i)i = r_0 \pmod K$. So we get μ linearly independent $(K-1)$-dimensional vectors $x(j) = (s^j(1), \ldots, s^j(n-1)) \in \{x \in \mathbb{Z}_+^{K-1} : \sum_{i=1}^{K-1} ix_i \equiv r_0 \pmod K\}$, $1 \le j \le \mu$, that satisfy $\pi x(j) = \gamma$. Therefore, $\pi x = \gamma$ is a supporting equality of a $(\mu-1)$-dimensional face of $P(C_{K,r_0})$. $\square$

The result of Theorem 15 can be used to measure the strength of a valid inequality for the group polyhedron by its supporting-face dimension. Observe that the converse of Theorem 15 is not always true because $x_i \ge 0$, $1 \le i \le K-1$, is a facet-defining inequality for $P(C_{K,r_0})$ whose corresponding $(\pi,\gamma) \notin S_{K,r_0}$.

In order to prove a lower bound l on the supporting-face dimension of a $\text{CPL}_3^=$-extreme inequality, it follows from Theorem 15 that it suffices to find some $l+1$ linearly independent constraints of S_{K,r_0} that are satisfied as equations. We next use this procedure on the set of $\text{CPL}_3^=$-extreme inequalities obtained in Section 3.

Let $f^\beta(u)$ be the group representation of the $\text{CPL}_3^=$ function induced by extreme point β, for $\beta = a, ..., r$. Let π^β be a $(K-1)$-dimensional vector given by $\pi_j^\beta = f^\beta(j)$ for $j = 1, ..., K-1$. For simplicity of notation in the remainder of this section, once β is fixed, π^β is denoted as π. We will refer to the following six discrete intervals: $J_1 = \{1, ..., r_0\}$, $J_2 = \{r_0+1, ..., r_0+z_1\}$, $J_3 = \{r_0+z_1+1, ..., r_0+2z_1\}$, $J_4 = \{r_0+2z_1+1, ..., K-2z_1\}$, $J_5 = \{K-2z_1+1, ..., K-z_1\}$, $J_6 = \{K-z_1+1, ..., K-1\}$.

It follows from their construction that $\mathrm{CPL}_3^=$ functions derived from extreme points (a-r), always satisfy several common linearly independent additive relations that are given in the following lemma.

Lemma 16 *There are $r_0 - 1$ linearly independent additive relations $\pi_1^\beta + \pi_i^\beta = \pi_{1+i}^\beta$ for $i = 1, ..., r_0 - 1$ satisfied by the inequality $\pi^\beta x \geq 1$ for $\beta = a, ..., r$ and 1 relation $\pi_{r_0}^\beta = \gamma = 1$.* □

Theorem 17 *For $\beta = a, ..., r$, let $dim(\pi^\beta)$ be the dimension of the face of $P(C_{K,r_0})$ induced by the inequality $\pi^\beta x \geq 1$ when K, r_0 and z_1 are in the validity ranges for extreme point β given in Table 2. Then*

$$\dim(\pi^a) \geq K - 2 \tag{15}$$

$$\dim(\pi^b) \geq K - 2 - max\{0, \lceil\frac{K + r_0}{2}\rceil + r_0 + 4z_1 - K - 1\} \tag{16}$$

$$\dim(\pi^c) \geq K - 2 - max\{0, \lceil\frac{K + r_0}{2}\rceil + r_0 + 2z_1 - K - 1\} \tag{17}$$

$$\dim(\pi^d) \geq K - z_1 - 1 - max\{0, -\lfloor\frac{K + r_0}{2}\rfloor + 4z_1 + r_0 - 1\} \tag{18}$$

$$\dim(\pi^e) \geq \lfloor\frac{K + r_0}{2}\rfloor + 2z_1 - 1 \tag{19}$$

$$\dim(\pi^f) \geq \lfloor\frac{K + r_0}{2}\rfloor + 7z_1 + r_0 - K - 1 \tag{20}$$

$$\dim(\pi^g) \geq K - z_1 - 1 \tag{21}$$

$$\dim(\pi^h) \geq K - 2 \tag{22}$$

$$\dim(\pi^i) \geq K - z_1 - 1 \tag{23}$$

$$\dim(\pi^j) \geq K - z_1 - 1 \tag{24}$$

$$\dim(\pi^k) \geq \lfloor\frac{K + r_0}{2}\rfloor + 2z_1 - 1 \tag{25}$$

$$\dim(\pi^l) \geq K - z_1 - 1 \tag{26}$$

$$\dim(\pi^m) \geq \lfloor\frac{K + r_0}{2}\rfloor + 2z_1 - 1 \tag{27}$$

$$\dim(\pi^n) \geq \lceil\frac{K + r_0}{2}\rceil + 2z_1 - 2 \tag{28}$$

$$\dim(\pi^o) \geq K - z_1 - 1 \tag{29}$$

$$\dim(\pi^p) \geq \lfloor\frac{K + r_0}{2}\rfloor + 2z_1 - 1 \tag{30}$$

$$\dim(\pi^q) \geq K - 2 \tag{31}$$

$$\dim(\pi^r) \geq K - 2 \tag{32}$$

Proof By construction, all cases satisfy the complementarity and subadditivity conditions in S_{K,r_0}. Lemma 16 provides r_0 relations for all cases. The remaining necessary relations for each case are listed below. For convenience, we will refer to the following simple sets of relations by (33), (34), and (35):

$$\pi_1 + \pi_i = \pi_{1+i} \tag{33}$$

$$\pi_i + \pi_{K-1} = \pi_{i-1} \tag{34}$$

$$\pi_i + \pi_{K+r_0-i} = \pi_{r_0}. \tag{35}$$

$dim(\pi^a):$ There are $K - r_0 - 1$ relations (34) for $i = r_0 + 1, ..., K - 1$.

$dim(\pi^b):$ There are $4z_1 - 2$ relations (34) for $i \in J_2 \cup J_3 \cup J_5 \cup J_6 \setminus \{r_0 + 2z_1\}$ and $K - 2z_1 - \lceil \frac{K+r_0}{2} \rceil + 1$ relations (35) for $i = \lceil \frac{K+r_0}{2} \rceil, ..., K - 2z_1$.

Additionally, by the validity condition for point b, $r_0 + 2z_1 \leq K - 4z_1 - r_0$. Therefore, for $i = r_0 + 2z_1, ..., K - 4z_1 - r_0$, there are $K - 2r_0 - 6z_1 + 1$ additive relations $\pi_i + \pi_{r_0+2z_1} = \pi_{i+r_0+2z_1}$. Of these, $min\{K - 2r_0 - 6z_1 + 1, \lceil \frac{r_0+K}{2} \rceil - r_0 - 2z_1\}$ are linearly independent from the above relations.

$dim(\pi^c):$ There are $2z_1 - 2$ relations (34) for $i \in J_2 \cup J_6 \setminus \{r_0 + z_1\}$ and $K - z_1 - \lceil \frac{r_0+K}{2} \rceil + 1$ relations (35) for $i = \lceil \frac{r_0+K}{2} \rceil, ..., K - z_1$.

Additionally, by the validity condition for point c, $r_0 + z_1 \leq K - r_0 - 2z_1$. Therefore, for $i = r_0 + z_1, ..., K - r_0 - 2z_1$, there are $K - 2r_0 - 3z_1 + 1$ relations $\pi_i + \pi_{r_0+z_1} = \pi_{i+r_0+z_1}$, of which $min\{K - 2r_0 - 3z_1 + 1, \lceil \frac{r_0+K}{2} \rceil - r_0 - z_1\}$ are linearly independent from the above relations.

$dim(\pi^d):$ There are $z_1 - 1$ relations (34) for $i \in J_6$ and $\lfloor \frac{K+r_0}{2} \rfloor - r_0$ relations (35) for $i = r_0 + 1, ..., \lfloor \frac{K+r_0}{2} \rfloor$. There is one relation $\pi_{r_0+z_1} + \pi_{K-r_0-2z_1} = \pi_{K-z_1}$.

Additionally, for $i = K - 2z_1 + 1 - \min\{K - 6z_1 - r_0 + 1, K - 2z_1 - \lfloor \frac{K+r_0}{2} \rfloor\}, ..., K - 2z_1$, there are $\min\{K - 6z_1 - r_0 + 1, K - 2z_1 - \lfloor \frac{K+r_0}{2} \rfloor\}$ relations $\pi_i + \pi_{K-2z_1} = \pi_{i-2z_1}$.

$dim(\pi^e):$ There are $2z_1 - 1$ relations (34) for $i \in J_5 \cup J_6$ and $\lfloor \frac{K+r_0}{2} \rfloor - r_0$ relations (35) for $i = r_0 + 1, ..., \lfloor \frac{K+r_0}{2} \rfloor$. There is one relation $2\pi_{r_0+2z_1} = \pi_{2r_0+4z_1}$.

$dim(\pi^f):$ There are $z_1 - 1$ relations (34) for $i \in J_6$ and $\lfloor \frac{K+r_0}{2} \rfloor - r_0$ relations (35) for $i = r_0 + 1, ..., \lfloor \frac{K+r_0}{2} \rfloor$.

Additionally, by the validity condition for point f, there are $r_0 + 6z_1 - K + 1$ additional independent relations $\pi_i + \pi_{K-2z_1} = \pi_{i-2z_1}$ for $i = K - 2z_1, ..., r_0 + 4z_1$.

$dim(\pi^g):$ There are $K - 2z_1 - r_0 - 1$ relations (34) for $i \in J_2 \cup J_4 \cup J_6$ and z_1 relations (35) for $i \in J_3$.

Additionally, by the validity condition for point g, $K - r_0 - 2z_1 \in J_4$. Therefore, there is one additional relation $\pi_{r_0+z_1} + \pi_{K-r_0-2z_1} = \pi_{K-z_1}$.

$dim(\pi^h):$ There are $K - 4z_1 - r_0$ relations (33) for $i \in J_4$ and $4z_1 - 1$ relations (34) for $i \in J_2 \cup J_3 \cup J_5 \cup J_6$.

$dim(\pi^i):$ There are $K - 4z_1 - r_0$ relations (33) for $i \in J_4 \cup \{r_0 + 2z_1\} \setminus \{K - 2z_1\}$, $2z_1 - 1$ relations (34) for $i \in J_2 \cup J_6$, and z_1 relations (35) for $i \in J_3$.

Finally, since for point i, $2r_0 + 3z_1 \leq K < 2r_0 + 4z_1$, we have that $K - r_0 - 2z_1 \in J_3$. Therefore, there is one additional independent additive relation $\pi_{K+z_1} + \pi_{K-r_0-2z_1} = \pi_{K-z_1}$.

$dim(\pi^j):$ There are $K - 2z_1 - r_0 - 1$ relations (34) for $i \in J_2 \cup J_4 \cup J_6$, and z_1 relations (35) for $i \in J_3$.

Finally, since for point j, $2r_0 + 3z_1 \leq K < 2r_0 + 4z_1$, we have that $K - r_0 - 2z_1 \in J_3$. Therefore, there is one additional additive additive relation $\pi_{K+z_1} + \pi_{K-r_0-2z_1} = \pi_{K-z_1}$.

$dim(\pi^k):$ There are $2z_1 - 1$ relations (33) for $i \in J_3 \cup J_5 \cup \{K - 2z_1\} \setminus \{r_0 + 2z_1, K - z_1\}$, $2z_1 - 1$ relations (34) for $i \in J_2 \cup J_6$, and $\lfloor \frac{K+r_0}{2} \rfloor - r_0 - 2z_1 + 1$ relations (35) for $i = r_0 + 2z_1, ..., \lfloor \frac{K+r_0}{2} \rfloor$.

Additionally, when ($r_0 > 0$ and $\max\{r_0 + 5z_1, 2r_0 + 3z_1\} \leq K < 2r_0 + 4z_1$), we have that $2r_0 + 2z_1 \in J_5$. Therefore, there is one additional independent additive relation $2\pi_{r_0+z_1} = \pi_{2r_0+2z_1}$. Moreover, when ($r_0 > 3z_1$ and $r_0 + 5z_1 \leq K < r_0 + 6z_1$), we have that $K - 4z_1 \in J_3$. Therefore, there is one additional independent additive relation $2\pi_{K-2z_1} = \pi_{K-4z_1}$.

$dim(\pi^l):$ There are $K - 2z_1 - r_0 - 1$ relations (34) for $i \in J_2 \cup J_4 \cup J_6$ and z_1 relations (35) for $i \in J_5 \setminus \{K - z_1\} \cup \{\lfloor \frac{K+r_0}{2} \rfloor\}$.

Additionally, if $2r_0 + 2z_1 \neq K$, there is a relation $2\pi_{r_0+z_1} = \pi_{2(r_0+z_1)}$. Otherwise, by the validity conditions for point l, $r = 2z_1$ and, therefore, $s_3 = s_1$, which gives the relation $\pi_1 + \pi_{r_0+z_1} = \pi_{r_0+z_1+1}$.

$dim(\pi^m):$ There are $2z_1$ relations (33) for $i \in J_3 \cup J_5 \cup \{r_0 + z_1, K - 2z_1\} \setminus \{r_0 + 2z_1, K - z_1\}$, $2z_1 - 1$ relations (34) for $i \in J_2 \cup J_6$, and $\lfloor \frac{K+r_0}{2} \rfloor - r_0 - 2z_1$ relations (35) for $i \in \{r_0 + 2z_1 + 1, ..., \lfloor \frac{K+r_0}{2} \rfloor\}$.

Additionally, by the validity conditions on point m, $2r_0 + 2z_1 \in J_4$. Therefore, by definition, $\pi_{2r_0+2z_1} = \pi_{r_0+z_1} + z_1 s_3 + r_0 s_4 = 2\pi_{r_0+z_1}$, giving an additional relation.

$dim(\pi^n):$ There are $2z_1$ relations (33) for $i \in J_3 \cup J_5 \cup \{r_0 + z_1, K - 2z_1\} \setminus \{r_0 + 2z_1, K - z_1\}$, $2z_1 - 1$ relations (34) for $i \in J_2 \cup J_6$, and 1 relations (35) for $i = \lfloor \frac{K+r_0}{2} \rfloor$.

The validity conditions for extreme point n imply that $(\lfloor \frac{K+r_0}{2} \rfloor + 2) + (K - 2z_1)) \pmod K \in J_4$. Therefore, for $i = \lfloor \frac{K+r_0}{2} \rfloor + 2), ..., K - 2z_1$, the relation $\pi_i + \pi_{K-2z_1} = \pi_{i-2z_1-1}$ is satisfied. Furthermore, this set of relations is linearly independent from those listed above.

$dim(\pi^o):$ There are $K - 4z_1 - r_0$ relations (33) for $i \in J_4 \cup \{r_0 + 2z_1\} \setminus \{K - 2z_1\}$, $2z_1 - 1$ relations (34) for $i \in J_2 \cup J_6$, and z_1 relations (35) for $i \in J_3$.

Additionally, by the validity condition for point o, $2r_0 + 2z_1 \in J_5$. Therefore, there is one additional independent additive relation $2\pi_{r_0+z_1} = \pi_{2r_0+2z_1}$.

$dim(\pi^p):$ There are $2z_1 - 1$ relations (33) for $i \in J_3 \cup J_5 \cup \{K - 2z_1\} \setminus \{r_0 + 2z_1, K - z_1\}$, $2z_1 - 1$ relations (34) for $i \in J_2 \cup J_6$, and $\lfloor \frac{K+r_0}{2} \rfloor - r_0 - 2z_1 + 1$ relations (35) for $i = r_0 + 2z_1, ..., \lfloor \frac{K+r_0}{2} \rfloor$.

Additionally, by the validity conditions for point p, $2r_0 + 2z_1 \leq K < 2r_0 + 3z_1$, $2r_0 + 2z_1 - 1 \in J_6$. Therefore, there is one additional independent additive relation $\pi_{r_0+z_1-1} + \pi_{r_0+z_1} = \pi_{2r_0+2z_1-1}$.

$dim(\pi^q):$ There are $2z_1$ relations (33) for $i \in J_3 \cup J_5 \cup \{r_0 + z_1, K - 2z_1\} \setminus \{r_0 + 2z_1, K - z_1\}$ and $K - 2z_1 - r_0 - 1$ relations (34) for $i \in J_2 \cup J_4 \cup J_6$.

$dim(\pi^r):$ There are $K - 2z_1 - r_0$ relations (33) for $i \in J_3 \cup J_4 \cup J_5 \cup \{r_0 + z_1\} \setminus \{K - z_1\}$ and $2z_1 - 1$ relations (34) for $i \in J_2 \cup J_6$. □

We observe that many of these lower bounds are close to $K - 2$, which is the dimension of facets for $P(C_{K,r_0})$. This shows that these inequalities, although not always facet-defining, are typically strong. In the following theorem, we describe conditions under which the inequalities described in Theorem 17 are facet-defining for $P(C_{K,r_0})$.

Theorem 18 *For the following choices of β and any values of K, r_0, and z_1 that are in the validity ranges for extreme point β given in Table 2 and satisfy the following conditions, the inequality $\pi^\beta x \geq 1$ is facet-defining for $P(C_{K,r_0})$:*

1. $\beta = a$
2. $\beta = b$ *and* $K \geq 3r_0 + 8z_1$
3. $\beta = c$ *and* $K \geq 3r_0 + 4z_1$
4. $\beta = d$ *and* $z_1 = 1$, $r_0 = 1$ *and* $K = 7$
5. $\beta = d$ *and* $z_1 = 1$, $r_0 = 2$ *and* $K = 8$
6. $\beta = d$ *and* $z_1 = 1$, $r_0 = 1$ *and* $K \geq 9$
7. $\beta = d$ *and* $r_0 = 2z_1$ *and* $K \geq r_0 + 8z_1$
8. $\beta = e$ *and* $r_0 = 1$ *and* $K = 4z_1 + 3$
9. $\beta = f$ *and* $r_0 \leq z_1$ *and* $K = r_0 + 5z_1$
10. $\beta = g$ *and* $K = 2r_0 + 4z_1$
11. $\beta = h$
12. $\beta = i$ *and* $z_1 = 1$ *and* $K = 2r_0 + 3$
13. $\beta = j$ *and* $K = 5, r_0 = 1, z_1 = 1$
14. $\beta = k$ *and* $K = r_0 + 5z_1$
15. $\beta = k$ *and* $K = 8, r_0 = 2, z_1 = 1$ *or* $K = 9, r_0 = 3, z_1 = 1$
16. $\beta = l$ *and* $r_0 = 2z_1$ *and* $6z_1 \leq K \leq 7z_1 - 1$
17. $\beta = n$ *and* $z_1 = 1$, $r_0 \geq 3$, *and* $K = r_0 + 6z_1$
18. $\beta = n$ *and* $K \geq r_0 + 8z_1$
19. $\beta = o$ *and* $K = 2r_0 + 2z_1$
20. $\beta = p$ *and* $K = 2r_0 + 2z_1$
21. $\beta = p$ *and* $K = r_0 + 4z_1 + 2$, $r_0 = 2z_1 + 1$ *and* $z_1 \geq 2$
22. $\beta = q$
23. $\beta = r$

Proof The restrictions on β, K, r_0, and z_1 given in cases (1)-(6), (11)-(13), (15), and (21)-(23), combined with the dimension results from Theorem 17, imply $dim(\pi^\beta) \geq K-2$.

Furthermore, the restrictions on β, K, r_0, and z_1 given in cases (14), (16), and (19)-(20), imply, in each of these cases, that π is a two-slope inequality. Therefore, Gomory and Johnson's two-slope theorem [10] implies that $\pi^\beta x \geq 1$ is facet-defining for $P(C_{K,r_0})$.

For the remaining cases, we provide $K-2$ linearly independent additive relations that show that $dim(\pi^\beta) \geq K-2$.

Case 7: There are $2z_1+r_0-1$ relations (33) for $i \in J_1 \cup J_3 \cup J_5 \cup \{r_0+z_1, K-2z_1\} \setminus \{r_0, r_0+2z_1, K-z_1\}$ and $2z_1-1$ relations (34) for $i \in J_2 \cup J_6$. By the condition on K, r_0, and z_1, $r_0+4z_1, K-4z_1+1, (K-2z_1)+(r_0+4z_1)-K$, and $(K-2z_1-1)+(K-4z_1+1)-K$ are all in $J_4 \cup \{r_0+2z_1\}$. Therefore, there are $K-6z_1-r_0+1$ additive relations $\pi_i + \pi_{K-2z_1} = \pi_{i-2z_1}$ for $i = r_0+4z_1, ..., K-2z_1$, and there are $2z_1-1$ additive relations $\pi_i + \pi_{K-2z_1-1} = \pi_{i-2z_1-1}$ for $i = K-4z_1+1, ..., K-2z_1-1$. These relations are linearly independent from the previous relations.

Case 8: There are z_1-1 relations $\pi_i+\pi_{K-1} = \pi_{i-1}$ for $i = K-z_1+1, \cdots, K-1$, there are $\lfloor \frac{K+r_0}{2} \rfloor - 1$ relations $\pi_i + \pi_{K+1-i} = \pi_1$ for $i = 2, \cdots, \lfloor \frac{K+r_0}{2} \rfloor$, and there is also one relation $\pi_{z_1+1} + \pi_{z_1+1} = \pi_{2z_1+2}$. Since $r_0 = 1$ and $K = 3r_0+2z_1$, the total number of additive relations is $\lfloor \frac{K+r_0}{2} \rfloor + z_1 - 1 = 2z_1+1 = K-2$. These $K-2$ relations are all linearly independent.

Case 9: There are r_0-1 relations (33) for $i \in J_1 \setminus \{r_0\}$ and $K-2z_1-r_0-1$ relations (34) for $i \in J_2 \cup J_4 \cup J_6$. For $i = K-2z_1, ..., K-z_1$, the relation $\pi_i + \pi_{K-2z_1} = \pi_{i-2z_1}$ is valid because $K-4z_1, ..., K-3z_1 \in J_3$. Also, for $i = K-2z_1+1, ..., K-z_1-1$, the relation $\pi_i + \pi_{K-2z_1+1} = \pi_{i-2z_1+1}$ is valid because $K-4z_1+2, ..., K-3z_1 \in J_3$. These $K-2$ relations are all linearly independent.

Case 10: In addition to the relations given in the proof of $dim(\pi^g)$, the additive relations $\pi_{r_0+z_1+i} + \pi_{r_0+2z_1-i-1} = \pi_{2r_0+3z_1-1}$ for $i = 1, ..., \lfloor \frac{z_1-1}{2} \rfloor$ and $\pi_{r_0+z_1+i} + \pi_{r_0+2z_1-i} = \pi_{2r_0+3z_1}$ for $i = 1, ..., \lfloor \frac{z_1}{2} \rfloor$ are valid and linearly independent from the previous relations.

Case 17: In addition to the relations given in the proof of $dim(\pi^n)$, the additive relation $2\pi_{K-2z_1} = \pi_{K-4z_1}$ is valid and linearly independent.

Case 18: The relations constructed for condition (7) also hold in this case.

□

In Table 4 we summarize the dimension and facet-defining results obtained in Theorems 17 and 18. In the column 'Face dimension' we give a lower bound on the dimension of the corresponding inequality. For each extreme point, the given lower bound is tight for some members of the family. In the columns '2-slope facet', '3-slope facet', and '4-slope facet,' we describe conditions under which the corresponding inequality is facet-defining as a two-, three- or four-slope inequality, respectively. We record the fact that there are no combinations of parameters for which a given inequality can be a two-slope by writing a 'NA' in the corresponding table entry. We proceed similarly when there are no combinations of parameters for which a given inequality can be a three-slope or a four-slope. Furthermore, if we do not know any conditions under which a three-slope inequality is facet-defining, we mark a '–' in the corresponding table entry. We proceed similarly for four-slope inequalities. Finally, under each of the conditions given in Table 4, we give the reference of the paper that first introduced and proved that the corresponding inequality is facet-defining. The case of two-slope inequalities is special because Gomory and Johnson [9] proved that any subadditive and complementary two-slope inequality is facet-defining for the group problem. However, their result does not give indications on how to construct subadditive and complementary two-slope inequalities. Therefore, the reference given in the table refers to the papers where the construction of the inequalities was first introduced. No reference is given for the classes first discovered in this paper.

In Figures 4-7 we give the group representation of a member of each of the families of functions that we proved to be facet-defining. We find several new families of facets whose structures are unlike those that were described in the literature. First, we obtain twelve different families of two-slope inequalities. Many of these inequalities are described in previous papers. They include (a), the Gomory mixed integer cut [8]; (e1),(f1),(o), and (p1), homomorphisms of the Gomory mixed integer cut [8] or k-cuts [3]; (h) and (r), two-peak functions [1]; (k1), Gomory and Johnson's two-slope function [11] and two-step MIR [4]. Some new functions include (l) and (q). These functions do not fit any of the known patterns of two-slope functions as neither their lower nor their upper peaks are symmetrical and aligned to the origin. We also obtain ten different families of three-slope inequalities. They include (b) and (c), the simple three-slope functions of [1]; and (d1) and (n1), the extension of three-slope functions presented in [11]. All other three-slope functions in the table are new. Observe that (n2) and (p2) have constructions similar to that presented in [11] but do not follow directly from that construction. Also (e2) and (f2) can be seen as slanted versions of homomorphisms of the Gomory mixed integer cut in which the lower peaks are aligned to the origin. Function (g) is different from other functions for the group problem as the linear segments on J_3 and J_5 are not aligned to the origin. Finally, (i) may be understood as a homomorphism of the Gomory mixed integer cut that is slightly modified in the intervals J_3 and J_5. We also obtain (d2), which is to our knowledge the first constructive family of four-slope functions for the group problem. Function (d2) is obtained by inserting two symmetric and narrow peaks inside a Gomory mixed integer cut.

4.3 New extreme inequalities for the infinite group problem

The study of the infinite group problem was initiated by Gomory and Johnson [9, 10]. However, it remained dormant until a recent paper by Gomory and Johnson [11] revived the attention to its theoretical and practical significance. Very few families of inequalities have been proven to be extreme for $PI(I, r_0)$. Exceptions include the two-slope inequalities that were proven to be extreme by Gomory and Johnson [10] in 1972. They also gave a family of extreme three-slope inequalities in 2003 [11]. Dey et al. [5] showed very recently that not all extreme inequalities for the infinite group problem are continuous and introduced several families of discontinuous extreme inequalities.

In this section, we give conditions under which the $\text{CPL}_3^=$-extreme inequalities are extreme for $PI(I, r_0)$. This study yields many new extreme inequalities for the infinite group problem. Because the convex hull of $PI(I, r_0)$ is not polyhedral, the concept of a facet of $PI(I, r_0)$ is not well-defined. Similarly, because the number of variables in the problem is infinite, it makes little sense to discuss the dimension of the face induced by a particular inequality. Therefore, in the remainder of this section, we explore only conditions for which inequalities are extreme for $PI(I, r_0)$.

Since $\text{CPL}_3^=$-extreme inequalities are minimal (Proposition 10), they are good candidates to be extreme for $PI(I, r_0)$. However, they are not always extreme and further conditions on their parameters are necessary. An important result to prove that an inequality is extreme was introduced recently by Gomory and Johnson [11]. It is known as Interval Lemma. Its scope has been extended in Dey et al. [5] where additional tools were developed to streamline proofs for extremality. In particular, these tools also work for discontinous functions. In the following two propositions, we recall some of the needed results from these papers.

Proposition 19 *(Interval Lemma) Let $U = [u_1, u_2] \subset I$, $V = [v_1, v_2] \subset I$, and $U + V = [u_1 + v_1, u_2 + v_2] \subset I$ be such that $u_1 \neq u_2$ and $v_1 \neq v_2$. If there exists a continuous real-valued function g defined over U, V and $U + V$ such that $g(u) + g(v) = g(u + v)$ $\forall u \in U$ and $v \in V$, then g must be a straight line with constant slope s over U, V and $U + V$.* □

The following result is key for proving extremality because it relates the continuity of a non-extreme function f and the continuity of the functions whose convex combination yields the non-extreme function.

Proposition 20 *Let $f : I \to \mathbb{R}_+$ be a piecewise linear, subadditive and valid function that satisfies $f(u) = au$ $\forall 0 \leq u \leq \hat{u}$, where $a > 0$ and $\hat{u} > 0$. Assume that $f = \lambda g + (1 - \lambda)h$, where $0 \leq \lambda \leq 1$ and g and h are subadditive valid inequalities. Then g and h are continuous at all points at which f is continuous.*
□

The following proposition is a fundamental result due to Johnson [14] that is very helpful in proving the extremality of a function.

Theorem 21 *If f is extreme in the space of subadditive valid inequalities and if f is minimal, then f is extreme in the space of valid inequalities.* □

Lemma 22 and Proposition 23 will be frequently used in the proof of our main result stated in Theorem 24.

Lemma 22 *Let $f(u)$ be a continuous, piecewise linear, subadditive function defined on I that satisfies $f(u) = au$ $\forall 0 \leq u \leq \hat{u}$, where $a > 0$ and $\hat{u} > 0$. Assume that $f(u) = \frac{1}{2}g(u) + \frac{1}{2}h(u)$, where $g(u)$ and $h(u)$ are valid subadditive functions defined on I, and assume that there are two nonempty intervals U and V such that $f(u) + f(v) = f(u + v)$ $\forall u \in U$ and $v \in V$. Then $g(u)$ (resp. $h(u)$) is a linear function over intervals U, V and $U + V$ with a constant slope.*

Proof By Proposition 20, we conclude that $g(u)$ and $h(u)$ are continuous functions. Since $f(u) + f(v) = f(u + v)$ for any $u \in U$ and $v \in V$, $(g(u) + g(v) - g(u+v)) + (h(u)+h(v) - h(u+v)) = 0$. Because $g(u)$ and $h(u)$ are subadditive, we obtain that $g(u) + g(v) = g(u + v)$ and $h(u) + h(v) = h(u + v)$ for arbitrary $u \in U$ and $v \in V$. Applying Proposition 19 to $g(u)$ and $h(u)$ easily yields the result. □

Proposition 23 *Let $f(u)$ be the group representation of a $CPL_3^=$-extreme inequality. Assume that all J_i have nonempty interiors for $i = 1, \dots, 6$ and let s_i be the slope of $f(u)$ in interval J_i. Assume also that $f(u) = \frac{1}{2}g(u) + \frac{1}{2}h(u)$, where $g(u)$ and $h(u)$ are subadditive valid functions. Then*

1. *$g(u) = h(u) = \frac{u}{r_0}$ for $u \in J_1$.*
2. *$h(u)$ (resp. $h(u)$) is linear over J_6 and $g(1) = 0$ (resp. $h(1) = 0$).*
3. *If $s_i = s_1$ for some $i \in \{2, \dots, 5\}$ then $g(u)$ (resp. $h(u)$) is linear over J_i and the slope of $g(u)$ (resp. $h(u)$) in J_i is the same as in J_1.*
4. *If $s_i = s_6$ for some $i \in \{2, \dots, 5\}$ then $g(u)$ (resp. $h(u)$) is linear over J_i and the slope of $g(u)$ (resp. $h(u)$) in J_i is the same as in J_6.*

Proof Note first that f is continuous and satisfies the assumptions of Lemma 22.

1. Let $U = V = [0, \frac{r_0}{2}]$. Because f is subadditive and is linear over J_1, we conclude from Lemma 22 that $g(u)$ (resp. $h(u)$) is linear over $U + V = J_1$. Because $g(u)$ (resp. $h(u)$) is a valid function, $g(0) = 0$ (resp. $h(0) = 0$) and $g(r_0) = 1$ (resp. $h(r_0) = 1$). The result follows.
2. Let $U = V = [1 - \frac{z_1}{2}, 1]$. Because f is subadditive and is linear over J_6, we conclude from Lemma 22 that $g(u)$ (resp. $h(u)$) is linear over $U + V = J_6$. Because $g(u)$ (resp. $h(u)$) is a valid function, $g(1) = 0$ (resp. $h(1) = 0$).
3. Let $i \in \{2, \dots, 5\}$ be such that $s_i = s_1$. Assume that $\overline{J}_i = [a, b]$. First, consider the case when $r_0 > (b - a)$. In this case, it is easily seen that $f(u) + f(a) = f(u + a)$ for $u \in U = [0, b - a]$. Therefore, it follows from the proof of Lemma 22 that $g(u + a) = g(u) + g(a)$ $\forall u \in U$, i.e., $g(u)$ is linear over J_i and the slope of $g(u)$ in J_i is the same as in J_1. The proof of the case when $r_0 < (b - a)$ is similar except that the argument needs to be repeated on $U = [0, r_0]$ and each of the intervals $[a, a + r_0]$, $[a+r_0, a+2r_0]$,..., $[a+(\tilde{k}-1)r_0, a+\tilde{k}r_0]$, $[b-r_0, b]$, where $\tilde{k}$ is the smallest integer such that $a + (\tilde{k} + 1)r_0 \geq b$. Hence, $g(u)$ is a linear function over each of the above intervals and the slopes of $g(u)$ in all those intervals are the same as that in J_1. Since $f(u)$ is continuous, by Proposition 20 $g(u)$ is also continuous. Therefore, $g(u)$ is linear over $\overline{J}_i = [a, b]$ and the slope of $g(u)$ in $\overline{J}_i$ is the same as that in J_1. The same result holds for $h(u)$. Thus, the proposition holds for $J_i = (a, b]$.
4. The proof is similar to that of 3. □

To our knowledge, there is no systematic approach that allows us to find conditions under which $\text{CPL}_3^=$-extreme inequalities are extreme for $PI(I, r_0)$.

However, some conditions can typically be deduced from the corresponding conditions on the finite group problem that are given in Section 4.2 or by applying limiting arguments on these conditions. The conditions in the following theorem were obtained in this way.

Theorem 24 *The inequality* $\sum_{g \in I} f^{\beta}(g)t(g) \geq 1$ *is extreme for* $PI(I, r_0)$ *under the following conditions:*

1. $\beta = a$: $0 < r_0 < 1$ *and* $0 \leq z_1 < 1$
2. $\beta = b$: $3r_0 + 8z_1 \leq 1$
3. $\beta = c$: $3r_0 + 4z_1 \leq 1$
4. $\beta = d$: $r_0 = 2z_1$ *and* $r_0 + 8z_1 \leq 1$
5. $\beta = f$: $r_0 \leq z_1$ *and* $r_0 + 5z_1 = 1$
6. $\beta = g$: $r_0 < z_1$ *and* $2r_0 + 4z_1 = 1$
7. $\beta = h$: $r_0 + 4z_1 \leq 1 < 2r_0 + 4z_1$
8. $\beta = k$: $r_0 \leq 2z_1$ *and* $r_0 + 5z_1 = 1$
9. $\beta = l$: $r_0 = 2z_1$ *and* $6z_1 \leq 1 < 7z_1$
10. $\beta = n$: $r_0 > 2z_1$ *and* $r_0 + 8z_1 \leq 1$
11. $\beta = o$: $r_0 \geq 2z_1$ *and* $2r_0 + 2z_1 = 1$
12. $\beta = p$: $r_0 > 2z_1$ *and* $2r_0 + 2z_1 = 1$
13. $\beta = q$: $r_0 > 2z_1$ $r_0 + 4z_1 \leq 1 < r_0 + 5z_1$
14. $\beta = r$: $r_0 > 2z_1$ *and* $r_0 + 4z_1 \leq 1 \leq 2r_0 + 2z_1$

Proof The proof of Cases 1, 7, 8, 9, 11, 12, 13, and 14 are extreme under the given conditions follows directly from the Gomory and Johnson's Two-Slope Theorem; see [10, 11].

For each of the remaining cases, by Theorem 21, it is sufficient to prove that the corresponding function $f^{\beta}(u)$ is extreme in the space of subadditive valid functions. We assume for contradiction that there exist two different subadditive valid functions $g(u)$ and $h(u)$ such that $f^{\beta}(u) = \frac{1}{2}g(u) + \frac{1}{2}h(u)$. We provide the desired contradiction in two steps. First we prove that $g(u)$ and $h(u)$ are continuous functions and are piecewise linear over the same intervals as $f(u)$. Next we prove that $g(u) = h(u) = f(u)$ for $u \in I$.

By Proposition 20, since $f^{\beta}(u)$ is piecewise linear and continuous for $\beta = b, c, d, f, g$ and n, the functions $g(u)$ and $h(u)$ are also continuous. By Proposition 23, $g(u)$ (resp. $h(u)$) is linear over the intervals J_1 and J_6.

Case 2: Because $s_2 = s_3 = s_5 = s_6$, we know from Proposition 23 that $g(u)$ (resp. $h(u)$) is linear over J_2, J_3, J_5 and J_6 with identical slope over these intervals. Now consider two intervals $U = [r_0 + 2z_1, \frac{1-r_0}{2} - 2z_1]$ and $V = [r_0 + 2z_1, \frac{1+r_0}{2}]$. It is easily seen that $f(u) + f(v) = f(u+v)$ for $u \in U$ and $v \in V$ and that $U \cup V \cup (U + V) = J_4$. Therefore, from Lemma 22, we have that $g(u)$ (resp. $h(u)$) is linear over J_4.

Case 3: Because $s_2 = s_6$, we know from Proposition 23 that $g(u)$ (resp. $h(u)$) is linear over J_2 and J_6 with identical slope over these intervals. Now, note that $s_3 = s_4 = s_5$.

Consider two intervals $U = [r_0 + z_1, \frac{1-r_0}{2} - z_1]$ and $V = [r_0 + z_1, \frac{1+r_0}{2}]$. We see that $f(u) + f(v) = f(u + v)$ for $u \in U$ and $v \in V$ and that $U \cup V \cup (U + V) = J_3 \cup J_4 \cup J_5$. Therefore, from Lemma 22, we have that $g(u)$ (resp. $h(u)$) is linear over $J_3 \cup J_4 \cup J_5$.

Case 4: Because $s_2 = s_6$, we know from Proposition 23 that $g(u)$ (resp. $h(u)$) is linear over J_2 and J_6 with identical slope over these intervals. Because $s_1 = s_3 = s_5$, $g(u)$ (resp. $h(u)$) is linear over J_1, J_3 and J_5 with same slope over these intervals. Consider the two intervals $U = [\frac{1+r_0}{2}, \frac{1+r_0}{2} + 2z_1]$ and $V = [\frac{1+r_0}{2} + 2z_1, 1 - 2z_1]$. We know that $f(u) + f(v) = f(u + v)$ for $u \in U$ and $v \in V$ and that $U \cup V \cup (U + V) = J_4$. By Lemma 22, $g(u)$ (resp. $h(u)$) is linear over J_4.

Case 5: When $r_0 = z_1$, $f(u)$ becomes a two-slope function whose extremality follows from Gomory and Johnson's Two-Slope Theorem. So we only prove the case when $r_0 < z_1$. In this case, $s_2 = s_4 = s_6$, so we know from Proposition 23 that $g(u)$ (resp. $h(u)$) is linear over J_2, J_4 and J_6 with the same slope over these intervals. Now consider two intervals $U = V = [1 - 2z_1, 1 - \frac{3}{2}z_1]$. It is easily seen that $f(u) + f(v) = f(u + v)$ for $u \in U$ and $v \in V$ and that $U + V = J_3$. Therefore, by Lemma 22 we have that $g(u)$ (resp. $h(u)$) is linear over J_3 and $[1 - 2z_1, 1 - \frac{3}{2}z_1]$ with the same slope on these two invervals. Now because $f(1 - 2z_1) + f(v) = f(1 - 2z_1 + v)$ for $v \in J_5$ and $f(1 - 2z_1 + v)$ is linear in y for $1 - 2z_1 + v \in J_3$, we see that $g(u)$ (resp. $h(u)$) is linear over J_5 and that the slope over J_5 is the same as that over J_3.

Case 6: Because $s_2 = s_4 = s_6$, we know from Proposition 23 that $g(u)$ (resp. $h(u)$) is linear over J_2, J_4 and J_6 with the same slope. Consider two intervals $U = V = [r_0 + z_1, r_0 + \frac{3}{2}z_1]$. It is easily seen that $f(u) + f(v) = f(u + v)$ for $u \in U$ and $v \in V$ and that $U + V = J_5$. Therefore, by Lemma 22, $g(u)$ (resp. $h(u)$) is linear over J_5 and $[r_0 + z_1, r_0 + \frac{3}{2}z_1]$ with the same slope. Because $f(r_0 + z_1) + f(v) = f(r_0 + z_1 + v)$ for $v \in J_3$ and that $f(r_0 + z_1 + v)$ is linear in y for $r_0 + z_1 + v \in J_5$, we see that $g(u)$ (resp. $h(u)$) is linear over J_3 and its slope over J_3 is the same as that over J_5.

Case 10: The proof is the same as in Case 4.

We now show for each of the above cases that $g(u) = f(u)$ for $u \in I$. This also proves that $f(u) = g(u) = h(u)$ for $u \in I$ and therefore provides the desired contradiction. Because we already established that g is piecewise linear over J_i for $i = 1, \dots, 6$, we introduce the following notation

$$g(u) = \alpha_i u + \beta_i \text{ for } u \in \overline{J}_i \text{ where } i = 1, \dots, 6.$$

Note that it follows from Proposition 23 that $\alpha_1 = s_1$.

Case 2: We know that $\alpha_2 = \alpha_3 = \alpha_5 = \alpha_6$. Note that $r_0 + 2z_1 \in \overline{J}_4$ and $2r_0 + 4z_1 \in \overline{J}_4$. Therefore, from the proof of Lemma 22 and the fact that $f(r_0 + 2z_1) + f(r_0 + 2z_1) = f(2r_0 + 4z_1)$, we conclude that $g(r_0 + 2z_1) +$

$g(r_0 + 2z_1) = g(2r_0 + 4z_1)$. This implies that $\beta_4 = 0$. Because g is a continuous function, $g(r_0) = 1$ and $g(1) = 0$, we have that

$$1 + \alpha_2(4z_1) + \alpha_4(1 - r_0 - 4z_1) = 0.$$

Moreover, because $\beta_4 = 0$, we have that

$$1 + \alpha_2(2z_1) = \alpha_4(r_0 + 2z_1).$$

Solving for α_2 and α_4, we obtain $\alpha_2 = \frac{1-2z_1}{2z_1(1+r_0)}$ and $\alpha_3 = \frac{1}{1+r_0}$. We conclude that $g(u) = f(u)$.

Case 3: We know that $\alpha_2 = \alpha_6$ and that $\alpha_3 = \alpha_4 = \alpha_5$. Note that, because g is linear over $J_3 \cup J_4 \cup J_5$, we have $\beta_3 = \beta_4$. Note also that $r_0 + z_1 \in \overline{J}_3$ and $2r_0 + 2z_1 \in \overline{J}_4$. Therefore, from the proof of Lemma 22 and the fact that $f(r_0 + z_1) + f(r_0 + z_1) = f(2r_0 + 2z_1)$, we conclude that $g(r_0 + z_1) + g(r_0 + z_1) = g(2r_0 + 2z_1)$. This implies that $\beta_3 = 0$. Because g is a continuous function, $g(r_0) = 1$ and $g(1) = 0$, we have that

$$1 + 2\alpha_2 z_1 + (1 - r_0 - 2z_1)\alpha_3 = 0.$$

Moreover, because $\beta_3 = 0$, we have that

$$1 + \alpha_2(z_1) = \alpha_3(r_0 + z_1).$$

Solving for α_2 and α_3, we obtain $\alpha_2 = \frac{z_1-1}{z_1(1+r_0)}$ and $\alpha_4 = \frac{1}{1+r_0}$. We conclude that $g(u) = f(u)$.

Case 4: We know that $\alpha_2 = \alpha_6$ and that $\alpha_1 = \alpha_3 = \alpha_5$. Note that $1-2z_1 \in \overline{J}_4$ and $2 - 4z_1 = 1 - 4z_1 \in \overline{J}_4$. Therefore, from the proof of Lemma 22 and the fact that $f(1 - 2z_1) + f(1 - 2z_1) = f(1 - 4z_1)$, we conclude that $g(1 - 2z_1) + g(1 - 2z_1) = g(1 - 4z_1)$. This implies that $\beta_4 = -\alpha_4$. Also it is easy to verify that $\frac{1+r_0}{2} \in \overline{J}_4$. Therefore, from the proof of Lemma 22 and the fact that $f(\frac{1+r_0}{2}) + f(\frac{1+r_0}{2}) = f(r_0) = 1$, we conclude that $g(\frac{1+r_0}{2}) + g(\frac{1+r_0}{2}) = g(r_0) = 1$. This implies that $\alpha_4 = \frac{1}{r_0-1}$. Because g is a continuous function, $g(r_0) = 1$, $g(1) = 0$, $\alpha_3 = \frac{1}{r_0}$ and $\alpha_4 = \frac{1}{r_0-1}$, we have that

$$1 + 2\alpha_2 z_1 + 2\frac{1}{r_0}z_1 + \frac{1}{r_0 - 1}(1 - r_0 - 4z_1) = 0.$$

This implies that $\alpha_2 = \frac{1+r_0}{r_0(r_0-1)}$. We conclude that $g(u) = f(u)$ since in this case $r_0 = 2z_1$.

Case 5: We know that $\alpha_2 = \alpha_4 = \alpha_6$ and that $\alpha_3 = \alpha_5$. Note that $r_0 + z_1 \in \overline{J}_3$ and that $2r_0 + 2z_1 \in \overline{J}_4$ since $r_0 \le z_1$ and $r_0 + 5z_1 = 1$. Therefore, from the proof of Lemma 22 and the fact that $f(r_0 + z_1) + f(r_0 + z_1) = f(2r_0 + 2z_1)$, we conclude that $g(r_0 + z_1) + g(r_0 + z_1) = g(2r_0 + 2z_1)$ which shows that

$$\alpha_2(r_0 - z_1) + \alpha_3 z_1 = 1.$$

Because g is a continuous function, $g(r_0) = 1$ and $g(1) = 0$, we have that

$$1 + (3z_1)\alpha_2 + 2\alpha_3 z_1 = 0.$$

Solving for α_2 and α_3 we obtain that $\alpha_2 = \frac{3}{3r_0-1}$ and that $\alpha_3 = \frac{3z_1-1}{(3r_0-1)z_1}$. These slopes are identical to those of $f(u)$ since $r_0+5z_1=1$. We conclude that $g(u)=f(u)$.

Case 6: We know that $\alpha_2=\alpha_4=\alpha_6$ and that $\alpha_3=\alpha_5$. Note that $r_0+z_1 \in \overline{J}_3$ and $2r_0+2z_1 = 1-2z_1 \in \overline{J}_5$. Therefore, from the proof of Lemma 22 and the fact that $f(r_0+z_1)+f(r_0+z_1)=f(2r_0+2z_1)$, we conclude that $g(r_0+z_1)+g(r_0+z_1)=g(2r_0+2z_1)$ which shows that $2\beta_3=\beta_5$. Because g is a continuous function, $g(r_0)=1$ and $g(1)=0$, we have that

$$\begin{aligned} 1+\alpha_2 z_1 &= \beta_3+\alpha_3(r_0+z_1) \\ 1+\alpha_2(1-r_0-2z_1)+\alpha_3(2z_1) &= 0 \\ \beta_5+\alpha_3(1-z_1)+\alpha_2 z_1 &= 0. \end{aligned}$$

Solving for α_2, α_3, β_3 and β_5, we obtain that $\beta_3 = \frac{1-4z_1}{z_1(1-12z_1)}$, $\beta_5 = \frac{2-8z_1}{z_1(1-12z_1)}$, $\alpha_2 = \frac{6}{1-12z_1}$ and $\alpha_3 = \frac{2-6z_1}{z_1(-1+12z_1)}$ since $2r_0+4z_1=1$. We conclude that that $g(u)=f(u)$.

Case 10: The proof is similar to Case 4. □

Note that there is an alternate derivation of the contradiction in the second part of the proof. We discuss this derivation now. Knowing that $g(u)$ and $h(u)$ are piecewise linear and continuous, it can be argued that $g(u)$ and $h(u)$ are $\mathrm{CPL}_3^=$ functions by showing that they are linear in the intervals $J_1, J_2, ..., J_6$ and that the slopes in the corresponding intervals match. Note that there is no need to verify that the parameters θ_1 and θ_2 for the functions $g(u)$ and $h(u)$ yield non-decreasing functions ϕ, since it can be verified from the case analysis in the Appendix that the constraints $\theta_1 \geq 0$, $\theta_2 \geq 0$ and $\theta_1+\theta_2 \leq \frac{1}{2}$ are always redundant in the description of $P\Theta_3^=(z_1)$ in all the cases. Therefore, when b, c, d, f, g and n are the extreme points of $P\Theta_3^=(z_1)$, they are actually the extreme points of the polyhedron

$$\begin{aligned} \tilde{P}\Theta_3^=(z_1) = \{(\theta_1,\theta_2)\in\mathbb{R}^2 \quad | \quad & \theta_2 \geq -\phi(r_0-z_1), \\ & 2\theta_1 \leq \phi(2r_0+2z_1), \\ & 2\theta_1+\theta_2 \geq \phi(r_0+3z_1), \\ & 2\theta_1+\theta_2 \leq \phi(2r_0+3z_1), \\ & 2\theta_1+2\theta_2 \geq \phi(r_0+4z_1), \\ & 2\theta_1+2\theta_2 \leq \phi(2r_0+4z_1), \\ & \theta_1-\theta_2 \geq 0\}. \end{aligned}$$

It follows that $f(u)=g(u)=h(u)$ since $f(u)$ is $\mathrm{CPL}_3^=$-extreme. This derivation is insightful because it implies that, provided that there are sufficiently many additive relations in $f(u)$ to prove that $g(u)$ and $h(u)$ are piecewise linear in the same intervals as $f(u)$, then the extremality of $f(u)$ in the set of $\mathrm{CPL}_3^=$ implies its extremality in the set of subadditive valid inequalities.

Next observe that not all the conditions for facets of the finite group problem naturally lead to the conditions for the infinite group problem. In particular, if the values of parameters r_0 and z_1 are some fixed numbers which are independent of the order of the finite group, then refining the grid changes the form of the function and extremality is lost.

It was shown in Dey et al. [5] that, under some conditions, the limiting function of a sequence of extreme functions on finer and finer grids is an extreme function for the infinite group problem. Note that not all the limiting functions of $\mathrm{CPL}_3^=$-extreme functions are continuous. The limiting functions of $\mathrm{CPL}_3^=$-extreme inequalities that are potentially interesting are those of (d3), (e), (i), (n2) and (p2). We observe that the limiting functions of (d3) and (e) are not of much interest to us since they result in inequalities for group problems with $r_0 = 0$. It is interesting to note that the limiting function of (n2) corresponds to the Gomory mixed integer cut (GMIC), and the limiting function of (p2) is a homomorphism of the GMIC with parameters $r_0 = \frac{1}{3}$ and $z_1 = \frac{1}{6}$. The only function whose limiting function is discontinuous is (i). This limit is described in Theorem 25 and is extreme. We note that this function can be seen as an improvement of the Gomory fractional cut and was first described by Letchford and Lodi [15].

Theorem 25 *The discontinuous function $f(u) = 2u$ for $0 \leq u \leq \frac{1}{2}$ and $f(u) = 2u - 1$ for $\frac{1}{2} < u < 1$ is extreme for $PI(I, \frac{1}{2})$.*

Proof The proof follows from Proposition 20 of Dey et al. [5]. □

The fact that extreme functions of the infinite group problem morph into one another has been discussed by Gomory and Johnson [11]. The above theorem illustrates that some continuous extreme functions may become discontinuous in the process.

Despite being a very simple family of superadditive valid functions, the $\mathrm{CPL}_3^=$-extreme functions provide an abundant supply of extreme inequalities for the infinite group problem, some of which are structurally different from previously known inequalities. For instance, the extreme inequalities (l) and (q) have different structures than the standard two-slope construction by Gomory and Johnson [11], and the extreme inequalities (f2) and (g) are two new three-slope families. As we further investigate other CPL_n-extreme functions, we expect to obtain a wider variety of extreme inequalities for $PI(I, r_0)$.

The extreme inequalities given in Theorem 24 can be easily used as cutting planes for solving integer programs. In fact, since these inequalities are valid for the infinite group problem, the only information we need to generate them is the fractional part of the variable coefficients. In particular, the determinants of LP-bases are not needed. To date, very few families of inequalities for general integer programs (and, most notably, the GMIC) have this nice property.

5 Conclusion and future research directions

In this paper, we used the approximate lifting scheme of Richard et al. [18] to obtain strong valid inequalities for general mixed-integer programs and group problems. We studied a specific family of continuous piecewise linear functions that are called $\mathrm{CPL}_3^=$ functions, and we characterized all of the extreme functions in this family. Using this general characterization, we obtained eighteen $\mathrm{CPL}_3^=$-extreme inequalities, including the Gomory mixed integer cut and representatives of all known two- and three-slope facets of the finite group problems. We analyzed the corresponding inequalities within the framework of finite and infinite group problems. For finite group problems, we derived new families of two- and three-slope facet-defining inequalities that are structurally different from those previously known. We also derived the first family of four-slope facets. For infinite group problems, we derived new families of two- and three-slope continuous extreme inequalities, as well as discontinuous extreme inequalities. These results are significant because they contribute to a better understanding of the polyhedral structure of group problems and of the diversity of their strong inequalities. More importantly, these results provide a large variety of new cuts for solving integer and mixed-integer programs.

This paper opens several directions for future research. Here, we derived closed-form expressions for several new strong inequalities through the analysis of the extreme points of $P\Theta_n(z)$. However, this study was limited to a small value of n. A further study of $P\Theta_n(z)$ for large n is possible but may require extensive case analysis. There are two ways to circumvent this difficulty. First, we may restrict the analysis to some subset of CPL_n functions by limiting the number of independent parameters, as we did for $\text{CPL}_3^=$. The second solution is to automate the case analysis using symbolic computing. We are currently pursuing this direction.

The empirical evaluation of the ideas proposed in this paper is another important direction in the future. The computational implementation should use the closed-form cutting planes obtained from the analytical study of the extreme points of $P\Theta_n(z)$ for small n, as well as the cutting planes obtained by numerically solving linear programs over $P\Theta_n(z)$ for large n. We believe that efficient separation procedures can be derived to reduce the solution time of general integer programs.

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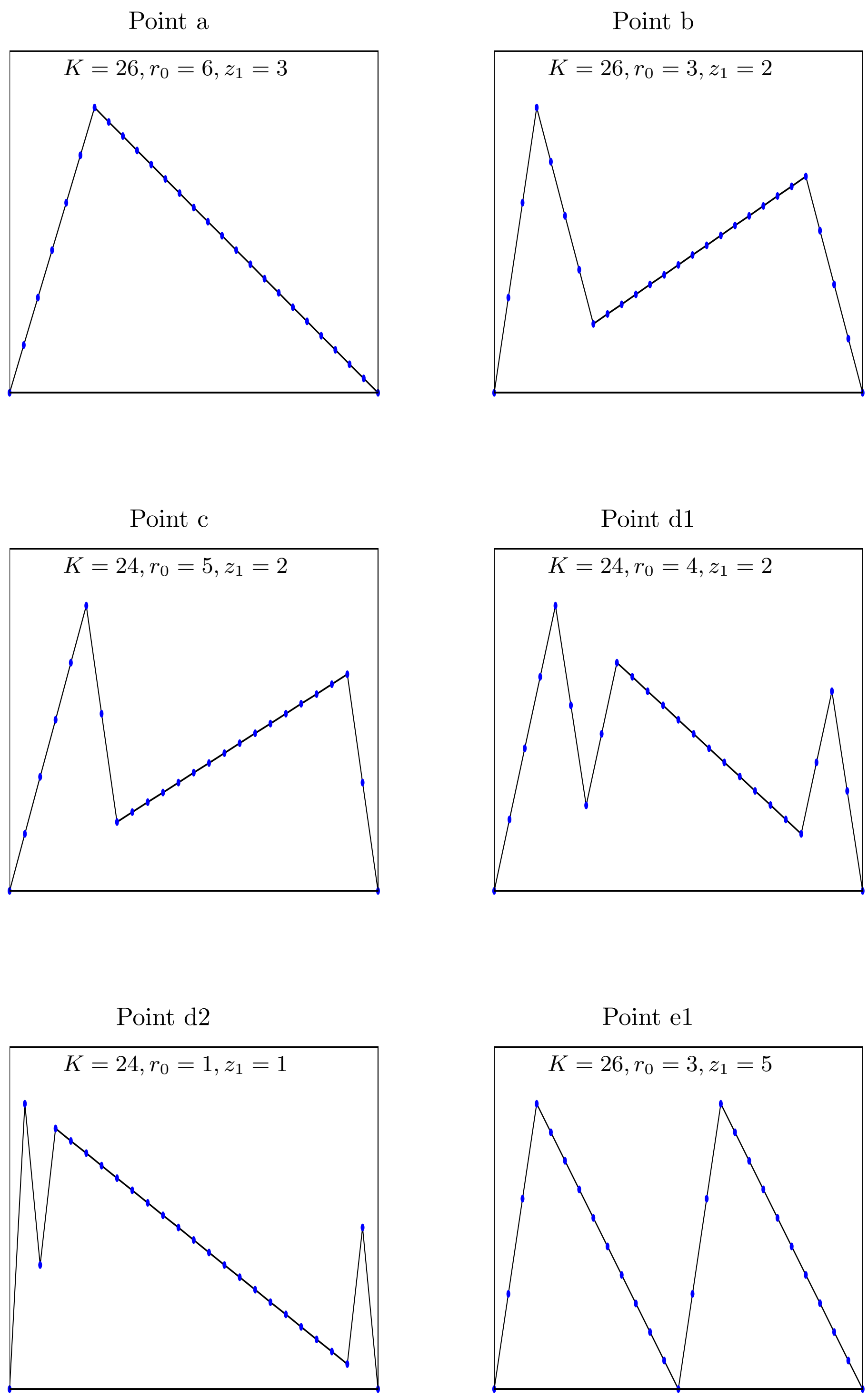

Figure 4: Group representations induced by extreme points $a - e$

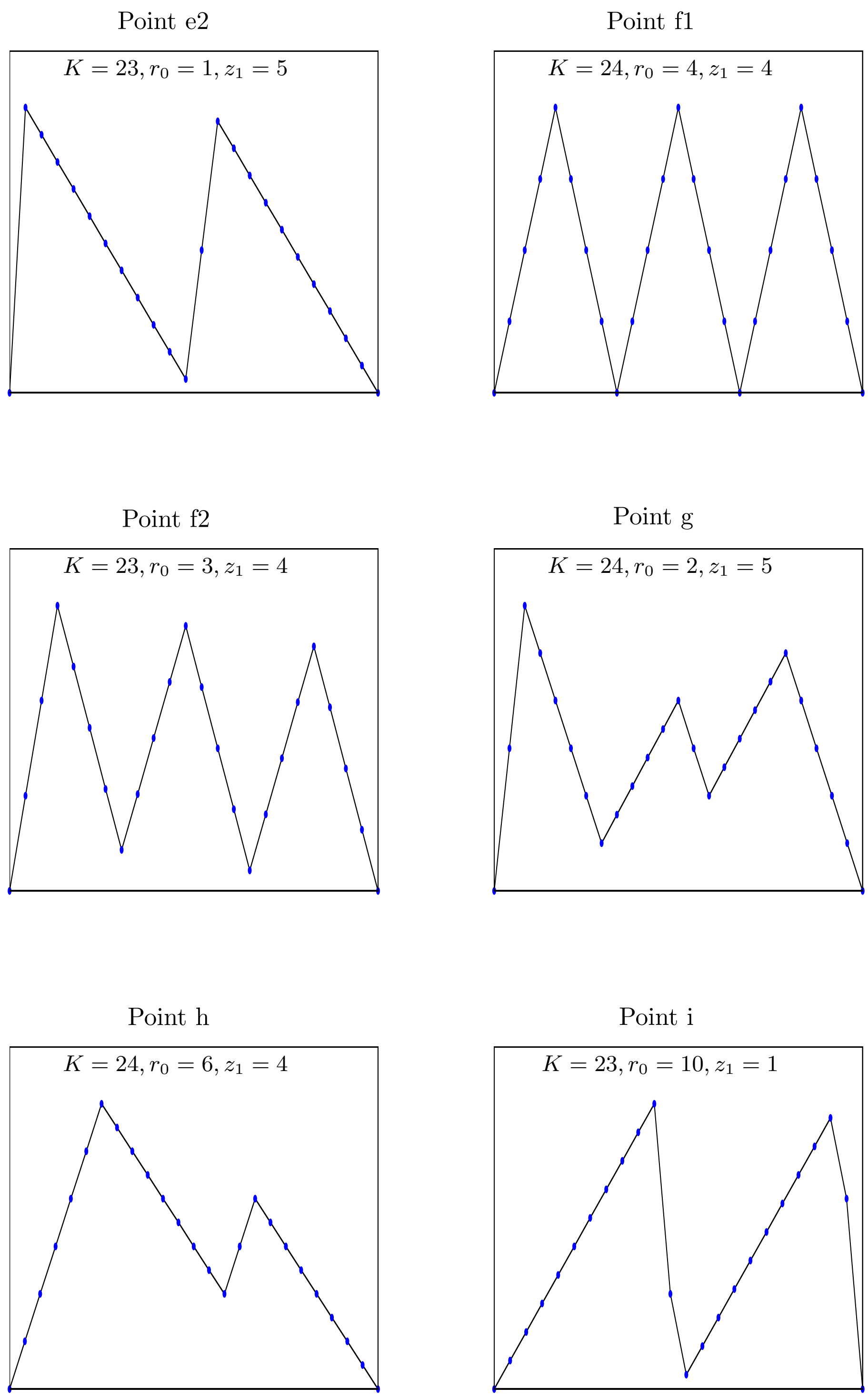

Figure 5: Group representations induced by extreme points $e - i$

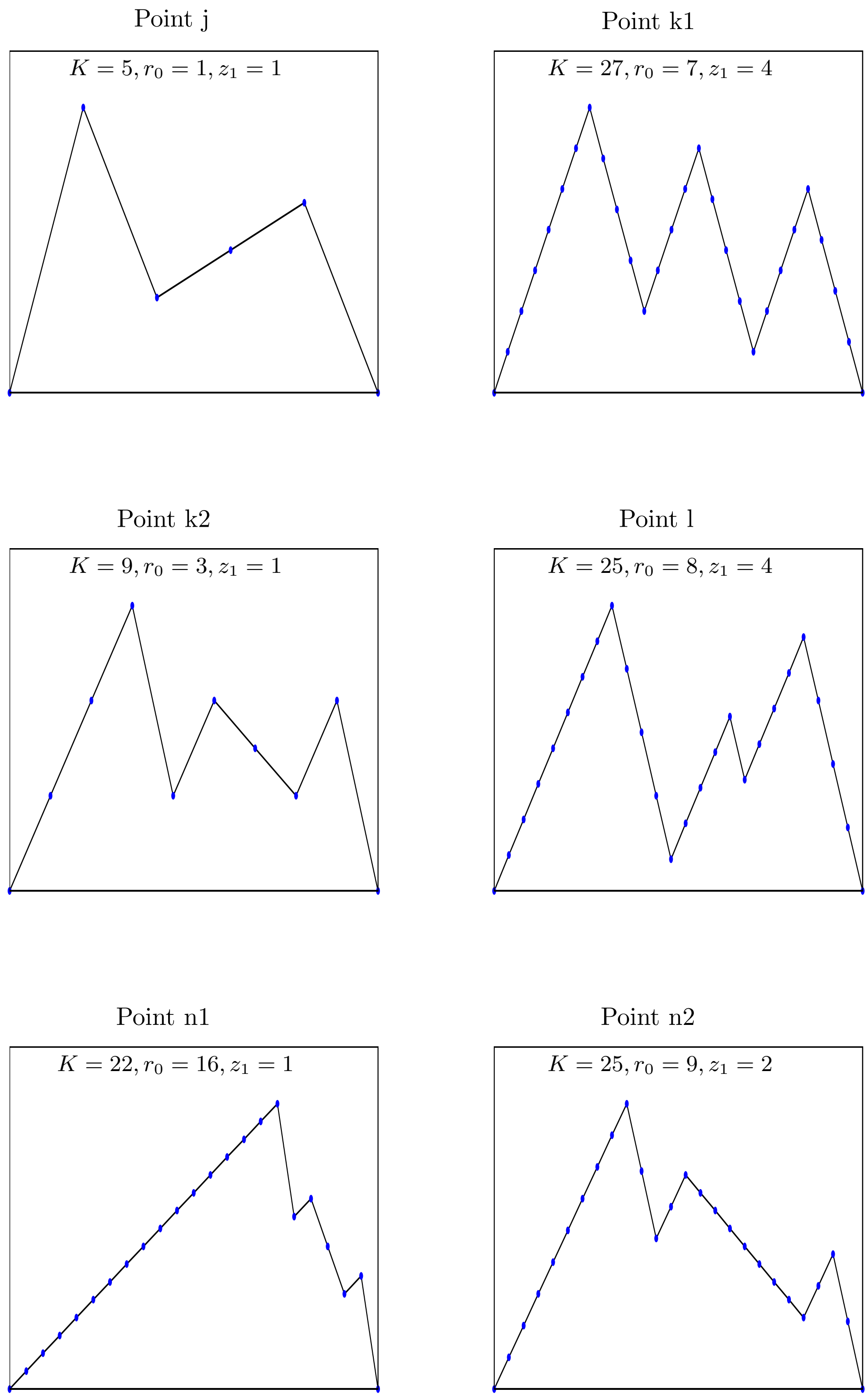

Figure 6: Group representations induced by extreme points $j - n$

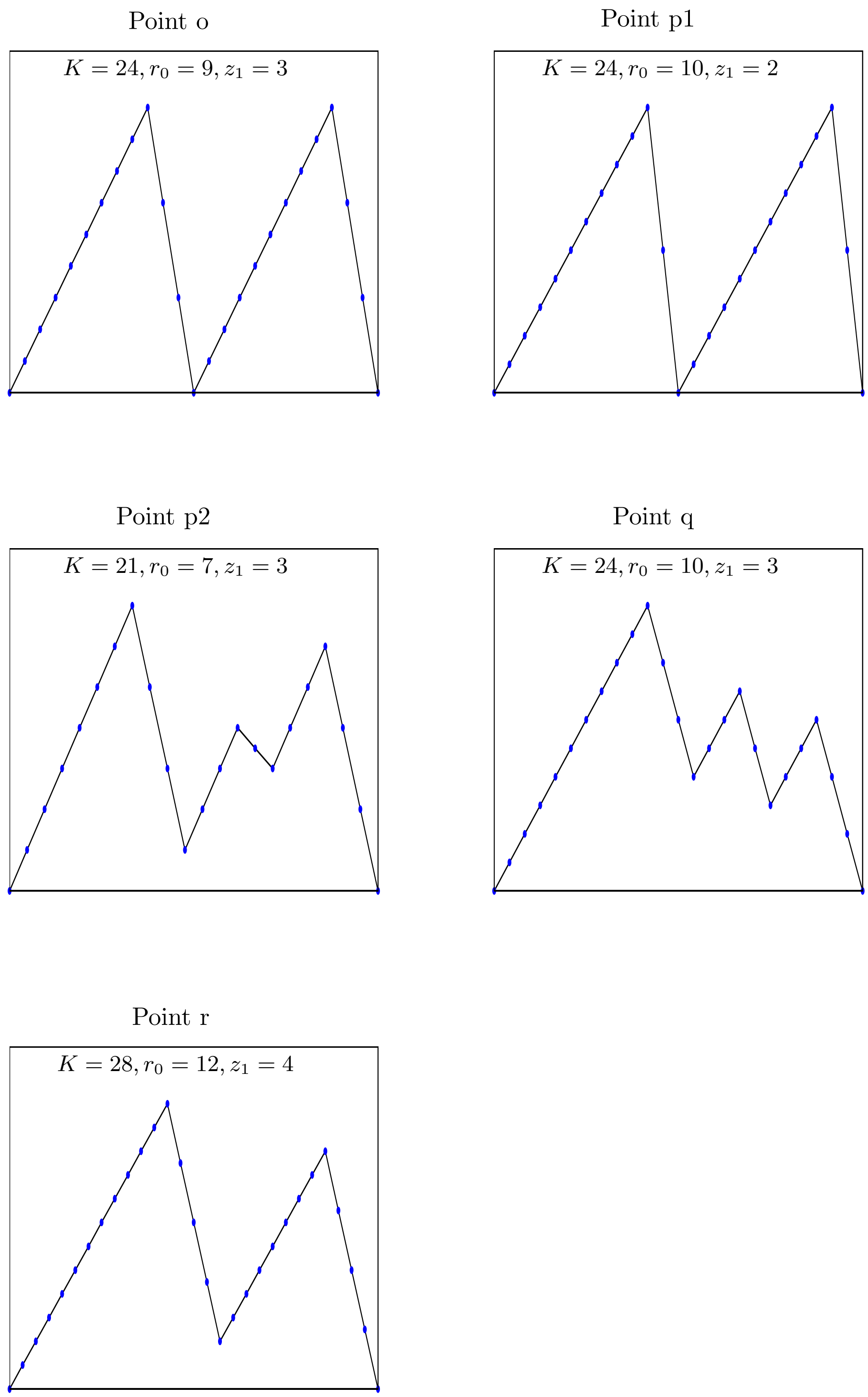

Figure 7: Group representations induced by extreme points $o - r$

	Case 1: $0 < r_0 < z_1$	Case 2: $r_0 = z_1$	Case 3: $z_1 < r_0 < 2z_1$
Subcase 1	$2r_0 + 6z_1 \leq K$	$2r_0 + 6z_1 \leq K$	$2r_0 + 6z_1 \leq K$
Subcase 2	$r_0 + 6z_1 \leq K < 2r_0 + 6z_1$	$2r_0 + 5z_1 \leq K < 2r_0 + 6z_1$	$2r_0 + 5z_1 \leq K < 2r_0 + 6z_1$
Subcase 3	$2r_0 + 5z_1 \leq K < r_0 + 6z_1$	$2r_0 + 4z_1 < K < 2r_0 + 5z_1$	$r_0 + 6z_1 \leq K < 2r_0 + 5z_1$
Subcase 4	$r_0 + 5z_1 \leq K < 2r_0 + 5z_1$	$K = 2r_0 + 4z_1$	$2r_0 + 4z_1 < K < r_0 + 6z_1$
Subcase 5	$2r_0 + 4z_1 < K < r_0 + 5z_1$	$r_0 + 4z_1 \leq K < 2r_0 + 4z_1$	$K = 2r_0 + 4z_1$
Subcase 6	$K = 2r_0 + 4z_1$	$-$	$r_0 + 5z_1 \leq K < 2r_0 + 4z_1$
Subcase 7	$r_0 + 4z_1 \leq K < 2r_0 + 4z_1$	$-$	$2r_0 + 3z_1 \leq K < r_0 + 5z_1$
Subcase 8	$-$	$-$	$r_0 + 4z_1 \leq K < 2r_0 + 3z_1$
	Case 4: $r_0 = 2z_1$	Case 5: $2z_1 < r_0 \leq 3z_1$	Case 6: $3z_1 < r_0 \leq 4z_1$
Subcase 1	$2r_0 + 6z_1 \leq K$	$2r_0 + 6z_1 \leq K$	$2r_0 + 6z_1 \leq K$
Subcase 2	$2r_0 + 5z_1 \leq K < 2r_0 + 6z_1$	$2r_0 + 5z_1 \leq K < 2r_0 + 6z_1$	$2r_0 + 5z_1 \leq K < 2r_0 + 6z_1$
Subcase 3	$2r_0 + 4z_1 < K < 2r_0 + 5z_1$	$2r_0 + 4z_1 < K < 2r_0 + 5z_1$	$2r_0 + 4z_1 < K < 2r_0 + 5z_1$
Subcase 4	$K = 2r_0 + 4z_1$	$K = 2r_0 + 4z_1$	$K = 2r_0 + 4z_1$
Subcase 5	$2r_0 + 3z_1 \leq K < 2r_0 + 4z_1$	$r_0 + 6z_1 \leq K < 2r_0 + 4z_1$	$2r_0 + 3z_1 \leq K < 2r_0 + 4z_1$
Subcase 6	$r_0 + 4z_1 \leq K < 2r_0 + 3z_1$	$2r_0 + 3z_1 \leq K < r_0 + 6z_1$	$r_0 + 6z_1 \leq K < 2r_0 + 3z_1$
Subcase 7	$-$	$r_0 + 5z_1 \leq K < 2r_0 + 3z_1$	$2r_0 + 2z_1 \leq K < r_0 + 6z_1$
Subcase 8	$-$	$2r_0 + 2z_1 \leq K < r_0 + 5z_1$	$r_0 + 5z_1 \leq K < 2r_0 + 2z_1$
Subcase 9	$-$	$r_0 + 4z_1 \leq K < 2r_0 + 2z_1$	$r_0 + 4z_1 \leq K < r_0 + 5z_1$
	Case 7: $r_0 > 4z_1$		
Subcase 1	$2r_0 + 6z_1 \leq K$		
Subcase 2	$2r_0 + 5z_1 \leq K < 2r_0 + 6z_1$		
Subcase 3	$2r_0 + 4z_1 < K < 2r_0 + 5z_1$		
Subcase 4	$K = 2r_0 + 4z_1$		
Subcase 5	$2r_0 + 3z_1 \leq K < 2r_0 + 4z_1$		
Subcase 6	$2r_0 + 2z_1 \leq K < 2r_0 + 3z_1$		
Subcase 7	$r_0 + 6z_1 \leq K < 2r_0 + 2z_1$		
Subcase 8	$r_0 + 5z_1 \leq K < r_0 + 6z_1$		
Subcase 9	$r_0 + 4z_1 \leq K < r_0 + 5z_1$		

Table 1: List of cases

Extreme point	θ_1	θ_2	Range of r_0	Range of K
a	$\frac{z_1}{K-r_0}$	$\frac{z_1}{K-r_0}$	all	all
b	$\frac{r_0+2z_1}{2K+2r_0}$	$\frac{r_0+2z_1}{2K+2r_0}$	all	$2r_0+6z_1 \leq K$
c	$\frac{r_0+z_1}{K+r_0}$	$\frac{z_1}{K+r_0}$	all	$2r_0+4z_1 < K$
d	$\frac{r_0+2z_1}{2K-2r_0}$	$\frac{2z_1-r_0}{2K-2r_0}$	$0 < r_0 \leq 2z_1$	$r_0+6z_1 \leq K$
e	$\frac{z_1}{K-2r_0}$	$\frac{z_1}{K-2r_0}$	all	$2r_0+4z_1 \leq K < 2r_0+6z_1$
f	$\frac{-Kz_1-Kr_0+6z_1r_0+r_0^2+4z_1^2}{4Kz_1+8z_1r_0-K^2+r_0^2}$	$\frac{z_1(2r_0+4z_1-K)}{4Kz_1+8z_1r_0-K^2+r_0^2}$	$0 < r_0 < 2z_1$	$\max\{r_0+5z_1, 2r_0+4z_1\} \leq K < r_0+6z_1$
g	$\frac{z_1}{K-3r_0}$	$\frac{z_1-r_0}{K-3r_0}$	$0 < r_0 < z_1$	$2r_0+4z_1 \leq K < r_0+5z_1$
h	$\frac{1}{4}$	$\frac{1}{4}$	all	$r_0+4z_1 \leq K < 2r_0+4z_1$
i	$\frac{K-2r_0-5z_1}{2(K-2r_0-6z_1)}$	$\frac{-z_1}{2(K-2r_0-6z_1)}$	all	$\max\{r_0+4z_1, 2r_0+3z_1\} \leq K < 2r_0+4z_1$
j	$\frac{z_1(2r_0+5z_1-K)}{-K^2+3Kr_0+5Kz_1-2r_0^2-7z_1r_0}$	$\frac{z_1(r_0+5z_1-K)}{-K^2+3Kr_0+5Kz_1-2r_0^2-7z_1r_0}$	$0 < r_0 < 2z_1$	$\max\{r_0+4z_1, 2r_0+3z_1\} \leq K < \min\{r_0+5z_1, 2r_0+4z_1\}$
k	$\frac{1}{3}$	0	$r_0 > 0$ $r_0 > 3z_1$	$\max\{r_0+5z_1, 2r_0+3z_1\} \leq K \leq 2r_0+4z_1$ $r_0+5z_1 \leq K < r_0+6z_1$
l	$\frac{z_1}{K-2r_0}$	$\frac{2z_1-r_0}{2K-4r_0}$	$0 < r_0 \leq 2z_1$	$r_0+4z_1 \leq K < 2r_0+3z_1$
m	$\frac{r_0}{K+r_0-4z_1}$	0	$r_0 > 2z_1$	$2r_0+4z_1 < K$
n	$\frac{2z_1}{K-r_0}$	0	$r_0 > 2z_1$	$r_0+6z_1 \leq K$
o	$\frac{z_1}{K-2r_0}$	$\frac{K-2r_0-2z_1}{2K-4r_0}$	$r_0 > z_1$	$\max\{r_0+4z_1, 2r_0+2z_1\} \leq K < 2r_0+3z_1$
p	$\frac{z_1}{K-2r_0}$	0	$r_0 > 2z_1$	$2r_0+2z_1 \leq K < 2r_0+3z_1$
q	$\frac{z_1}{K-r_0-2z_1}$	0	$r_0 > 2z_1$	$r_0+4z_1 \leq K < r_0+5z_1$
r	$\frac{1}{2}$	0	$r_0 > 2z_1$	$r_0+4z_1 \leq K \leq 2r_0+2z_1$

Table 2: Extreme points of $P\Theta_3^{=}(z_1)$ in all cases

Ext. pnt	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$
a	$\frac{1}{r_0}$	$\frac{1}{r_0-K}$	s_2	s_2	s_2	s_2
b	$\frac{1}{r_0}$	$\frac{2z_1-K}{2z_1(K+r_0)}$	s_2	$\frac{1}{K+r_0}$	s_2	s_2
c	$\frac{1}{r_0}$	$\frac{z_1-K}{z_1(K+r_0)}$	$\frac{1}{K+r_0}$	s_3	s_3	s_2
d	$\frac{1}{r_0}$	$\frac{2z_1+K}{2z_1(r_0-K)}$	$\frac{K-2z_1}{2z_1(K-r_0)}$	$\frac{1}{r_0-K}$	s_3	s_2
e	$\frac{1}{r_0}$	$\frac{2}{2r_0-K}$	s_2	$\frac{(K-2r_0-8z_1)}{(K-2r_0)(-K+r_0+4z_1)}$	s_2	s_2
f	$\frac{1}{r_0}$	$\frac{z_1r_0+8z_1^2-Kr_0-6z_1K+K^2}{z_1(r_0^2+4z_1K+8z_1r_0-K^2)}$	$\frac{r_0+8z_1-2K}{r_0^2+4z_1K+8z_1r_0-K^2}$	$\frac{r_0+8z_1-K}{r_0^2+4z_1K+8z_1r_0-K^2}$	s_3	s_2
g	$\frac{1}{r_0}$	$\frac{3}{3r_0-K}$	$\frac{K-3z_1}{z_1(K-3r_0)}$	s_2	s_3	s_2
h	$\frac{1}{r_0}$	$\frac{4z_1-K}{4z_1r_0}$	s_2	s_1	s_2	s_2
i	$\frac{1}{r_0}$	$\frac{-7z_1K+4z_1r_0+12z_1^2+K^2-2Kr_0}{2r_0z_1(-K+2r_0+6z_1)}$	$\frac{-3K+4r_0+12z_1}{2r_0(-K+2r_0+6z_1)}$	s_1	s_3	s_2
j	$\frac{1}{r_0}$	$\frac{-K+2r_0+7z_1}{K^2-3Kr_0-5z_1K+2r_0^2+7z_1r_0}$	$\frac{-2K+2r_0+7z_1}{K^2-3Kr_0-5z_1K+2r_0^2+7z_1r_0}$	s_2	s_3	s_2
k	$\frac{1}{r_0}$	$\frac{3z_1-K}{3z_1r_0}$	s_1	$\frac{2K-3r_0-12z_1}{3r_0(K-r_0-4z_1)}$	s_1	s_2
l	$\frac{1}{r_0}$	$\frac{2}{2r_0-K}$	$\frac{K-4z_1-K}{2z_1(K-2r_0)}$	s_2	s_3	s_2
m	$\frac{1}{r_0}$	$\frac{-z_1K-z_1r_0+4z_1^2+Kr_0}{r_0z_1(-K-r_0+4z_1)}$	s_1	$\frac{-r_0+4z_1}{r_0(-K-r_0+4z_1)}$	s_1	s_2
n	$\frac{1}{r_0}$	$\frac{K+r_0}{r_0(r_0-K)}$	s_1	$\frac{1}{r_0-K}$	s_1	s_2
o	$\frac{1}{r_0}$	$\frac{2}{2r_0-K}$	$\frac{4z_1K-4z_1r_0-K^2+2Kr_0}{2r_0z_1(K-2r_0)}$	s_1	s_3	s_2
p	$\frac{1}{r_0}$	$\frac{2}{2r_0-K}$	s_1	$\frac{2z_1K-10z_1r_0+Kr_0}{r_0(K-2r_0)(4z_1-K+r_0)}$	s_1	s_2
q	$\frac{1}{r_0}$	$\frac{r_0+2z_1}{r_0(-K+r_0+2z_1)}$	s_1	s_2	s_1	s_2
r	$\frac{1}{r_0}$	$\frac{2z_1-K}{2z_1r_0}$	s_1	s_1	s_1	s_2

Table 3: Slopes of $CPL_3^{=}$-extreme functions $f(r)$

Extreme point	Face dimension	2-slope facet	3-slope facet	4-slope facet
a	$= K - 2$	always [8]	NA	NA
b	$\geq K - 2 - \max\{0, \lceil \frac{K+r_0}{2} \rceil + r_0 + 4z_1 - K - 1\}$	NA	$K \geq 3r_0 + 8z_1$ [1]	NA
c	$\geq K - 2 - \max\{0, \lceil \frac{K+r_0}{2} \rceil + r_0 + 2z_1 - K - 1\}$	NA	$K \geq 3r_0 + 4z_1$ [1]	NA
d	$\geq K - z_1 - 1 - \max\{0, -\lfloor \frac{K+r_0}{2} \rfloor + 4z_1 + r_0 - 1\}$	NA	1) $r_0 = 2z_1,\ K \geq r_0 + 8z_1$ [12] 3) $K = 8, r_0 = 2, z_1 = 1$ [1]	2) $r_0 = z_1 = 1,\ K \geq 9$ 4) $K = 7, r_0 = 1, z_1 = 1$ [1]
e	$\geq \lfloor \frac{K+r_0}{2} \rfloor + 2z_1 - 1$	1) $K = 2r_0 + 4z_1$ [8]	2) $r_0 = 1,\ K = 3r_0 + 4z_1$	NA
f	$\geq \lfloor \frac{K+r_0}{2} \rfloor + 7z_1 + r_0 - K - 1$	1) $r_0 = z_1$, $K = r_0 + 5z_1$ [8]	2) $r_0 \leq z_1 - 1,\ K = r_0 + 5z_1$	NA
g	$\geq K - z_1 - 1$	NA	$K = 2r_0 + 4z_1$	-
h	$= K - 2$	always [1]	NA	NA
i	$\geq K - z_1 - 1$	NA	$z_1 = 1,\ K = 2r_0 + 3z_1$	NA
j	$\geq K - z_1 - 1$	NA	$K = 5, r_0 = 1, z_1 = 1$ [1]	NA
k	$\geq \lfloor \frac{K+r_0}{2} \rfloor + 2z_1 - 1$	1) $K = r_0 + 5z_1$ [4, 11]	2) $K = 9, r_0 = 3, z_1 = 1$ [1] 3) $K = 8, r_0 = 2, z_1 = 1$ [1]	NA
l	$\geq K - z_1 - 1$	$r_0 = 2z_1$	-	NA
m	$\geq \lfloor \frac{K+r_0}{2} \rfloor + 2z_1 - 1$	NA	-	NA
n	$\geq \lceil \frac{K+r_0}{2} \rceil + 2z_1 - 2$	NA	1) $r_0 \geq 2z_1 + 1,\ K \geq 8z_1 + r_0$ [11] 2) $z_1 = 1,\ r_0 \geq 3,\ K = 6z_1 + r_0$	NA
o	$\geq K - z_1 - 1$	$K = 2r_0 + 2z_1$ [8]	-	NA
p	$\geq \lfloor \frac{K+r_0}{2} \rfloor + 2z_1 - 1$	1) $K = 2r_0 + 2z_1$ [8]	2) $r_0 = 2z_1 + 1,\ K = 3r_0,\ z_1 \geq 2$	NA
q	$= K - 2$	always	NA	NA
r	$= K - 2$	always [1]	NA	NA

Table 4: High-dimension faces and facets from the extreme points of $P\Theta_3^{=}(z_1)$