An inexact primal-dual path following algorithm for convex quadratic SDP

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Abstract

We propose primal-dual path-following Mehrotra-type predictor-corrector methods for solving convex quadratic semidefinite programming (QSDP) problems of the form: $\min_X\{\frac{1}{2}X \bullet \mathcal{Q}(X) + C \bullet X : \mathcal{A}(X) = b, X \succeq 0\}$, where $\mathcal{Q}$ is a self-adjoint positive semidefinite linear operator on $\mathcal{S}^n$, $b \in R^m$, and $\mathcal{A}$ is a linear map from $\mathcal{S}^n$ to R^m. At each interior-point iteration, the search direction is computed from a dense symmetric indefinite linear system (called the augmented equation) with dimension $m+n(n+1)/2$. Such linear systems are very large when n is larger than a few hundreds and can only be solved by iterative methods. We propose three classes of preconditioners for the augmented equation, and show that the corresponding preconditioned matrices have favorable asymptotic eigenvalue distributions for fast convergence under suitable non-degeneracy assumptions. We are able to solve the augmented equation efficiently via the preconditioned symmetric quasi-minimal residual iterative method with the preconditioners constructed. Numerical experiments on a variety of QSDPs with matrices of dimensions up to 1600 are performed and the computational results show that our methods are efficient and robust.

1 Introduction

We consider the following convex quadratic semidefinite program (QSDP):

$$\begin{array}{ll} (QSDP) & \min_X \ \frac{1}{2}X \bullet \mathcal{Q}(X) + C \bullet X \\ & \mathcal{A}(X) \ = \ b, \quad X \ \succeq \ 0, \end{array} \tag{1}$$

where $\mathcal{Q} : \mathcal{S}^n \to \mathcal{S}^n$ is a given self-adjoint positive semidefinite linear operator in $\mathcal{S}^n$ (the space of $n \times n$ symmetric matrices endowed with the standard trace inner product denoted by "$\bullet$"), $\mathcal{A} : \mathcal{S}^n \to \boldsymbol{R}^m$ is a linear map, and $b \in \boldsymbol{R}^m$. The notation $X \succeq 0$ means that X is positive semidefinite.

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We use the following notation and terminologies. For an integer n, we let $\bar{n} = n(n+1)/2$. Given $U \in \boldsymbol{R}^{q\times l}, V \in \boldsymbol{R}^{q\times n}$, the symmetrized Kronecker product $U \circledast V$ is the linear map from $\boldsymbol{R}^{n\times l}$ to $\mathcal{S}^q$ defined by $U \circledast V(M) = (VMU^T + UM^TV^T)/2$. For $U \in \boldsymbol{R}^{p\times l}$ and $V \in \boldsymbol{R}^{q\times n}$, the Kronecker product $U \otimes V$ is the liner map from $\boldsymbol{R}^{n\times l}$ to $\boldsymbol{R}^{q\times p}$ defined by $U \otimes V(M) = VMU^T$; see [16, p. 254]. We use $U \circ V$ to denote the Hadamard product of two matrices U, V with the same dimensions. The set of symmetric positive semidefinite (definite) matrices is denoted by $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$). We use $\|\cdot\|_2$ to denote the vector 2-norm or matrix 2-norm, and $\|\cdot\|_F$ to denote the Frobenius norm. The notation $x = \Theta(\nu)$ means that there exist constants $c_1, c_2 > 0$ independent of ν such that $c_1\nu \le x \le c_2\nu$. We denote the identity matrix or operator of dimension d by I_d.

Given a self-adjoint linear operator $\mathcal{V}$ defined on a finite dimensional inner product space, $\lambda_j(\mathcal{V})$ denotes the jth eigenvalue of $\mathcal{V}$ (sorted in ascending order). We denote the set of eigenvalues of $\mathcal{V}$ by $\mathrm{eig}(\mathcal{V})$. The largest and smallest eigenvalues of $\mathcal{V}$ in magnitudes are denoted by $\lambda_{\max}(\mathcal{V})$ and $\lambda_{\min}(\mathcal{V})$, respectively. The condition number of $\mathcal{V}$ is denoted by $\kappa(\mathcal{V}) := |\lambda_{\max}(\mathcal{V})|/|\lambda_{\min}(\mathcal{V})|$. For two matrices P and Q, $[P; Q]$ denotes the matrix obtained by appending Q to the last row of P. For a linear map $\mathcal{T} : (\mathcal{X}, \bullet) \to (\mathcal{Y}, \bullet)$, where $\mathcal{X} = \boldsymbol{R}^{k\times l}$ or $\mathcal{S}^l$, and $\mathcal{Y} = \boldsymbol{R}^{p\times q}$ or $\mathcal{S}^q$, we define $\|\mathcal{T}\| = \max\{\|\mathcal{T}(M)\|_F : \|M\|_F \le 1\}$. The adjoint of $\mathcal{T}$ is denoted by $\mathcal{T}^T$. The null space of $\mathcal{T}$ is denoted by $\mathcal{N}(\mathcal{T})$. The matrix representation of $\mathcal{T}$ with respect to the canonical orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}$ is denoted by $\mathrm{mat}(\mathcal{T})$. We will typically identify $\mathcal{T}$ with $\mathrm{mat}(\mathcal{T})$ and a phrase such as "the matrix $\mathcal{T}$" means the matrix representation of $\mathcal{T}$. Note that $\|\mathcal{T}\| = \|\mathrm{mat}(\mathcal{T})\|_2$. For self-adjoint linear operators $\mathcal{S}, \mathcal{T} : (\mathcal{X}, \bullet) \to (\mathcal{X}, \bullet)$, the notation $\mathcal{S} \preceq \mathcal{T}$ means that $M \bullet \mathcal{S}(M) \le M \bullet \mathcal{T}(M)$ for all $M \in \mathcal{X}$. For a self-adjoint linear operator $\mathcal{V}$ defined on a finite dimensional inner product space, note that $\|\mathcal{V}\| = |\lambda_{\max}(\mathcal{V})|$, and if $\mathcal{V}$ is invertible, then $\|\mathcal{V}^{-1}\| = 1/|\lambda_{\min}(\mathcal{V})|$.

The Lagrangian dual problem for (1) is given as follows:

$$\begin{array}{ll} (QSDD) & \max_{X,y,S} \ -\frac{1}{2}X \bullet \mathcal{Q}(X) + b^T y \\ & \mathcal{A}^T(y) - \mathcal{Q}(X) + S \ = \ C, \quad S \ \succeq \ 0. \end{array} \tag{2}$$

Let p be the rank of $\mathcal{Q}$. By considering the Cholesky factorization of $\mathcal{Q}$, it is readily shown that (1) can be reformulated as a standard semidefinite-quadratic-linear programming (SQLP) by introducing an additional p linear constraints and $p+1$ variables. Unfortunately, the computational cost required to solve the reformulated problem grows at least like $\Theta((m+p)^3)$ and the memory requirement grows like $\Theta((m+p)^2)$. Thus unless $m+p$ is small, it not viable to solve (QSDP) by reformulating it into a standard SQLP.

The problem (QSDP) can also be reformulated as a semidefinite linear complementarity problem (SDLCP) [18]. However, the computational cost at each interior-point iteration for the SDLCP problem is the same as that for (QSDP) because both need to solve linear systems with coefficient matrices having the same dimensions and numerical properties. In [18], polynomial iteration complexities of some theoretical path-following and potential reduction methods were established. But as far as we are aware of, there is very little research on the efficient numerical computation of the solution of (QSDP) or the SDLCP problem derived from it.

In [25], a theoretical interior-point algorithm based on reducing a primal-dual potential function was proposed for (QSDP) and (QSDD). The algorithm has an iteration

complexity of $O(\sqrt{n}\log(1/\epsilon))$ for computing an ϵ-optimal solution. At each iteration, the search direction is computed from a *dense* augmented system of dimension $\bar{n}+m$. As the linear system is generally very large, the authors suggested using the conjugate gradient (CG) method to compute an approximate direction. However, as the focus of [25] was on establishing polynomial iteration complexity, the conditioning of the augmented matrix was not analyzed. Preconditioning for the CG method was also not discussed although it is crucial to precondition the method for it to be practical. There is also no numerical implementation to test the performance of the proposed method.

A prime example of (QSDP) is the nearest correlation matrix problem, where given a data matrix $K \in \mathcal{S}^n$ and a self-adjoint linear operator $\mathcal{L}$ on $\mathcal{S}^n$, we want to solve

$$\min_X \Big\{ \frac{1}{2}\|\mathcal{L}(X-K)\|_F^2 \,:\, \mathrm{diag}(X) = e, X \succeq 0 \Big\}. \tag{3}$$

Here e is the vector of ones. The QSDP resulting from (3) has $\mathcal{Q} = \mathcal{L}^2$ and $C = -\mathcal{L}^2(K)$. Previous research on QSDP were mainly on the nearest correlation matrix problem with the special choice $\mathcal{L} = U^{1/2} \circledast U^{1/2}$ (hence $\mathcal{Q} = U \circledast U$) for a given $U \in \mathcal{S}^n_+$. One of the earliest work on such a special case for (3) was by Higham [15] who proposed a modified alternating projection solution method. The paper [15] also briefly considered the problem (3) with $\mathcal{L}(X) = \Sigma \circ X$ for a given $\Sigma \in S^n \cap \boldsymbol{R}^{n\times n}_+$ (correspondingly, $\mathcal{Q}(X) = U \circ X$, with $U = \Sigma \circ \Sigma$). However, as the latter problem is considerably more difficult to solve, no practical solution method was proposed in [15].

Subsequent works on (3) for the special case where $\mathcal{L} = I$ (hence $\mathcal{Q} = I$), but extendable to the case $\mathcal{L} = U^{1/2} \circledast U^{1/2}$, include Anjos et al. [4], Malick [22], and Sun et al. [27]. The latter two papers used a Lagrangian dual approach that relied critically on the assumption that $\mathcal{Q} = U \circledast U$ to derive an analytical formula for the projection of $\boldsymbol{R}^{n\times n}$ onto $\mathcal{S}^n_+$ with respect to the norm $\|U^{1/2}(\cdot)U^{1/2}\|_F$. The recent paper [31] for (QSDP) also focused on the special case $\mathcal{Q} = U \circledast U$. It proposed efficient methods for computing the search direction at each interior-point iteration for (QSDP) by applying the symmetric quasi-minimal residual method with two suitably designed preconditioners to the $m \times m$ Schur complement equation. All the previous techniques for solving (QSDP) with $\mathcal{Q} = U \circledast U$, however, do not extend to the case of a general positive semidefinite self-adjoint linear operator $\mathcal{Q}$. The last statement holds true even if $\mathcal{Q}$ is a diagonal operator where $\mathcal{Q}(X) = U \circ X$ for some $U \in \mathcal{S}^n \cap \boldsymbol{R}^{n\times n}_+$.

The problem (QSDP) also arises from the nearest Euclidean distance matrix (EDM) problem. In [3], the EDM problem was solved by a primal-dual interior-point method for which the search direction at each iteration was computed from a linear system with dimension n^2 by a direct method. However, it appears that the direct approach in [3] is computationally viable only for small problems, say with the dimension of X less than a hundred.

While the search direction at each interior-point iteration for (QSDP) can be computed from the $m \times m$ Schur complement equation when $\mathcal{Q} = U \circledast U$ [31], the corresponding direction for a general QSDP for which $\mathcal{Q}$ does not have the special form $U \circledast U$ must be computed from a *dense* augmented equation with dimension $\bar{n}+m$. This is because reducing the augmented equation to the Schur complement equation is not viable due to the excessive memory ($\Theta(n^4)$ bytes) and computational cost ($\Theta(n^6)$ flops) required. Unfortunately, the augmented equation is generally very large, even for a moderate n. For example, $n = 500$ would lead to a system of dimension at least

125000. Thus it is not possible to solve the augmented equation by a direct method on a personal computer when n is say more than a hundred. The only other alternative is to use a preconditioned Krylov subspace iterative method. But because the augmented and Schur complement equations are quite different in structure, the preconditioning techniques described in [31] are not applicable to the augmented equation. The main purpose of this paper is to analyze the spectral property of the augmented matrix and to propose suitable preconditioners for the augmented equation so that the search direction at each interior-point iteration can be computed efficiently.

In this paper, we are primarily interested in QSDP problems for which following assumption holds:

Assumption A1. The linear map $\mathcal{A}$ is sparse or structured in the sense that if A_k denotes $\mathcal{A}^T$ acting the kth unit vector of $\boldsymbol{R}^m$, then the matrices $A_1, \ldots, A_m$ are either sparse or low-rank. We assume that m is a moderate number, say less than 5000, so that a matrix of the form $\mathcal{A}U \circledast V\mathcal{A}^T$ and its Cholesky factorization can be computed at a moderate cost under the assumption on $\mathcal{A}$. The dimension of the primal matrix X is restricted to the range of say less than 2000 so that its full eigenvalue decomposition can be computed at a moderate cost.

Our inexact primal-dual path-following method for (1) and (2) follows the same framework as the primal-dual path-following method with Mehrotra's predictor-corrector appeared in [31]. The key difference is in the computation of search directions. Here, at each interior-point iteration, the search direction is computed from a *dense* augmented equation of dimension $m + \bar{n}$, similar to the one appeared in [25]. We apply a preconditioned symmetric quasi-minimal residual (PSQMR) method [10] to solve the augmented equation.

Our contributions in this paper are as follows. We first analyze the asymptotic spectral property of the augmented matrix arising at each interior-point iteration for (QSDP). Then we design three classes of preconditioners for the augmented matrix. Under suitable conditions including nondegeneracy of the optimal solution, the preconditioned matrices are shown to have favorable asymptotic spectral distributions to accelerate the convergence of the PSQMR method used to solve the augmented equation. But note that for one of the classes of preconditioners, no nondegeneracy assumption is needed. We also addressed numerous implementation issues to make our inexact primal-dual interior-point method for (QSDP) practical.

Preconditioning for the augmented equations arising from interior-point methods for *sparse* linearly constrained convex quadratic programming (LCCQP) problems of the form

$$\min \left\{ \frac{1}{2} x^T Q x + c^T x \ : \ Ax = b, x \in \boldsymbol{R}^n_+ \right\} \tag{4}$$

has been studied in [8]. The preconditioners constructed in [8] are based on the constrained preconditioners proposed in [17], but the $(1,1)$ block $-X^{-1}S - Q$ of the augmented matrix is approximated by $-X^{-1}S - \mathrm{diag}(Q)$, where X and S are positive definite diagonal matrices for LCCQP. A reader who is familiar with semidefinite programming would realize that the augmented systems in LCCQP and QSDP are different in a fundamental way. While the augmented matrices in LCCQP are sparse if A and Q are sparse, the augmented matrices in QSDP however are typically dense even if $\mathcal{A}$ and $\mathcal{Q}$ are sparse. Thus the construction of preconditioners for the augmented

matrices in QSDP is considerably more difficult and the analysis and computation involved are more complex. Just like [8], two of the preconditioners we constructed are inspired by the constrained preconditioners in [17].

The problem (QSDP) is actually a generalization of (4) just like a linear SDP is a generalization of a linear program. We note that when the matrix variable X in (QSDP) is restricted to a diagonal matrix of the form $X = \mathrm{diag}(x)$, then (QSDP) reduces to (4) with $c = \mathrm{diag}(C)$, and the kth column of A^T is given by $\mathrm{diag}(\mathcal{A}^T(e_k))$, and $Q_{ij} = e_j^T(\mathcal{Q}(e_i e_i^T))e_j$. Based on this observation, many of the results derived for (QSDP) in this paper can be modified to suit (4). However, in the interest of keeping the paper coherent, we will not separately state the corresponding results for (4).

The paper is organized as follows. In the next section, we derive the augmented equation from which the search direction at each interior-point iteration for (QSDP) is computed. In Section 3, the asymptotic spectrum and conditioning of the augmented matrix are analyzed. This motivates the construction of preconditioners for the augmented matrix in Section 4. Three classes of preconditioners are constructed, and the asymptotic spectra of the associated preconditioned matrices are analyzed. In Section 5, we construct preconditioners for the augmented matrix during the initial phase of the interior-point iteration, and discuss the construction of symmetrized Kronecker product approximations for linear operators. Section 6 presents numerical experiments to test the performance of our inexact interior-point methods that employ iterative solvers with appropriate preconditioners to solve the augmented equation at each iteration. In the last section we give the conclusion.

2 Computation of search direction

Our interior-point method for (QSDP) is a primal-dual path-following method with Mehrotra's predictor-corrector. It is based on the perturbed KKT conditions associated with the primal-dual pair (1) and (2), which are given by

$$-\mathcal{Q}(X) + \mathcal{A}^T(y) + S = C, \quad \mathcal{A}(X) = b, \quad XS = \nu I, \quad X, S \succ 0, \tag{5}$$

where $\nu > 0$ is a parameter that is to be driven to zero explicitly.

Let $\rho \geq 0$ be a given constant. We may observe that by adding $-\rho\mathcal{A}^T\mathcal{A}(X) = -\rho\mathcal{A}^T b$ to the first condition in (5), we get an equivalent condition:

$$-\mathcal{Q}_\rho(X) + \mathcal{A}^T(y) + S = C_\rho, \tag{6}$$

where $\mathcal{Q}_\rho := \mathcal{Q} + \rho\mathcal{A}^T\mathcal{A}$ and $C_\rho := C - \rho\mathcal{A}^T b$. Thus we can replace the first condition in (5) by (6). The motivation for considering (6) instead of the first condition in (5) is given in Remark 3.1.

The framework of our interior-point algorithm for (QSDP) is described in Algorithm IP-QSDP in [31]. At a given iterate (X, y, S) with $X, S \succ 0$ (positive definite), the search direction $(\Delta X, \Delta y, \Delta S)$ at each interior-point iteration is the solution of the following symmetrized Newton system:

$$\begin{array}{rrrcl} -\mathcal{Q}_\rho(\Delta X) & + \mathcal{A}^T(\Delta y) & + \Delta S = R_d & = & C_\rho - S - \mathcal{A}^T y + \mathcal{Q}_\rho(X) \\ \mathcal{A}(\Delta X) & & = R_p & = & b - \mathcal{A}(X) \\ \mathcal{F}_S \Delta X & & + \mathcal{F}_X \Delta S = R_c & = & \sigma\mu I - H_K(XS), \end{array} \tag{7}$$

where $\mathcal{F}_X$ and $\mathcal{F}_S$ are linear operators on $\mathcal{S}^n$ that depend on the symmetrization scheme $H_K(\cdot)$ chosen, with K being the symmetrization matrix; for more details, see for example [29]. Here $\mu = X \bullet S/n$, and $\sigma \in (0,1)$ is the centering parameter.

By eliminating ΔS, we get the following augmented equation with dimension $\bar{n}+m$:

$$\underbrace{\begin{bmatrix} -\mathcal{K}_\rho & \mathcal{A}^T \\ \mathcal{A} & 0 \end{bmatrix}}_{\mathcal{B}_\rho} \begin{bmatrix} \Delta X \\ \Delta y \end{bmatrix} = \begin{bmatrix} R_a \\ R_p \end{bmatrix}, \tag{8}$$

where $\mathcal{K}_\rho = \mathcal{F}_X^{-1}\mathcal{F}_S + \mathcal{Q}_\rho$ and $R_a = R_d - \mathcal{F}_X^{-1}R_c$. In this paper, we will consider only the Nesterov-Todd (NT) symmetrization scheme [29] for which $\mathcal{F}_X^{-1}\mathcal{F}_S = W^{-1} \circledast W^{-1}$, where W is the unique symmetric positive definite matrix satisfying $WSW = X$. But note that the preconditioning strategies that we will describe later are also applicable to the purely primal (purely dual) scheme for which $\mathcal{F}_X^{-1}\mathcal{F}_S = X^{-1} \circledast X^{-1}$ $(S \circledast S)$.

By further eliminating ΔX from (8), we get the Schur complement equation with dimension m below:

$$\underbrace{\mathcal{A}\mathcal{K}_\rho^{-1}\mathcal{A}^T}_{M_\rho} \Delta y \;=\; R_p + \mathcal{K}_\rho^{-1}R_a. \tag{9}$$

When $\mathcal{Q}$ has the special form $U \circledast U$ and $\rho = 0$, the Schur complement matrix M_ρ can be computed at a cost of at most $4mn^3 + m^2n^2$ floating point operations (flops); see [31]. This is done by exploiting the fact that $\mathcal{K}_\rho^{-1}$ has an analytical expression and $\mathcal{K}_\rho^{-1}(V)$ can be computed with $\Theta(n^3)$ flops for any given $V \in \mathcal{S}^n$. However, for a general $\mathcal{Q}$, the corresponding $\mathcal{K}_\rho^{-1}$ does not have an analytical expression and the computation of M_ρ requires the inversion of $\mathcal{K}_\rho$ (with dimension $\bar{n}$) that costs $\Theta(n^6)$ flops. Such a cost is prohibitively expensive unless n is small, say less than 100. Thus for a general $\mathcal{Q}$, computing the search direction via the Schur complement equation (9) is not a viable approach. The practical alternative is to use the augmented equation (8).

However, the square linear system (8), with dimension $(\bar{n}+m)$, is a large system even for a moderate n. In addition, the system is typically dense since $\mathcal{F}_X^{-1}\mathcal{F}_S$ is typically so. Thus unless $\bar{n}+m$ is of moderate size, it is impossible to solve (8) by a direct method, and iterative methods are the only alternatives. In this paper, we use the PSQMR method as the iterative solver for (8). The unpreconditioned PSQMR method is mathematically equivalent to the minimal residual (MINRES) method. It has the same work and storage requirements as MINRES and usually converges in about the same number of iterations. But the PSQMR method has the advantage that it allows the use symmetric indefinite preconditioners whereas MINRES allows only symmetric positive definite preconditioners.

As iterative methods do not solve a linear system exactly (modulo rounding errors), we need the following proposition to gauge the quality of the computed direction.

Proposition 2.1. *Suppose the residual in computed direction $(\Delta X, \Delta y)$ for (8) is given by $(\eta_1\,,\,\eta_2)$, and that ΔS is computed exactly from the first equation in (7) based on the computed $(\Delta X, \Delta y)$. Then the residual of the direction $(\Delta X, \Delta y, \Delta S)$ with respect to (7) is given by $(0\,,\,\eta_2\,,\,-\mathcal{F}_X(\eta_1))$.*

Proof. We omit the proof since it is straightforward. ∎

In the numerical experiments in Section 6, we deem a direction $(\Delta X, \Delta y)$ computed by an iterative solver from (8) to be sufficiently accurate if

$$\max\{\|\eta_2\|_2, \|\mathcal{F}_X(\eta_1)\|_F\} \;\le\; 0.01\max\{\|R_d\|_F, \|R_p\|_2, \|R_c\|_F\}. \tag{10}$$

We end this section by stating some basic facts concerning a 2×2 block symmetric indefinite system. Note that the inversion of a 2×2 block matrix of the form $G = [-U,\, V^T;\, V,\, W]$ is at the heart of the computation of search directions. When the (1,1) block U and the associated Schur complement matrix $Y := W + VU^{-1}V^T$ are invertible, the inverse of G can be computed from the following analytical expression whose proof can be found for example in [28, p. 389]:

$$G^{-1} \;=\; \begin{bmatrix} -U^{-1} + U^{-1}V^TY^{-1}VU^{-1} & U^{-1}V^TY^{-1} \\ Y^{-1}VU^{-1} & Y^{-1} \end{bmatrix}. \tag{11}$$

Using the above analytical formula, the matrix-vector product $G^{-1}[x;\, y]$ can be computed efficiently through the following procedure:

Compute $w_1 = U^{-1}x$;
Compute $w_2 = Y^{-1}(Vw_1 + y)$;
Compute $w_3 = U^{-1}\left(V^Tw_2 - x\right)$;
Set $G^{-1}[x;\, y] = [w_3;\, w_2]$.

Observe that if Y is pre-computed, then computing $G^{-1}[x;\, y]$ require the solution of two linear systems involving U and one linear system involving Y.

Lemma 2.1. *Suppose U is symmetric positive definite, W is symmetric positive semidefinite, and V has full row rank. Let $G = [-U,\, V^T;\, V,\, W]$. Then the following results hold:*

$$\max\{\|U\|, \|V\|, \|W\|\} \;\le\; \|G\| \;\le\; 2\max\{\|U\|, \|V\|, \|W\|\},$$
$$\|Y^{-1}\| \;\le\; \|G^{-1}\| \;\le\; 2\max\{\|U^{-1}\|,\, \|Y^{-1}\|\},$$

where $Y = W + VU^{-1}V^T$.

Proof. We shall proof only the second inequality since the first follows readily from the inequality, $\|G\|^2 \le \|U\|^2 + 2\|V\|^2 + \|W\|^2$, proven in [5]. It is readily shown that

$$G^{-1} \;=\; \begin{bmatrix} U^{-1/2}\Big(\Pi - I\Big)U^{-1/2} & U^{-1}V^TY^{-1} \\ Y^{-1}VU^{-1} & Y^{-1} \end{bmatrix},$$

where $\Pi = U^{-1/2}V^TY^{-1}VU^{-1/2}$. By noting that $0 \preceq \Pi \preceq I$, the required result can be proven easily. ∎

3 Conditioning of the augmented matrix $\mathcal{B}_\rho^k$

We made the following assumptions on (1) and (2).
Assumption A2. The problems (QSDP) and (QSDD) are strictly feasible and that $\mathcal{A}$ is surjective. The last condition implies that $m \leq \bar{n}$.

Assumption A2 is necessary and sufficient conditions for the existence and uniqueness of solutions (X^ν, y^ν, S^ν) of the central path equations (5). Also, these solutions converge to some optimal solution (X^*, y^*, S^*) as ν tends to zero; see Halicka, de Klerk, and Roos [13], and Luo, Sturm, and Zhang [21]. We further assume that (X^*, y^*, S^*) satisfies the following assumption.
Assumption A3. Strict complementarity holds for (X^*, y^*, S^*) in the sense of Alizadeh, Haeberly, and Overton [2]. Thus the ranks of X^* and S^* sum to n.

Suppose $\{\nu_k\}$ is a monotonically decreasing sequence such that $\lim_{k\to\infty} \nu_k = 0$. Let the coefficient matrices in (8) and (9) corresponding to (X^ν, y^ν, S^ν) be $\mathcal{B}_\rho^\nu$ and M_ρ^ν, respectively. For simplicity of notation, we write $\mathcal{B}_\rho^k$, M_ρ^k, X^k, S^k, etc., for $\mathcal{B}_\rho^{\nu_k}$, $M_\rho^{\nu_k}$, X^{ν_k}, S^{ν_k}, and so on. Since X^k and S^k commute, there is an orthogonal matrix P^k that simultaneously diagonalizes X^k and S^k so that

$$X^k = P^k \Lambda^k (P^k)^T, \quad S^k = P^k \Sigma^k (P^k)^T,$$

where the eigenvalue matrices

$$\Lambda^k = \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k), \quad \Sigma^k = \mathrm{diag}(\sigma_1^k, \ldots, \sigma_n^k)$$

satisfy $\lambda_i^k \sigma_i^k = \nu_k$, and the eigenvalues are ordered such that

$$\lambda_1^k \geq \cdots \geq \lambda_n^k > 0, \quad 0 < \sigma_1^k \leq \cdots \leq \sigma_n^k.$$

Let P^* be a limit point of the set $\{P^k\}$. We refine the sequence if necessary so that $\{P^k\}$ converges to P^*. Then P^* is an orthogonal matrix that simultaneously diagonalizes X^* and S^* with

$$X^* = P^* \Lambda^* (P^*)^T, \quad S^* = P^* \Sigma^* (P^*)^T, \tag{12}$$

where

$$\Lambda^* = \mathrm{diag}(\lambda_1^*, \ldots, \lambda_n^*), \quad \Sigma^* = \mathrm{diag}(\sigma_1^*, \ldots, \sigma_n^*)$$

satisfy $\lambda_i^* \sigma_i^* = 0$, and

$$\lambda_1^* \geq \cdots \geq \lambda_r^* > \lambda_{r+1}^* = \cdots = \lambda_n^* = 0, \quad 0 = \sigma_1^* = \cdots = \sigma_{n-s}^* < \sigma_{n-s+1}^* \leq \cdots \leq \sigma_n^*.$$

Here r and s are the ranks of X^* and S^*, respectively.

We are assuming that (X^*, y^*, S^*) satisfies the strict complementarity condition, i.e., $r + s = n$. To analyze the spectrum of $\mathcal{B}_\rho^k$, we will identify the inner product space $\mathcal{S}^n$ with the space $\mathcal{S}^r \times \mathbf{R}^{r\times s} \times \mathcal{S}^s$ as follows. For an element $X \in \mathcal{S}^n$, consider the partition $X = [X_1, X_2; X_2^T, X_3]$, where $X_1 \in \mathcal{S}^r$, $X_2 \in \mathbf{R}^{r\times s}$, and $X_3 \in \mathcal{S}^s$. Then X is identified with the element $[X_1; X_2; X_3]$ in $\mathcal{S}^r \times \mathbf{R}^{r\times s} \times \mathcal{S}^s$. The notation $[X_1; X_2; X_3]$ means that the objects X_1, X_2, X_3 are placed in a column format. The

space $\mathcal{S}^r \times \mathbf{R}^{r\times s} \times \mathcal{S}^s$ is endowed with the inner product $[X_1; X_2; X_3] \bullet [Y_1; Y_2; Y_3] = X_1 \bullet Y_1 + 2X_2 \bullet Y_2 + X_3 \bullet Y_3$ so that the identification of $\mathcal{S}^n$ with $\mathcal{S}^r \times \mathbf{R}^{r\times s} \times \mathcal{S}^s$ is an isometry. Note the factor of 2 in front of the inner product $X_2 \bullet Y_2$. Thus the space $\mathbf{R}^{r\times s}$ is $\boldsymbol{R}^{r\times s}$ but with the inner product for $X_2, Y_2 \in \mathbf{R}^{r\times s}$ given by $2X_2 \bullet Y_2$.

Let P_1^* and P_2^* be the submatrices denoting the first r and the last $n-r$ columns of P^*, respectively. Let $\mathcal{P}^* = P^* \circledast P^*$. Based on the identification of $\mathcal{S}^n$ with $\mathcal{S}^r \times \mathbf{R}^{r\times s} \times \mathcal{S}^s$, $\mathcal{P}^*$ can be partitioned as follows. For $X = [X_1, X_2; X_2^T, X_3]$, we have

$$\begin{aligned} \mathcal{P}^*(X) &= P_1^* X_1 (P_1^*)^T + P_1^* X_2 (P_2^*)^T + P_2^* X_2^T (P_1^*)^T + P_2^* X_3 (P_2^*)^T \\ &= \mathcal{P}_1^*(X_1) + \mathcal{P}_2^*(X_2) + \mathcal{P}_3^*(X_3), \end{aligned} \tag{13}$$

where $\mathcal{P}_1^* = P_1^* \circledast P_1^* : \mathcal{S}^r \to \mathcal{S}^n$, $\mathcal{P}_2^* = 2P_1^* \circledast P_2^* : \mathbf{R}^{r\times s} \to \mathcal{S}^n$, and $\mathcal{P}_3^* = P_2^* \circledast P_2^* : \mathcal{S}^s \to \mathcal{S}^n$. We may write $\mathcal{P}^* = [\mathcal{P}_1^*, \mathcal{P}_2^*, \mathcal{P}_3^*]$ so that $\mathcal{P}^*(X) = [\mathcal{P}_1^*, \mathcal{P}_2^*, \mathcal{P}_3^*][X_1; X_2; X_3]$.

Similarly, we let $\widetilde{\mathcal{A}}^* := \mathcal{A}\mathcal{P}^* = [\widetilde{\mathcal{A}}_1^*, \widetilde{\mathcal{A}}_2^*, \widetilde{\mathcal{A}}_3^*]$ with $\widetilde{\mathcal{A}}_j^* = \mathcal{A}\mathcal{P}_j^*$ for $j = 1, 2, 3$, and let $\widetilde{\mathcal{Q}}_\rho^* = (\mathcal{P}^*)^T \mathcal{Q}_\rho \mathcal{P}^*$ with $(\widetilde{\mathcal{Q}}_\rho^*)_{ij} = (\mathcal{P}_i^*)^T \mathcal{Q}_\rho \mathcal{P}_j^*$ for $i, j = 1, 2, 3$. We define $\widetilde{\mathcal{Q}}^*$ and $(\widetilde{\mathcal{Q}}^*)_{ij}$ similarly.

Based on the partitions of $\mathcal{P}^*$ and $\widetilde{\mathcal{A}}^*$, we can define the degeneracy of the optimal solution (X^*, y^*, S^*) as follows.

Definition 3.1. *Suppose the optimal solution (X^*, y^*, S^*) satisfies the strict complementarity condition.*

(a) The solution X^* is said to be primal nondegenerate [2] if the linear map $[\widetilde{\mathcal{A}}_1^*, \widetilde{\mathcal{A}}_2^*] : \mathcal{S}^r \times \mathbf{R}^{r\times s} \to \boldsymbol{R}^m$ defined by $[\widetilde{\mathcal{A}}_1^*, \widetilde{\mathcal{A}}_2^*][U; V] = \mathcal{A}(P_1^* U (P_1^*)^T) + \mathcal{A}(P_1^* V (P_2^*)^T + P_2^* V^T (P_1^*)^T)$ is surjective (this is an equivalent definition: see Theorem 6 in [2]).

(b) The solution S^* is said to be dual nondegenerate [2] if the linear map $\widetilde{\mathcal{A}}_1^* : \mathcal{S}^r \to \boldsymbol{R}^m$ defined by $\widetilde{\mathcal{A}}_1^*(U) = \mathcal{A}(P_1^* U (P_1^*)^T)$ is injective (this is again an equivalent definition: see Theorem 9 in [2]).

The importance of nondegeneracy of the optimal solution will become clear when we analyze the asymptotic spectral distribution of $\{\mathcal{B}_\rho^k\}$ and those of the preconditioned matrices later.

Note that for the QSDP arising from (3), the optimal solution (X^*, y^*, S^*) is always primal nondegenerate; see [31].

On the central path $\{(X^\nu, y^\nu, S^\nu) : \nu > 0\}$, and under Assumptions A2 and A3, the eigenvalue decomposition of $(W^k)^{-1}$, where W^k is the NT scaling matrix corresponding to (X^k, y^k, S^k), must have the following form:

$$(W^k)^{-1} \;=\; P^k D^k (P^k)^T \;=\; P_1^k D_1^k (P_1^k)^T \;+\; P_2^k D_2^k (P_2^k)^T, \tag{14}$$

where $D_1^k = \mathrm{diag}(d_1^k) \in \boldsymbol{R}^{r\times r}$, $P_1^k \in \boldsymbol{R}^{n\times r}$ correspond to the small eigenvalues of order $\Theta(\sqrt{\nu_k})$, and $D_2^k = \mathrm{diag}(d_2^k) \in \boldsymbol{R}^{s\times s}$, $P_2^k \in \boldsymbol{R}^{n\times s}$ correspond to the large eigenvalues of the order $\Theta(1/\sqrt{\nu_k})$. Recall that the notation $\gamma = \Theta(\sqrt{\nu_k})$ means that there are constants $c_1, c_2 > 0$ such that $c_1\sqrt{\nu_k} \le \gamma \le c_2\sqrt{\nu_k}$ for all $k = 1, 2, \ldots$.

It is easily shown that the following eigenvalue decomposition holds [29]:

$$(W^k)^{-1} \circledast (W^k)^{-1} \;=\; (P^k \circledast P^k)(D^k \circledast D^k)(P^k \circledast P^k)^T \;=:\; \mathcal{P}^k \mathcal{D}^k (\mathcal{P}^k)^T, \tag{15}$$

where $\mathcal{P}^k = P^k \circledast P^k$ and $\mathcal{D}^k = D^k \circledast D^k$. Let $\mathcal{D}_1^k = D_1^k \circledast D_1^k$, $\mathcal{D}_2^k = D_2^k \otimes D_1^k$ and $\mathcal{D}_3^k = D_2^k \circledast D_2^k$. Then the diagonal entries of the matrix representations of $\mathcal{D}_1^k$, $\mathcal{D}_2^k$,

and $\mathcal{D}_3^k$ consist of $\bar{r}$, rs, and $\bar{s}$ eigenvalues of $(W^k)^{-1} \circledast (W^k)^{-1}$ of the orders $\Theta(\nu_k)$, $\Theta(1)$, and $\Theta(1/\nu_k)$, respectively. We assume that there are constants $\underline{\tau}, \overline{\tau} > 0$ such that $\underline{\tau} I \preceq \mathcal{D}_2^k \preceq \overline{\tau} I$ for all $k = 1, 2, \ldots$. Without lost of generality, we may choose $\overline{\tau}$ large enough and $\underline{\tau}$ small enough so that $0 \preceq \mathcal{D}_1^k \preceq \overline{\tau} I$ and $\mathcal{D}_3^k \succeq \underline{\tau} I$ for all $k = 1, 2, \ldots$, respectively. Note that we may write $\mathcal{D}$ as $\mathcal{D} = \mathrm{diag}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$.

Consider the partition $\mathcal{P}^k = [\mathcal{P}_1^k, \mathcal{P}_2^k, \mathcal{P}_3^k]$ with $\mathcal{P}_1^k = P_1^k \circledast P_1^k$, $\mathcal{P}_2^k = 2P_1^k \circledast P_2^k$ and $\mathcal{P}_3^k = P_2^k \circledast P_2^k$ corresponding to $\mathcal{D}_1^k$, $\mathcal{D}_2^k$, and $\mathcal{D}_3^k$, respectively. Let $\widetilde{\mathcal{A}}^k = \mathcal{A}\mathcal{P}^k$, with the partition $\widetilde{\mathcal{A}}^k = [\widetilde{\mathcal{A}}_1^k, \widetilde{\mathcal{A}}_2^k, \widetilde{\mathcal{A}}_3^k] = [\mathcal{A}\mathcal{P}_1^k, \mathcal{A}\mathcal{P}_2^k, \mathcal{A}\mathcal{P}_3^k]$ that conforms to that of $\mathrm{diag}(\mathcal{D}_1^k, \mathcal{D}_2^k, \mathcal{D}_3^k)$. Suppose $\widetilde{\mathcal{Q}}^k = (\mathcal{P}^k)^T \mathcal{Q} \mathcal{P}^k$. Then $\widetilde{\mathcal{Q}}_\rho^k := (\mathcal{P}^k)^T \mathcal{Q}_\rho \mathcal{P}^k = \widetilde{\mathcal{Q}}^k + \rho(\widetilde{\mathcal{A}}^k)^T \widetilde{\mathcal{A}}^k$. Let $(\widetilde{\mathcal{Q}}_\rho^k)_{ij} = (\mathcal{P}_i^k)^T \mathcal{Q}_\rho \mathcal{P}_j^k$ for $i, j = 1, 2, 3$. We define $\widetilde{\mathcal{Q}}_{ij}^k$ similarly. Note that since $\mathcal{P}^k \to \mathcal{P}^*$ as $k \to \infty$, we have $\widetilde{\mathcal{A}}^k \to \widetilde{\mathcal{A}}^*$ and $\widetilde{\mathcal{Q}}_\rho^k \to \widetilde{\mathcal{Q}}_\rho^*$ as $k \to \infty$.

Based on the eigenvalue decomposition (15), it is readily shown that the augmented matrix $\mathcal{B}_\rho^k$ in (8) has the following decomposition:

$$\mathcal{B}_\rho^k \;=\; \begin{bmatrix} \mathcal{P}^k & 0 \\ 0 & I_m \end{bmatrix} \underbrace{\begin{bmatrix} -D^k \circledast D^k - \widetilde{\mathcal{Q}}_\rho^k & (\widetilde{\mathcal{A}}^k)^T \\ \widetilde{\mathcal{A}}^k & 0 \end{bmatrix}}_{\widetilde{\mathcal{B}}_\rho^k} \begin{bmatrix} (\mathcal{P}^k)^T & 0 \\ 0 & I_m \end{bmatrix}. \tag{16}$$

By considering the following partition

$$(\mathcal{P}^k)^T \mathcal{K}_\rho^k \mathcal{P}^k \;=\; D^k \circledast D^k + \widetilde{\mathcal{Q}}_\rho^k \;=\; \begin{bmatrix} \mathcal{D}_1^k + (\widetilde{\mathcal{Q}}_\rho^k)_{11} & (\widetilde{\mathcal{Q}}_\rho^k)_{12} & (\widetilde{\mathcal{Q}}_\rho^k)_{13} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{21} & \mathcal{D}_2^k + (\widetilde{\mathcal{Q}}_\rho^k)_{22} & (\widetilde{\mathcal{Q}}_\rho^k)_{23} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{31} & (\widetilde{\mathcal{Q}}_\rho^k)_{32} & \mathcal{D}_3^k + (\widetilde{\mathcal{Q}}_\rho^k)_{33} \end{bmatrix}, \tag{17}$$

it is easy to see that the matrix $\widetilde{\mathcal{B}}_\rho^k$ in (16) can be written as follows:

$$\widetilde{\mathcal{B}}_\rho^k \;=\; \begin{bmatrix} -\mathcal{D}_1^k - (\widetilde{\mathcal{Q}}_\rho^k)_{11} & -(\widetilde{\mathcal{Q}}_\rho^k)_{12} & -(\widetilde{\mathcal{Q}}_\rho^k)_{13} & (\widetilde{\mathcal{A}}_1^k)^T \\ -(\widetilde{\mathcal{Q}}_\rho^k)_{21} & -\mathcal{D}_2^k - (\widetilde{\mathcal{Q}}_\rho)_{22}^k & -(\widetilde{\mathcal{Q}}_\rho^k)_{23} & (\widetilde{\mathcal{A}}_2^k)^T \\ -(\widetilde{\mathcal{Q}}_\rho^k)_{31} & -(\widetilde{\mathcal{Q}}_\rho^k)_{32} & -\mathcal{D}_3^k - (\widetilde{\mathcal{Q}}_\rho)_{33}^k & (\widetilde{\mathcal{A}}_3^k)^T \\ \widetilde{\mathcal{A}}_1^k & \widetilde{\mathcal{A}}_2^k & \widetilde{\mathcal{A}}_3^k & 0 \end{bmatrix}. \tag{18}$$

We note that it is possible to have the case where $r = n$ and $s = 0$. In that case, D_2^k is a null matrix for k sufficiently large, and correspondingly $\mathcal{D}_2^k, \mathcal{D}_3^k$ and $\mathcal{P}_2^k, \mathcal{P}_3^k$ are null maps. Similarly, $(\widetilde{\mathcal{Q}}_\rho^k)_{ij}$, $i, j = 1, 2, 3$, are null maps except for $(\widetilde{\mathcal{Q}}_\rho^k)_{11}$. As we shall see from the following proposition, the norm $\|\mathcal{B}_\rho^k\|$ behaves differently for $r = n$ and $r < n$.

Proposition 3.1. *Let r and s be the ranks of the optimal solution X^* and S^*, respectively.*
(a) Suppose $r = n$ and $s = 0$. Then

$$\limsup_{k \to \infty} \|\mathcal{B}_\rho^k\| \;\le\; 2 \max\{\|\mathcal{A}\|, \|\mathcal{Q}_\rho\|\}.$$

(b) Suppose $r < n$ and $s = n - r > 0$. Then

$$\|\mathcal{B}_\rho^k\| \;\ge\; \max\{\|\mathcal{Q}_\rho\|, \|\mathcal{D}_3^k\|\} \;=\; \Theta(1/\nu_k).$$

Hence $\|\mathcal{B}_\rho^k\| \to \infty$ *as* $k \to \infty$.

Proof. (a) In this case, $\mathcal{D}^k = \Theta(\nu_k)$, and the required result follows from Lemma 2.1. (b) In this case, $\mathcal{D}_3^k$ is not a null operator. By Lemma 2.1 and the fact that $\mathcal{D}_3^k = \Theta(1/\nu_k)$, the required result follows. □

Before we analyze the conditioning of $\mathcal{B}_\rho^k$, let us state the following lemmas which will be used in the analysis.

Lemma 3.1. *Given* $U \in \mathcal{S}^p$, $W \in \mathcal{S}^q$ *and* $V \in \boldsymbol{R}^{q\times p}$. *A matrix of the form* $[U, V^T; V, W]$ *is positive definite if and only if either (i)* U *and* $W - VU^{-1}V^T$ *are positive definite; or (ii)* W *and* $U - V^TW^{-1}V$ *are positive definite.*

Proof. We omit the proof since it is easy. □

Lemma 3.2. *Suppose that Assumptions A2 and A3 hold for the optimal solution* (X^*, y^*, S^*).
(a) Suppose that $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$. *Then we have*

$$\Upsilon_\rho^* := \begin{bmatrix} (\widetilde{\mathcal{Q}}_\rho^*)_{11} & (\widetilde{\mathcal{Q}}_\rho^*)_{12} \\ (\widetilde{\mathcal{Q}}_\rho^*)_{21} & \underline{\tau} I + (\widetilde{\mathcal{Q}}_\rho^*)_{22} \end{bmatrix} \succ 0. \tag{19}$$

(b) Suppose that $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$. *Let*

$$M_\rho^* = [\widetilde{\mathcal{A}}_1^*, \widetilde{\mathcal{A}}_2^*](\Upsilon_\rho^*)^{-1}[\widetilde{\mathcal{A}}_1^*, \widetilde{\mathcal{A}}_2^*]^T. \tag{20}$$

Suppose further that primal nondegeneracy holds for (X^*, y^*, S^*). *Then* M_ρ^* *is positive definite.*

Proof. (a) Since the sub-matrix of maps, $[(\widetilde{\mathcal{Q}}_\rho^*)_{11}, (\widetilde{\mathcal{Q}}_\rho^*)_{12}; (\widetilde{\mathcal{Q}}_\rho^*)_{21}, (\widetilde{\mathcal{Q}}_\rho^*)_{22}]$, of $\widetilde{\mathcal{Q}}_\rho$ is positive semidefinite, we have $\underline{\tau} I + (\widetilde{\mathcal{Q}}_\rho^*)_{22} - (\widetilde{\mathcal{Q}}_\rho^*)_{21}(\widetilde{\mathcal{Q}}_\rho^*)_{11}^{-1}(\widetilde{\mathcal{Q}}_\rho^*)_{12} \succeq \underline{\tau} I$. By Lemma 3.1, Υ_ρ^* in (19) is positive definite.

(b) Under the assumption that X^* is primal nondegenerate, the linear map $[\widetilde{\mathcal{A}}_1^*, \widetilde{\mathcal{A}}_2^*]$ is surjective. From here, it is easy to show that M_ρ^* is positive definite. □

Remark 3.1. *(a) If* $\widetilde{\mathcal{Q}}_{11}^*$ *is positive definite on* $\mathcal{N}(\sqrt{\rho}\widetilde{\mathcal{A}}_1^*) = \mathcal{N}(\rho(\widetilde{\mathcal{A}}_1^*)^T\widetilde{\mathcal{A}}_1^*)$, *then* $(\widetilde{\mathcal{Q}}_\rho^*)_{11}$ *is positive definite. This result follows from the fact that if* $U, V \in \mathcal{S}_+^p$ *are matrices such that* U *is positive definite on* $\mathcal{N}(V)$, *then* $U + V \succ 0$. *As we can see from the previous fact, it is easier for* $(\widetilde{\mathcal{Q}}_\rho^*)_{11}$ *to be positive definite compared to* $\widetilde{\mathcal{Q}}_{11}^*$. *This explains why we used the condition in (6) instead of the first condition in (5).*

(b) If either $\rho > 0$ *and* (X^*, y^*, S^*) *is dual nondegenerate, or* $\widetilde{\mathcal{Q}}_{11}^*$ *is positive definite, then* $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$. *The proof is as follows. If* $\widetilde{\mathcal{Q}}_{11}^*$ *is positive definite, clearly* $(\widetilde{\mathcal{Q}}_\rho^*)_{11} = \widetilde{\mathcal{Q}}_{11}^* + \rho(\widetilde{\mathcal{A}}_1^*)^T\widetilde{\mathcal{A}}_1^*$ *is positive definite. On the other hand, if* $\rho > 0$ *and* (X^*, y^*, S^*) *is dual nondegenerate, then* $\widetilde{\mathcal{A}}_1^*$ *is injective and this in turn implies that* $\rho(\widetilde{\mathcal{A}}_1^*)^T\widetilde{\mathcal{A}}_1^*$ *is positive definite. The latter also implies that* $(\widetilde{\mathcal{Q}}_\rho^*)_{11}$ *is positive definite, even if* $\mathcal{Q} = 0$ *in the case of a linear SDP.*

(c) If $\mathcal{Q}$ *is positive definite, then* $\widetilde{\mathcal{Q}}_{11}^*$ *is automatically positive definite since* $\mathrm{eig}(\widetilde{\mathcal{Q}}_{11}^*) \subset \mathrm{eig}(\widetilde{\mathcal{Q}}^*) = \mathrm{eig}(\mathcal{Q})$.

Let

$$\Upsilon_\rho^k \;=\; \left[\begin{array}{cc} \mathcal{D}_1^k + (\widetilde{\mathcal{Q}}_\rho^k)_{11} & (\widetilde{\mathcal{Q}}_\rho^k)_{12} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{21} & \mathcal{D}_2^k + (\widetilde{\mathcal{Q}}_\rho^k)_{22} \end{array}\right]. \tag{21}$$

Note that Υ_ρ^k is positive definite, and

$$\left[\begin{array}{cc} (\widetilde{\mathcal{Q}}_\rho^k)_{11} & (\widetilde{\mathcal{Q}}_\rho^k)_{12} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{21} & \underline{\tau} I + (\widetilde{\mathcal{Q}}_\rho^k)_{22} \end{array}\right] \;\preceq\; \Upsilon_\rho^k \;\preceq\; \overline{\tau} I + \left[\begin{array}{cc} (\widetilde{\mathcal{Q}}_\rho^k)_{11} & (\widetilde{\mathcal{Q}}_\rho^k)_{12} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{21} & \underline{\tau} I + (\widetilde{\mathcal{Q}}_\rho^k)_{22} \end{array}\right]. \tag{22}$$

Recall that M_ρ^k denotes the Schur complement matrix in (9) corresponding to (X^k, y^k, S^k). We have the following theorem concerning $\{\|M_\rho^k\|\}$ and $\{\|(M_\rho^k)^{-1}\|\}$.

Theorem 3.1. *Suppose that Assumptions A2 and A3 hold for the optimal solution (X^*, y^*, S^*). Suppose further that $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$. Then the following results hold.*

(a) There exists a positive constant c such that $\limsup_{k\to\infty} \|M_\rho^k\| \le c\|M_\rho^*\|$.

(b) Suppose in addition that primal nondegeneracy holds for (X^*, y^*, S^*). Then there exists a positive constant c such that $\limsup_{k\to\infty} \|(M_\rho^k)^{-1}\| \le c\|(M_\rho^*)^{-1}\|$.

Proof. Let

$$J_\rho^k \;=\; \Upsilon_\rho^k - \left[\begin{array}{c} (\widetilde{\mathcal{Q}}_\rho^k)_{13} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{23} \end{array}\right] \Big(\mathcal{D}_3^k + (\widetilde{\mathcal{Q}}_\rho^k)_{33}\Big)^{-1} \left[\begin{array}{cc} (\widetilde{\mathcal{Q}}_\rho^k)_{31} & (\widetilde{\mathcal{Q}}_\rho^k)_{32} \end{array}\right].$$

Note that

$$\lim_{k\to\infty} [(\widetilde{\mathcal{Q}}_\rho^k)_{13};\, (\widetilde{\mathcal{Q}}_\rho^k)_{23}]\Big(\mathcal{D}_3^k + (\widetilde{\mathcal{Q}}_\rho^k)_{33}\Big)^{-1}[(\widetilde{\mathcal{Q}}_\rho^k)_{31},\, (\widetilde{\mathcal{Q}}_\rho^k)_{32}] = 0,$$

and Υ_ρ^k satisfies (22). Since the 2×2 block matrix in (22) converges to Υ_ρ^* in (19) as $k \to \infty$, and $\Upsilon_\rho^* \succ 0$ under the assumption that $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$, thus there are positive constants c_1, c_2 such that $c_1^{-1}\Upsilon_\rho^* \preceq J_\rho^k \preceq c_2^{-1}\Upsilon_\rho^*$ for sufficiently large k. This implies that $c_1(\Upsilon_\rho^*)^{-1} \succeq (J_\rho^k)^{-1} \succeq c_2(\Upsilon_\rho^*)^{-1}$ for all k sufficiently large.

(a) Observe that $M_\rho^k = \widetilde{\mathcal{A}}^k(D^k \circledast D^k + \widetilde{\mathcal{Q}}_\rho^k)^{-1}(\widetilde{\mathcal{A}}^k)^T$. By applying the formula in (11) to the matrix in (17), we have

$$(D^k \circledast D^k + \widetilde{\mathcal{Q}}_\rho^k)^{-1} \;=\; \left[\begin{array}{cc} (J_\rho^k)^{-1} & 0 \\ 0 & 0 \end{array}\right] + O(\nu_k\|(J_\rho^k)^{-1}\|).$$

Thus, for k sufficiently large,

$$\begin{aligned} M_\rho^k \;&=\; [\widetilde{\mathcal{A}}_1^k,\, \widetilde{\mathcal{A}}_2^k](J_\rho^k)^{-1}[\widetilde{\mathcal{A}}_1^k,\, \widetilde{\mathcal{A}}_2^k]^T + O(\nu_k\|(J_\rho^k)^{-1}\|\,\|\mathcal{A}\|^2) \\ &\preceq\; c_1[\widetilde{\mathcal{A}}_1^k,\, \widetilde{\mathcal{A}}_2^k](\Upsilon_\rho^*)^{-1}[\widetilde{\mathcal{A}}_1^k,\, \widetilde{\mathcal{A}}_2^k]^T + O(\nu_k\|(\Upsilon_\rho^*)^{-1}\|\,\|\mathcal{A}\|^2). \end{aligned} \tag{23}$$

Since the first matrix on the right-hand side converges to M_ρ^* defined in (20), the required result follows.

(b) From (23), we have that for k sufficiently large,

$$M_\rho^k \;\succeq\; c_2[\widetilde{\mathcal{A}}_1^k,\, \widetilde{\mathcal{A}}_2^k](\Upsilon_\rho^*)^{-1}[\widetilde{\mathcal{A}}_1^k,\, \widetilde{\mathcal{A}}_2^k]^T + O(\nu_k\|(\Upsilon_\rho^*)^{-1}\|\,\|\mathcal{A}\|^2).$$

Since the first matrix on the right-hand side converges to M_ρ^*, and M_ρ^* is positive definite under the assumption of primal nondegeneracy, thus $\liminf_{k\to\infty}\lambda_{\min}(M_\rho^k) \geq c_2\lambda_{\min}(M_\rho^*) > 0$. From here, the required result follows. □

We have the following theorem for $\{\|(\mathcal{B}_\rho^k)^{-1}\|\}$.

Theorem 3.2. *Under the assumptions stated in Theorem 3.1(b), we have*

$$\limsup_{k\to\infty}\|(\mathcal{B}_\rho^k)^{-1}\| \;\leq\; c\max\{\|(\Upsilon_\rho^*)^{-1}\|\,,\,\|(M_\rho^*)^{-1}\|\}, \tag{24}$$

for some constant $c > 0$.

Proof. By Lemma 2.1, we have

$$\|(\mathcal{B}_\rho^k)^{-1}\| \;\leq\; 2\max\{\|(\mathcal{K}_\rho^k)^{-1}\|,\,\|(M_\rho^k)^{-1}\|\}.$$

Now

$$(\mathcal{P}^k)^T(\mathcal{K}_\rho^k)^{-1}\mathcal{P}^k \;=\; (D^k \circledast D^k + \widetilde{\mathcal{Q}}_\rho^k)^{-1} \;=\; \begin{bmatrix} (J_\rho^k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + O(\nu_k\|(J_\rho^k)^{-1}\|),$$

where J_ρ^k is defined as in the proof of Theorem 3.1. By Theorem 3.1, we have positive constants c_1, c_2 such that for k sufficiently large,

$$c_1(\Upsilon_\rho^*)^{-1} \succeq (J_\rho^k)^{-1} \succeq c_2(\Upsilon_\rho^*)^{-1}.$$

Thus $\|(\mathcal{K}_\rho^k)^{-1}\| = \|(J_\rho^k)^{-1}\|(1+O(\nu_k)) \leq c_1\|(\Upsilon_\rho^*)^{-1}\|(1+O(\nu_k))$ for all k sufficiently large. This implies that $\limsup_{k\to\infty}\|(\mathcal{K}_\rho^k)^{-1}\| \leq c_1\|(\Upsilon_\rho^*)^{-1}\|$. By Theorem 3.1(b), we have $\limsup_{k\to\infty}\|(M_\rho^k)^{-1}\| \leq c_2\|(M_\rho^*)^{-1}\|$ for some constant $c_2 > 0$. From here, the required result in (24) follows. □

Proposition 3.2. *Suppose that Assumptions A2 and A3 hold for the optimal solution (X^*, y^*, S^*) and $\mathcal{Q}_\rho \succ 0$. If $S^* = 0$, then*

$$\limsup_{k\to\infty}\|M_\rho^k\| \;\leq\; \|\mathcal{A}\mathcal{Q}_\rho^{-1}\mathcal{A}^T\| \tag{25}$$

$$\limsup_{k\to\infty}\|(M_\rho^k)^{-1}\| \;\leq\; \|(\mathcal{A}\mathcal{Q}_\rho^{-1}\mathcal{A}^T)^{-1}\| \tag{26}$$

$$\limsup_{k\to\infty}\|(\mathcal{B}_\rho^k)^{-1}\| \;\leq\; 2\max\{\|\mathcal{Q}_\rho^{-1}\|\,,\,\|(\mathcal{A}\mathcal{Q}_\rho^{-1}\mathcal{A}^T)^{-1}\|\}. \tag{27}$$

Proof. Under the assumption of strict complementarity, $S^* = 0$ implies that X^* has full rank. In this case, $\mathcal{D}^k = \Theta(\nu_k)$. Now, $M_\rho^k = \widetilde{\mathcal{A}}^k(\mathcal{D}^k + \widetilde{\mathcal{Q}}_\rho^k)^{-1}(\widetilde{\mathcal{A}}^k)^T = \mathcal{A}\mathcal{Q}_\rho^{-1}\mathcal{A}^T + O(\nu_k)$. Thus the result in (25) follows. Since $\mathcal{A}\mathcal{Q}_\rho^{-1}\mathcal{A}^T$ is nonsingular, the required result in (26) follows readily. The result in (27) can be proven similarly. □

Example 1. To illustrate the asymptotic result in Theorem 3.2 and later results on the spectra of preconditioned matrices, we consider an example of (QSDP) arising from the nearest correlation matrix problem (3). We take $n = 30$.

The matrix K in (3) is generated as in [15] from the MATLAB function `gencorrmat.m` described below:

```
function [B,E] = gencorrmat(n);
   tmp  = 10.^[-4: 4/(n-1): 0];
   beta = n * tmp/sum(tmp);
   B = gallery('randcorr',beta);
   tmp  = randn(n); E = (tmp+tmp')/2;
   E = E/norm(E,'fro');
```

We take $K =$ `B + 1e-4*E`. Observe that a random symmetric perturbation of Frobenius norm 10^{-4} is added to the random correlation matrix `B` generated from the MATLAB function `gallery.m`.

The linear operator $\mathcal{Q}$ associated with (3) is chosen to be the positive definite operator $\mathcal{Q}(X) = U \circ X$, where $U \in \mathcal{S}^n \cap \boldsymbol{R}^{n\times n}_{++}$ is randomly generated in MATLAB via the segment: `tmp = rand(n); U = (tmp + tmp')/2`.

For all the numerical examples in this paper, we take $\rho = 0$. All the experiments are run in MATLAB version 7 on a 3.0 GHz Pentium IV PC with 2GB of physical memory.

Figure 1 shows the spectra of $\mathcal{B}^k_\rho$ and M^k_ρ corresponding to the iterate (X^k, y^k, S^k) with complementarity gap $X^k \bullet S^k/n = 1.9 \times 10^{-10}$. For this example, X^* has full rank and $S^* = 0$. This can be deduced from the fact that $\|S^k\| = 3.1 \times 10^{-7}$ and $\lambda_{\min}(X^k) = 8.2 \times 10^{-4}$.

From the eigenvalues shown in the left-hand side plot of Figure 1, it is clear that $\|\mathcal{B}^k_\rho\| = |\lambda_{\max}(\mathcal{B}^k_\rho)|$ and $\|(\mathcal{B}^k_\rho)^{-1}\| = 1/|\lambda_{\min}(\mathcal{B}^k_\rho)|$ remain bounded as $k \to \infty$. This observation is consistent with the asymptotic result in Propositions 3.1 and 3.2. The right-hand side plot shows the eigenvalues of the Schur complement matrix M^k_ρ. Again, $\|M^k_\rho\|$ and $\|(M^k_\rho)^{-1}\| = 1/\lambda_{\min}(M^k_\rho)$ remain bounded as $k \to \infty$, and this is consistent with the result in Proposition 3.2.

Examples 2 and 3. The last example is not very interesting since $\mathcal{B}^k_\rho$ remains well conditioned as $k \to \infty$. The next 2 examples we consider are similar to Example 1. For easy reference, the problem characteristics are summarized Table 1, together with information on the optimal solutions of the problems.

Figure 2 shows the spectra of $\mathcal{B}^k_\rho$ and M^k_ρ corresponding to the final iterates for Examples 2 and 3. From the eigenvalues shown in the left-hand side plots, it is clear that $\|\mathcal{B}^k_\rho\|$ will tend to ∞ as $k \to \infty$. On the other hand, $\|(\mathcal{B}^k_\rho)^{-1}\| = 1/|\lambda_{\min}(\mathcal{B}^k_\rho)|$ remains bounded as $k \to \infty$. This observation is consistent with the asymptotic result in Theorem 3.2. The right-hand side plots in Figure 2 show the eigenvalues of the Schur complement matrices M^k_ρ. Again, $\|(M^k_\rho)^{-1}\| = 1/\lambda_{\min}(M^k_\rho)$ remains bounded as $k \to \infty$, and this is consistent with the result in Theorem 3.1(b).

4 Preconditioners for the augmented matrix $\mathcal{B}^k_\rho$

The convergence rate of a Krylov subspace method such as the minimum residual method (MINRES) applied to the matrix $\mathcal{B}^k_\rho$ depends in a complicated manner on

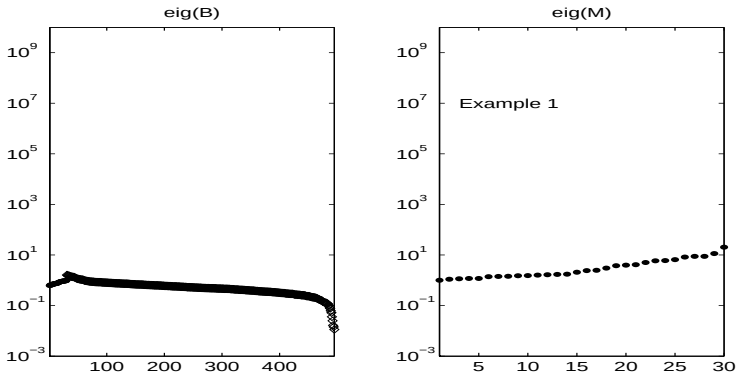

Figure 1: The spectra of $\mathcal{B}_\rho^k$ and M_ρ^k for the final iterate in Example 1. The solid dots correspond to positive eigenvalues, and the diamonds correspond to negative eigenvalues. The x-coordinates are the indices of the eigenvalues, while the y-coordinates are the magnitudes of the eigenvalues in logarithmic scale.

Table 1: Two examples of QSDP with $n = 30$ arising from the nearest correlation matrix problem (3) with resulting $\mathcal{Q}(X) = U \circ X$.

	Example 2	Example 3
K	`B + E`	`B + 1e2*E`
U	`tmp = rand(n);` U `= (tmp+tmp')/2`	`tmp = rand(n);` U `= (tmp+tmp')/2`
$\kappa(\mathcal{Q})$	92.9	92.9
$\|\mathcal{Q}\|$	1	1
$r = \text{rank}(X^*)$ $s = \text{rank}(S^*)$	20 10	5 25
optimal solution (X^*, y^*, S^*)	primal non-degenerate dual degenerate	primal non-degenerate dual non-degenerate
complementarity gap of final iterate	3.3×10^{-10}	3.6×10^{-9}

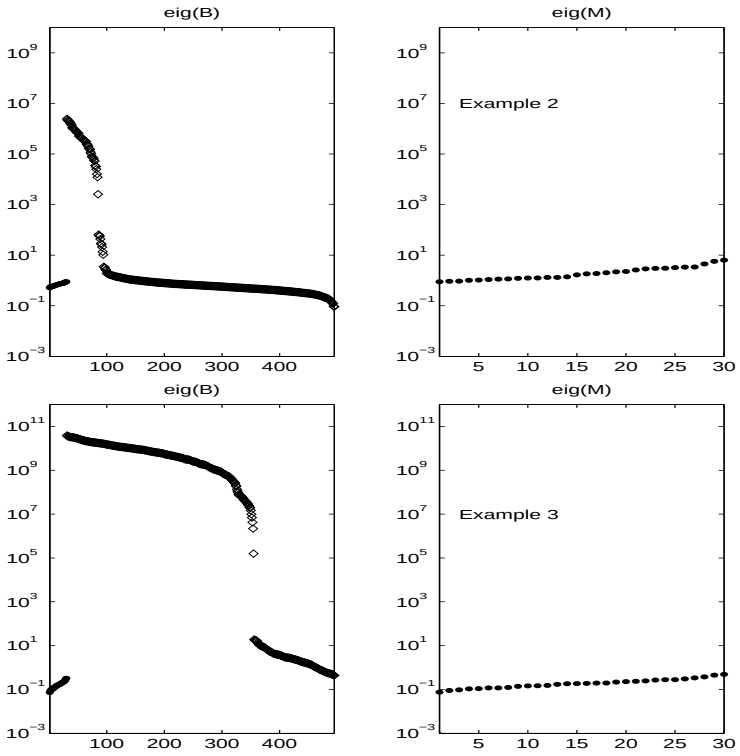

Figure 2: The spectra of $\mathcal{B}_\rho^k$ and M_ρ^k for the final iterates in Examples 2 and 3. The solid dots correspond to positive eigenvalues, and the diamonds correspond to negative eigenvalues.

the eigenvalue distribution of the matrix; see [34]. Even though there is no precise estimate of the convergence rate based on the condition number $\kappa(\mathcal{B}_\rho^k)$ like the case of a symmetric positive definite matrix, the condition number is generally still a reasonable measure of the convergence rate – the smaller the condition number is, the faster the convergence.

Our purpose now is to design preconditioners for $\mathcal{B}_\rho^k$ so that the condition numbers of the preconditioned matrices are bounded independent of ν_k.

We have shown in the previous section that the sequence of condition numbers $\{\kappa(\mathcal{B}_\rho^k)\}_{k=1}^\infty$ is generally unbounded except when the optimal solution is primal non-degenerate, and $S^* = 0$ and $\mathrm{rank}(X^*) = n$. Even if $\{\kappa(\mathcal{B}_\rho^k)\}_{k=1}^\infty$ is bounded, $\kappa(\mathcal{B}_\rho^k)$ can still be large when $\mathcal{Q}_\rho$ is badly conditioned. It is well known that for an iterative method on (8) to achieve an acceptable convergence rate for it to be practical, a good preconditioner is generally needed to accelerate the convergence. Thus preconditioning for $\mathcal{B}_\rho^k$ is generally still needed even if its condition number is bounded as a function of k.

Let $\mathcal{H}_1^k : \mathcal{S}^r \to \mathcal{S}^r$, $\mathcal{H}_2^k : \mathbf{R}^{r\times s} \to \mathbf{R}^{r\times s}$, and $\mathcal{H}_3^k : \mathcal{S}^s \to \mathcal{S}^s$ be given self-adjoint linear operators that are positive definite approximations of $\mathcal{D}_1^k + (\widetilde{\mathcal{Q}}_\rho^k)_{11}$, $\mathcal{D}_2^k + (\widetilde{\mathcal{Q}}_\rho^k)_{22}$, and $\mathcal{D}_3^k + (\widetilde{\mathcal{Q}}_\rho^k)_{33}$, respectively. We will discuss specific choices of $\mathcal{H}_1^k$, $\mathcal{H}_2^k$, $\mathcal{H}_3^k$ later when the needs arise. For later discussions, we define the following linear operator $\mathcal{E}^k : \mathcal{S}^n \to \mathcal{S}^n$:

$$\mathcal{E}^k(U) \;=\; \begin{bmatrix} (\mathcal{H}_1^k)^{-1}(U_1) & (\mathcal{H}_2^k)^{-1}(U_2) \\ ((\mathcal{H}_2^k)^{-1}(U_2))^T & (\mathcal{H}_3^k)^{-1}(U_3) \end{bmatrix}.$$

4.1 Part I

In this subsection, we assume that $\mathcal{H}_1^k$, $\mathcal{H}_2^k$, $\mathcal{H}_3^k$ satisfy the following conditions for all $k = 1, 2, \ldots$:

$$\underline{\sigma}_1 I \preceq \mathcal{H}_1^k \preceq \overline{\sigma}_1 I, \quad \underline{\sigma}_2 I \preceq \mathcal{H}_2^k \preceq \overline{\sigma}_2 I, \quad \mathcal{D}_3^k \preceq \mathcal{H}_3^k \preceq \mathcal{D}_3^k + \overline{\sigma}_3 I, \tag{28}$$

for some positive constants $\underline{\sigma}_1, \overline{\sigma}_1$, $\underline{\sigma}_2, \overline{\sigma}_2$, and $\overline{\sigma}_3$. Let

$$\underline{\theta}^k \;=\; \min\{1, \lambda_{\min}(\mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2}\, \Upsilon_\rho^k\, \mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2})\}, \tag{29}$$

$$\overline{\theta}^k \;=\; \max\{1, \lambda_{\max}(\mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2}\, \Upsilon_\rho^k\, \mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2})\}, \tag{30}$$

where Υ_ρ^k is defined in (21). We have the following asymptotic results for $\{\underline{\theta}^k\}$ and $\{\overline{\theta}^k\}$.

Lemma 4.1. *Suppose that Assumptions A2 and A3 hold for the optimal solution (X^*, y^*, S^*). Suppose further that $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$. Then*

$$\liminf_{k\to\infty} \underline{\theta}_k \;\geq\; \min\left\{1, \min(\overline{\sigma}_1^{-1}, \overline{\sigma}_2^{-1})\lambda_{\min}(\Upsilon_\rho^*)\right\} \;>\; 0, \tag{31}$$

$$\limsup_{k\to\infty} \overline{\theta}_k \;\leq\; \max\left\{1, \max(\underline{\sigma}_1^{-1}, \underline{\sigma}_2^{-1})\lambda_{\max}\left(\overline{\tau} I + \Upsilon_\rho^*\right)\right\} \;<\; \infty. \tag{32}$$

Thus there exist constants $c_1, c_2 > 0$ such that for k sufficiently large,

$$c_1 < \underline{\theta}^k \leq \overline{\theta}^k < c_2.$$

Proof. By noting that $\underline{\sigma}_j I \preceq \mathcal{H}_j^k \preceq \overline{\sigma}_j I$ for $j = 1, 2$, and $\underline{\tau} I \preceq \mathcal{D}_2^k \preceq \overline{\tau} I$ for all k, we have

$$\mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2}\, \Upsilon_\rho^k\, \mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2} \;\succeq\; \min(\overline{\sigma}_1^{-1}, \overline{\sigma}_2^{-1}) \begin{bmatrix} (\widetilde{\mathcal{Q}}_\rho^k)_{11} & (\widetilde{\mathcal{Q}}_\rho^k)_{12} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{21} & \underline{\tau} I + (\widetilde{\mathcal{Q}}_\rho^k)_{22} \end{bmatrix},$$

where the last matrix converges to the matrix Υ_ρ^* in (19) as $k \to \infty$. Under the assumption that $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$, Υ_ρ^* is positive definite, and hence the result in (31) follows.

On the other hand, we have

$$\mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2}\, \Upsilon_\rho^k\, \mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2} \;\preceq\; \max(\underline{\sigma}_1^{-1}, \underline{\sigma}_2^{-1}) \left(\overline{\tau} I + \begin{bmatrix} (\widetilde{\mathcal{Q}}_\rho^k)_{11} & (\widetilde{\mathcal{Q}}_\rho^k)_{12} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{21} & \underline{\tau} I + (\widetilde{\mathcal{Q}}_\rho^k)_{22} \end{bmatrix} \right),$$

where the last matrix again converges to Υ_ρ^* as $k \to \infty$. From here, the inequality in (32) follows easily. ∎

The first preconditioner we consider is the following block diagonal matrix whose partition conforms to that of $\mathcal{B}_\rho^k$:

$$\Phi_{\rho,\pm}^k \;=\; \begin{bmatrix} \pm \mathcal{P}^k \mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k, \mathcal{H}_3^k)(\mathcal{P}^k)^T & 0 \\ 0 & \widehat{S}_\rho^k \end{bmatrix}, \tag{33}$$

where $\widehat{S}_\rho^k$ is a symmetric positive definite approximation of the Schur complement matrix M_ρ^k in (9). An example would be $\widehat{S}_\rho^k = I_m$. We assume that $\widehat{S}_\rho^k$ satisfies the following condition: $\underline{\alpha} I \preceq \widehat{S}_\rho^k \preceq \overline{\alpha} I$ for all $k = 1, 2, \ldots$, for some constants $\underline{\alpha}, \overline{\alpha} > 0$. The motivation for assuming $\mathcal{H}_3^k \succeq \mathcal{D}_3^k$ in (28) for the (1,1) block of $\Phi_{\rho,\pm}^k$ is to annulate the effect of large eigenvalues of $\mathcal{D}^k$ on the conditioning of $\mathcal{B}_\rho^k$.

Block diagonal preconditioners for *sparse* 2×2 block symmetric indefinite systems have been studied extensively. In particular, the paper [24] gave a compelling theoretical basis for considering block diagonal preconditioners. We shall not give the literature review here but refer the reader to the recent survey paper [6]. As far as we are aware of, this paper is the first attempt to apply a block diagonal preconditioner to a *dense* 2×2 block symmetric indefinite system.

Given $[X; y] \in \mathcal{S}^n \times \mathbb{R}^m$, it is easy to see that $(\Phi_{\rho,\pm}^k)^{-1}[X; y]$ can be evaluated efficiently through the steps given in Figure 3.

Compute $\widehat{X} = (P^k)^T X P^k$
Compute $(\Phi_{\rho,\pm}^k)^{-1}[X; y] = [\pm P^k(\mathcal{E}^k(\widehat{X}))(P^k)^T;\, (\widehat{S}_\rho^k)^{-1} y]$.

Figure 3: Computation of $(\Phi_{\rho,\pm}^k)^{-1}[X; y]$ for a given $[X; y] \in \mathcal{S}^n \times \mathbb{R}^m$. This involves four $n \times n$ matrix-matrix multiplications that cost $4n^3$ flops; see [23].

It is well known that the convergence rate of a preconditioned Krylov subspace iterative method such as PSQMR is determined primarily by the spectral distribution and

condition number of the preconditioned matrix. Thus it is of great interest to estimate the spectrum of the preconditioned matrix. We will first establish an asymptotic result for the spectrum of $(\Phi^k_{\rho,+})^{-1}\mathcal{B}^k_\rho$.

Theorem 4.1. *(a) Suppose that Assumptions A2 and A3 hold. Let $r = rank(X^*)$ and $s = n - r$. Then the preconditioned matrix $(\Phi^k_{\rho,+})^{-1}\mathcal{B}^k_\rho$ has $\bar{s}$ eigenvalues equal to $-1+O(\sqrt{\nu_k}\max\{\|\mathcal{Q}_\rho\|, \|\mathcal{A}\|\})$, the rest of the eigenvalues are $O(\sqrt{\nu_k}\max\{\|\mathcal{Q}_\rho\|, \|\mathcal{A}\|\})$ perturbations of those of the following matrix*

$$G^k_{\rho,+} \;:=\; \begin{bmatrix} -N^k_\rho & (L^k)^T \\ L^k & 0 \end{bmatrix},$$

where

$$\begin{aligned} N^k_\rho &= \mathrm{diag}(\mathcal{H}^k_1, \mathcal{H}^k_2)^{-1/2}\Upsilon^k_\rho\,\mathrm{diag}(\mathcal{H}^k_1, \mathcal{H}^k_2)^{-1/2} \\ L^k &= \Big[(\widehat{S}^k_\rho)^{-1/2}\widetilde{\mathcal{A}}^k_1(\mathcal{H}^k_1)^{-1/2},\, (\widehat{S}^k_\rho)^{-1/2}\widetilde{\mathcal{A}}^k_2(\mathcal{H}^k_2)^{-1/2}\Big]. \end{aligned}$$

Let $S^k_\rho = [\widetilde{\mathcal{A}}^k_1,\, \widetilde{\mathcal{A}}^k_2](\Upsilon^k_\rho)^{-1}[\widetilde{\mathcal{A}}^k_1,\, \widetilde{\mathcal{A}}^k_2]^T$. The spectrum of $G^k_{\rho,+}$ is contained in following set on the real line:

$$\Big[-\overline{\theta}^k(\sigma^k_{\max}+1)/2,\, -\underline{\theta}^k(\sigma^k_{\min}+1)/2\Big] \cup \Big[-\overline{\theta}^k,\, -\underline{\theta}^k\Big] \cup \Big[\underline{\theta}^k(\sigma^k_{\min}-1)/2,\, \overline{\theta}^k(\sigma^k_{\max}-1)/2\Big], \quad (34)$$

where $\sigma^k_{\max} = \sqrt{1+4\lambda_{\max}((\widehat{S}^k_\rho)^{-1}S^k_\rho)}$ and $\sigma^k_{\min} = \sqrt{1+4\lambda_{\min}((\widehat{S}^k_\rho)^{-1}S^k_\rho)}$.

(b) Under the assumptions stated in Theorem 3.1(b), there exist positive constants c_1, c_2, c_3, c_4 such that for k sufficiently large,

$$0 < c_1 < \underline{\theta}^k \le \overline{\theta}^k < c_2, \quad 1 < c_3 < \sigma^k_{\min} \le \sigma^k_{\max} < c_4.$$

Thus for k sufficiently large,

$$\mathrm{eig}(G^k_{\rho,+}) \;\subset\; [-c_2(c_4+1),\, -c_1(c_3+1)] \cup [-c_2,\, -c_1] \cup [c_1(c_3-1),\, c_2(c_4-1)].$$

Proof. For simplicity, we drop the superscript k in this proof. Note that the spectrum of $\Phi^{-1}_{\rho,+}\mathcal{B}_\rho$ is the same as that of $\Phi^{-1/2}_{\rho,+}\mathcal{B}_\rho\Phi^{-1/2}_{\rho,+}$.

(a) It is readily shown that $\Phi^{-1/2}_{\rho,+}\mathcal{B}_\rho\Phi^{-1/2}_{\rho,+}$ is orthogonally similar to the following matrix:

$$\begin{bmatrix} -\mathrm{diag}(\mathcal{H}_1, \mathcal{H}_2)^{-1/2}\Upsilon_\rho\,\mathrm{diag}(\mathcal{H}_1, \mathcal{H}_2)^{-1/2} & 0 & \mathcal{H}_1^{-1/2}\widetilde{\mathcal{A}}_1^T\widehat{S}_\rho^{-1/2} \\ & 0 & \mathcal{H}_2^{-1/2}\widetilde{\mathcal{A}}_2^T\widehat{S}_\rho^{-1/2} \\ 0 \quad 0 & -I_{\bar{s}} & 0 \\ \widehat{S}_\rho^{-1/2}\widetilde{\mathcal{A}}_1\mathcal{H}_1^{-1/2} \quad \widehat{S}_\rho^{-1/2}\widetilde{\mathcal{A}}_2\mathcal{H}_2^{-1/2} & 0 & 0 \end{bmatrix} + O(\sqrt{\nu}\max\{\|\mathcal{Q}_\rho\|, \|\mathcal{A}\|\}).$$

Thus the eigenvalues of $\Phi^{-1/2}_{\rho,+}\mathcal{B}_\rho\Phi^{-1/2}_{\rho,+}$ are $O(\sqrt{\nu}\max\{\|\mathcal{Q}_\rho\|, \|\mathcal{A}\|\})$ perturbations of the above matrix, whose eigenvalues are either -1 or those of the matrix $G_{\rho,+}$. Now, we have

$$G_{\rho,+} \;=\; \underbrace{\begin{bmatrix} \mathrm{diag}(\mathcal{H}_1, \mathcal{H}_2)^{-1/2}\Upsilon_\rho^{1/2} & 0 \\ 0 & I \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -I_{\bar{r}+rs} & \Upsilon_\rho^{-1/2}[\widetilde{\mathcal{A}}_1,\, \widetilde{\mathcal{A}}_2]^T\widehat{S}_\rho^{-1/2} \\ \widehat{S}_\rho^{-1/2}[\widetilde{\mathcal{A}}_1,\, \widetilde{\mathcal{A}}_2]\Upsilon_\rho^{-1/2} & 0 \end{bmatrix}}_{U} V^T.$$

By a theorem due to Ostrowski [16, p. 225], we have $\lambda_j(G_{\rho,+}) = \theta_j\,\lambda_j(U)$ for $j = 1,\ldots,\bar{r}+rs+m$, where θ_j satisfies the inequalities: $\underline{\theta} \le \theta_j \le \overline{\theta}$.

It is not difficult to show that U has -1 as an eigenvalue with multiplicity $\bar{r}+rs-m$, and the remaining $2m$ eigenvalues are given by

$$\tfrac{1}{2}\Big(-1 \pm \sqrt{1+4\lambda_j(\widehat{S}_\rho^{-1/2} S_\rho \widehat{S}_\rho^{-1/2})}\Big) \qquad j = 1,\ldots,m.$$

It is now easy to show that the spectrum of $G_{\rho,+}$ is contained in the set given in (34).

(b) Under the assumptions of Theorem 3.1(b), the existence of positive constants c_1 and c_2 is guaranteed by Lemma 4.1. The existence of positive constants c_3 and c_4 will follow if can prove the following results:

$$\liminf_{k\to\infty} \lambda_{\min}((\widehat{S}_\rho^k)^{-1} S_\rho^k) \;\ge\; \underline{\alpha}\,\lambda_{\min}\Big([\widetilde{\mathcal{A}}_1^*,\,\widetilde{\mathcal{A}}_2^*]\,(\overline{\tau} I + \Upsilon_\rho^*)^{-1}\,[\widetilde{\mathcal{A}}_1^*,\,\widetilde{\mathcal{A}}_2^*]^T\Big) \;>\; 0 \quad (35)$$

$$\limsup_{k\to\infty} \lambda_{\max}((\widehat{S}_\rho^k)^{-1} S_\rho^k) \;\le\; \overline{\alpha}\,\lambda_{\max}\Big([\widetilde{\mathcal{A}}_1^*,\,\widetilde{\mathcal{A}}_2^*](\Upsilon_\rho^*)^{-1}[\widetilde{\mathcal{A}}_1^*,\,\widetilde{\mathcal{A}}_2^*]^T\Big) \;<\; \infty. \quad (36)$$

Noting that $\underline{\alpha} I \preceq \widehat{S}_\rho^k \preceq \overline{\alpha} I$ and $\underline{\tau} I \preceq \mathcal{D}_2^k \preceq \overline{\tau} I$ for all k, the first inequality in (35) is straightforward to prove. Under the assumptions of Theorem 3.1(b), $[\widetilde{\mathcal{A}}_1^*,\,\widetilde{\mathcal{A}}_2^*]$ is surjective and Υ_ρ^* is positive definite, from here, the second inequality in (35) follows. We can prove (36) similarly. □

Next, we shall establish an asymptotic result for the spectrum of $(\Phi_{\rho,-}^k)^{-1}\mathcal{B}_\rho^k$.

Theorem 4.2. *(a) Suppose that Assumptions A2 and A3 hold. Let $r = rank(X^*)$ and $s = n - r$. Then the preconditioned matrix $(\Phi_{\rho,-}^k)^{-1}\mathcal{B}_\rho^k$ has $\bar{s}$ eigenvalues equal to $1 + O(\sqrt{\nu_k}\max\{\|\mathcal{Q}_\rho\|, \|\mathcal{A}\|\})$, the rest of the eigenvalues are $O(\sqrt{\nu_k}\max\{\|\mathcal{Q}_\rho\|, \|\mathcal{A}\|\})$ perturbations of the eigenvalues of*

$$G_{\rho,-}^k \;=\; \begin{bmatrix} N_\rho^k & -(L^k)^T \\ L^k & 0 \end{bmatrix},$$

where N_ρ^k and L^k are defined as in Theorem 4.1(a). The complex eigenvalues (with non-zero imaginary parts) of $G_{\rho,-}^k$ are contained in the region

$$\left\{x + iy : \frac{1}{2}\underline{\theta}^k \le x \le \frac{1}{2}\overline{\theta}^k,\; |y| \le \|L^k\|\right\}.$$

The real eigenvalues of $G_{\rho,-}^k$ are contained in the interval $[0,\, \overline{\theta}^k]$.

(b) Under the assumptions stated in Theorem 3.1(b), there exist positive constants c_1, c_2, c_3 such that for k sufficiently large, the complex eigenvalues of $G_{\rho,-}^k$ are contained in the region:

$$\Big\{x + iy \;:\; c_1 \le x \le c_2,\; |y| \le c_3\Big\}.$$

The real eigenvalues of $G_{\rho,-}^k$ are contained in the interval $[0,\, 2c_2]$.

Proof. For simplicity, we drop the superscript k in this proof.

(a) It is readily shown that $\Phi_{\rho,-}^{-1}\mathcal{B}_\rho$ is similar to the following matrix:

$$\begin{bmatrix} \operatorname{diag}(\mathcal{H}_1,\mathcal{H}_2)^{-1/2}\Upsilon_\rho \operatorname{diag}(\mathcal{H}_1,\mathcal{H}_2)^{-1/2} & \begin{matrix} 0 & -\mathcal{H}_1^{-1/2}\widetilde{\mathcal{A}}_1^T\widehat{S}_\rho^{-1/2} \\ 0 & -\mathcal{H}_2^{-1/2}\widetilde{\mathcal{A}}_2^T\widehat{S}_\rho^{-1/2} \end{matrix} \\ \begin{matrix} 0 & 0 \\ \widehat{S}_\rho^{-1/2}\widetilde{\mathcal{A}}_1\mathcal{H}_1^{-1/2} & \widehat{S}_\rho^{-1/2}\widetilde{\mathcal{A}}_2\mathcal{H}_2^{-1/2} \end{matrix} & \begin{matrix} I_{\bar{s}} & 0 \\ 0 & 0 \end{matrix} \end{bmatrix} + O(\sqrt{\nu}\max\{\|\mathcal{Q}_\rho\|,\|\mathcal{A}\|\}).$$

Thus the eigenvalues of $\Phi_{\rho,-}^{-1}\mathcal{B}_\rho$ are $O(\sqrt{\nu}\max\{\|\mathcal{Q}_\rho\|,\|\mathcal{A}\|\})$ perturbations of the above matrix, whose eigenvalues are either 1 or those of the matrix $G_{\rho,-}$. The remaining result in part (a) follows from Proposition 2.11 in [7] and the definitions of $\underline{\theta}$ and $\overline{\theta}$ in (29) and (30), respectively.

(b) Under the assumptions of Theorem 3.1(b), the existence of positive constants c_1 and c_2 is guaranteed by Lemma 4.1. Since $\|L\| \le \underline{\alpha}^{-1/2}\max\{\underline{\sigma}_1^{-1/2},\underline{\sigma}_2^{-1/2}\}\|\mathcal{A}\| =: c_3$, the lemma is proved. ∎

Remark 4.1. *Unlike the matrix $G_{\rho,+}^k$ in Theorem 4.1, we are not able to show that the real eigenvalues of $G_{\rho,-}^k$ in Theorem 4.2 are bounded away from zero even with the assumptions stated in Theorem 3.1(b). However, even though the asymptotic result for the spectral distribution of $(\Phi_{\rho,-}^k)^{-1}\mathcal{B}_\rho^k$ is weaker than that for $(\Phi_{\rho,+}^k)^{-1}\mathcal{B}_\rho^k$, numerical results in [26] showed that a preconditioner analogous to $\Phi_{\rho,-}^k$ is typically more effective than one that is analogous to $\Phi_{\rho,+}^k$. For the numerical experiments in Section 6, our empirical experience (which we do not report in this paper) confirmed that $\Phi_{\rho,-}^k$ is indeed more effective than $\Phi_{\rho,+}^k$.*

Finally, we illustrate the asymptotic result in Theorem 4.1 using the problems in Examples 2 and 3. In this case, we take

$$\mathcal{H}_1^k \;=\; \mathcal{D}_1^k + \|\mathcal{Q}_\rho\| I_{\bar{r}}, \quad \mathcal{H}_2^k \;=\; \mathcal{D}_2^k + \|\mathcal{Q}_\rho\| I_{rs}, \quad \mathcal{H}_3^k \;=\; \mathcal{D}_3^k, \quad \widehat{S}_\rho^k \;=\; I_m. \tag{37}$$

Observe that we have approximated $\mathcal{Q}_\rho$ by $\|\mathcal{Q}_\rho\| I_{\bar{n}}$, and hence

$$\begin{aligned} (\widetilde{\mathcal{Q}}_\rho^k)_{11} &= (\mathcal{P}_1^k)^T\mathcal{Q}_\rho(\mathcal{P}_1^k) \approx \|\mathcal{Q}_\rho\| I_{\bar{r}} \\ (\widetilde{\mathcal{Q}}_\rho^k)_{22} &= (\mathcal{P}_2^k)^T\mathcal{Q}_\rho(\mathcal{P}_2^k) \approx \|\mathcal{Q}_\rho\| I_{rs}. \end{aligned}$$

From the eigenvalues shown in the right-hand side plots in Figure 4, it is clear that for k sufficiently large, $\operatorname{eig}((\Phi_{\rho,+}^k)^{-1}\mathcal{B}_\rho^k)$ is contained in an interval of the form $[-c_1,-c_2]\cup[c_3,c_4]$ with $c_1,c_2,c_3,c_4>0$ independent of k. This observation is consistent with the result in Theorem 4.1(b) by noting that the optimal solutions for both examples are primal non-degenerate and $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$ (a consequence of the fact that $\mathcal{Q} \succ 0$).

4.2 Part II

Our construction of the next preconditioner start with the observation that a matrix of the form given below can be inverted at a moderate cost under Assumption A1 when

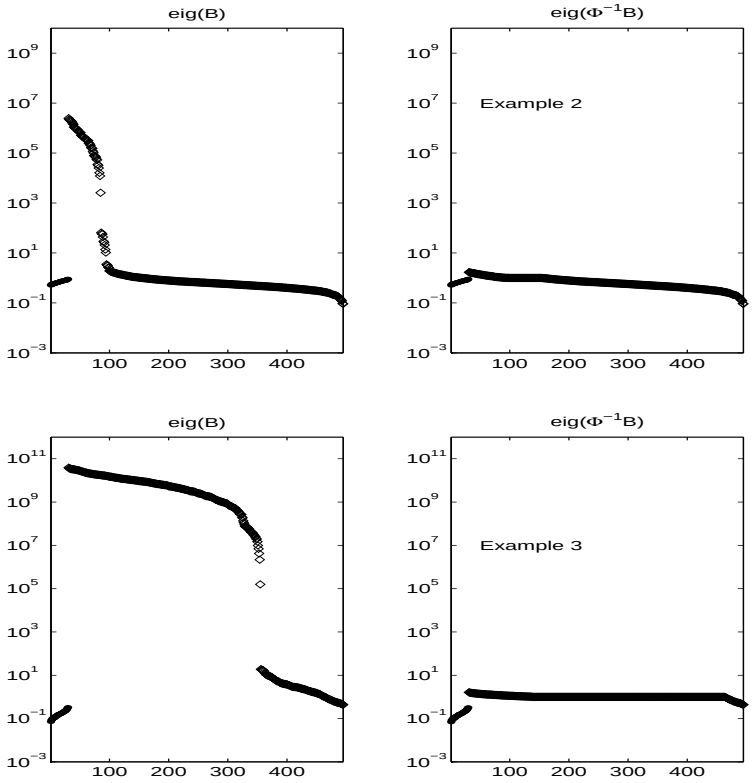

Figure 4: Same as Fig. 2, but for the spectra of $\mathcal{B}_\rho^k$ and $(\Phi_{\rho,+}^k)^{-1}\mathcal{B}_\rho^k$ for Examples 2 and 3.

m is not too large (say, less than 5000):

$$\Omega^k \;=\; \left[\begin{array}{cc} -(W^k)^{-1} \circledast (W^k)^{-1} & \mathcal{A}^T \\ \mathcal{A} & 0 \end{array}\right]. \tag{38}$$

The reason why $(\Omega^k)^{-1}[X;y]$ can be evaluated at a moderate cost is because the (1,1) block of Ω^k can be inverted easily and the corresponding Schur complement matrix $S_\Omega^k := \mathcal{A}W^k \circledast W^k\mathcal{A}^T$ can also be computed at a moderate cost under Assumption A1. Given $[X;y] \in \mathcal{S}^n \times \mathbb{R}^m$, $(\Omega^k)^{-1}[X;y]$ can be evaluated efficiently through the steps given in Figure 5.

Compute $\widehat{X} = W^k X W^k$;
Compute $u = \mathcal{A}(\widehat{X})$;
Compute $v = (S_\Omega^k)^{-1}(u+y)$;
Compute $U = W^k(\mathcal{A}^T v - X)W^k$;
Compute $(\Omega^k)^{-1}[X;y] = [U;\, v]$.

Figure 5: Computation of $(\Omega^k)^{-1}[X;y]$ for a given $[X;y] \in \mathcal{S}^n \times \mathbb{R}^m$. This involves four $n \times n$ matrix-matrix multiplications that cost $4n^3$ flops.

Let $\widehat{\mathcal{Q}}_\rho^k = [\mathcal{Q}_\rho^k,\ 0;\ 0,\ 0]$. Since $\mathcal{B}_\rho^k = \Omega^k - \widehat{\mathcal{Q}}_\rho^k$, it is natural to consider using Ω^k as the preconditioner for $\mathcal{B}_\rho^k$. Unfortunately, when the complementarity gap ν_k is small, Ω^k may have small eigenvalues of the order $\Theta(\nu_k)$ (which would happen if (X^*, y^*, S^*) is primal degenerate) and these small eigenvalues may blow up the norm of $(\Omega^k)^{-1}\widehat{\mathcal{Q}}_\rho^k$. Thus, from the equation $(\Omega^k)^{-1}\mathcal{B}_\rho^k = I - (\Omega^k)^{-1}\widehat{\mathcal{Q}}_\rho^k$, it is not clear whether Ω^k would be a good preconditioner for $\mathcal{B}_\rho^k$. By carefully analyzing the block structure of the preconditioned matrix, we will show later that when the optimal solution (X^*, y^*, S^*) is dual nondegenerate, Ω^k can in fact be an effective preconditioner for $\mathcal{B}_\rho^k$.

The advantage of Ω^k compared to $\Phi_{\rho,\pm}^k$ in the last subsection is that unlike the latter, the former does not require the eigenvalue decomposition of $(W^k)^{-1}$ in its construction. As we shall see later, the eigenvalue decomposition of $(W^k)^{-1}$ is used in the analysis of the eigenvalue distribution of the preconditioned matrix $(\Omega^k)^{-1}\mathcal{B}_\rho^k$, but it is not used computationally.

Before we proceed further, we state a lemma that will be used in the analysis of the spectrum of the preconditioned matrix.

Lemma 4.2. *Suppose $U \in \mathcal{S}^p$ is symmetric positive definite, and $V \in \mathbb{R}^{m\times p}$ has full row rank. Let $G = [-U,\, V^T; V,\, 0]$. Suppose $\widehat{U}$ is a symmetric positive definite approximation of U, and we consider $\widehat{G} = [-\widehat{U},\, V^T; V,\, 0]$ as a preconditioner for G. Then $\widehat{G}^{-1}G$ has $2m$ eigenvalues located at 1, and the remaining $p-m$ eigenvalues are those of the matrix $Z^T\widehat{U}^{-1/2}U\widehat{U}^{-1/2}Z$, where $Z \in \mathbb{R}^{p\times(p-m)}$ is a matrix whose columns form an orthonormal basis of $\mathcal{N}(V\widehat{U}^{-1/2})$.*

Proof. See Theorem 2 in [30]. ∎

The analysis of the spectrum of $(\Omega^k)^{-1}\mathcal{B}_\rho^k$ depends on the following lemma concerning the null space of $\widetilde{\mathcal{A}}^k(\mathcal{D}^k)^{-1/2}$.

Lemma 4.3. *Let* $\left\{Z^l = [Z_1^l;\, Z_2^l;\, Z_3^l]\right\}_{l=1}^{\bar{n}-m}$ *be an orthonormal set in* $\mathcal{S}^r \times \mathbf{R}^{r\times s} \times \mathcal{S}^s$ *that form a basis of* $\mathcal{N}(\widetilde{\mathcal{A}}^k(\mathcal{D}^k)^{-1/2})$. *Let* $\mathcal{Z}_1^k : \boldsymbol{R}^{\bar{n}-m} \to \mathcal{S}^r$ *be the linear map defined by* $\mathcal{Z}_1^k y = \sum_{l=1}^{\bar{n}-m} y_l Z_1^l$. *We define* $\mathcal{Z}_2^k, \mathcal{Z}_3^k$ *similarly. If* (X^*, y^*, S^*) *is dual nondegenerate, then for* k *sufficiently large,*

$$(\mathcal{Z}_1^k)^T(\mathcal{D}_1^k)^{-1}\mathcal{Z}_1^k \;\preceq\; \|(\widetilde{\mathcal{A}}_1^k)^\dagger\|^2\|\mathcal{A}\|^2\Big((\mathcal{Z}_2^k)^T(\mathcal{D}_2^k)^{-1}\mathcal{Z}_2^k + (\mathcal{Z}_3^k)^T(\mathcal{D}_3^k)^{-1}\mathcal{Z}_3^k\Big),$$

where $(\widetilde{\mathcal{A}}_1^k)^\dagger$ *is the pseudo-inverse of* $\widetilde{\mathcal{A}}_1^k$.

Proof. The condition that (X^*, y^*, S^*) is dual nondegenerate implies that $\widetilde{\mathcal{A}}_1^*$ is injective. Since $\lim_{k\to\infty}\widetilde{\mathcal{A}}_1^k = \widetilde{\mathcal{A}}_1^*$, thus for k sufficiently large, $\widetilde{\mathcal{A}}_1^k$ is injective, and $(\widetilde{\mathcal{A}}_1^k)^\dagger = \Big((\widetilde{\mathcal{A}}_1^k)^T\widetilde{\mathcal{A}}_1^k\Big)^{-1}(\widetilde{\mathcal{A}}_1^k)^T$.

Now, $Z^l \in \mathcal{N}(\widetilde{\mathcal{A}}^k(\mathcal{D}^k)^{-1/2})$ for $l = 1, \ldots, \bar{n}-m$ implies that

$$\widetilde{\mathcal{A}}_1^k(\mathcal{D}_1^k)^{-1/2}\mathcal{Z}_1^k \;=\; -[\widetilde{\mathcal{A}}_2^k,\, \widetilde{\mathcal{A}}_3^k]\;[(\mathcal{D}_2^k)^{-1/2}\mathcal{Z}_2^k;\, (\mathcal{D}_3^k)^{-1/2}\mathcal{Z}_3^k].$$

Thus for k sufficiently large,

$$(\mathcal{D}_1^k)^{-1/2}\mathcal{Z}_1^k \;=\; -(\widetilde{\mathcal{A}}_1^k)^\dagger[\widetilde{\mathcal{A}}_2^k,\, \widetilde{\mathcal{A}}_3^k]\,[(\mathcal{D}_2^k)^{-1/2}\mathcal{Z}_2^k;\, (\mathcal{D}_3^k)^{-1/2}\mathcal{Z}_3^k]. \tag{39}$$

By noting that $[\widetilde{\mathcal{A}}_2^k,\, \widetilde{\mathcal{A}}_3^k]^T((\widetilde{\mathcal{A}}_1^k)^\dagger)^T(\widetilde{\mathcal{A}}_1^k)^\dagger[\widetilde{\mathcal{A}}_2^k,\, \widetilde{\mathcal{A}}_3^k] \preceq \|(\widetilde{\mathcal{A}}_1^k)^\dagger\|^2\|\mathcal{A}\|^2 I$, the equation in (39) implies that

$$(\mathcal{Z}_1^k)^T(\mathcal{D}_1^k)^{-1}\mathcal{Z}_1^k \;\preceq\; \|(\widetilde{\mathcal{A}}_1^k)^\dagger\|^2\|\mathcal{A}\|^2[(\mathcal{D}_2^k)^{-1/2}\mathcal{Z}_2^k;\, (\mathcal{D}_3^k)^{-1/2}\mathcal{Z}_3^k]^T[(\mathcal{D}_2^k)^{-1/2}\mathcal{Z}_2^k;\, (\mathcal{D}_3^k)^{-1/2}\mathcal{Z}_3^k].$$

Hence the required result in the lemma follows. ∎

Theorem 4.3. *Let* $\mathcal{Z}_j^k$, $j = 1, 2, 3$, *be defined as in Lemma 4.3 for* $\mathcal{N}(\widetilde{\mathcal{A}}^k(\mathcal{D}^k)^{-1/2})$. *Let* $\mathcal{Z}^k = [\mathcal{Z}_1^k;\, \mathcal{Z}_2^k;\, \mathcal{Z}_3^k]$ *and*

$$\boldsymbol{G}_\rho^k \;=\; (\mathcal{Z}^k)^T(\mathcal{D}^k)^{-1/2}(\mathcal{D}^k + \widetilde{\mathcal{Q}}_\rho^k)(\mathcal{D}^k)^{-1/2}\mathcal{Z}^k.$$

We have the following results.

(a) The preconditioned matrix $(\Omega^k)^{-1}\mathcal{B}_\rho^k$ *has* $2m$ *eigenvalues equal to* 1, *and the remaining* $\bar{n}-m$ *eigenvalues are those of the matrix* $\boldsymbol{G}_\rho^k$.
(b) Suppose the optimal solution (X^*, y^*, S^*) *is dual nondegenerate. Then for* k *sufficiently large,*

$$\mathrm{eig}(\boldsymbol{G}_\rho^k) \subset 1 + \Big[0,\, \|\mathcal{Q}_\rho\|\Big(\|(\widetilde{\mathcal{A}}_1^k)^\dagger\|^2\|\mathcal{A}\|^2 + 1\Big)\max(\underline{\tau}^{-1}, \Theta(\nu_k))\Big].$$

Proof. (a) The result follows from Lemma 4.2.
(b) For the rest of the proof, we assume that k is sufficiently large. Now we have

$$\boldsymbol{G}_\rho^k \;=\; I + (\mathcal{Z}^k)^T(\mathcal{D}^k)^{-1/2}\widetilde{\mathcal{Q}}_\rho^k(\mathcal{D}^k)^{-1/2}\mathcal{Z}^k.$$

Observe that the second term on the right-hand side is positive semidefinite. Thus $\boldsymbol{G}_\rho^k \succeq I$. Now consider the term $(\mathcal{Z}^k)^T(\mathcal{D}^k)^{-1/2}\widetilde{\mathcal{Q}}_\rho^k(\mathcal{D}^k)^{-1/2}\mathcal{Z}^k$ on the right-hand side. Since $0 \preceq \widetilde{\mathcal{Q}}_\rho^k \preceq \|\mathcal{Q}_\rho\| I$, we have

$$\begin{aligned}
(\mathcal{Z}^k)^T(\mathcal{D}^k)^{-1/2}\widetilde{\mathcal{Q}}_\rho^k(\mathcal{D}^k)^{-1/2}\mathcal{Z}^k \;&\preceq\; \|\mathcal{Q}_\rho\|(\mathcal{Z}^k)^T(\mathcal{D}^k)^{-1}\mathcal{Z}^k \\
&= \|\mathcal{Q}_\rho\|\sum_{j=1}^{3}(\mathcal{Z}_j^k)^T(\mathcal{D}_j^k)^{-1}\mathcal{Z}_j^k \\
&\preceq \|\mathcal{Q}_\rho\|\,(\|(\widetilde{\mathcal{A}}_1^k)^\dagger\|^2\|\mathcal{A}\|^2+1)\Big((\mathcal{Z}_2^k)^T(\mathcal{D}_2^k)^{-1}\mathcal{Z}_2^k+(\mathcal{Z}_3^k)^T(\mathcal{D}_3^k)^{-1}\mathcal{Z}_3^k\Big).
\end{aligned}$$

Now $\underline{\tau} I \preceq \mathcal{D}_2^k$ and $\Theta(1/\nu_k)I \preceq \mathcal{D}_3^k$ implies that $(\mathcal{Z}_j^k)^T(\mathcal{D}_j^k)^{-1}\mathcal{Z}_j^k \preceq \max(\underline{\tau}^{-1},\Theta(\nu_k))(\mathcal{Z}_j^k)^T\mathcal{Z}_j^k$ for $j=2,3$. Together with the fact that $\sum_{j=1}^3(\mathcal{Z}_j^k)^T\mathcal{Z}_j^k = I$, we have

$$(\mathcal{Z}^k)^T(\mathcal{D}^k)^{-1/2}\widetilde{\mathcal{Q}}_\rho^k(\mathcal{D}^k)^{-1/2}\mathcal{Z}^k \preceq \|\mathcal{Q}_\rho\|\,(\|(\widetilde{\mathcal{A}}_1^k)^\dagger\|^2\|\mathcal{A}\|^2+1)\max(\underline{\tau}^{-1},\Theta(\nu_k))\; I.$$

From the above, the required result follows readily. □

To illustrate the asymptotic result in Theorem 4.3(b), we consider again the problems in Examples 2 and 3. The spectra of $(\Omega^k)^{-1}\mathcal{B}_\rho^k$ for Examples 2 and 3 are plotted in Figure 6. From the eigenvalues shown in the right-hand side plots in Figure 6, it is clear that for k sufficiently large, $\mathrm{eig}((\Omega^k)^{-1}\mathcal{B}_\rho^k)$ for Example 3 is contained in an interval of the form $[c_1,c_2]$ with $c_1,c_2>0$ independent of k. This observation is consistent with the result in Theorem 4.3(b) by noting that the optimal solution for Example 3 is dual nondegenerate. On the other hand, the optimal solution for Example 2 is dual degenerate, thus the spectrum of $(\Omega^k)^{-1}\mathcal{B}_\rho^k$ shown in Figure 6 is not contained in a finite interval as $k\to\infty$.

4.3 Part III

The preconditioner Ω^k in (38) is simpler to construct than $\Phi_{\rho,\pm}^k$ in (33) in that it does not require the partition of the eigenspace of $(W^k)^{-1}$, and hence the eigenvalue decomposition of $(W^k)^{-1}$ is not needed. Despite the advantage just mentioned, and that the preconditioned matrix $(\Omega^k)^{-1}\mathcal{B}_\rho^k$ has favorable asymptotic spectrum when the optimal solution is dual nondegenerate, the preconditioner Ω^k however, does not take into account the presence of $\mathcal{Q}_\rho$ in the (1,1) block of $\mathcal{B}_\rho^k$. Such an omission makes Ω^k potentially less effective than a preconditioner that also takes $\mathcal{Q}_\rho$ into account.

The next preconditioner we propose has the same off-diagonal blocks as Ω^k, but also take into account the presence of $\mathcal{Q}_\rho$ in the (1,1) block. As we shall see below, it is constructed by suitably modifying the eigenvalues of $(W^k)^{-1}\circledast(W^k)^{-1} = \mathcal{P}^k\mathrm{diag}(\mathcal{D}_1^k,\mathcal{D}_2^k,\mathcal{D}_3^k)(\mathcal{P}^k)^T$.

We consider the following preconditioner:

$$\Psi_\rho^k \;=\; \begin{bmatrix} -\mathcal{P}^k\,\Gamma_\rho^k\,(\mathcal{P}^k)^T & \mathcal{A}^T \\ \mathcal{A} & 0 \end{bmatrix}, \quad \text{where } \Gamma_\rho^k = \mathrm{diag}(\mathcal{H}_1^k,\mathcal{H}_2^k,\mathcal{H}_3^k). \tag{40}$$

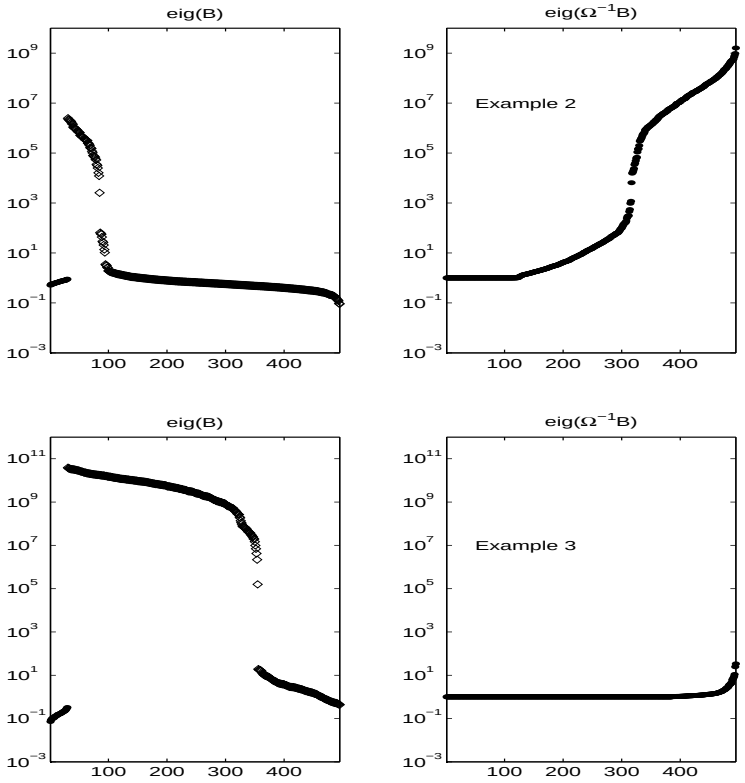

Figure 6: Same as Fig. 2 but for the spectra of $\mathcal{B}_\rho^k$ and $(\Omega^k)^{-1}\mathcal{B}_\rho^k$ for Examples 2 and 3.

For the above preconditioner, we typically want to choose the linear operators $\mathcal{H}_1^k$ on $\mathcal{S}^r$, $\mathcal{H}_2^k$ on $\mathbf{R}^{r\times s}$, and $\mathcal{H}_3^k$ on $\mathcal{S}^s$ to have the following forms:

$$\begin{aligned}\mathcal{H}_1^k &= H_1^k \circledast H_1^k, \quad H_1^k \in \mathcal{S}^r_{++},\\ \mathcal{H}_2^k &= \widehat{H}_2^k \otimes H_2^k, \quad H_2^k \in \mathcal{S}^r_{++}, \widehat{H}_2^k \in \mathcal{S}^s_{++}\\ \mathcal{H}_3^k &= H_3^k \circledast H_3^k, \quad H_3^k \in \mathcal{S}^s_{++}.\end{aligned} \tag{41}$$

The motivation for considering the operators in (41) will become apparent when we derive the Schur complement matrix associated with Ψ_ρ^k in the ensuing paragraph.

For the choices of $\mathcal{H}_1^k$, $\mathcal{H}_2^k$, and $\mathcal{H}_3^k$ in (41), the Schur complement matrix associated with Ψ_ρ^k is given by

$$\begin{aligned}S_\Psi^k &= \mathcal{A}\mathcal{P}^k(\Gamma_\rho^k)^{-1}(\mathcal{P}^k)^T\mathcal{A}^T &(42)\\ &= \widetilde{\mathcal{A}}_1^k(\mathcal{H}_1^k)^{-1}(\widetilde{\mathcal{A}}_1^k)^T + \widetilde{\mathcal{A}}_2^k(\mathcal{H}_2^k)^{-1}(\widetilde{\mathcal{A}}_2^k)^T + \widetilde{\mathcal{A}}_3^k(\mathcal{H}_3^k)^{-1}(\widetilde{\mathcal{A}}_3^k)^T\\ &= \mathcal{A}V_1^k \circledast V_1^k\mathcal{A}^T + 2\mathcal{A}V_2^k \circledast \widehat{V}_2^k\mathcal{A}^T + \mathcal{A}V_3^k \circledast V_3^k\mathcal{A}^T, &(43)\end{aligned}$$

where

$$\begin{aligned}V_1^k = P_1^k(H_1^k)^{-1}(P_1^k)^T, \quad V_2^k = P_1^k(H_2^k)^{-1}(P_1^k)^T\\ \widehat{V}_2^k = P_2^k(\widehat{H}_2^k)^{-1}(P_2^k)^T, \quad V_3^k = P_2^k(H_3^k)^{-1}(P_2^k)^T.\end{aligned}$$

In this case, the computation of S_Ψ^k is similar to that for the Schur complement matrix in a linear SDP employing the NT direction. In particular, any sparsity structure present in $\mathcal{A}$ can be exploited in the computation just as in a linear SDP employing the NT direction; see [11]. The formula in (43) for $\mathcal{S}_\Psi^k$ holds only for the special choices of $\mathcal{H}_1^k$, $\mathcal{H}_2^k$ and $\mathcal{H}_3^k$ in (41). Without using the special choices, $\mathcal{S}_\Psi^k$ must be computed from (42) for which the computation is akin to that for the Schur complement matrix of a linear SDP employing the AHO direction [1]. It is well known that the latter computation cannot exploit sparsity in $\mathcal{A}$ as effectively as the computation using (43). This explains the rationale for the special choices of $\mathcal{H}_1^k$, $\mathcal{H}_2^k$ and $\mathcal{H}_3^k$ in (41).

Given $[X; y] \in \mathcal{S}^n \times \boldsymbol{R}^m$, $(\Psi_\rho^k)^{-1}[X; y]$ can be evaluated efficiently through the steps given in Figure 7. Note that if (X^*, y^*, S^*) is primal degenerate (i.e., $[\widetilde{\mathcal{A}}_1^*, \widetilde{\mathcal{A}}_2^*]$ is not surjective), then Ψ_ρ^k may be nearly singular when k is large.

Theorem 4.4. *Suppose that Assumptions A2 and A3 hold. Let* $\left\{Z^l = [Z_1^l; Z_2^l; Z_3^l]\right\}_{l=1}^{\bar{n}-m}$ *be an orthonormal set in* $\mathcal{S}^r \times \mathbf{R}^{r\times s} \times \mathcal{S}^s$ *that form a basis of* $\mathcal{N}(\widetilde{\mathcal{A}}^k(\Gamma_\rho^k)^{-1/2})$. *Let* $\mathcal{Z}_1^k : \boldsymbol{R}^{\bar{n}-m} \to \mathcal{S}^r$ *be the linear map defined by* $\mathcal{Z}_1^k y = \sum_{l=1}^{\bar{n}-m} y_l Z_1^l$. *We define* $\mathcal{Z}_2^k, \mathcal{Z}_3^k$ *similarly. Suppose* $\mathcal{Z}^k = [\mathcal{Z}_1^k; \mathcal{Z}_2^k; \mathcal{Z}_3^k]$.
Consider the matrix

$$\boldsymbol{G}_\rho^k = (\mathcal{Z}^k)^T(\Gamma_\rho^k)^{-1/2}(\mathcal{D}^k + \widetilde{\mathcal{Q}}_\rho^k)(\Gamma_\rho^k)^{-1/2}\mathcal{Z}^k.$$

The preconditioned matrix $(\Psi_\rho^k)^{-1}\mathcal{B}_\rho^k$ *has* $2m$ *eigenvalues located at* 1. *The remaining* $\bar{n} - m$ *eigenvalues are those of the matrix* $\boldsymbol{G}_\rho^k$. *In addition, we have the following results.*

Compute $\widehat{X} = (P^k)^T X P^k$;
Compute $u = \mathcal{A}(P^k(\mathcal{E}^k(\widehat{X}))(P^k)^T)$;
Compute $v = (S_\Psi^k)^{-1}(u + y)$;
Compute $U = (P^k)^T(\mathcal{A}^T v - X)P^k$;
Compute $(\Psi_\rho^k)^{-1}[X; y] = [P^k(\mathcal{E}^k(U))(P^k)^T;\, v]$.

Figure 7: Computation of $(\Psi_\rho^k)^{-1}[X; y]$ for a given $[X; y] \in \mathcal{S}^n \times I\!R^m$. This involves eight $n \times n$ matrix-matrix multiplications that cost $8n^3$ flops.

(b) Suppose that $\mathcal{H}_1^k, \mathcal{H}_2^k, \mathcal{H}_3^k$ in Γ_ρ^k satisfy the condition in (28) and $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$. Then there exist positive constants c_1, c_2 such that for k sufficiently large,

$$\mathrm{eig}(\boldsymbol{G}_\rho^k) \;\subset\; [c_1,\, c_2]. \tag{44}$$

(c) Suppose that $\mathcal{H}_1^k, \mathcal{H}_2^k, \mathcal{H}_3^k$ in Γ_ρ^k satisfy the condition in (28) and $\mathcal{D}^k \preceq \Gamma_\rho^k \preceq \mathcal{D}^k + \widetilde{\mathcal{Q}}_\rho^k$. Then

$$\mathrm{eig}(\boldsymbol{G}_\rho^k) \;\subset\; 1 + \Big[0,\, \|\mathcal{Q}_\rho\| \max\{\underline{\sigma}_1^{-1}, \underline{\sigma}_2^{-1}, \Theta(\nu_k)\}\Big]. \tag{45}$$

(d) Suppose $\underline{\beta}(\mathcal{D}^k + \widetilde{\mathcal{Q}}_\rho^k) \preceq \Gamma_\rho^k \preceq \overline{\beta}(\mathcal{D}^k + \widetilde{\mathcal{Q}}_\rho^k)$ for some constants $\underline{\beta}, \overline{\beta} > 0$. Then

$$\mathrm{eig}(\boldsymbol{G}_\rho^k) \;\subset\; \Big[\overline{\beta}^{-1},\, \underline{\beta}^{-1}\Big]. \tag{46}$$

Proof. (a) The first result follows from Lemma 4.2.

(b) It is readily shown that $\boldsymbol{G}_\rho^k$ can be written as follows:

$$\boldsymbol{G}_\rho^k \;=\; (\mathcal{Z}^k)^T \begin{bmatrix} \mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2} \Upsilon_\rho^k \,\mathrm{diag}(\mathcal{H}_1^k, \mathcal{H}_2^k)^{-1/2} & 0 \\ 0 & I_{\bar{s}} \end{bmatrix} \mathcal{Z}^k \;+\; O(\sqrt{\nu_k}\|\mathcal{Q}_\rho\|),$$

where Υ_ρ^k is defined in (21). Thus, using the definitions of $\underline{\theta}^k$ and $\overline{\theta}^k$ in (29) and (30), we have

$$\mathrm{eig}(\boldsymbol{G}_\rho^k) \;\subset\; [\underline{\theta}^k,\, \overline{\theta}^k] + O(\sqrt{\nu_k}\|\mathcal{Q}_\rho\|).$$

By Lemma 4.1, the required result in (44) follows.

(c) To prove (45), note that

$$\boldsymbol{G}_\rho^k \;=\; I + (\mathcal{Z}^k)^T(\Gamma_\rho^k)^{-1/2}(\widetilde{\mathcal{Q}}_\rho^k - \mathcal{V}_\rho^k)(\Gamma_\rho^k)^{-1/2}(\mathcal{Z}^k)^T,$$

where $\mathcal{V}_\rho^k := \Gamma_\rho^k - \mathcal{D}^k \succeq 0$. Since $\widetilde{\mathcal{Q}}_\rho^k \succeq \mathcal{V}_\rho^k$, it is clear that $\boldsymbol{G}_\rho^k \succeq I$. On the other hand, since $\mathcal{V}_\rho^k \succeq 0$, we have $\widetilde{\mathcal{Q}}_\rho^k - \mathcal{V}_\rho^k \preceq \widetilde{\mathcal{Q}}_\rho^k \preceq \|\mathcal{Q}_\rho\| I$, and hence

$$\boldsymbol{G}_\rho^k \;\preceq\; I + (\mathcal{Z}^k)^T(\Gamma_\rho^k)^{-1/2}\widetilde{\mathcal{Q}}_\rho^k(\Gamma_\rho^k)^{-1/2}\mathcal{Z}^k \;\preceq\; I + \|\mathcal{Q}_\rho\|(\mathcal{Y}^k)^T\mathcal{Y}^k,$$

where $\mathcal{Y}^k := [\mathcal{Y}_1^k; \mathcal{Y}_2^k; \mathcal{Y}_3^k] = (\Gamma_\rho^k)^{-1/2}\mathcal{Z}^k = [(\mathcal{H}_1^k)^{-1/2}\mathcal{Z}_1^k; (\mathcal{H}_2^k)^{-1/2}\mathcal{Z}_2^k; (\mathcal{H}_3^k)^{-1/2}\mathcal{Z}_3^k]$. Now

$$\begin{aligned} I &= (\mathcal{Z}^k)^T\mathcal{Z}^k = (\mathcal{Y}^k)^T\Gamma_\rho^k\mathcal{Y}^k = (\mathcal{Y}_1^k)^T\mathcal{H}_1^k\mathcal{Y}_1^k + (\mathcal{Y}_2^k)^T\mathcal{H}_2^k\mathcal{Y}_2^k + (\mathcal{Y}_3^k)^T\mathcal{H}_3^k\mathcal{Y}_3^k \\ &\succeq \underline{\sigma}_1(\mathcal{Y}_1^k)^T\mathcal{Y}_1^k + \underline{\sigma}_2(\mathcal{Y}_2^k)^T\mathcal{Y}_2^k + \Theta(1/\nu_k)(\mathcal{Y}_3^k)^T\mathcal{Y}_3^k \\ &\succeq \min\{\underline{\sigma}_1, \underline{\sigma}_2, \Theta(1/\nu_k)\}\ (\mathcal{Y}^k)^T\mathcal{Y}^k. \end{aligned}$$

Hence

$$\boldsymbol{G}_\rho^k \ \preceq\ I + \|\mathcal{Q}_\rho\|(\mathcal{Y}^k)^T\mathcal{Y}^k \ \preceq\ I + \|\mathcal{Q}_\rho\|\max\{\underline{\sigma}_1^{-1}, \underline{\sigma}_2^{-1}, \Theta(\nu_k)\}I.$$

From here, (45) follows readily.

(d) Since $\underline{\beta}(\Gamma_\rho^k)^{-1/2}(\mathcal{D}^k+\widetilde{\mathcal{Q}}_\rho^k)(\Gamma_\rho^k)^{-1/2} \preceq I \preceq \overline{\beta}(\Gamma_\rho^k)^{-1/2}(\mathcal{D}^k+\widetilde{\mathcal{Q}}_\rho^k)(\Gamma_\rho^k)^{-1/2}$, and $(\mathcal{Z}^k)^T\mathcal{Z}^k = I$, we have

$$\overline{\beta}^{-1}I \ \preceq\ \boldsymbol{G}_\rho^k \ \preceq\ \underline{\beta}^{-1}I.$$

From here, the required result follows. □

Remark 4.2. *(a) Observe that unlike Theorem 4.1(b), Theorem 4.4(b) does not need to assume primal nondegeneracy for the optimal solution (X^*, y^*, S^*). Also, Theorem 4.4(b) does not need the optimal solution to be dual nondegenerate in contrast to Theorem 4.3(b).*

(b) The result in Theorem 4.4(b) guarantees that the sequence $\{\lambda_{\min}((\Psi_\rho^k)^{-1}\mathcal{B}_\rho^k)\}$ is bounded away from 0, even though $\{\lambda_{\min}(\mathcal{B}_\rho^k)\}$ is converging to 0 when (X^*, y^*, S^*) is primal degenerate. Such a result is possible because $\{\lambda_{\min}(\Psi_\rho^k)\}$ itself is converging to 0 when (X^*, y^*, S^*) is primal degenerate.

(c) As an example, we show how Theorem 4.4(b) can be modified to suit the LCCQP problem (4). Note that for a LCCQP problem, $\mathcal{D}_2^k$ does not exit, $\mathcal{D}^k = \mathrm{diag}(\mathcal{D}_1^k, \mathcal{D}_3^k)$ with $\mathrm{diag}(\mathcal{D}_1^k) = \Theta(\nu_k)$, $\mathrm{diag}(\mathcal{D}_3^k) = \Theta(1/\nu_k)$, and $\mathcal{P}^k$ is a permutation matrix. $\widetilde{\mathcal{A}}^k$ is actually A but with its columns permuted according to the partition in $\mathcal{D}^k$. Similarly, $\widetilde{\mathcal{Q}}_\rho^k$ is $Q_\rho := Q + \rho A^TA$ but with its rows and columns permuted according to the partition in $\mathcal{D}^k$. Assuming that $\mathcal{H}_1^k$ and $\mathcal{H}_3^k$ satisfy the condition in (28) and $(\widetilde{\mathcal{Q}}_\rho)_{11} \succ 0$, then there exist positive constants c_1, c_2 such that $\mathrm{eig}(\Psi_\rho^k)^{-1}\mathcal{B}_\rho^k) \subset \{1\} \cup [c_1, c_2]$ for k sufficiently large. We note that such a result is established under a weaker condition than the one required in Corollary 4.5 of [8], which assumed Q to be positive definite.

We consider again the problems in Examples 2 and 3 to illustrate the asymptotic result in Theorem 4.4(b). We take $\mathcal{H}_1^k$, $\mathcal{H}_2^k$, and $\mathcal{H}_3^k$ as in (37). The spectra of $(\Psi_\rho^k)^{-1}\mathcal{B}_\rho^k$ for Examples 2 and 3 are plotted in Figure 8. From the eigenvalues shown in the right-hand side plots of Figure 8, it is clear that for k sufficiently large, $\mathrm{eig}((\Psi_\rho^k)^{-1}\mathcal{B}_\rho^k)$ is contained in an interval of the form $[c_1, c_2]$ with $c_1, c_2 > 0$ independent of k. This observation is consistent with the result in Theorem 4.4(b).

Table 2 summarizes the asymptotic results for the spectra of the preconditioned matrices corresponding to the preconditioners we have constructed so far.

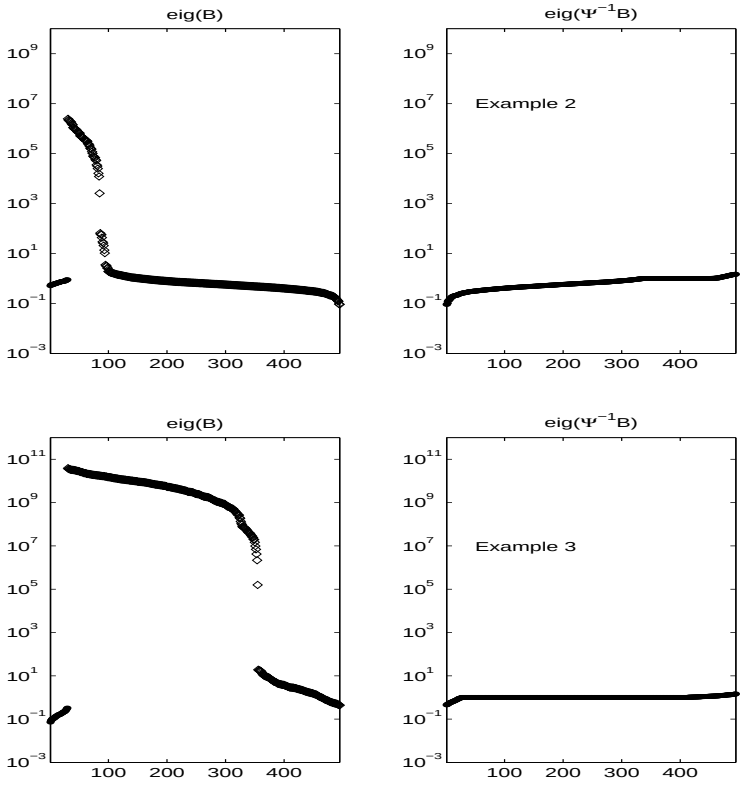

Figure 8: Same as Fig. 2 but for the spectra of $\mathcal{B}_\rho^k$ and $(\Psi_\rho^k)^{-1}\mathcal{B}_\rho^k$ for Examples 2 and 3.

Table 2: Under Assumptions 2 and 3, we have the following results on the various sequences indexed by k.

	Additional assumption	Result: for k sufficiently large
$\|\mathcal{B}_\rho^k\|$	$(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$	Bounded if $r = n$ where $r = \mathrm{rank}(X^*)$; unbounded if $r < n$.
$\|(\mathcal{B}_\rho^k)^{-1}\|$	$(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$	Bounded if (X^*, y^*, S^*) is primal nondegenerate; unbouded otherwise.
$\kappa((\Phi_{\rho,+}^k)^{-1}\mathcal{B}_\rho^k)$	(28) holds and $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$	Bounded if (X^*, y^*, S^*) is primal nondegenerate; unbouded otherwise.
$\kappa((\Omega^k)^{-1}\mathcal{B}_\rho^k)$		Bounded if (X^*, y^*, S^*) is dual nondegenerate; unbouded otherwise.
$\kappa((\Psi_\rho^k)^{-1}\mathcal{B}_\rho^k)$	(28) holds and $(\widetilde{\mathcal{Q}}_\rho^*)_{11} \succ 0$ or $\mathcal{D}^k \preceq \Gamma_\rho^k \preceq \mathcal{D}^k + \widetilde{\mathcal{Q}}_\rho^k$	Bounded.

5 Preconditioners for the initial phase of the interior-point iteration

The preconditioners in Section 4 produce favorable asymptotic spectral distributions for the preconditioned matrices under suitable nondegeneracy assumptions. However, during the initial phase of the interior-point iteration, these preconditioners may not be the most effective ones for $\mathcal{B}_\rho^k$. This is especially true when the complementarity gap ν_k is relatively large and there is no clear separation of the eigenvalues of $(W^k)^{-1}$ into two distinct clusters as we had relied upon in the asymptotic analysis.

The preconditioner Ω^k in (38) is simple to construct and it is favorable for $\mathcal{B}_\rho^k$ asymptotically when the optimal solution is dual nondegenerate. But it does not take into account the presence of $\mathcal{Q}_\rho$. A better choice than Ω^k would be the following preconditioner that also takes $\mathcal{Q}_\rho$ into account:

$$\mathcal{V}_\rho^k \;=\; \left[\begin{array}{cc} -V_\rho^k \circledast V_\rho^k & \mathcal{A}^T \\ \mathcal{A} & 0 \end{array} \right], \tag{47}$$

where V_ρ^k is constructed as follows. First we find a positive semidefinite symmetrized Kronecker product approximation, say $\Delta_\rho \circledast \Delta_\rho$, of $\mathcal{Q}_\rho$. Then consider the decomposition of the sum: $(\widehat{W}^k)^{-1}\circledast(\widehat{W}^k)^{-1}+\Delta_\rho\circledast\Delta_\rho = (P^k\circledast P^k)(\widehat{D}^k\circledast\widehat{D}^k+\widetilde{\Delta}_\rho^k\circledast\widetilde{\Delta}_\rho^k)(P^k\circledast P^k)^T$, where $\widetilde{\Delta}_\rho^k = (P^k)^T\Delta_\rho P^k$, and $\widehat{D}^k = \mathrm{diag}(D_1^k+I, D_2^k)$. (Recall that $D^k = \mathrm{diag}(D_1^k, D_2^k)$ is the diagonal matrix of eigenvalues of $(W^k)^{-1}$. The identity matrix in $\widehat{D}^k$ can of course be replaced by any non-negative scalar multiple of I. The motivation for adding I to D_1^k is to make $\widehat{D}^k$ positive definite when ν_k tends to zero.) For the linear operator $\widehat{D}^k \circledast \widehat{D}^k + \widetilde{\Delta}_\rho^k \circledast \widetilde{\Delta}_\rho^k$ in the decomposition, find a positive definite symmetrized Kronecker product approximation $\widetilde{V}_\rho^k \circledast \widetilde{V}_\rho^k$. Finally, take $V_\rho^k = P^k\widetilde{V}_\rho^k(P^k)^T$. For the matrix $\mathcal{V}_\rho^k$ in (47), the computation of $(\mathcal{V}_\rho^k)^{-1}[X;y]$ for a given $[X;y]$ is similar to that for $(\Omega^k)^{-1}[X;y]$, and this can be done efficiently under Assumption A1.

An alternative to $\mathcal{V}_\rho^k$ in (47) during the initial phase of the interior-point iteration is the following preconditioner:

$$\left[\begin{array}{cc} -\mathrm{diag}((W^k)^{-1}) \circledast \mathrm{diag}((W^k)^{-1}) - \mathrm{diag}(\mathcal{Q}_\rho) & \mathcal{A}^T \\ \mathcal{A} & 0 \end{array} \right]. \tag{48}$$

The above is analogous to the preconditioner proposed in [8] for the LCCQP problem (4) in $\boldsymbol{R}^n$. Unfortunately, it is not competitive at all compared to $\mathcal{V}_\rho^k$ for the test problems we are considering in Section 6. An important distinction between the preconditioner in (48) and its counterpart for LCCQP is that for the latter, the contribution by the interior-point iterate is fully incorporated into the (1,1) block of the preconditioner, but for the former, only a diagonal approximation of the operator $(W^k)^{-1} \circledast (W^k)^{-1}$ is incorporated. Such a diagonal approximation appears to be too weak to make (48) an effective preconditioner for $\mathcal{B}_\rho^k$, even when the complementarity gap ν_k is relatively large.

The preconditioner (48) for (QSDP) is actually not the correct analog of the preconditioner proposed in [8] for LCCQP. The correct version should be the following:

$$\left[\begin{array}{cc} -\mathcal{P}^k(\mathcal{D}^k + \mathrm{diag}(\widetilde{\mathcal{Q}}_\rho^k))(\mathcal{P}^k)^T & \mathcal{A}^T \\ \mathcal{A} & 0 \end{array} \right]. \tag{49}$$

(The above is the same as taking $\mathcal{H}_j^k = \mathcal{D}_j^k + \mathrm{diag}((\widetilde{\mathcal{Q}}_\rho^k)_{jj})$ for $j = 1, 2, 3$ in Ψ_ρ^k.) However, the cost of forming (49) is generally very expensive since at least $\Theta(n^4)$ flops is needed to construct $\mathrm{diag}(\widetilde{\mathcal{Q}}_\rho^k)$ in general. As such, the preconditioner (49) is typically not a practical choice.

5.1 A strategy for constructing symmetrized Kronecker product approximations

The construction of preconditioners in the preceding sections relied to some extent on our ability to find a suitable symmetrized Kronecker product approximation of $\mathcal{Q}_\rho$ or submatrices of $\widetilde{\mathcal{Q}}_\rho^k$. However, very little is known about such an approximation problem. Our aim in this subsection is to provide a strategy for constructing a symmetrized Kronecker product approximation.

Contrary to the symmetrized Kronecker product approximation problem, the problem of finding the nearest (in Frobenius norm) Kronecker product (NKP) approximation of a linear operator defined on $\boldsymbol{R}^{n\times n}$ has been studied in [33, Section 6]. (We refer the reader to the references therein for earlier work on the subject.) It is shown in [33] that the NKP approximation can be derived from the best rank-one approximation of a permuted version of the matrix representation of the linear operator with respect to the canonical basis of $\boldsymbol{R}^{n\times n}$. The special case of finding the NKP approximation of a sum of Kronecker products is studied in [19], wherein the solution is shown to be readily obtainable by solving a small non-convex optimization problem. It is shown recently in [31] that the latter solution can actually be found analytically by solving a small eigenvalue problem.

By applying our knowledge on the NKP problem, a natural route for constructing a suitable symmetrized Kronecker product approximation of a linear operator $\mathcal{T}$ defined on $\mathcal{S}^n$ is as follows. We first extend the linear operator to $\widehat{\mathcal{T}} : R^{n\times n} \to R^{n\times n}$ defined by $\widehat{\mathcal{T}}(U) = \mathcal{T}((U + U^T)/2)$. Suppose $U \otimes V$ is the NKP approximation of $\widehat{\mathcal{T}}$. Then $U \circledast V$ can be used as a symmetrized Kronecker product (SKP) approximation of $\mathcal{T}$. Moreover, the error in the SKP approximation is no larger than that for the NKP approximation in the following sense. Suppose the matrix representations of $\mathcal{T}$ and $\widehat{\mathcal{T}}$ in the canonical orthonormal bases of $\mathcal{S}^n$ and $\boldsymbol{R}^{n\times n}$ are denoted by $\mathrm{mat}(\mathcal{T})$ and $\mathrm{mat}(\widehat{\mathcal{T}})$, respectively. We define $\mathrm{mat}(U \otimes V)$ and $\mathrm{mat}(U \circledast V)$ similarly. Then $\mathrm{mat}(\mathcal{T} - U \circledast V) = \Pi^T \mathrm{mat}(\widehat{\mathcal{T}} - U \otimes V)\,\Pi$, where $\Pi \in \boldsymbol{R}^{n^2\times \bar{n}}$ has orthonormal columns; see [29]. As a result, $\|\mathrm{mat}(\mathcal{T} - U \circledast V)\|_F \le \|\mathrm{mat}(\widehat{\mathcal{T}} - U \otimes V)\|_F$.

Note that in the special case where $\mathcal{Q}$ is a diagonal operator defined by

$$\mathcal{Q}(X) \;=\; U \circ X,$$

where $U \in \mathcal{S}^n \cap \boldsymbol{R}_+^{n\times n}$ is given, a positive semidefinite SKP approximation for $\mathcal{Q}$ can be constructed readily as follows. Consider the best rank-one approximation uu^T of U such that $\|U - uu^T\|_F = \min$, then $\mathrm{diag}(u) \circledast \mathrm{diag}(u)$ is a positive semidefinite SKP approximation of $\mathcal{Q}$. The fact that the preceding SKP is positive semidefinite is a consequence of the Perron-Frobenius Theorem, which ensures that u can be chosen in $\boldsymbol{R}_+^n$ when the entries of U are non-negative; see [16, Theorem 8.3.1].

For the preconditioner Ψ_ρ^k in (40), a procedure for constructing an approximation of the form $\mathcal{H}_1^k = H_1^k \circledast H_1^k$ for $\mathcal{D}_1^k + (\widetilde{\mathcal{Q}}_\rho^k)_{11}$ in (41) is as follows.

Procedure SKPA:

(i) Construct a positive semidefinite SKP approximation $\Delta_\rho \circledast \Delta_\rho$ for $\mathcal{Q}_\rho$. Thus $(\widetilde{\mathcal{Q}}_\rho^k)_{11} \approx (P_1^k)^T \Delta_\rho P_1^k \circledast (P_1^k)^T \Delta_\rho P_1^k$.

(ii) From the sum of positive semidefinite Kronecker products, $D_1^k \otimes D_1^k + (P_1^k)^T \Delta_\rho P_1^k \otimes (P_1^k)^T \Delta_\rho P_1^k$, construct the NKP approximation $H_1^k \otimes H_1^k$ by using the method described in [19]. It is shown in [31] that H_1^k can be chosen to be positive semidefinite.

(iii) Take $\mathcal{H}_1^k = H_1^k \circledast H_1^k$.

The same procedure can similarly be applied to the construction of the approximations: $\mathcal{H}_2^k = \widehat{H}_2^k \otimes H_2^k$ for $\mathcal{D}_2^k + (\widetilde{\mathcal{Q}}_\rho^k)_{22}$, and $\mathcal{H}_3^k = H_3^k \circledast H_3^k$ for $\mathcal{D}_3^k + (\widetilde{\mathcal{Q}}_\rho^k)_{33}$.

6 Numerical experiments

Before we present the details of the numerical experiments, we should emphasize that our purpose here is not conduct extensive numerical testing but to demonstrate that the algorithms we have proposed for solving (QSDP) are correct and they can potentially be efficient and robust. These algorithms may be adopted as prototypes from which more sophisticated and tailor-made algorithms can be designed for solving QSDPs coming from real-world applications such as the nearest correlation matrix problem (3) with real-world data matrix K.

As we mentioned before, we set the parameter ρ in (6) to zero throughout the experiments.

To evaluate the performance of our interior-point algorithms for solving (QSDP) and the effectiveness of the preconditioners constructed in the preceding sections for solving the augmented equation (8), we consider the following classes of test problems:

E1. Quadratic SDPs arising from the nearest correlation matrix problem (3) with $\mathcal{Q}(X) = U \circ X$. Recall that in this case, $\mathcal{A}(X) = \mathrm{diag}(X)$, and b is the vector of all ones. We generate the matrix K in (3) in the same way as Higham did in [15, p.340]. It is generated from the MATLAB function "`[B,E] = gencorrmat(n)`" described in Section 3. We take $K = \mathtt{B} + \mathtt{E}$. Note that we make the perturbation to the random correlation matrix `B` larger than `1e-4*E` considered in [15] so as to generate QSDPs with augmented matrices $\mathcal{B}_\rho^k$ that become increasing ill-conditioned as k increases. As we have already seen from Example 1, if the perturbation is small, then the resulting QSDP has augmented matrices $\mathcal{B}_\rho^k$ that are always well conditioned.

The matrix U is generated randomly as follows: `tmp = rand(n); U = (tmp+tmp')/2`. We generate 5 test problems with $n = 100, 200, 400, 800, 1600$. For these problems, the condition numbers of $\mathcal{Q}$ range from 7.67×10^2 to 4.37×10^3. The norms $\|\mathcal{Q}\|$ are approximately 1 for all problems.

E2. Same as E1 but $K = \mathtt{B} + \mathtt{1e2} * \mathtt{E}$. In this case, `B` has negligible contribution to K. The norms $\|\mathcal{Q}\|$ are approximately 1 for all problems.

E3. Same as E1 but the matrix U is given by `tmp = rand(n); U = 5*(tmp+tmp')`. In this case, the quadratic form $X \bullet \mathcal{Q}(X)$ has more weight in the objective function of (QSDP). The norms $\|\mathcal{Q}\|$ are approximately 10 for all problems.

E4. Same as E1 but $K = \texttt{B} + \texttt{1e2} * \texttt{E}$, and U is given by `tmp = rand(n); U = 5*(tmp+tmp')`. The norms $\|\mathcal{Q}\|$ are approximately 10 for all problems.

E5. Same as E1 but with additional linear constraints added. Let J_n be a randomly chosen subset of $\{(i,j) : 1 \le i < j,\ j = 1,\ldots,n\}$. We set $X_{ij} = 0$ for all $(i,j) \in J_n$. The purpose of this example is to study the effect of having a more complicated linear map $\mathcal{A}(X)$ compared to the diagonal map $\mathrm{diag}(X)$. The condition numbers of $\mathcal{Q}$ range from 6.10×10^2 to 1.16×10^4. The norms $\|\mathcal{Q}\|$ are approximately 1 for all problems.

Again, we generate 5 test problems with $n = 100, 200, 400, 800, 1600$. The number of additional linear constraints added are $120, 225, 397, 772, 1566$, respectively.

E6–E8. Same as E2–E4 but with additional linear constraints similar to those in E5 added.

We use 5 variants of Algorithm IP-QSDP described in [31] to solve each test problem:

A0. Algorithm IP-QSDP with search direction computed from (8) by a direct solver.

A1. Algorithm IP-QSDP with search direction computed from (8) by PSQMR with no preconditioning.

A2. Algorithm IP-QSDP with search direction computed from (8) by PSQMR with the preconditioner at each iteration chosen as follows: it is taken to be $\mathcal{V}_\rho^k$ in (47) if $\kappa(W^k) \le 10^3$; otherwise, it is taken to be $\Phi_{\rho,-}^k$ in (33). We note that even though the asymptotic result for the spectrum of $(\Phi_{\rho,-}^k)^{-1}\mathcal{B}_\rho^k$ is weaker than the corresponding result for $(\Phi_{\rho,+}^k)^{-1}\mathcal{B}_\rho^k$, however, the preconditioner $\Phi_{\rho,-}^k$ is typically more effective than $\Phi_{\rho,+}^k$. As such, we only consider $\Phi_{\rho,-}^k$ in the numerical experiments.

A3. Algorithm IP-QSDP with search direction computed from (8) by PSQMR using the preconditioner $\mathcal{V}_\rho^k$ in (47). Recall that $\mathcal{V}_\rho^k$ is a variant of the preconditioner Ω^k in (38). Even though we do not have any asymptotic result for the spectrum of the preconditioned matrix $(\mathcal{V}_\rho^k)^{-1}\mathcal{B}_\rho^k$. We found that $\mathcal{V}_\rho^k$ is typically a much more effective preconditioner than Ω^k. For this reason, we shall not consider Ω^k in the numerical experiments.

A4. Algorithm IP-QSDP with search direction computed from (8) by PSQMR with the preconditioner at each iteration chosen as follows: it is taken to be $\mathcal{V}_\rho^k$ in (47) if $\kappa(W^k) \le 10^3$; otherwise, it is taken to be Ψ_ρ^k in (40).

When $\Phi_{\rho,-}^k$ is chosen as the preconditioner, the eigenvalues of $(W^k)^{-1}$ is partitioned according to the threshold value of 1. The linear operators $\mathcal{H}_1^k$, $\mathcal{H}_2^k$, $\mathcal{H}_3^k$, and $\widehat{S}_\rho^k$ in $\Phi_{\rho,-}^k$ are chosen as in (37).

When Ψ_ρ^k is chosen as the preconditioner, the eigenvalues of $(W^k)^{-1}$ is also partitioned according to the threshold value of 1. The linear operators $\mathcal{H}_1^k$, $\mathcal{H}_2^k$, and $\mathcal{H}_3^k$ in Ψ_ρ^k are chosen to have the forms in (41) and they are constructed from Procedure SKPA described in the last section with $\Delta_\rho = \sqrt{\|\mathcal{Q}_\rho\|}I_n$. Note that such a choice of Δ_ρ is particularly attractive because the resulting constituent matrices $H_1^k, H_2^k, \widehat{H}_2^k, H_3^k$ in $\mathcal{H}_1^k$, $\mathcal{H}_2^k$, $\mathcal{H}_3^k$ are diagonal matrices.

When $\mathcal{V}_\rho^k$ is chosen as the preconditioner, the operator $\mathcal{Q}(X) = U \circ X$ is approximated by the symmetrized Kronecker product $\Delta \circledast \Delta$, with $\Delta = \mathrm{diag}(u)$, where $u \in \boldsymbol{R}_+^n$ is the vector such that $\|U - uu^T\|_F = \min$.

For the test problems in E1–E8, the main computational cost at each PSQMR step lies in $n \times n$ matrix-matrix multiplications, and they are summarized in Table 3. For these problems, evaluating $\mathcal{A}(X)$, $\mathcal{A}^T y$, $\mathcal{Q}(X)$ cost at most $O(n^2)$ flops, and these terms are ignored.

Table 3: The main computational cost at each PSQMR step for various algorithms when solving the problems in E1–E8.

A1	$2n^3$
A2, A3	$6n^3$
A4	$6n^3$ if the preconditioner is $\mathcal{V}_\rho^k$; $10n^3$ if the preconditioner is Ψ_ρ^k.

We implemented the algorithms in MATLAB (version 7.0) and the experiments were conducted on a Pentium 4 3.0GHz PC with 2GB of RAM. We stopped the algorithms when the accuracy measure ϕ defined by

$$\phi \;=\; \max\left\{ \frac{X \bullet S}{1 + |\mathrm{pobj}| + |\mathrm{dobj}|}\,,\; \frac{\|R_p\|_2}{1 + \|b\|_2}\,,\; \frac{\|R_d\|_F}{1 + \|C\|_F} \right\} \tag{50}$$

is less than 10^{-7}, or when the algorithms did not improve both the duality gap and infeasibilities. In (50), "pobj" and "dobj" denote the primal and dual objective values, respectively.

The stopping criterion used to solve the system (8) is described in (10). We also set the maximum number of PSQMR steps allowed to solve each augmented system to 1000. Moreover, we stopped the interior-point iteration when PSQMR hit the maximum number of steps allowed. The last condition indicated that the linear system was becoming very ill-conditioned and there was little to be gained in continuing the interior-point iteration unless the maximum number of PSQMR steps allowed was increased.

The initial iterate for all the algorithms was taken to be

$$X^0 \;=\; \frac{n}{\sqrt{2}} I, \quad y^0 = 0, \quad S^0 \;=\; \sqrt{n} I.$$

The performance results of our algorithms are given in Table 4. The columns corresponding to "`it`" give the number of interior-point iterations required to solve each problem, whereas the columns "`qmr`" give the average number of PSQMR steps required to solve each of the two linear systems (8) during the computation of the predictor and corrector directions at each interior-point iteration. Note that we did not run Algorithm A1 for some of the larger problems and the corresponding entries in Table 4 are left blank. The entries in bold mean that the algorithms were terminated because the PSQMR solver hit the maximum number of steps allowed.

Table 4 contains a variety of information that we extract and summarize below.

1. The number of interior-point iterations required by our proposed primal-dual interior-point method grows very modestly with the problem dimension n. In all the test problems, the number of iterations required is less than 20.

2. Solving the augmented equation (8) via a direct solver is extremely expensive. For the problems E1-100, ..., E8-100, it is at least 100 times more expensive than using an iterative solver with an appropriate preconditioner, say Ψ_ρ^k.

3. Based on the stopping criterion (10) for the PSQMR method used to solve (8), we can see that Algorithms A2, A3 and A4 took about the same number of interior-point iterations to converge compared to Algorithm A0 that uses a direct method. This indicates that the inexact search directions are computed to sufficient accuracy, and thus the residual errors do not degrade the outer iterations.

4. Algorithm A1, which employs no preconditioner, is not able to solve most of the test problems to the required accuracy in ϕ, defined in (50). All our test problems have ill-conditioned augmented matrices $\mathcal{B}_\rho^k$ when the complementarity gaps $X^k \bullet S^k/n$ are small.

5. Algorithm A3, which employs $\mathcal{V}_\rho^k$ in (47) as the preconditioner, performed fairly well on the test problems in E1, E3, E5, and E7, but less so for the problems in E2, E4, E6 and E8. We should note that the optimal solutions for all the test problems are dual degenerate. The latter violates one of the condition in Theorem 4.3(b), and so there is no theoretical basis to expect $\mathcal{V}_\rho^k$ to be an effective preconditioner for $\mathcal{B}_\rho^k$.

6. The preconditioners $\Phi_{\rho,-}^k$ and Ψ_ρ^k are very effective for $\mathcal{B}_\rho^k$, as attested by the good performance of Algorithms A2 and A4. We note that for the QSDPs arising from (3), the optimal solutions are always primal non-degenerate, and so the good performance of Algorithm A2 that involves $\Phi_{\rho,-}^k$ is consistent with Theorem 4.2(b).

7. Comparing Algorithms A2 and A4, we see that the preconditioner Ψ_ρ^k is more effective (in terms of the number of PSQMR steps) than $\Phi_{\rho,-}^k$ for the test problems in E1, E3, E5, and E7. For the problems E1-1600, E3-1600, E5-1600, and E7-1600, the average numbers of PSQMR steps required to solve (8) for Algorithm A4 is between 37% to 75% of the corresponding numbers for Algorithm A2. However, because evaluating $(\Psi_\rho^k)^{-1}[X;y]$ requires two times the number of $n \times n$ matrix-matrix multiplications for $(\Phi_{\rho,-}^k)^{-1}[X;y]$, the savings in computation time are not as impressive as the reductions in the number of PSQMR steps.

   Notice that $\Phi_{\rho,-}^k$ is a much more effective preconditioner than Ψ_ρ^k for the problems in E2, E4, E6, and E8. For these test problems, the choices $\mathcal{H}_1^k$, $\mathcal{H}_2^k$ and $\mathcal{H}_3^k$ in (37) used for $\Phi_{\rho,-}^k$ is much more effective than the choices in (41) constructed from Procedure SKPA for Ψ_ρ^k. Of course, one can adopt the choices in (37) for Ψ_ρ^k, but as mentioned before, the resulting Schur complement matrix S_Ψ^k is much more costly to compute.

8. Based on the computational costs listed in Table 3 and the average number of PSQMR steps reported in Table 4, we see that for Algorithms A2 and A4, the computational costs required to solve (8) for the problems in E1–E8 at each interior-point iteration are at most $35 \times 6n^3$ and $58 \times 10n^3$ flops, respectively.

These numbers are far smaller than the cost of $n^6/24$ flops needed by a direct solver used to compute the search direction in Algorithm A0.

9. We should emphasize that the dimension of the linear system (8) that is solved at each interior-point iteration is $m + n(n+1)/2$, and the system is **dense**. Thus for $n = 1600$, the dimension is more than 1.28 millions. It is quite remarkable that an iterative solver with appropriate preconditioning can solve such a large and ill-conditioned linear system in less than a hundred iterative steps. Because the preconditioning steps can be evaluated at moderate costs, the savings in the iterative steps enabled the test problems, E1-1600,. . . ,E8-1600, to be solved in a few hours on a desktop computer.

7 Conclusion

We proposed an inexact primal-dual path-following Mehrotra-type predictor-corrector method for solving convex quadratic SDP problems. We suggested computing the search direction at each iteration from the augmented equation by an iterative solver with preconditioners properly designed to overcome the ill-conditioning of the augmented matrix. The preconditioners are shown to produce favorable asymptotic eigenvalue distributions for the associated preconditioned systems to achieve fast convergence under suitable nondegeneracy assumptions on the optimal solution. For one of the classes of preconditioners, no nondegeneracy assumption on the optimal solution is needed.

Numerical experiments on a variety of convex QSDPs with matrices of dimension up to 1600 showed that our inexact interior-point method is quite efficient and robust. For the test problems considered in this paper, our inexact interior-point method can solve each QSDPs at a cost that is at most $\Theta(mn^3) + \Theta(m^2n^2) + \Theta(m^3)$.

However, the computational complexity at each iteration of our interior-point method is inherently higher than first-order methods such as the spectral bundle method of Helmberg and Rendl [14] and the nonlinear-programming based method of Burer, Monteiro, and Zhang [9], so it is of interest to ask whether such methods can be extended to QSDPs. It is also of interest to investigate whether variants of augmented Lagrangian dual methods can be applied to QSDPs and whether such methods are competitive to our inexact interior-point methods. The performance of our inexact interior-point methods for QSDPs may serve as the benchmark for evaluating other (possibly more efficient) algorithms in the future.

While extensive testing on large QSDPs coming from practical problems remain to be done, we have provided the essential computational framework for such exploration to be possible.

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Table 4: Performance of the Algorithms A0–A4 on the problem sets E1–E8.

		A0 (direct)			A1 (I)				A2 ($\mathcal{V}_\rho$, $\Phi_{\rho,-}$)				A3 ($\mathcal{V}_\rho$)				A4 ($\mathcal{V}_\rho$, Ψ_ρ)			
	n	it	ϕ	Time	it	ϕ	Time	qmr	it	ϕ	Time	qmr	it	ϕ	Time	qmr	it	ϕ	Time	qmr
E1	100	12	9.5 -9	46:47	11	**1.9 -6**	0:15	167.3	11	7.3 -8	0:06	22.8	11	7.1 -8	0:07	25.5	12	3.1 -8	0:07	15.6
	200				12	**6.9 -7**	1:21	159.5	12	5.2 -8	0:26	19.9	13	1.0 -8	0:55	40.5	12	9.3 -8	0:22	11.8
	400				13	**2.1 -7**	8:16	166.6	13	4.4 -8	3:11	24.1	13	7.7 -8	2:37	17.7	13	3.0 -8	2:39	13.2
	800				13	**2.0 -7**	36:47	122.0	13	8.4 -8	18:50	21.5	13	6.1 -8	13:55	13.9	13	6.5 -8	13:29	10.0
	1600								14	1.4 -8	2:53:35	28.2	14	9.4 -9	1:59:34	17.0	13	9.5 -8	1:38:05	10.5
E2	100	9	7.0 -8	34:26	7	**2.8 -4**	0:11	206.9	9	5.7 -8	0:03	11.8	9	5.9 -8	0:07	37.4	9	7.5 -8	0:03	9.1
	200				8	**4.2 -4**	1:32	279.2	10	4.6 -8	0:18	15.6	10	4.7 -8	1:07	67.1	10	3.9 -8	0:30	20.9
	400				8	**9.0 -4**	6:38	221.3	11	5.1 -8	1:50	14.9	11	5.2 -8	12:14	122.5	11	5.4 -8	3:38	23.1
	800				8	**5.0 -4**	33:15	185.3	11	9.6 -8	12:39	16.2	11	8.9 -8	1:05:24	103.3	11	9.8 -8	36:36	36.8
	1600								12	6.4 -8	1:44:14	18.0	12	8.9 -8	7:11:33	91.8	12	5.5 -8	5:12:48	42.5
E3	100	9	6.3 -8	34:27	8	**1.1 -5**	0:13	205.8	9	5.7 -8	0:05	24.6	9	6.5 -8	0:10	51.1	10	9.0 -9	0:06	19.6
	200				9	**2.1 -5**	1:29	236.9	10	1.8 -8	0:34	34.3	10	1.4 -8	0:52	50.6	10	1.3 -8	0:31	20.9
	400				9	**9.3 -7**	6:20	185.9	10	4.5 -8	2:38	26.2	10	5.1 -8	2:28	22.9	10	5.7 -8	2:35	18.2
	800				10	**1.0 -6**	40:22	179.4	11	1.6 -8	21:58	32.0	11	1.6 -8	17:38	23.6	11	2.0 -8	21:34	21.0
	1600								11	8.8 -8	2:10:28	26.6	11	6.5 -8	1:32:44	16.8	11	9.8 -8	2:20:41	19.9
E4	100	10	4.2 -8	38:45	7	**4.8 -3**	0:16	309.3	9	8.8 -8	0:04	17.7	9	8.2 -8	0:09	46.6	9	8.6 -8	0:04	12.4
	200				7	**5.0 -3**	1:30	311.5	10	5.6 -8	0:20	18.4	10	5.0 -8	1:19	80.0	10	5.1 -8	0:33	22.3
	400				7	**4.3 -3**	7:21	282.6	10	8.0 -8	1:59	18.7	10	9.3 -8	8:24	91.0	10	6.8 -8	3:02	21.4
	800				7	**2.0 -3**	37:01	239.4	11	3.1 -8	16:10	22.1	10	9.9 -8	56:37	98.1	11	3.0 -8	54:07	57.5
	1600								11	4.0 -8	2:01:02	24.5	11	3.6 -8	9:52:44	140.5	11	3.6 -8	4:18:52	38.1
E5	100	11	3.8 -8	43:00	9	**8.9 -5**	0:14	195.7	11	4.9 -8	0:07	31.4	11	4.8 -8	0:14	61.4	11	4.8 -8	0:07	20.9
	200				10	**5.2 -5**	1:14	175.3	12	4.1 -8	0:35	28.3	12	4.2 -8	0:59	46.6	12	4.4 -8	0:38	21.0
	400				11	**2.1 -5**	7:02	167.2	12	6.5 -8	3:20	27.8	13	1.3 -8	4:46	36.1	12	7.0 -8	2:26	12.8
	800				12	**1.0 -5**	53:55	200.7	13	5.0 -8	25:05	30.5	13	3.6 -8	23:12	26.6	13	3.6 -8	19:52	15.3
	1600								13	6.8 -8	2:52:59	30.6	13	4.8 -8	2:14:04	21.5	13	8.7 -8	2:01:02	13.0
E6	100	10	5.4 -8	39:17	7	**1.1 -3**	0:14	246.9	10	2.8 -8	0:04	17.6	10	3.9 -8	0:13	62.2	10	2.6 -8	0:04	10.9
	200				7	**3.3 -4**	1:02	212.0	10	3.9 -8	0:18	15.3	10	4.7 -8	1:08	66.3	10	5.0 -8	0:30	20.2
	400				8	**1.0 -3**	7:35	253.4	11	5.8 -8	1:57	16.0	11	6.0 -8	9:06	88.4	11	4.3 -8	4:49	31.1
	800				8	**8.3 -4**	35:10	196.3	11	7.6 -8	17:10	23.9	11	7.3 -8	1:20:13	127.0	11	9.2 -8	34:07	33.8
	1600								12	3.0 -8	2:00:09	21.7	12	6.5 -8	7:29:30	95.4	12	6.2 -8	6:23:31	52.4

Table 4: Performance of the Algorithms A0–A4 on the problem sets E1–E8.

		A0 (direct)			A1 (I)				A2 ($\mathcal{V}_\rho$, $\Phi_{\rho,-}$)				A3 ($\mathcal{V}_\rho$)				A4 ($\mathcal{V}_\rho$, Ψ_ρ)			
	n	it	ϕ	Time	it	ϕ	Time	qmr	it	ϕ	Time	qmr	it	ϕ	Time	qmr	it	ϕ	Time	qmr
E7	100	9	4.0 -8	35:06	7	**4.6 -4**	0:18	326.5	9	3.9 -8	0:06	32.2	9	3.9 -8	0:12	61.4	9	3.9 -8	0:07	25.8
	200				7	**2.2 -4**	1:14	253.1	9	8.1 -8	0:29	31.8	9	9.7 -8	0:38	39.8	9	8.2 -8	0:28	20.8
	400				8	**1.5 -4**	7:31	250.7	10	2.5 -8	3:15	33.6	10	2.6 -8	5:21	55.2	10	4.4 -8	2:50	19.4
	800				9	**1.4 -4**	50:54	255.7	10	9.4 -8	21:02	33.9	11	9.6 -9	38:20	57.5	10	9.9 -8	16:50	17.2
	1600								11	3.0 -8	2:45:05	35.4	11	3.5 -8	2:58:11	37.6	11	3.1 -8	2:19:04	19.3
E8	100	10	4.5 -8	39:14	6	**7.3 -3**	0:12	251.1	10	3.3 -8	0:05	22.8	10	3.6 -8	0:16	76.3	10	3.3 -8	0:05	13.0
	200				6	**4.7 -3**	1:00	237.5	10	5.7 -8	0:23	20.8	10	4.8 -8	1:22	81.8	10	5.3 -8	0:33	21.9
	400				7	**4.8 -3**	8:00	307.7	10	7.5 -8	2:14	21.4	10	7.1 -8	7:58	85.0	10	7.4 -8	4:27	31.8
	800				7	**3.0 -3**	40:28	262.3	11	3.0 -8	21:36	31.4	10	9.7 -8	1:08:50	119.5	11	3.4 -8	49:28	52.1
	1600								11	3.6 -8	2:11:56	27.1	11	3.8 -8	8:49:57	124.3	11	3.2 -8	5:07:31	45.7